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CHARACTERIZATIONS OF THE RHR AND MIT ORDERINGS AND THE DRHR AND IMIT CLASSES OF LIFE DISTRIBUTIONS

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Two well-known orders that have been introduced and studied in reliability theory are defined via stochastic comparison of inactivity time: the reversed hazard rate order and the mean inactivity time order. In this article, some characterization results of those orders are given. We prove that, under suitable conditions, the reversed hazard rate order is equivalent to the mean inactivity time order. We also provide new characterizations of the decreasing reversed hazard rate (increasing mean inactivity time) classes based on variability orderings of the inactivity time of k-out-of-n system given that the time of the $(n-k+1)$st failure occurs at or sometimes before time $t \geq 0$. Similar conclusions based on the inactivity time of the component that fails first are presented as well. Finally, some useful inequalities and relations for weighted distributions related to reversed hazard rate (mean inactivity time) functions are obtained.

1. INTRODUCTION AND MOTIVATION

Stochastic comparisons between probability distributions play a fundamental role in probability, statistics, and some related areas, such as reliability theory, survival

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analysis, economics, and actuarial science. As a result, several more refined orders have been comprehensively discussed in the literature. In this section, we present some of these orders (see Müller and Stoyan [26] and Fernandez-Ponce, Kochar, and Munoz-Pérez [16]).

Let X and Y be two random variables having distribution functions F and G, respectively, and denote by $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ their respective survival functions; F^{-1} and G^{-1} are corresponding right continuous inverses. A random variable X is said to be smaller than a random variable Y in the stochastic order (denoted by $X \leq_{\mathrm{ST}} Y$) if and only if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

Recall also that a random variable X is said to be smaller than Y in right spread order (denoted by $X \leq_{\mathrm{RS}} Y$) if and only if

$$\int_{F^{-1}(p)}^{\infty} \bar{F}(x)\,dx \leq \int_{G^{-1}(p)}^{\infty} \bar{G}(x)\,dx \quad \text{for all } p \in (0,1),$$

where $F^{-1}(p) = \inf\{t : F(t) \geq p\}$.

For any random variable X, let

$$X_t = [X - t \mid X > t], \qquad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. An important ordering that relates the residual lifetimes is the hazard rate order: X is said to be smaller than Y in hazard rate order (denoted by $X \leq_{\mathrm{HR}} Y$) if $X_t \leq_{\mathrm{ST}} Y_t$.

Let us observe that when X and Y represent lifetimes of systems or components (i.e., X and Y are nonnegative), the hazard rate order is more informative than either the stochastic order or the right spread order, since it compares the underlying systems at a time t in contrast to the global comparison offered by the order $\leq_{\mathrm{ST}}$ or $\leq_{\mathrm{RS}}$. However, it is reasonable to presume that in many realistic situations, the random variables are not necessarily related only to the future, but they can also refer to the past. In fact, in many reliability problems, it is of interest to consider variables of the kind $X_{(t)} = [t - X \mid X \leq t]$, for fixed $t > 0$, having distribution function $F_{(t)}(s) = P[t - X \leq s \mid X \leq t]$, and known in literature as the inactivity time (see Chandra and Roy [13], Di Crescenzo and Longobardi [15], Li and Lu [23], Nanda, Singh, Misra, and Paul [28], Kayid and Ahmad [19], and Ahmad, Kayid, and Pellerey [1]).

Two well-known orders that have been introduced and studied in reliability theory are defined via stochastic comparison of inactivity time: the reversed hazard rate order and the mean inactivity time order, whose definitions are recalled here.

Definition 1.1: *Let X and Y be two nonnegative random variables.*

(i) X is smaller than Y in the reversed hazard rate order (denoted by $X \leq_{\mathrm{RHR}} Y$) if

$$X_{(t)} \geq_{\mathrm{ST}} Y_{(t)} \quad \textit{for all } t \geq 0,$$

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(ii) X is smaller than Y in the mean inactivity time order (denoted by $X \leq_{\text{MIT}} Y$*) if and only if*

$$E[X_{(t)}] \geq E[Y_{(t)}] \quad \textit{for all } t \geq 0. \tag{1.1}$$

Related to the above two orderings, two classes of life distributions have been introduced and studied in the literature. These are the decreasing reversed hazard rate (DRHR) and increasing mean inactivity time (IMIT) classes of life distributions (cf. Block, Savits, and Singh [11], Nanda et al. [28], Kayid and Ahmad [19], and Ahmad et al. [1]). Their definitions are also recalled here.

Definition 1.2: *A nonnegative random variable X is said to be*

(i) a decreasing reversed hazard rate (denoted by $X \in$ *DRHR) if and only if*

$$X_{(s)} \leq_{\text{ST}} X_{(t)} \quad \textit{for all } 0 < s < t, \tag{1.2}$$

(ii) an increasing mean inactivity time (denoted by $X \in$ *IMIT) if and only if*

$$E[X_{(t)}] \textit{ is increasing in } t \geq 0.$$

Another condition equivalent to (1.2) is (Sengupta and Nanda [34])

$$X \in \text{DRHR} \Leftrightarrow X_{(s)} \leq_{\text{HR}} X_{(t)} \quad \text{for all } 0 < s < t. \tag{1.3}$$

The IMIT class can be characterized by way of the right spread order observing that (Ahmad et al. [1])

$$X \in \text{IMIT} \Leftrightarrow X_{(s)} \leq_{\text{RS}} X_{(t)} \quad \text{for all } 0 < s < t.$$

A wide range of distributions happen to be DRHR. These include two-parameter Weibull, gamma, Makeham, Pareto, log-normal, and linear failure rate distributions. In addition, the linear failure rate and the Makeham distributions are also included in the IMIT class. The following implications among some of the above-mentioned orders and nonparametric classes are well known (see Nanda et al. [28] and Kayid and Ahmad [19]):

$$X \leq_{\text{RHR}} Y \Rightarrow X \leq_{\text{MIT}} Y$$

and

$$\text{DRHR} \Rightarrow \text{IMIT}.$$

The MIT order and the IMIT class were introduced recently by Nanda et al. [28] and Chandra and Roy [13], respectively, whereas Kayid and Ahmad [19] and Ahmad et al. [1] studied some characterizations and preservation results for this order and introduced a life test for the new class. From the definition of mean inactivity time order it could be thought that the results one can obtain on this order follow directly from the results on the mean residual life order (MRL) and the fact that $X \leq_{\text{MIT}} Y$ if and only if $-X \geq_{\text{MRL}} -Y$. However, this is not generally true since this property cannot be useful when one assumes nonnegativity of X and Y, which is assumed for most of the known results on the MRL order. Such equivalence can be applied only in the case that X and Y have a common finitely bounded support $[0, l]$, $l \in R^+$, since, in that case, we can write it as $X \le_{\mathrm{MIT}} Y$ if and only if $l - X \ge_{\mathrm{MRL}} l - Y$. However, we always consider the case of unbounded supports (which are more common in real problems). A similar remark applies to the results on the IMIT property (and DMRL property). In fact, for example, it is well known that a nonnegative random variable X is DMRL if and only if $X \ge_{\mathrm{MRL}} X_t$ for all $t \ge 0$, whereas it is not true that X is IMIT if and only if $X \le_{\mathrm{MIT}} X_{(t)}$ for all $t \ge 0$ (for the definition of the MRL order and DMRL class, we refer to Alzaid [2], among others).

Some authors have made efforts to investigate the characterizations of several stochastic orders as well as several nonparametric life classes. The purpose of the current investigation is to provide some characterization results of the RHR (MIT) orders. In Section 2, some new results related to the RHR order and Laplace transform order are given. In that section, we prove that, under suitable condition, the RHR rate order is equivalent to the MIT order. In Section 3, we provide new characterizations of the DRHR (IMIT) classes based on variability orderings of the inactivity time of a k-out-of-n system given that the time of the $(n-k+1)$st failure occurs at or sometimes before time $t \ge 0$. Similar conclusions based on the inactivity time of the component that fails first are presented as well. Finally, in the rest of that section, we establish some useful inequalities and relations for weighted distributions related to RHR and MIT functions.

Throughout the article, we will use the term *increasing* in place of *nondecreasing*, and *decreasing* in place of *nonincreasing*. Moreover, all integrals and expectations are implicitly assumed to exist wherever they are given.

2. CHARACTERIZATIONS OF RHR AND MIT ORDERS

First, we give a characterization of the RHR order by means of the Laplace transform order. Let X be a nonnegative random variable with distribution function F; the Laplace–Stieltjes transform of $\bar{F}$ is given by

$$L_X(s) = \int_0^\infty e^{-su}\bar{F}_X(u)\,du, \qquad s > 0.$$

Given two random variables X and Y, X is said to be smaller than Y in the Laplace transform order (denoted by $X \le_{\mathrm{Lt}} Y$) if

$$L_X(s) \le L_Y(s) \quad \text{for all } s > 0.$$

Interpretations, properties, and applications of this order in reliability theory and actuarial science can be found in Alzaid, Kim, and Proschan [3], Klefsjo [20], and Denuit [14], among others. To state and prove Theorem 2.1, we need the following result.

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PROPOSITION 2.1: *Let X and Y be two continuous nonnegative random variables with distribution functions F and G, respectively; then for all $t \geq 0$ and $s > 0$,*

$$X_{(t)} \geq_{\mathrm{Lt}} Y_{(t)} \Leftrightarrow \frac{\int_0^t e^{su} F(u)\, du}{\int_0^t e^{su} G(u)\, du} \text{ is decreasing in } t \geq 0.$$

PROOF: First let us observe that, for all $t \geq 0$,

$$L_{X_{(t)}}(s) = \int_0^t \frac{e^{su} F(u)\, du}{e^{st} F(t)} = \frac{\int_0^t e^{su} F(u)\, du}{(\partial/\partial t)\left(\int_0^t e^{su} F(u)\, du\right)}.$$

Therefore, given $s > 0$,

$$\begin{aligned} X_{(t)} \geq_{Lt} Y_{(t)} &\quad \text{for all } t \geq 0 \\ &\Leftrightarrow L_{X_{(t)}}(s) \geq L_{Y_{(t)}}(s) \quad \text{for all } t \geq 0 \\ &\Leftrightarrow \frac{\int_0^t e^{su} F(u)\, du}{(\partial/\partial t)\left(\int_0^t e^{su} F(u)\, du\right)} \geq \frac{\int_0^t e^{su} G(u)\, du}{(\partial/\partial t)\left(\int_0^t e^{su} G(u)\, du\right)} \quad \text{for all } t \geq 0 \\ &\Leftrightarrow \frac{\int_0^t e^{su} F(u)\, du}{\int_0^t e^{su} G(u)\, du} \quad \text{is decreasing in } t \geq 0, \end{aligned}$$

and this completes the proof. ■

THEOREM 2.1: *Let X and Y be two continuous nonnegative random variables. Then for all $t \geq 0$,*

$$X \leq_{\mathrm{RHR}} Y \Leftrightarrow X_{(t)} \geq_{\mathrm{Lt}} Y_{(t)}.$$

PROOF: The implication $X \leq_{\mathrm{RHR}} Y \Rightarrow X_{(t)} \geq_{\mathrm{Lt}} Y_{(t)}$ follows from Theorem 3.1 of Nanda et al. [28] together with Theorem 3.B.6 of Shaked and Shanthikumar [35]. We now need to prove the implication $X_{(t)} \geq_{\mathrm{Lt}} Y_{(t)} \Rightarrow X \leq_{\mathrm{RHR}} Y$. Observing that, for fixed $s \geq 0$ and $t \geq 0$, we have

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$$\int_0^t e^{su} F(u)\,du = \frac{1}{s} e^{st}\left[F(t) - e^{-st}\int_0^t e^{su}\,dF(u)\right],$$

then

$$\frac{\displaystyle\int_0^t e^{su} F(u)\,du}{\displaystyle\int_0^t e^{su} G(u)\,du} = \frac{F(t) - e^{-st}\displaystyle\int_0^t e^{su}\,dF(u)}{G(t) - e^{-st}\displaystyle\int_0^t e^{su}\,dG(u)}. \tag{2.1}$$

Using the dominated convergence theorem together with (2.1) and following the proof of Proposition 2.2 of Belzunce, Gao, Hu, and Pellerey [9], the theorem follows. ■

Let us now denote the expected values of the random variables $X_{(t)}$ and $Y_{(t)}$ by $\mu(t)$ and $\beta(t)$, respectively, where

$$\mu(t) = \int_0^t \frac{F(u)\,du}{F(t)}, \qquad t > 0,$$

and

$$\beta(t) = \int_0^t \frac{G(u)\,du}{G(t)}, \qquad t > 0.$$

The reversed hazard order implies the mean inactivity time order, but the converse is not necessarily true (see Nanda et al. [28]). The next result, however, gives a condition under which $X \leq_{\mathrm{MIT}} Y$ if and only if $X \leq_{\mathrm{RHR}} Y$.

THEOREM 2.2: *Let X and Y be two nonnegative continuous random variables with differentiable MIT functions μ and β, respectively. Suppose that $\mu(t)/\beta(t)$ is increasing in t. Then*

$$X \leq_{\mathrm{MIT}} Y \Leftrightarrow X \leq_{\mathrm{RHR}} Y.$$

PROOF: Let X have a RHR function $r(t) = f(t)/F(t)$ and MIT function $\mu(t)$. Similarly, define $q(t)$ and $\beta(t)$ to be the analogous functions for Y. First, note that the mean inactivity time function is differentiable over $\{t : P(X < t)\}$ and

$$r(t) = \frac{1 - \mu'(t)}{\mu(t)},$$

where μ' denotes the derivative of μ. Similarly,

$$q(t) = \frac{1 - \beta'(t)}{\beta(t)}.$$

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Using the monotonicity property of $\mu(t)/\beta(t)$, together with (1.1), implies that

$$r(t) = \frac{1}{\mu(t)} + \left(\frac{-\mu'(t)}{\mu(t)}\right) \leq \frac{1}{\beta(t)} + \left(\frac{-\beta'(t)}{\beta(t)}\right) = q(t);$$

that is, $X \leq_{\mathrm{RHR}} Y$. ■

In many instances in the analysis of system reliability, for the purpose of comparison of two distribution functions, one is often concerned with properties of a system life distribution, which can be guaranteed from properties of component life distributions; general closure properties of stochastic orders with respect to reliability operations have been studied extensively. In addition to the theoretical interest in the problem, closure properties furnish both rules to choose among various types of replacement and maintenance policies and useful bounds when the exact distribution of the system is not available in closed form. Actually, it is a well-known fact that some stochastic orders are preserved under the formation of a k-out-of-n system of independent and identically distributed (i.i.d.) functions, whereas others are preserved under the formation of simple parallel or series systems. In the following example, we show that the MIT order is not preserved by a k-out-of-n system of i.i.d. functions.

Example 2.1: Let X and Y have distribution functions as follows:

$$F(x) = \exp(-x), \qquad x \geq 0,$$

and

$$G(x) = \exp(-x^2), \qquad x \geq 0.$$

It is easily seen that

$$\frac{1}{F(t)} \int_0^t F(x)\, dx \geq \frac{1}{G(t)} \int_0^t G(x)\, dx \quad \text{for all } x \geq 0,$$

implying that $X \leq_{\mathrm{MIT}} Y$.

For $n = 2$,

$$\int_0^\infty F_{X_{(1:2)}}(u)\, du = \int_0^\infty [F(u)]^2\, du = \frac{1}{2} \tag{2.2}$$

and

$$\int_0^\infty G_{Y_{(1:2)}}(u)\, du = \int_0^\infty [G(u)]^2\, du = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{2.3}$$

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Therefore, from (2.2) and (2.3), it follows that

$$\frac{1}{F_{X_{(1:2)}}(x)}\int_0^x F_{X_{(1:2)}}(u)\,du \le \frac{1}{G_{Y_{(1:2)}}(x)}\int_0^x G_{Y_{(1:2)}}(u)\,du,$$

thus implying that $X_{(1:2)}$ is not less than $Y_{(1:2)}$ in the MIT ordering.

As shown in Example 2.1, in general, the MIT ordering is not preserved by a k-out-of-n system of i.i.d. functions. However, under an additional stronger assumption on the variables involved (i.e., $X \le_{\text{RHR}} Y$), such closure can be satisfied.

3. CHARACTERIZATIONS OF DRHR AND IMIT CLASSES

Many technical systems or subsystems have a k-out-of-n structure. These so-called k-out-of-n systems consist of n components of the same kind. The entire system is working if at least k of its n components are operating. It fails if $n-k+1$ or more components fail. Hence, a k-out-of-n system breaks down at the time of the $(n-k+1)$st component failure. Important particular cases of k-out-of-n systems are parallel and series systems corresponding to $k=1$ and $k=n$, respectively. Practical examples of k-out-of-n systems are an aircraft with four engines that will not crash if at least 2 out of its 4 engines remain functioning or a satellite that will have enough power to send signals if not more than 4 out of its 10 batteries are discharged.

In reliability theory, the lifetime of a k-out-of-n system is usually described by the $(n-k+1)$st-order statistic $X_{n-k+1,n}$ from the sample $X_1,\ldots,X_n$, where the random variable X_i represents the lifetime or failure time of the ith component of the system, $1 \le i \le n$. Many authors have paid attention to behaviors of nonparametric life classes based on the parallel (series) systems and the k-out-of-n systems (see Barlow and Proschan [6], Belzunce, Franco, and Ruiz [8], Li and Zuo [25], Pellerey and Petakos [33], and Li and Chen [22]).

Assume now that the component with lifetime $X_{n-k+1,n}$ fails and the system can be considered as a black box in the sense that the exact failure time of $X_{n-k+1,n}$ in it is unknown. Motivated by this, we assume that at time t the system is not working and in fact it has failed at time t or sometimes before time t. In fact, it is of special interest to estimate the time that has elapsed since the $(n-k+1)$st component failure. This is quite useful for estimation of the survival function for left-censored data. As a result, some authors centered their attention on investigating the behavior of some nonparametric life classes based on the inactivity time of k-out-of-n systems given that the time of the $(n-k+1)$st failure occurs at time $t \ge 0$. These characterizations are meaningful because they help in the understanding of the intrinsic meaning of reliability properties that are involved (see, e.g., Andersen, Borgan, Gill, and Keiding [4], Kalbfleisch and Lawless [18], Xu and Li [36], and Asadi [5]).

The inactivity time of a k-out-of-n system given that the $(n-k+1)$st failure occurs at or some time before time $t \ge 0$ is represented by the following conditional random variable:

$$\begin{aligned}\mathrm{ITS}_{n-k+1,n,(t)} &= (t - X_{n-k+1,n} | X_{n-k+1,n} \leq t)\\ &= \min\{(X_1)_{(t)},\ldots,(X_k)_{(t)}\}\\ &= \min_{1\leq i\leq n-k+1} (X_i)_{(t)},\end{aligned}$$

and the inactivity time of the unit that fails at first is

$$\begin{aligned}\mathrm{ITP}_{1,n,(t)} &= (t - X_{1,n} | X_{n-k+1,n} \leq t),\\ &= \max\{(X_1)_{(t)},\ldots,(X_{n-k+1})_{(t)}\}\\ &= \max_{1\leq i\leq n-k+1} (X_i)_{(t)},\end{aligned}$$

where $(X_1)_{(t)},\ldots,(X_{n-k+1})_{(t)}$ are i.i.d. copies of $X_{(t)}$.

In this way, using stochastic comparisons, we will characterize the DRHR and the IMIT classes of life distributions by the stochastic ordering of the inactivity time of the k-out-of-n system, given that the $(n-k+1)$st failure has occurred at different times. The following results give such characterizations in terms of the inactivity time of the first failure component as well as the inactivity time of k-out-of-n systems.

THEOREM 3.1: *Let X be a continuous nonnegative random variable with finite left end point of the support l_X and $F(l_X) = 0$.*

(a) If X is IMIT, then

$$\mathrm{ITP}_{1,n,(s)} \leq_{\mathrm{RS}} \mathrm{ITP}_{1,n,(t)} \quad \text{for all } s < t \leq l_X$$

(b) If $\mathrm{ITS}_{k,n,(s)} \leq_{\mathrm{RS}} \mathrm{ITS}_{k,n,(t)}$, *for all $s < t \leq l_X$, then X is IMIT.*

PROOF: According to Theorem 3.1 in Ahmad et al. [1], for $i = 1,2,\ldots,n$, X_i is IMIT if and only if

$$(X_i)_{(s)} \leq_{\mathrm{RS}} (X_i)_{(t)} \quad \text{for all } s < t \leq l_X. \tag{3.1}$$

(a) In view of the preservation property of the right spread order under the taking of the maximum of i.i.d. components (Kochar, Li, and Shaked [21, Thm. 5.1(b)]), it follows from (3.1) that

$$\max\{(X_1)_{(s)},\ldots,(X_k)_{(s)}\} \leq_{\mathrm{RS}} \max\{(X_1)_{(t)},\ldots,(X_k)_{(t)}\} \quad \text{for all } s < t \leq l_X.$$

Thus, we arrive at the desired result.

(b) Since the right spread order has the reversed preservation property under the taking of the minimum of i.i.d. components (Li and Yam [24, Thm. 3.3(i)]), we have

$$\min\{(X_1)_{(s)},\ldots,(X_k)_{(s)}\} \leq_{\mathrm{RS}} \min\{(X_1)_{(t)},\ldots,(X_k)_{(t)}\} \quad \text{for all } s < t \leq l_X,$$

which implies (3.1). Hence, X is IMIT. ■

THEOREM 3.2:

(a) X is DRHR if and only if

$$\mathrm{ITP}_{1,n,(s)} \le_{\mathrm{HR}} \mathrm{ITP}_{1,n,(t)} \quad \textit{for all } t > s \ge 0.$$

(b) X is DRHR if and only if

$$\mathrm{ITS}_{n-k+1,n,(s)} \le_{\mathrm{HR}} \mathrm{ITS}_{n-k+1,n,(s)} \quad \textit{for all } t > s \ge 0.$$

PROOF: First, note that the survival function of $\mathrm{ITS}_{n-k+1,n,(t)}$ and the distribution function of the $\mathrm{ITP}_{1,n,(t)}$ can be represented respectively as

$$P(\mathrm{ITS}_{n-k+1,n,(t)} > x) = \left(\frac{F(t-x)}{F(t)}\right)^{n-k+1} = [\bar{F}_{(t)}(x)]^{n-k+1}$$

and

$$P(\mathrm{ITP}_{1,n,(t)} \le x) = \left(1 - \frac{F(t-x)}{F(t)}\right)^{n-k+1} = [F_{(t)}(x)]^{n-k+1}.$$

Since X is DRHR if and only if

$$\left(\frac{F(s-x)}{F(s)}\right)^{n-k+1} \le \left(\frac{F(t-x)}{F(t)}\right)^{n-k+1} \quad \text{for all } t > s \ge 0,$$

$$\mathrm{ITP}_{1,n,(s)} \le_{\mathrm{ST}} \mathrm{ITP}_{1,n,(t)} \quad \text{for all } t > s \ge 0;$$

hence, from (1.2) and (1.3), we obtain part (a). Similarly, part (b) can be established. ■

Suppose now that the system is composed of independent but not identical components. In this case, the survival function of $\mathrm{ITS}_{n-k+1,n,t}$ can be expressed as

$$P(\mathrm{ITS}_{n-k+1,n,(t)} > x) = \prod_{j=1}^{n-k+1} \bar{F}_{i_j,(t)}(x),$$

In the next result, we investigate the inactivity time of the k-out-of-n systems given that the $(n-k+1)$st failure occurs at time $t \ge 0$.

PROPOSITION 3.1: *For any integer $1 \le k < n$, if X_i's, $i = 1,2,\ldots,n$, are all DRHR, then*

$$\mathrm{ITS}_{n-k+1,n,(s)} \le_{\mathrm{ST}} \mathrm{ITS}_{n-k+1,n,(t)} \quad \textit{for all } t > s \ge 0. \tag{3.2}$$

PROOF: From the DRHR property, we have, for all $t > s \ge 0$,

$$X_{i_j,(s)} \le_{\mathrm{ST}} X_{i_j,(t)}, \qquad j = 1,2,\ldots,n-k+1.$$

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Thus, for all $X > 0$ and $t \geq s \geq 0$,

$$\bar{F}_{i_j,(s)}(x) \leq \bar{F}_{i_j,(t)}(x), \qquad j = 1,2,\ldots,n-k+1.$$

Hence,

$$\prod_{j=1}^{n-k+1} \bar{F}_{i_j,(s)}(x) \leq \prod_{j=1}^{n-k+1} \bar{F}_{i_j,(t)}(x).$$

This is just (3.2). ■

In the rest of this section, we establish some useful inequalities and relations for weighted distributions related to RHR and MIT functions. Let X be a nonnegative random variable with distribution function F and probability density function (pdf) f. The weighted distribution of X or the pdf of the weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(X))}, \tag{3.3}$$

where $0 < E(w(x)) < \infty$.

The distribution given in (3.3) arises naturally when one subsamples from the original distribution with large sample size, say N, given a chance proportional to $w(x)$ to observation x. The weighted distribution with weight function $w(x) = x$ is called the length-biased distribution. Statistical applications of weighted distributions, especially to the analysis of data relating to human populations and ecology, can be found in Patil and Rao [31,32]. Pakes, Sapatinas, and Fosam [30] studied relations between length-biased distributions and infinite divisibility. Jain, Singh, and Bagai [17], Nanda and Jain [27], Bartoszewicz and Skolimowska [7], and Belzunce, Navarro, Ruiz, and del Aguila [10] studied relations of weighted distributions with classes of life distributions. Navarro, del Aguila, and Ruiz [29] developed characterizations through reliability measures from weighted distributions. Some known and important distributions in statistics and applied probability may be expressed as weighted distributions (e.g., truncated distributions, the equilibrium renewal distribution, distributions of order statistics, distributions in proportional hazards, and proportional reversed hazards models); for more details, see Broderick and Olusegun [12] and Bartoszewicz and Skolimowska [7].

The weighted distribution function can be expressed as

$$\begin{aligned} F_w(x) &= \int_0^x \frac{w(u)F'(u)\,du}{E(w(X))} \\ &= \frac{1}{E(w(X))}\left\{w(x)F(x) + \int_0^x w'(u)F(u)\,du\right\} \\ &= \frac{F(x)[w(x)+M_F(x)]}{E(w(X))}, \end{aligned}$$

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where

$$M_F(x) = \int_0^x \frac{w'(u)F(u)\,du}{F(x)}.$$

Note that if $w'(x) > 0$, then $M_F(x) \geq 0$ for all $x \geq 0$. The reversed hazard function of the weighted distribution F_w is given by

$$\begin{aligned} r_{F_w}(x) &= \frac{f_w(x)}{F_w(x)} \\ &= \frac{w(x)f(x)}{F(x)[w(x) + M_F(x)]} \\ &= \frac{w(x)r_F(x)}{[w(x) + M_F(x)]}, \end{aligned}$$

where $r_F(x)$ is the reversed hazard rate function of X. Bartoszewicz and Skolimowska [7] obtained the following result.

PROPOSITION 3.2: *Let w be a monotone left continuous function. If $w(x)r_F(x)$ is decreasing, then $F_w(x)$ is DRHR.*

The next result provides inequalities for the lower bound of the inactivity time distributions for large values x from the use of the information about the reversed hazard function of the weighted distribution function.

THEOREM 3.3: *Let F_w be a weighted distribution function with increasing weight function $w(x)$ and pdf $f_w > 0$ for $x \geq x_0$ and let X be the original random variable. If the reversed hazard function is such that $r_{F_w}(x) \geq c/x$ for all $x \geq x_0$, where c is a real positive number, then*

$$P[x - X \leq xt \mid X < t] \leq 1 - (1 - t)^c,$$

for all $t < 0$ and $x \geq x_0$.

PROOF: Let $w(x)$ be increasing in x; then the reversed hazard function of the distribution function F of X satisfies

$$r_F(x) \geq r_{F_w}(x) \geq \frac{c}{x} \qquad \text{for all } x > x_0.$$

Now, for $x \geq x_0$ and $t < 0$,

$$\int_{x(1-t)}^{x} r_F(u)\,du \geq \int_{x(1-t)}^{x} r_{F_w}(u)\,du \geq c\int_{x(1-t)}^{x} \frac{du}{u} \geq 1 - (1 - t)^c.$$

The last inequality follows from algebraic arguments; that is, $\ln a \geq 1 - a^{-1}$ for all $a > 0$. ■

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PROPOSITION 3.3: *Suppose that, for all $t > 0$,*

$$\int_a^b \frac{dF_w(u)}{F_w(u)} \leq \int_{a-t}^{b-t} \frac{dF_w(u)}{F_w(u)};$$

then for all $x \in [a, b)$ such that $x - t > a$,

$$\mu_{F_w}(x - t) \leq \mu_{F_w}(x).$$

PROOF: First, note that the MIT function of the weighted distribution function F_w can be expressed as

$$\begin{aligned} \mu_{F_w}(x) &= \int_0^x \frac{F_w(u)\, du}{F_w(x)} \\ &= \int_0^x \frac{F_w(u)[w(u) + M_F(u)]\, du}{F_w(x)[w(x) + M_F(x)]}. \end{aligned}$$

The result follows from the observation that

$$\begin{aligned} \mu_{F_w}(x) &= \int_a^x \frac{F_w(u)\, du}{F_w(x)} \\ &\geq \int_{a+t}^x \frac{F_w(u)\, du}{F_w(x)} \\ &\geq \int_a^{x-t} \frac{F_w(u)\, du}{F_w(x-t)} \\ &= \mu_{F_w}(x - t), \end{aligned}$$

where the second inequality follows from the assumption and, hence, the proof is completed. ■

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