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Homology TQFT's and the Alexander–Reidemeister Invariant of 3-Manifolds via Hopf Algebras and Skein Theory

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Abstract. We develop an explicit skein-theoretical algorithm to compute the Alexander polynomial of a 3-manifold from a surgery presentation employing the methods used in the construction of quantum invariants of 3-manifolds. As a prerequisite we establish and prove a rather unexpected equivalence between the topological quantum field theory constructed by Frohman and Nicas using the homology of $U(1)$-representation varieties on the one side and the combinatorially constructed Hennings TQFT based on the quasitriangular Hopf algebra $\mathcal{N} = \mathbb{Z}/2 \ltimes \bigwedge^* \mathbb{R}^2$ on the other side. We find that both TQFT's are $SL(2,\mathbb{R})$-equivariant functors and, as such, are isomorphic. The $SL(2,\mathbb{R})$-action in the Hennings construction comes from the natural action on $\mathcal{N}$ and in the case of the Frohman–Nicas theory from the Hard–Lefschetz decomposition of the $U(1)$-moduli spaces given that they are naturally Kähler. The irreducible components of this TQFT, corresponding to simple representations of $SL(2,\mathbb{Z})$ and $Sp(2g,\mathbb{Z})$, thus yield a large family of homological TQFT's by taking sums and products. We give several examples of TQFT's and invariants that appear to fit into this family, such as Milnor and Reidemeister Torsion, Seiberg–Witten theories, Casson type theories for homology circles *à la* Donaldson, higher rank gauge theories following Frohman and Nicas, and the $\mathbb{Z}/p\mathbb{Z}$ reductions of Reshetikhin–Turaev theories over the cyclotomic integers $\mathbb{Z}[\zeta_p]$. We also conjecture that the Hennings TQFT for quantum-$\mathfrak{sl}_2$ is the product of the Reshetikhin–Turaev TQFT and such a homological TQFT.

1 Introduction

In recent years much energy has been put into finding new ways to describe and compute classical invariants of 3-manifolds using the tools and structures developed in the relatively new area of quantum topology. In this paper we will establish another such relation between quantum and classical invariants which emerged in quite different guises in recent research in 3-dimensional topology.

The classical invariant of a 3-manifold M we are interested in here is its Alexander polynomial $\Delta(M) \in \mathbb{Z}[H_1(M)]$. It is closely related and in most cases identical to the Reidemeister Milnor Torsion $r(M)$, see [38] and [45]. More recently, Meng and Taubes [37] show that this invariant is also equal to the Seiberg Witten invariant for 3-manifolds. Turaev [47] proves a refined version of this theorem by comparing the behavior of both invariants under surgery.

On the side of the quantum invariants we consider the formalism used for the Hennings invariant of 3-manifolds [14]. This invariant is motivated by and follows the same principles as the Witten–Reshetikhin–Turaev invariant, which is developed

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in [50], [43] and [48], in the sense that it assigns algebraic data to a surgery presentation for M. The innovation of the Hennings approach is that it starts directly from a possibly non-semisimple Hopf algebra $\mathcal{A}$ rather than its semisimple representation theory. This formalism is refined by Kauffman and Radford in [15]. Also Kuperberg [27] gives a construction that assigns data directly from a Hopf algebra to a Heegaard presentation of M.

In this article we discover and explain in detail the relation between the Hennings theory for a certain 8-dimensional Hopf algebra $\mathcal{N}$ and the (reduced) Alexander polynomial $\Delta_\varphi(M) \in \mathbb{Z}[t, t^{-1}]$ for the cyclic covering given by an epimorphism $\varphi\colon \pi_1(M) \to \mathbb{Z}$. As a consequence, we have at our disposal the entire combinatorial machinery of the Hennings formalism in order to evaluate the Alexander polynomial from surgery diagrams. Particularly, we are able to develop from this an efficient skein-theoretical algorithm. The method of relating these two very differently defined theories is based itself on a quite unexpected equivalence of more refined structures.

More precisely, it turns out that underlying both invariants is the structure of a topological quantum field theory (TQFT). The notion of a TQFT, which can be thought of as a fiber functor on a category of cobordisms, was first cast into a mathematical axiomatic framework by Atiyah [1]. Typically (or by definition) all quantum invariants extend to TQFT's on 3-manifolds with boundaries. In the case of the semisimple theories generalizing the Witten–Reshetikhin–Turaev invariant these TQFT's are described in great detail in [46]. In our context we need the non-semisimple version as it is worked out for the Hennings invariant in [19] and in full generality in [25].

On the side of the classical invariants Frohman and Nicas [8] managed to give an interpretation of the Alexander polynomial of knot complements in the setting of TQFT's. In particular, they construct a TQFT $\mathcal{V}^{\mathrm{FN}}$, which assigns to every surface Σ as a vector space the cohomology ring $H^*\big(J(\Sigma)\big)$ of the $U(1)$-representation variety $J(\Sigma) = \mathrm{Hom}\big(\pi_1(\Sigma), U(1)\big)$. The morphisms are constructed in the style of the Casson invariant from the intersection numbers of representation varieties for a given Heegaard splitting of a cobordism. The Alexander polynomial is thus given as the Lefschetz trace over $\mathcal{V}^{FN}(C_\Sigma)$, where Σ is an arbitrary Seifert surface and C_Σ is the 3-dimensional cobordisms from Σ to itself, obtained by cutting away a neighborhood of Σ.

The unexpected upshot is that this functor $\mathcal{V}^{\mathrm{FN}}$ is isomorphic to the Hennings TQFT $\mathcal{V}_\mathcal{N}$ for the non-semisimple Hopf algebra $\mathcal{N} \cong \mathbb{Z}/2 \ltimes \bigwedge^* \mathbb{R}^2$. The realization of the abelian gauge field theory by a specific Hopf algebra is not at all obvious since $\mathcal{V}^{\mathrm{FN}}$ and $\mathcal{V}_\mathcal{N}$ are defined in entirely different ways. In fact the isomorphism between these functors on the vectors spaces mixes up the degrees of exteriors algebras in still puzzling ways. For these reason the proof is rather explicit and computational.

Nonetheless, it can be seen quite easily that it is not possible to realize $\mathcal{V}^{\mathrm{FN}}$ as a semisimple theory. Particularly, $\mathcal{V}^{\mathrm{FN}}$ represents Dehn twists by matrices of the form $1+N$ where N is nilpotent. Furthermore, the invariant vanishes on $S^1 \times S^2$. Yet, in the semisimple theories from [46], Dehn twists are represented by semisimple matrices D with $D^n = 1$ and the invariant on $S^1 \times S^2$ is never zero.

Once $\mathcal{V}^{\mathrm{FN}}$ and thus the Alexander polynomial Δ_φ are translated into the language of the Hennings formalism for the Hopf algebra $\mathcal{N}$ we are in the position to develop a skein theory for the computation of Δ_φ. The skein identities reflect algebraic relations in $\mathcal{N}$. We derive from this a step by step recipe for the computation of the Alexander polynomial.

Another intriguing feature of the two TQFT's is that both of them admit natural equivariant $\mathrm{SL}(2,\mathbb{R})$-actions that have very different origins but are, nevertheless, intertwined by the isomorphism between them. In the case of $\mathcal{V}^{\mathrm{FN}}$ the $\mathrm{SL}(2,\mathbb{R})$-action on $H^*\big(J(\Sigma)\big)$ is given by the Hard Lefschetz decomposition of the cohomology ring that arises from a Kähler structure on $J(\Sigma)$. For $\mathcal{V}_{\mathcal{N}}$ this action is derived from an $\mathrm{SL}(2,\mathbb{R})$-actions on $\mathcal{N}$ as a Hopf algebra. As a consequence $H^*\big(J(\Sigma)\big)$ carries a non-standard ring-structure induced by that of $\mathcal{N}^{\otimes g}$, which, as opposed to the standard one, is compatible with the Hard Lefschetz $\mathrm{SL}(2,\mathbb{R})$-action.

Let us summarize the content and the main results of this paper in better order and detail. In Section 2 we recall relevant notions that characterize topological quantum field theories, such as (non)semisimplicity. Section 3 reviews the construction of the functor $\mathcal{V}^{\mathrm{FN}}$ of Frohman and Nicas and its values on basic cobordisms. In Section 4 we describe a convenient set of generators of the mapping class groups as combinations of Dehn twists and tangles, and determine their actions on homology. Section 5 introduces the basic rules for the construction of a Hennings TQFT as well as a method that allows us to construct TQFT's even from *non-modular* Hopf algebras or categories. In Section 6 we give the precise definition of $\mathcal{N}$ as a quasi triangular Hopf algebra in the sense of Drinfel'd together with the $\mathrm{SL}(2,\mathbb{R})$-action on it. The vector spaces and the basic morphisms of the associated Hennings TQFT are computed in Section 7 using standard tangle presentations. We prove $\mathrm{SL}(2,\mathbb{R})$-covariance and single out an index 2 subcategory of framed cobordisms that naturally yields a *real* valued TQFT. For later applications we also determine the *categorical* Hopf algebra that is canonically associated to this TQFT. The nilpotent braided structure of $\mathcal{N}$ is then used in Section 8 to develop a skein theory for the evaluation of tangle diagrams. The pivotal equivalence of TQFT's that relates this theory to the Alexander polynomial is given by a natural isomorphism of functors as follows. This is proven in Section 9 by explicit comparison of generating morphism.

Theorem 1 *There is an* $\mathrm{SL}(2,\mathbb{R})$*-equivariant isomorphism*

$$\xi\colon \mathcal{V}_{\mathcal{N}}^{(2)} \xrightarrow{\ \bullet\simeq\ } \mathcal{V}^{\mathrm{FN}},$$

where both TQFT's are "non-semisimple", $\mathbb{Z}/2\mathbb{Z}$-projective functors from the category $\mathcal{C}ob_3^{\bullet}$ of surfaces with one boundary component and relative cobordisms to the category of real $\mathrm{SL}(2,\mathbb{R})$*-modules.*

The Hard Lefschetz $\mathrm{SL}(2,\mathbb{R})$ action on the cohomology of the $U(1)$ moduli spaces and its covariance with $\mathcal{V}^{\mathrm{FN}}$ are described more precisely in Section 10. The fact that ξ is an $\mathrm{SL}(2,\mathbb{R})$-equivariant transformation is proven. Moreover, we describe the canonical decompositions of the TQFT and the Alexander polynomial according to their dual $\mathrm{SL}(2,\mathbb{R})$-representations. The summands are irreducible TQFT's for which the mapping class groups are represented by fundamental weight representations of the symplectic groups $\mathrm{Sp}(2g, \mathbb{Z})$. In Section 11 we use the equivalence from Section 9 and the skein theory for tangles from Section 12 to lay out an explicit algorithm, based on a skein theory that extends the Alexander–Conway calculus, for the computation of $\Delta_{\varphi}(M)$.

Theorem 2 *Let $\mathcal{L}$ be a framed link and $\mathcal{Z} \subset \mathcal{L}$ a distinguished component that has zero framing and algebraic linking number zero with all other components. Let $M_{\mathcal{L}}$ be the 3-manifold obtained by surgery along $\mathcal{L}$ and $\varphi_{\mathcal{Z}} : \pi_1(M) \to \mathbb{Z}$ the linking number with $\mathcal{Z}$.*

Then $\Delta_{\varphi_{\mathcal{Z}}}(M_{\mathcal{L}}) \in \mathbb{Z}[t, t^{-1}]$ can be computed systematically as follows:

- *Use the skein relations from Proposition 15 to unknot the special strand $\mathcal{Z}$.*
- *Put the new configuration into a standard form as depicted in Figure 15, yielding a tangle $\mathcal{T}$.*
- *Use the skein relations from Theorem 7 and framing relations from Figure 13 to decompose $\mathcal{T}^{\#}$ into elementary diagrams as described in in Theorem 8.*
- *Translate the elementary tangle diagrams into Hopf algebra diagrams as in* (95).
- *Go through the steps of Proposition 14 to assign polynomials to each component of a diagram.*
- *Take products over components and sums over elementary diagrams.*

The calculus described here for the evaluation of tangle diagrams is precisely the one used to compute the morphisms for the TQFT functors from Theorem 1 via tangle surgery presentations of cobordisms.

Another application of the equivalence established in Theorem 1 arises from the observation that every TQFT $\mathcal{V}$ on $\mathcal{C}ob_3^{\bullet}$ naturally implies a braided Hopf algebra structure $\mathcal{H}_{\mathcal{V}}$ on $\mathcal{N}_0 := \mathcal{V}(\Sigma_{1,1})$. Now, the cohomology ring $H^*\Big(J\big(\Sigma_g, U(1)\big)\Big) \cong \bigwedge^* H_1(\Sigma_g)$ already has a canonical structure $\mathcal{H}_{\mathrm{ext}}$ of a $\mathbb{Z}/2$-graded Hopf algebra induced by the group structure on $J\big(\Sigma_g, U(1)\big)$. It is easy to see that $\mathcal{H}_{\mathrm{ext}}$ is *not* compatible with the Lefschetz $\mathrm{SL}(2, \mathbb{R})$-action. However, the braided Hopf algebra structure $\mathcal{H}_{\mathcal{V}^{\mathrm{FN}}}$ inherited from the TQFT's in Theorem 1 is naturally $\mathrm{SL}(2, \mathbb{R})$-variant, and, furthermore, equivalent to $\mathcal{H}_{\mathrm{ext}}$:

Theorem 3 *For any choice of an integral Lagrangian decomposition, $H_1(\Sigma_g, \mathbb{Z}) = \Lambda \oplus \Lambda^*$, and volume forms, $\omega_\Lambda \in \bigwedge^g \Lambda$ and $\omega_{\Lambda^*} \in \bigwedge^g \Lambda^*$, the space $H^*\big(J(\Sigma_g)\big)$ admits a canonical structure $\mathcal{H}_\Lambda$ of a $\mathbb{Z}/2$-graded Hopf algebra. It coincides with the braided Hopf algebra structure induced by $\mathcal{V}^{\mathrm{FN}}$ and is isomorphic to the canonical structure $\mathcal{H}_{\mathrm{ext}}$.*

In particular, $\Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_\Lambda\Big)$ is commutative and cocommutative in the graded sense, with unit ω_{Λ^*}, integral ω_Λ, and primitive elements given by $a \wedge \omega_{\Lambda^*}$ and $i_z^* \omega_{\Lambda^*}$ for $a \in H_1(\Sigma)$ and $z \in H^1(\Sigma)$.

The structure $\mathcal{H}_\Lambda$ is, furthermore, compatible with the Hard–Lefschetz $\mathrm{SL}(2, \mathbb{R})$*-action. Specifically, this action is the Howe dual to the action of* $\mathrm{SL}(g, \mathbb{Z})$ *on the Lagrangian subspace in the group of Hopf automorphisms:*

$$\mathrm{SL}(2, \mathbb{R})_{\text{Lefsch.}} \times \mathrm{SL}(\Lambda) \subset \mathrm{GL}(2g, \mathbb{R}) = \mathrm{Aut}\Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_\Lambda\Big).$$

In Section 13 we discuss the appearance of these TQFT's in other contexts. To this end let us denote by $\mathcal{V}^{(j)}$ the irreducible component of $\mathcal{V}^{\mathrm{FN}}$ dual to the j-dimensional $\mathrm{SL}(2,\mathbb{R})$-representation. A detailed description of it is given in Theorem 12. Choose for a closed 3-manifold M with Betti number $\beta_1(M) \geq 1$ a surjection $\varphi\colon H_1(M) \twoheadrightarrow \mathbb{Z}$ (which would be canonical for homology circles as given by 0-surgeries on knots). A series of invariants for the pair (M,φ) can now be constructed by choosing any two-sided, embedded surface $\Sigma \subset M$ that is dual to φ, and considering the cobordism $C_\Sigma\colon \Sigma \to \Sigma$ obtained by removing an open tubular neighborhood of Σ from M. The *j-th (fundamental) Alexander Character* is now defined to be the integer

$$\Delta_\varphi^{(j)}(M) = \mathrm{trace}\left(\mathcal{V}^{(j)}(C_\Sigma)\right), \tag{1}$$

which is easily seen to depend only on φ but not the choice of Σ. Besides the Alexander Polynomial also two other invariants invariant I^{SW} and I^{DC} depending on this data have been constructed by Donaldson in [5] from a Seiberg–Witten Theory and an $\mathrm{SO}(3)$-Casson-type gauge theory respectively. Let us also denote by λ_L the Lescop Invariant [29]. As specified in the next theorem all of these invariants are in fact linear combinations of the (fundamental) Alexander Characters.

Theorem 4 **(Mostly Corollaries to [8], [5], [29], [24])**

$$\Delta_\varphi(M) = \sum_{j\geq 1} [j]_{-t} \cdot \Delta_\varphi^{(j)}(M) \tag{2}$$

$$I_\varphi^{\mathrm{DC}}(M) = \sum_{j\geq 2} \binom{j+1}{3} \cdot \Delta_\varphi^{(j)}(M) \tag{3}$$

$$I_{d,\varphi}^{\mathrm{SW}}(M) = \sum_{j\geq d+2} \left[\!\left[\left(\frac{j-d}{2}\right)^2 \right]\!\right] \cdot \Delta_\varphi^{(j)}(M) \tag{4}$$

$$\lambda_L(M) = \sum_{j\geq 1} (-1)^{j-1} \frac{j(2j^2-3)}{12} \cdot \Delta_\varphi^{(j)}(M). \tag{5}$$

Here we denoted $[j]_q = \frac{q^j - q^{-j}}{q-q^{-1}}$ and by $[\![x]\!]$ the largest integer $\leq x$. We further review in how far the higher $\mathrm{PSU}(n)$ knot invariants $I_{k,n,\varphi}^{\mathrm{FN}}$ of Frohman and Nicas [9] turn out to be polynomial expressions in the Alexander Characters. As products of characters are associated to tensor products of TQFT's and their decompositions into irreducible components, it is natural to consider the corresponding higher, irreducible Alexander Characters $\Delta^{(\gamma)}$. We conjecture that the $I_{k,n,\varphi}^{\mathrm{FN}}$ are linear combinations of the $\Delta^{(\gamma)}$ with coefficients in $\mathbb{N} \cup \{0\}$ as it is the case for I^{DC} and I^{SW}.

Moreover, we explain how the irreducible p-modular reductions $\overline{\overline{\mathcal{V}}}_p^{(j)}$ over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of the $\mathcal{V}^{(j)}$ relate to the irreducible factors of the $\mathbb{Z}[\zeta_p] \to \mathbb{F}_p$ of the Reshetikhin–Turaev TQFT's at a p-th root of unity ζ_p. We finally give evidence that the TQFT from Theorem 1 is essentially the missing tensor factor that relates the semisimple and the non-semisimple TQFT constructions for $U_q(\mathfrak{sl}_2)$ following Reshetikhin–Turaev and Hennings respectively.

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2 Topological Quantum Field Theory

We start with the definition of a TQFT as a functor as proposed by Atiyah [1], largely suppressing a more detailed discussion of the tensor structures.

For every integer, $g \geq 0$, choose a compact, oriented model surface, Σ_g, of genus g, and to a tuple of integers $\underline{g} = (g_1, \ldots, g_n)$ associate the ordered union $\Sigma_{\underline{g}} := \Sigma_{g_1} \sqcup \cdots \sqcup \Sigma_{g_n}$. A *cobordism* is a collection, $\mathbf{M} = (M, \phi_\#, \Sigma_{g_\#})$, of the following:

A compact, oriented 3-manifold, M, whose boundary is divided into two components $\partial M = -\partial_{\text{in}} M \sqcup \partial_{\text{out}} M$, two standard surfaces $\Sigma_{\underline{g}_{\text{in}}}$ and $\Sigma_{\underline{g}_{\text{out}}}$, and two orientation preserving homeomorphisms $\phi_{\text{in}} \colon \Sigma_{\underline{g}_{\text{in}}} \xrightarrow{\sim} \partial_{\text{in}} M$ and $\phi_{\text{out}} \colon \Sigma_{\underline{g}_{\text{out}}} \xrightarrow{\sim} \partial_{\text{out}} M$.

We say two cobordisms, $\mathbf{M}$ and $\mathbf{M}'$, are equivalent if they have the same "in" and "out" standard surfaces, and there is a homeomorphism $h \colon M \xrightarrow{\sim} M'$, such that $h \circ \phi_\# = \phi'_\#$.

Let $\mathcal{C}ob_3$ be the category of cobordisms in dimension 2 + 1, which has the standard surfaces as objects and equivalence classes of cobordisms as morphism. The composition of morphisms is defined via gluing over boundary components using the coordinate maps to the same standard surfaces. In addition, $\mathcal{C}ob_3$ has a tensor product given by disjoint unions of surfaces and cobordisms.

A *Topological Quantum Field Theory* (TQFT) is a functor, $\mathcal{V} \colon \mathcal{C}ob_3 \longrightarrow \text{Vect}(\mathbb{K})$, from the category of cobordisms to the category of vector spaces over a field $\mathbb{K}$.

Let us recall next some generalizations of the definition given in [1] that will be relevant for our purposes. By $\mathcal{C}ob_3^{2\,\text{fr}}$ we denote the category of 2-framed cobordisms, where we fixed some standard framings on the model surfaces Σ_g, see [21]. A *2-framed TQFT* is now a functor $\mathcal{V} \colon \mathcal{C}ob_3^{2\,\text{fr}} \longrightarrow \text{Vect}(\mathbb{K})$. The category of 2-framed cobordisms can be understood as a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}ob_3^{2\,\text{fr}} \longrightarrow \mathcal{C}ob_3 \longrightarrow 1 \tag{6}$$

of the ordinary cobordism category, if restricted to connected cobordisms. Hence, an irreducible 2-framed TQFT yields a *projective TQFT* since $\mathbb{Z}$ is presented as a scalar. See [21] for further descriptions of this extension in terms of signatures of bounding 4-manifolds.

For a group G, we introduce the notion of a *G-equivariant TQFT*. It is a functor, $\mathcal{V} \colon \mathcal{C}ob_3 \longrightarrow G\text{-mod}_{\mathbb{K}}$, from the category of cobordisms to the category of finite dimensional G-modules over a field $\mathbb{K}$. This means that the linear map associated to any cobordism commutes with the action of G on the vector spaces of the respective boundary components.

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Recall also from [20] that a *half-projective* or *non-semisimple* TQFT is one in which functoriality is weakened and replaced by the composition law $\mathcal{V}(MN) = 0^{\mu(M,N)}\mathcal{V}(M)\mathcal{V}(N)$. Here $\mu(M,N) = b(MN) - b(M) - b(N) \in \mathbb{Z}^{+,0}$, where $b(M)$ is the number of components of M minus half the number of components of ∂M. Note that $0^0 = 1$.

We often call a cobordism for which all (rational) homology comes from the homology of the boundary *(rationally) homologically trivial* (r.h.t). More precisely, we mean by this that $i_*\colon H_1(\partial M,\mathbb{Q}) \to H_1(M,\mathbb{Q})$ is onto. Typical examples of r.h.t. cobordisms are the ones in (8) and (9) below and closed rational homology spheres. Examples of cobordisms that are not r.h.t. are any connected sums with closed manifolds M with $\beta_1(M) \geq 1$. We find the following vanishing property:

Lemma 1 **([20])** *If $\mathcal{V}$ is a non-semisimple TQFT, then for any cobordism M,*

$$\textit{if } \mathcal{V}(M) \neq 0 \textit{ then } M \textit{ is r.h.t.}$$

We further introduce $\mathcal{C}ob_3^{\bullet}$, the category of cobordisms for which the surfaces are connected and have exactly one boundary component. As objects we thus use model surfaces $\Sigma_{g,1}$, such that $\Sigma_{g+1,1}$ is obtained from $\Sigma_{g,1}$ by gluing in a torus, $\Sigma_{1,2}$, with two boundary components. Thus, we have a presentation

$$\Sigma_{g,1} = \underbrace{\Sigma_1\#\cdots\#\Sigma_1\#\Sigma_{1,1}}_{g} \quad \text{with inclusions} \quad \Sigma_{g,1} \subset \Sigma_{g+1,1}. \tag{7}$$

Instead of ordinary cobordisms we then consider *relative* ones. We finally introduce categories of cobordisms with combinations of these properties such as $\mathcal{C}ob_3^{2\,\mathrm{fr},\bullet}$, the category of 2-framed, relative cobordisms.

For any homeomorphism, $\psi \in \mathrm{Homeo}^+(\Sigma_g)$, of a surface to itself we define the cobordism

$$\mathbf{I}_\psi = (\Sigma_g \times [0,1], id \sqcup \psi, \Sigma_g \sqcup \Sigma_g). \tag{8}$$

The morphism $[\mathbf{I}_\psi]$ depends only on the isotopy class $\{\psi\}$ of ψ, and the resulting map $\Gamma_g \to \mathrm{Aut}(\Sigma_g)\colon \{\psi\} \mapsto [\mathbf{I}_\psi]$ from the mapping class group to the group of invertible cobordisms on Σ_g is an isomorphism, see [25]. Consequently, every TQFT defines a representation of the mapping class group $\Gamma_g \to \mathrm{GL}\big(\mathcal{V}(\Sigma_g)\big) : \{\psi\} \mapsto \mathcal{V}([\mathbf{I}_\psi])$.

Moreover, let us introduce special cobordisms

$$\mathbf{H}_{\mathbf{g}}^+ := (H_g^+, id \sqcup id, \Sigma_g \sqcup \Sigma_{g+1}), \tag{9}$$

where H_g^+ is obtained by adding a full 1-handle to the cylinder $\Sigma_g \times [0,1]$ at two discs in $\Sigma_g \times 1$. This is done in a way compatible with the choice of the model surfaces in equation (7). Another cobordism H_g^- is built by gluing a 2-handle into the thickened surface $\Sigma_{g+1} \times [0,1]$ along a curve b_{g+1} which lies in the added torus from (7) and has geometric intersection number one with the meridian of the 1-handle added by H_g^+. From this we obtain a cobordism $\mathbf{H}_\mathbf{g}^- = (H_g^-, \Sigma_{g+1} \sqcup \Sigma_g)$ in the opposite direction, with the property that $\mathbf{H}_\mathbf{g}^- \circ \mathbf{H}_\mathbf{g}^+$ is equivalent to the identity.

Basic Morse theory implies a Heegaard decomposition as follows for any cobordism:

$$\mathbf{M} \cong \mathbf{H}^-_{\mathbf{g_2}} \circ \mathbf{H}^-_{\mathbf{g_2+1}} \circ \cdots \circ \mathbf{H}^-_{\mathbf{N-1}} \circ \mathbf{I}_\psi \circ \mathbf{H}^+_{\mathbf{N-1}} \circ \cdots \circ \mathbf{H}^+_{\mathbf{g_1+1}} \circ \mathbf{H}^+_{\mathbf{g_1}}, \tag{10}$$

where $\psi \in \mathrm{Homeo}^+(\Sigma_N)$. Hence, a TQFT is completely determined by the induced representations of the mapping class groups and the maps $\mathcal{V}([\mathbf{H}_g^+])$ and $\mathcal{V}([\mathbf{H}_g^-])$. Therefore, any two TQFT's coinciding on the basic generators from (8) and (9) have to be equal.

3 The Frohman–Nicas TQFT for $U(1)$

Let us review the basic steps in the construction of the topological quantum field theory $\mathcal{V}^{\mathrm{FN}}$ as given in [8] via intersection theory of $U(1)$-representation varieties.

For a compact, connected manifold X its $U(1)$-representation variety is defined as

$$J(X) := \mathrm{Hom}\big(\pi_1(X), U(1)\big) \cong H^1\big(X, U(1)\big). \tag{11}$$

Observe that $J(X)$ is a manifold of dimension $\beta_1(X)$. Specifically, it is a torus if $H_1(X, \mathbb{Z})$ is torsion free, and a discrete group if $\beta_1(X) = 0$.

The vector space associated to a surface $\Sigma_{\underline{g}}$ is given by $\mathcal{V}^{\mathrm{FN}}(\Sigma_{\underline{g}}) = H^*\big(J(\Sigma_{g_1}) \times \cdots \times J(\Sigma_{g_N}), \mathbb{R}\big)$.

We consider first cobordisms, M, between surfaces, $\partial_{\mathrm{in}} M$ and $\partial_{\mathrm{out}} M$, that are rationally homologically trivial in the sense of Section 2. In this case the map $j\colon J(M) \to J(\partial_{\mathrm{in}} M) \times J(\partial_{\mathrm{out}} M)$ is a half dimensional immersion. Thus the top form $\pm[J(M)]$ defines (up to sign) a middle dimensional homology class in $H_*\big(J(\partial_{\mathrm{in}} M), \mathbb{R}\big) \otimes H_*\big(J(\partial_{\mathrm{out}} M), \mathbb{R}\big)$. Using Poincaré Duality and the coordinate maps of the cobordism, the latter space is isomorphic to the space of linear maps from $\mathcal{V}^{\mathrm{FN}}(\Sigma_{\underline{g}_{\mathrm{in}}})$ to $\mathcal{V}^{\mathrm{FN}}(\Sigma_{\underline{g}_{\mathrm{out}}})$. $\mathcal{V}^{\mathrm{FN}}(M)$, for a homologically trivial cobordism M, is now the linear map associated to $j_*\big(\pm[J(M)]\big)$.

In the general case Frohman and Nicas define $\mathcal{V}^{\mathrm{FN}}(M)$ via a Heegaard splitting of M as in (10), and consider the intersection number of representation varieties of the elementary thick surfaces with handles separated by the Heegaard surface. In the case where $H_1(\partial M, \mathbb{R}) \to H_1(M, \mathbb{R})$ is not onto, *i.e.*, M is not homologically trivial, these varieties no longer transversely intersect so that $\mathcal{V}^{\mathrm{FN}}(M) = 0$.

Regarding the composition structure, $\mathcal{V}^{\mathrm{FN}}$ has a couple of nonstandard properties. For one, functoriality fails to hold when M and N are homologically trivial but $M \circ N$ is not. Moreover, the orientation of the classes $\pm[J(M)]$ and cycles cannot be chosen consistently with composition so that a sign-projectivity persists. Recall, however, that a 2-framed TQFT is really defined on the $\mathbb{Z}$-extensions of cobordisms given in (6).

Lemma 2 *$\mathcal{V}^{\mathrm{FN}}$ is a non-semisimple, $\mathbb{Z}/2\mathbb{Z}$-projective TQFT in the sense of Section 2.*

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The mechanism by which the universal $\mathbb{Z}$-extension is factored into a $\mathbb{Z}/2\mathbb{Z}$-extension is explained further for the quantum theory in Lemma 10 and Proposition 6 of Section 7. At least indirectly, we have thus related the orientation ambiguities in [8] to the usual framing ambiguities of quantum theories.

Now, in the $U(1)$ case, $J(X)$ has a group structure itself, which induces a coalgebra structure on the cohomology ring so that $H^*\big(J(X)\big)$ is endowed with a canonical Hopf algebra structure $\mathcal{H}_{\mathrm{ext}}$. If $H_1(X)$ is torsion free then $H^*\big(J(X)\big)$ is connected and we obtain a natural isomorphism $H^*\big(J(X)\big) \cong \bigwedge^* H_1(X)$ of $\mathbb{Z}/2$-graded Hopf algebras, and $H_1(X)$ is the space of primitive elements. Hence, we can write for the vector spaces:

$$\mathcal{V}^{\mathrm{FN}}(\Sigma_{\underline{g}}) = \bigwedge^* H_1(\Sigma_{\underline{g}}). \tag{12}$$

The representation of the mapping class group $\Gamma_{\underline{g}}$ on this space is given by the obvious action

$$\mathcal{V}^{\mathrm{FN}}([\mathbf{I}_\psi]) = \bigwedge^* [\psi] \quad \forall\{\psi\} \in \Gamma_g. \tag{13}$$

Here, $[\psi] \in \mathrm{Sp}\big(H_1(\Sigma_{\underline{g}})\big)$ is the natural, induced action on homology. For a connected surface Σ_g we have the associated short exact sequence

$$1 \to \mathcal{J}_g \longrightarrow \Gamma_g \xrightarrow{\psi\mapsto[\psi]} \mathrm{Sp}(2g,\mathbb{Z}) \to 1, \tag{14}$$

where $\mathcal{J}_g$ is the Torelli group.

Let $\mathbf{H}_\mathbf{g}^+$ be the cobordism as defined in (9), and let $[a_{g+1}]$ be a generator of $\ker\big(H_1(\Sigma_{g+1},\mathbb{Z}) \to H_1(H_{g+},\mathbb{Z})\big)$ seen as an element of $H_1(\Sigma_{g+1},\mathbb{R})$. It is represented by the meridian a_{g+1} of the added handle. In a slight variation of the Frohman–Nicas formalism we see that the associated linear map is given as

$$\mathcal{V}^{\mathrm{FN}}(\mathbf{H}_\mathbf{g}^+)\colon \bigwedge^* H_1(\Sigma_g) \longrightarrow \bigwedge^* H_1(\Sigma_{g+1}) : \alpha \mapsto i_*(\alpha) \wedge [a_{g+1}]. \tag{15}$$

Here we use the fact that $H_1(\Sigma_{g,1}) = H_1(\Sigma_g)$ so that the inclusion of surfaces in (7) implies also an inclusion $i_*\colon H_1(\Sigma_g) \subset H_1(\Sigma_{g+1})$.

Let $\mathbf{H}_\mathbf{g}^-$ be the cobordism obtained by gluing a 2-handle along b_{g+1} as defined above. We note that $H_1(\Sigma_{g+1}) = H_1(\Sigma_g) \oplus \langle [a_{g+1}], [b_{g+1}]\rangle$, so that $\bigwedge^* H_1(\Sigma_{g+1})$ is the direct sum of spaces $V_1 \oplus V_a \oplus V_b \oplus V_{a\wedge b}$ where $V_x = [x_{g+1}] \wedge \bigwedge^* H_1(\Sigma_g)$. The linear map associated in [8] to $\mathbf{H}_\mathbf{g}^-$ acts on V_a as

$$\mathcal{V}^{\mathrm{FN}}(\mathbf{H}_\mathbf{g}^-)\colon V_a \longrightarrow \bigwedge^* H_1(\Sigma_g) : i_*(\alpha) \wedge [a_{g+1}] \mapsto \alpha \tag{16}$$

and is zero on all other summands.

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4 The Mapping Class Groups and their Actions on Homology

The mapping class group $\Gamma_{g,1} = \pi_0\big(\mathrm{Homeo}^+(\Sigma_{g,1})\big)$ on a model surface $\Sigma_{g,1}$ is generated by the right handed Dehn twists along oriented curves a_j, b_j, and c_j, as depicted in Figure 1. We denote them by capital letters $A_j, B_j, C_j \in \Gamma_{g,1}$ respectively. In fact we only need the A_2 of the A_j's to generate $\Gamma_{g,1}$. A presentation of $\Gamma_{g,1}$ in these generators is given by Wajnryb [49]. For our purposes we prefer the set $\{A_j, D_j, S_j\}$ of generators defined as follows:

(17) $$D_j := A_j^{-1}A_{j+1}^{-1}C_j \quad \text{and} \quad S_j := A_jB_jA_j \quad \text{for } j = 1, \dots, g.$$

In [36] a tangle presentation of $\Gamma_{g,1}$ is given using the results in [49]. The same presentation results from the tangle presentation of $\mathcal{C}ob_3^{2\,\mathrm{fr},\bullet}$ in [21, Proposition 14], which extends to the central extension $1 \to \mathbb{Z} \to \Gamma_{g,1}^{2\,\mathrm{fr}} \to \Gamma_{g,1} \to 1$ that stems from the 2-framing of cobordisms. The framed tangles associated to our preferred generators are given in Figures 2, 3, and 4. We use an empty circle to indicate a right handed 2π-twist on the framing of a strand as in Figure 2, and a full circle for a left handed one as in Figure 5. Note that the extra 1-framed circle in Figure 4 does not change the 3-cobordism in $\mathcal{C}ob_3^\bullet$ but shifts its 2-framing in $\mathcal{C}ob_3^{2\,\mathrm{fr},\bullet}$ by one.

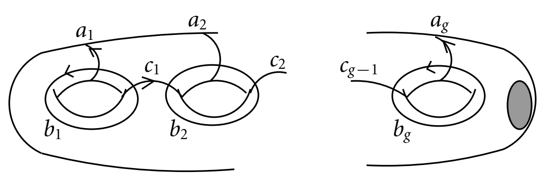

Figure 1: Curves on $\Sigma_{g,1}$.

$\Gamma_{g,1}^{2\,\mathrm{fr}}$ can then be thought of as the sub-group of tangles generated by these diagrams, modulo isotopies, 2-handle slides, the σ-move and the Hopf link move; see [21].

For later purposes we give the explicit action of these generators on $H_1(\Sigma_g, \mathbb{Z}) = H_1(\Sigma_{g,1}, \mathbb{Z})$ in the sense of (14). Suppose $p, f \subset \Sigma_{g,1}$ are two transverse, oriented curves. We denote by P the Dehn twist along p, by $[P] \in \mathrm{Sp}(2g, \mathbb{Z})$ its action on homology, and by $[p]$ and $[f]$ the respective homology classes. We have

(18) $$[P].[f] = [f] + ([p] \cdot [f])[p].$$

Here $([p] \cdot [f]) \in \mathbb{Z}$ is the algebraic intersection number of p with f, counting +1 for a crossing if the tangent vectors of p, f form an oriented basis and -1 if the basis has opposite orientation.

A basis for $H_1(\Sigma_g)$ is given by $\{[a_1], \dots, [a_g], [b_1], \dots, [b_g]\}$, and intersection numbers can be read off Figure 1. For example a_j intersects b_j in only one point, where $[a_j] \cdot [b_j] = +1$ since b_j follows a_j counter clockwise at the crossing. Hence

(19) $$[A_j].[b_j] = [b_j] + [a_j] \quad \text{and} \quad [A_j].[x] = [x] \quad \text{for all other basis vectors.}$$

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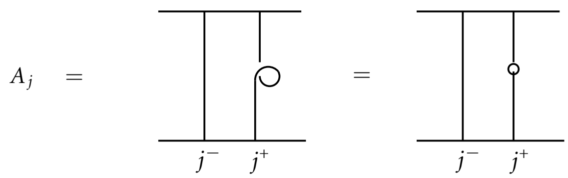


Figure 2: Tangle for A_j.

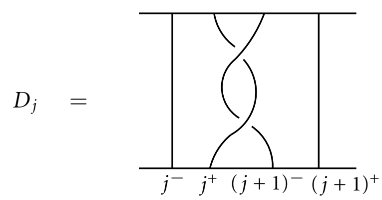


Figure 3: Tangle for D_j.

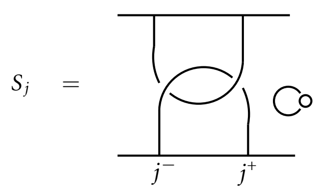


Figure 4: Tangle for S_j.

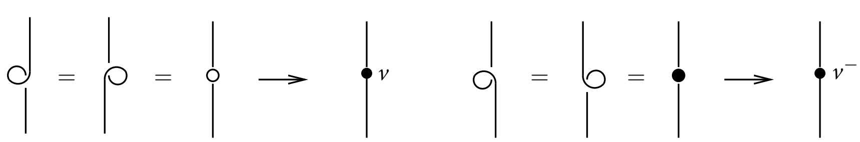


Figure 5: Twist assignments.

Similarly, we have that $[C_j]$ only acts on $[b_j]$ and $[b_{j+1}]$ with $[C_j].[b_j] = [b_j] + [c_j]$ and $[C_j].[b_{j+1}] = [b_{j+1}] - [c_j]$. Substituting $[c_j] = [a_j] - [a_{j+1}]$, and using the definition of D_j in (17) and (19) we compute

$$[D_j].[b_j] = [b_j] - [a_{j+1}] \quad \text{and} \quad [D_j].[b_{j+1}] = [b_{j+1}] - [a_j], \tag{20}$$

and, again, $[D_j].[x] = [x]$ for all other basis vectors $[x]$ of $H_1(\Sigma_1, \mathbb{Z})$. Finally, we find $[B_j].[a_j] = [a_j] - [b_j]$ so that

$$[S_j].[a_j] = -[b_j] \quad \text{and} \quad [S_j].[b_j] = [a_j] \tag{21}$$

and $[S_j].[x] = [x]$ otherwise.

The above action can be identified with specific generators of the Lie algebra $\mathfrak{sp}(2g, \mathbb{R})$ as follows:

$$\begin{aligned} [A_j] &= I_{2g} + E_{j,-j} = I_{2g} + e_{2\epsilon_j} = \exp(e_{2\epsilon_j}) \\ [B_j] &= I_{2g} - E_{-j,j} = I_{2g} - f_{2\epsilon_j} = \exp(-f_{2\epsilon_j}) \\ [D_j] &= I_{2g} - E_{j,-(j+1)} - E_{j+1,-j} = I_{2g} - e_{\epsilon_j+\epsilon_{j+1}} = \exp(-e_{\epsilon_j+\epsilon_{j+1}}). \end{aligned} \tag{22}$$

The conventions and notations for the weights ϵ_j and the matrices $E_{i,j}$ are taken from [12, Chapter 2.3]. Hence, the natural representation on $\mathrm{Sp}(2g, \mathbb{Z})$ clearly lifts to the fundamental representation of $\mathrm{Sp}(2g, \mathbb{R})$.

Finally, there is an $\mathrm{Sp}(2g, \mathbb{Z})$-invariant 2-form, which is unique up to signs and given in our basis as:

$$\omega_g := \sum_{j=1}^{g} [a_j] \wedge [b_j] \in \bigwedge^2 H_1(\Sigma_g) = H^2\big(J(\Sigma_g)\big). \tag{23}$$

It is identical to twice the Kähler metric form in $H^2\big(J(\Sigma_g)\big)$, see Section 10 and [13].

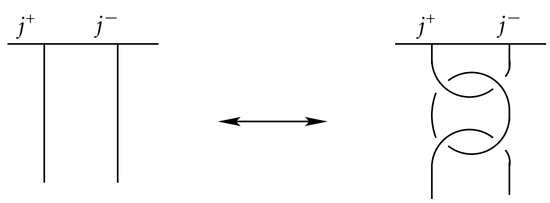
5 Hennings TQFT's

In [14] Hennings describes a calculus that allows us to compute an invariant, $\mathcal{V}^H_{\mathcal{A}}(M)$, for a closed 3-manifold, M, starting from a surgery presentation, $M = S^3_{\mathcal{L}}$, by a framed link, $\mathcal{L} \subset S^3$, and a quasitriangular Hopf algebra $\mathcal{A}$. It is obtained by inserting and moving elements of $\mathcal{A}$ along the strands of a projection of $\mathcal{L}$ and evaluating them against integrals. This procedure was refined by Kauffman and Radford [15] permitting unoriented links and simplifying the evaluation and proofs substantially. $\mathcal{V}^H_{\mathcal{A}}$ turns out to be a special case of the invariant given by Lyubashenko [31], which is constructed from general abelian categories. In [19, Theorem 14] we generalize the Hennings procedure to tangles and cobordisms and thus construct a topological quantum field theory $\mathcal{V}^H_{\mathcal{A}}$ for any modular Hopf algebra $\mathcal{A}$. In turn $\mathcal{V}^H_{\mathcal{A}}$ is derived as a special case of the general TQFT construction by Lyubashenko and the author in [25].

The TQFT in [19] was formulated as a contravariant functor, $\mathcal{V}^*_{\mathcal{A}}\colon \mathcal{C}ob_3^{\bullet} \to \mathrm{Vect}(\mathbb{K})$, where $\mathcal{V}^*_{\mathcal{A}}(\Sigma_{g,1}) = \mathcal{A}^{\otimes g}$. In this section we will give the rules for construction for the covariant version, defined by $\mathcal{V}_{\mathcal{A}}(M) = (f^{\otimes g})^{-1}\big(\mathcal{V}^*_{\mathcal{A}}(M)\big)^* f^{\otimes g}$, where $f\colon \mathcal{A} \to \mathcal{A}^*\colon x \mapsto \mu\big(S(x)\cdot \_\big)$. We generalize [19] further by allowing Hopf algebras, $\mathcal{A}$, that are not modular, at the expense of reducing the vector space by a canonical projection.

Let M be a 2-framed cobordism between two model surfaces, Σ_{g_1} and Σ_{g_2}. As in [21] we associate to the homeomorphism class of M an equivalence class of framed tangle diagrams. The projection of a representative tangle, T_M, in $\mathbb{R}\times[0,1]$ has $2g_1$ endpoints $1^- < 1^+ < 2^- < \cdots < g_1^- < g_1^+$ in the top line $\mathbb{R}\times 1$ and $2g_2$ endpoints $1^- < 1^+ < 2^- < \cdots < g_2^- < g_2^+$ in the bottom line $\mathbb{R}\times 0$. Besides closed components ($\cong S^1$) the tangle can have components with boundary ($\cong [0,1]$). An interval component, J, of the tangle can either run between points j^- and j^+ at the top line or between j^- and j^+ at the bottom line. As a forth possibility we admit pairs of components, I and J, of which each starts at the top line and ends at the bottom line and cobords a pair $\{j^-, j^+\}$ to a pair $\{k^-, k^+\}$. The equivalences of tangles are generated by isotopies, 2-handle slides (second Kirby move) over closed components, the addition and removal of an isolated Hopf link in which one component has 0-framing, and additional boundary moves, called σ- and τ-Moves, see [21]. For later purposes we also depict here the σ-Move:

(24)



The next ingredient is a unimodular, ribbon Hopf algebra, $\mathcal{A}$, in the sense of [42], over a perfect field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. In particular, $\mathcal{A}$ is a *quasitriangular* Hopf algebra as introduced by Drinfel'd [6]. This means there exists an element $\mathcal{R} = \sum_j e_j \otimes f_j \in \mathcal{A}^{\otimes 2}$, called the *R-matrix*, which fulfills several natural conditions. As in [6] we define the element $u = \sum_j S(f_j)e_j$, which implements the square of the antipode S by $S^2(x) = uxu^{-1}$. A *ribbon* Hopf algebra is now a quasitriangular Hopf algebra with a group-like element, G, such that G also implements S^2 and $G^2 = uS(u)^{-1}$. From this we define the ribbon element $v := u^{-1}G$, which is central in $\mathcal{A}$. Furthermore, it satisfies the equation

(25) $$\mathcal{M} = \mathcal{R}^{\dagger}\mathcal{R} = \Delta(v^{-1})v \otimes v,$$

where $(a\otimes b)^{\dagger} = b \otimes a$ is the transposition of tensor factors.

Now, any finite dimensional Hopf algebra contains a *right integral*, which is an element $\mu \in \mathcal{A}^*$ characterized by the equation:

(26) $$(\mu \otimes id_{\mathcal{A}})\big(\Delta(x)\big) = 1\cdot\mu(x).$$

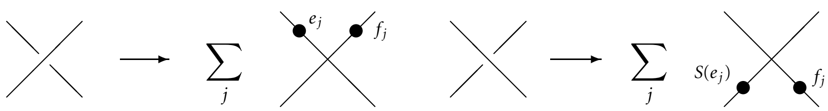
Its existence and uniqueness (up to scalar multiplication) has been proven in [28]. The adjective "unimodular" implies that

$$\mu(xy) = \mu\big(S^2(y)x\big) \quad \text{and} \quad \mu\big(S(x)\big) = \mu(G^2x), \tag{27}$$

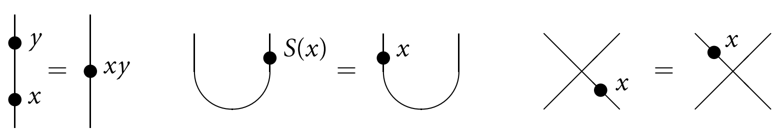
see [42]. For the remainder of this article we will also assume the following normalizations:

$$\mu \otimes \mu(\mathcal{M}) = 1 \quad \text{and} \quad \mu(v)\mu(v^{-1}) = 1. \tag{28}$$

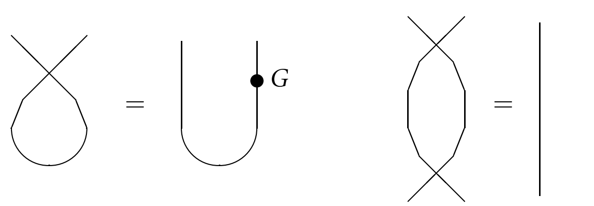
The next step in the Hennings procedure is to replace the tangle projection T_M with distinguished over and under crossings by a formal linear combination of copies of the projection T_M in which we do not distinguish between over and under crossings but decorate segments of the resulting planar curve with elements of $\mathcal{A}$. Specifically, we replace an over crossing by an indefinite crossing and insert at the two incoming pieces the elements occurring in the R-matrix, and similarly for an under crossing, as indicated in the following diagrams.

(29) 

The elements on the segments of the planar diagram can then be moved along the connected components according to the following rules.

(30) 

Finally, every diagram can be untangled using the local moves given below, and the usual planar third Reidemeister move. In particular, undoing a closed curve in the diagram yields an extra overall factor G^d, where G is the group-like element defined above and d the Whitney number of the curve.

(31) 

The assignments that result from this for the left and right ribbon 2π-twists are summarized in Figure 5. Note that in the assignment on the right hand side the full circle on the left side stands for a left handed twist for the framing, while the fat dot on the right hand side indicates a decoration of the strand by the element v^{-1}.

It is clear that after application of these types of manipulations to any decorated diagram we eventually obtain a set of disjoint, planar curves which can be one of four types. For each of these types we describe next the evaluation rule that leads to the definition of a linear map $\mathcal{V}^{\#}(T_M)$:

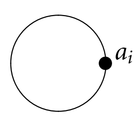
Components of the first type are closed circles decorated with one element $a_i \in \mathcal{A}$ on the right side. To this we associate the number $\mu(a_i) \in \mathbb{K}$.

Next, we may have an arc at the bottom line of the diagram connecting points p'_k and q'_k with one decoration $b_k \in \mathcal{A}$ at the left strand. To this we associate the vector $b_k \in \mathcal{A}^{(k)}$ in the k-th copy of the tensor product $\mathcal{A}^{\otimes g_2}$.

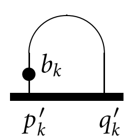
Thirdly, for an arc at the top line between points p_j and q_j with decoration $c_j \in \mathcal{A}$ on the right we assign the linear form $l_{c_j} \colon \mathcal{A}^{(j)} \to \mathbb{K}$ given by $l_{c_j}(x) = \mu\big(S(x)c_j\big)$ on the j-th copy of the tensor product $\mathcal{A}^{\otimes g_1}$.

Finally, we may have pairs of straight strands that connect a pair $\{p_j, q_j\}$ to the pair $\{p'_k, q'_k\}$, carrying decorations, a and b. In case the strands are parallel, that is, one connects p_j to p'_k and the other q_j to q'_k, we assign a linear map $T_{a,b} \colon \mathcal{A}^{(j)} \to \mathcal{A}^{(k)}$ between the j-th copy of $\mathcal{A}^{\otimes g_1}$ to the k-th copy of $\mathcal{A}^{\otimes g_2}$, by $T_{a,b}(x) = axS(b)$.

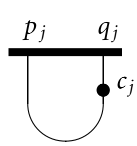
If the connecting strands cross over we apply in addition the endomorphism $K(x) = G^{-1}S(x)$ on the k-th copy $\mathcal{A}^{(k)}$ for a crossing right at the bottom line. It is quite useful to summarize these rules also pictorially as follows:

(32) 

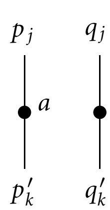
$\longrightarrow$ $\mu(a_i)$

(33) 

$\longrightarrow$ $b \colon \mathbb{K} \longrightarrow \mathcal{A}^{(k)} : x \mapsto b_k$

(34) 

$\longrightarrow$ $l_{c_j} \colon \mathcal{A}^{(j)} \longrightarrow \mathbb{K} : x \mapsto \mu\big(S(x)c_j\big)$

(35) 

$\longrightarrow$ $T_{a,b} \colon \mathcal{A}^{(j)} \longrightarrow \mathcal{A}^{(k)} : x \mapsto axS(b)$

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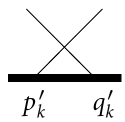

$$\longrightarrow \qquad K\colon \mathcal{A}^{(k)} \longrightarrow \mathcal{A}^{(k)} : x \mapsto G^{-1}S(x)$$

From these rules for evaluating diagrams we obtain a linear map $\mathcal{A}^{\otimes g_1} \to \mathcal{A}^{\otimes g_2}$ for any decorated planar tangle. For a given tangle T_M we denote by $\mathcal{V}^{\#}(T_M)$ the sum of all of these maps associated to the sum of decorated diagrams for T_M. Thus, if we consider, for simplicity, a tangle T_M without components of the fourth type, and denote by a_i^ν, b_j^ν and c_k^ν the respective elements of the ν-th summand of the same untangled curve of T_M, this linear map can be expressed as

$$\mathcal{V}^{\#}(T_M) := \sum_\nu \mu(a_1^\nu)\cdots\mu(a_N^\nu) b_1^\nu \otimes \cdots \otimes b_{g_2}^\nu l_{a_1^\nu} \otimes \cdots \otimes l_{a_{g_1}^\nu}.$$

For tangles with strand pairs that connect top and bottom pairs we insert the operators $T_{a,b}$ in the respective positions.

Lemma 3 *The linear maps $\mathcal{V}^{\#}(T_M)$ are well defined, (covariantly) functorial under the composition of tangles, and they commute with the adjoint action of $\mathcal{A}$ on $\mathcal{A}^{\otimes g}$. They are also invariant under isotopies and the following moves:*

(1) *2-handle slides of any type of strand over a closed component of T_M;*
(2) *adding/removing an isolated Hopf link for which one component has 0-framing and the other framing 0 or 1.*

Proof The fact that the construction procedure for a given diagram is unambiguous is almost straight forward, except that one has to pay attention to the positioning of the resulting elements. Details for closed links can be found in [16]. Functoriality is easily checked from the rules of construction. The fact that the maps are $\mathcal{A}$-equivariant follows from the fact that it is a special case of the categorical construction in [25] and the fact that $f\colon \mathcal{A} \to \mathcal{A}^*$ intertwines the adjoint with the coadjoint action. Invariance under isotopies follows, as in [14] or [15], from the properties of the R-matrix of a quasitriangular Hopf algebra. In the same articles the 2-handle slide is directly related to the defining equation (26) of the right integral, see also [31] for the categorical version of the argument. Invariance under the Hopf link moves is a direct consequence of the normalizations in (28), since they imply that the Hennings invariants on the Hopf links are all one. ∎

In order to describe the reduction procedure that allows us to define a TQFT also for non-modular Hopf algebras we introduce the operators associated to the diagrams in Figure 6, the left being isotopic to the one in Figure 4. The double crossing is replaced by the elements m_j^+, n_j^+ from $\mathcal{M} = \sum_j m_j^+ \otimes n_j^+$, as defined in (25). The transformation $S^+\colon \mathcal{A} \to \mathcal{A}$ is readily worked out to be

$$S^+(x) = \sum_j \mu\big(S(x)m_j^+\big)\, n_j^+. \tag{36}$$

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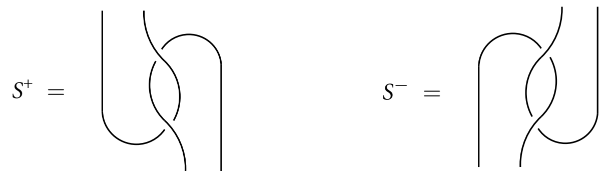

Figure 6: $S^{\pm}$-transformations.

The formula for S^- follows analogously, substituting $\mathcal{M}$ for $\mathcal{M}^{-1} = \sum_j m_j^- \otimes n_j^-$. We consider next the result Π of stacking the two tangles in Figure 6 on top of each other:

Lemma 4 *Let* $\Pi := S^+ \circ S^- = S^- \circ S^+$, *and denote* $\Pi^{(j)} = 1 \otimes \cdots \otimes 1 \otimes \Pi \otimes 1 \otimes \cdots \otimes 1$, *with* Π *occurring in the* j*-th tensor position.*

(1) Π *is an idempotent that commutes with the adjoint action of* $\mathcal{A}$.
(2) $\mathcal{V}^\#(T_M)\Pi^{(j)} = \mathcal{V}^\#(T_M)$ *if the* j*-th top index pair in* T_M *is attached to a top ribbon in* T_M. *(Analogously for bottom ribbons).*
(3) $\Pi^{(k)}\mathcal{V}^\#(T_M) = \mathcal{V}^\#(T_M)\Pi^{(j)}$ *if* T_M *has a through pair connecting the* j*-th top pair to the* k*-th bottom pair.*

Proof For (1) note that the picture for Π consists of two arcs that are connected by a circle. Stacking Π on top of itself we obtain the picture for Π^2 by functoriality in Lemma 3. The resulting tangle is the chain of circles C_j and arcs $A_{t/b}$ depicted on the left of Figure 7. By (1) of Lemma 3 we may use 2-handle slides to manipulate this picture. We first slide C_1 over C_3, and then A_b over C_2. The result is the tangle for Π and a separate Hopf link. The value of the latter, however, is 1 by (28). Hence, $\Pi^2 = \Pi$.

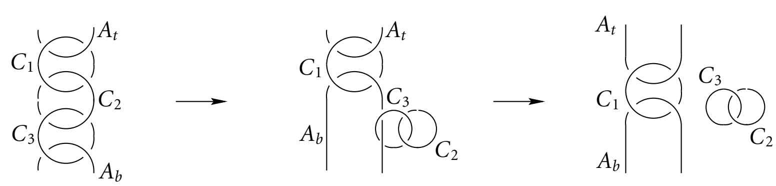

Figure 7: Π is idempotent.

Equivariance with respect to the action of $\mathcal{A}$ is immediate from Lemma 3.

For (2), we repeat an argument from [25]. Suppose τ is a top component and η any band connecting two intervals I_i in τ in an orientation-preserving way. To this we associated the surgered diagram in which the component τ is replaced by

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the union τ_η of three components. They are obtained by cutting away the intervals I_i from τ and inserting the other two edges of η at the endpoints ∂I_i as indicated in Figure 8. Furthermore, we insert a 0-framed annulus A around η. Sliding any

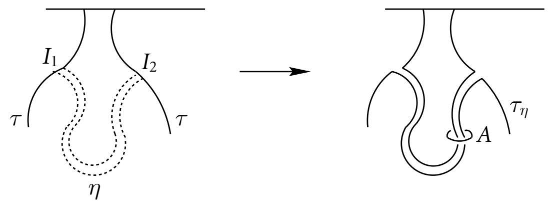

Figure 8: η-surgery.

other component over A at an arbitrary point along η has the effect of just moving it through η at this point. Moreover, we can slide a ± 1-framed annulus K over A so that it surrounds the two parallel strands in τ_η, and then slide the two strands over K. The effect is the same as putting a 2π-twist into η. These two operations allow us to move any band η to any other band η' such that τ_η and $\tau_{\eta'}$ are related by a sequence of two handle slides.

Now, adding the picture of Π to the top-component τ of a tangle T_M is the same as surgering τ along a straight band parallel and close to the interval between the attaching points of τ at the top line. We replace this η by a small planar arc at τ separate from the rest of the tangle. Surgery along this corresponds to linking a Hopf link to τ, as $C_2 \cup C_3$ is linked to A_b in the middle of Figure 7, and consequently can be removed by the same argument. The proofs for the formulas for bottom and through strands are entirely analogous. ∎

Set $\Pi^\# = \Pi^{\otimes g}$ when acting on $\mathcal{A}^{\otimes g}$. It follows now easily from Lemma 4 that $\mathcal{V}^\#(T_M)\Pi^\# = \Pi^\#\mathcal{V}^\#(T_M)$ for all T_M. Thus each $\mathcal{V}^\#(T_M)$ maps the image of $\Pi^\#$ to itself so that we can define the restriction

$$\mathcal{V}(T_M) := \mathcal{V}^\#(T_M)|_{im(\Pi^\#)} : \mathcal{V}_{\mathcal{A}}(\Sigma_{g_1,1}) \longrightarrow \mathcal{V}_{\mathcal{A}}(\Sigma_{g_2,1}), \tag{37}$$

where the vector spaces are given as

$$\mathcal{V}_{\mathcal{A}}(\Sigma_{g,1}) = \Pi^\#\big(\mathcal{V}^\#(\Sigma_g)\big) = \mathcal{A}_0^{\otimes g} \quad \text{with} \quad \mathcal{A}_0 = \Pi(\mathcal{A}). \tag{38}$$

Theorem 5 *The assignment $\mathcal{V}$ as given in (37) yields a well defined, 2-framed, relative, $\mathcal{A}$-equivariant topological quantum field theory*

$$\mathcal{V}_{\mathcal{A}} : \mathcal{C}ob_3^{2\,\mathrm{fr},\bullet} \longrightarrow \mathcal{A}\text{-}\mathrm{mod}_{\mathbb{K}} \subset \mathrm{Vect}(\mathbb{K}).$$

Using the invariance functor $\mathrm{Inv} = \mathrm{Hom}(1, \_): \mathcal{A}\text{-mod} \to \mathrm{Vect}(\mathbb{K})$ *we obtain an ordinary 2-framed TQFT for closed surfaces as*

$$\mathcal{V}^0_{\mathcal{A}} := \mathrm{Inv} \circ \mathcal{V}_{\mathcal{A}} : \mathcal{C}ob_3^{2\,\mathrm{fr}} \longrightarrow \mathrm{Vect}(\mathbb{K}).$$

Proof We recall from [21, Proposition 12] that two presentations, T_M and T'_M, of a framed, relative cobordism $M \in \mathcal{C}ob_3^{2\,\mathrm{fr},\bullet}$ are related by the moves described in Lemma 3 and the so called σ-moves, which consist of adding the picture of Π to a pair of points at the top or bottom line of the diagram. From $\mathcal{V}(T_M)\Pi^{(j)} = \mathcal{V}^\#(T_M)\Pi^\#\Pi^{(j)} = \mathcal{V}^\#(T_M)\Pi^\#$ we see that $\mathcal{V}(T_M)$ is invariant under this move. Hence, $\mathcal{V}(T_M)$ only depends on the cobordism represented by T_M and we can write $\mathcal{V}_{\mathcal{A}}(M) := \mathcal{V}(T_M)$.

Due to the equivariance of Π also $\mathcal{A}_0$ from (38) is invariant under the adjoint action of $\mathcal{A}$, and the restricted maps commute with the action of $\mathcal{A}$ as well. Functoriality of $\mathcal{V}$ follows from functoriality of $\mathcal{V}^\#$ and the fact that $\Pi^\#$ commutes with $\mathcal{V}^\#$.

Since each $\mathcal{V}(M)$ commutes with the action of $\mathcal{A}$ they also map the $\mathcal{A}$-invariant subspaces $\mathcal{V}^0(\Sigma_g) := \mathrm{Inv}\big(\mathcal{V}(\Sigma_{g,1})\big)$ to themselves. This implements the additional τ-move [21] needed to represent cobordisms between closed surfaces. ■

6 The Algebra $\mathcal{N}$

The Hopf algebra $\mathcal{N}$ we will define in this section is the same as the algebra A_2 described by Radford in Example 1 of Section 4.1 in [41]. The quasitriangular structure that we endow $\mathcal{N}$ with is essentially distilled from the one of $U_{-1}(\mathfrak{sl}_2)$.

Let $\mathbb{E} \cong \mathbb{R}^2$ be the Euclidean plane, and consider the 8-dimensional algebra

$$\mathcal{N} := \mathbb{Z}/2 \ltimes \bigwedge^* \mathbb{E}. \tag{39}$$

The generator of $\mathbb{Z}/2$ is denoted by K, with $K^2 = 1$, and we write $x^K = KxK$ for any $x \in \mathcal{N}$. We thus have relations $w'w = -ww'$ and $w^K := KwK = -w$ for all $w, w' \in \mathbb{E}$.

Lemma 5 *$\mathcal{N}$ is a Hopf algebra with coproducts*

$$\Delta(K) = K \otimes K \quad \textit{and} \quad \Delta(w) = w \otimes 1 + K \otimes w \quad \forall w \in \mathbb{E}. \tag{40}$$

Proof The fact that $\Delta\colon \mathcal{N} \to \mathcal{N}^{\otimes 2}$ is a coassociative homomorphism is readily verified. The antipode is given by

$$S(K) = K \quad \text{and} \quad S(w) = -Kw, \quad \forall w \in \mathbb{E}. \tag{41}$$ ■

We note the following formulas for the adjoint action and antipode:

$$\mathrm{ad}(w)(x) = wx - x^K w, \quad S^2(x) = x^K \quad \forall x \in \mathcal{N}, w \in \mathbb{E}. \tag{42}$$

Let us pick a non-zero element $\rho \in \bigwedge^2 \mathbb{E} \subset \mathcal{N}$, and for this define a form $\mu_0 \in \mathcal{N}^*$ as follows:

$$\begin{gathered} \mu_0(\rho) = 1, \quad \mu_0(K\rho) = 0, \\ \text{and} \quad \mu_0(K^\delta x) = 0 \,\forall x \in \bigwedge^j \mathbb{E}, \text{ whenever } j, \delta \in \{0,1\}. \end{gathered} \tag{43}$$

Lemma 6 *μ_0 is a right (and left) integral on $\mathcal{N}$. Moreover,*

$$\lambda_0 := (1+K)\rho \quad \textit{with} \quad \mu_0(\lambda_0) = 1 \tag{44}$$

is a two sided integral in $\mathcal{N}$.

Proof Straightforward verification of (26). The defining equation for a two sided integral in $\mathcal{N}$ is $x\lambda_0 = \lambda_0 x = \epsilon(x)\lambda_0$, which is also readily found. ■

Next, we fix a basis $\{\theta, \bar{\theta}\}$ for $\mathbb{E}$. We define an R-matrix, $\mathcal{R} \in \mathcal{N} \otimes \mathcal{N}$, by the formula

$$\mathcal{R} := (1 \otimes 1 + \theta \otimes K\bar{\theta}) \cdot \mathcal{Z}, \quad \text{where } \mathcal{Z} := \frac{1}{2}\sum_{i,j=0}^{1} (-1)^{ij} K^i \otimes K^j. \tag{45}$$

Lemma 7 *The element $\mathcal{R}$ makes $\mathcal{N}$ into a quasitriangular Hopf algebra.*
Moreover, $\mathcal{N}$ is a ribbon Hopf algebra with unique balancing element $G = K$.

Proof Quasitriangularity follows from a straightforward verification of the axioms in [6]. We compute the special element $u^{-1} = \sum_j f_j S^2(e_j) = K(1+\bar{\theta}\theta)$ for which $uS(u)^{-1} = uu^{-1} = 1$, so that $G = K$ is a valid and unique choice. The ribbon element is then given by

$$v := 1 + \rho \quad \text{with } \rho := \bar{\theta}\theta. \tag{46}$$ ■

For the monodromy matrix, as defined in (25), we obtain:

$$\mathcal{M} = 1 + K\bar{\theta} \otimes \theta + \theta K \otimes \bar{\theta} - \rho \otimes \rho. \tag{47}$$

Setting $T = K\bar{\theta} \otimes \theta + \theta K \otimes \bar{\theta}$ we compute $T^2 = -2\rho \otimes \rho$ and $T^3 = 0$ so that $\mathcal{M} = \exp(T)$. Hence we can also compute p-th powers of the monodromy matrix:

$$\mathcal{M}^p = \exp(pT) = 1 + pT + \frac{p^2}{2}T^2. \tag{48}$$

With μ_0 as defined in (43), and for ρ as in (46) we find $\mu_0 \otimes \mu_0(\mathcal{M}) = \mu_0(v)\mu_0(v^{-1}) = -1$. Hence, in order to fulfill (28) we need to use the renormalized integrals

$$\mu = i\mu_0, \quad \lambda = \frac{1}{i}\lambda_0, \quad \text{with } i = \sqrt{-1}. \tag{49}$$

For these choices we compute the $S^{\pm}$-transformations assigned to (36) as follows:

$$\begin{aligned} \frac{1}{i}S^{\pm}(w) &= \mp w \quad \forall w \in \mathbb{E}, & \frac{1}{i}S^{\pm}(\rho) &= 1 \\ \frac{1}{i}S^{\pm}(Kx) &= 0 \quad \forall x \in \bigwedge\nolimits^{*} \mathbb{E}, & \frac{1}{i}S^{\pm}(1) &= -\rho. \end{aligned} \tag{50}$$

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This implies that the projector Π from Lemma 4 has kernel $\ker(\Pi) = \{Kw : w \in \bigwedge^* \mathbb{E}\}$ and image

$$\mathcal{N}_0 = \mathrm{im}(\Pi) = \bigwedge^* \mathbb{E}. \tag{51}$$

From (42) we see that $\mathcal{N}_0$ acts trivially on itself so that the action of $\mathcal{N}$ factors through the obvious $\mathbb{Z}/2\mathbb{Z} = \mathcal{N}/\mathcal{N}_0$-action.

Finally, we note that $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{E}$ and, hence, also on $\mathcal{N}$, assuming K is $\mathrm{SL}(2,\mathbb{R})$-invariant.

Lemma 8 $\mathrm{SL}(2,\mathbb{R})$ *acts on* $\mathcal{N}$ *by Hopf algebra automorphisms.*

The ribbon element v*, the monodromy* $\mathcal{M}$*, and the two integrals are invariant under this action.*

Proof The fact that $\mathrm{SL}(2,\mathbb{R})$ yields algebra automorphisms is obvious by construction. Linearity of coproduct and antipode in w in (40) and (41) imply that this is, in fact, a Hopf algebra homomorphism. v and λ are invariant since $\mathrm{SL}(2,\mathbb{R})$ acts trivially on $\mathbb{E} \wedge \mathbb{E}$. Invariance of $\mathcal{M}$ follows then from (25). ■

Note that $\mathcal{R}$ itself is *not* $\mathrm{SL}(2,\mathbb{R})$-invariant.

7 The Hennings TQFT for $\mathcal{N}$

From (51) and (37) we see that the vector spaces of the Hennings TQFT for the algebra from (39) are given as

$$\mathcal{V}_{\mathcal{N}}(\Sigma_g) := \Big(\bigwedge^* \mathbb{E}\Big)^{\otimes g} \quad \text{with } \dim\big(\mathcal{V}_{\mathcal{N}}(\Sigma_g)\big) = 4^g. \tag{52}$$

We now compute the action of the mapping class group generators from the tangles in Figures 2, 3, and 4. From the extended Hennings rules it is clear that the pictures for both A_j and S_j result in actions only on the j-th factor in the tensor product in (52). For A_j we use the presentation from Figure 2 and the rules from Figure 5 and (35) to obtain the linear map $\mathbb{A}(x) := x \cdot v$.

The extra 1-framed circle in Figure 4 results in an extra factor $\mu(v) = i$, since an empty circle corresponds to an insertion of v. The action on the j-th factor is thus given by an application of $\mathbb{S} := iS^+|_{\mathcal{N}_0}$ so that

$$\mathbb{S}(\rho) = -1, \quad \mathbb{S}(1) = \rho, \quad \text{and} \quad \mathbb{S}(w) = w,\ \forall w \in \mathbb{E}. \tag{53}$$

Similarly, D_j acts only on the j-th and the $(j+1)$-st factors of $\mathcal{N}_0^{\otimes g}$. From (35) and the formula for $\mathcal{M}^{-1}$ in (47) we compute for the action on these two factors

$$\mathbb{D}\colon \mathcal{N}_0^{\otimes 2} \to \mathcal{N}_0^{\otimes 2}, \quad x \otimes y \mapsto x \otimes y + x\theta \otimes \bar{\theta}y - x\bar{\theta} \otimes \theta y - x\rho \otimes \rho y. \tag{54}$$

The generators of the mapping class group Γ_g are thus represented as follows:

$$\mathcal{V}_{\mathcal{N}}(\mathbf{I}_{A_j}) = I^{\otimes j-1} \otimes \mathbb{A} \otimes I^{\otimes g-j}, \quad \mathcal{V}_{\mathcal{N}}(\mathbf{I}_{S_j}) = I^{\otimes j-1} \otimes \mathbb{S} \otimes I^{\otimes g-j}$$
$$\text{and} \quad \mathcal{V}_{\mathcal{N}}(\mathbf{I}_{D_j}) = I^{\otimes j-1} \otimes \mathbb{D} \otimes I^{\otimes g-j-1}. \tag{55}$$

Let us also compute the linear maps associated to the cobordisms $\mathbf{H}_g^{\pm}$ from (9). Their tangle presentations follow from [21] and have the forms given in Figure 9.

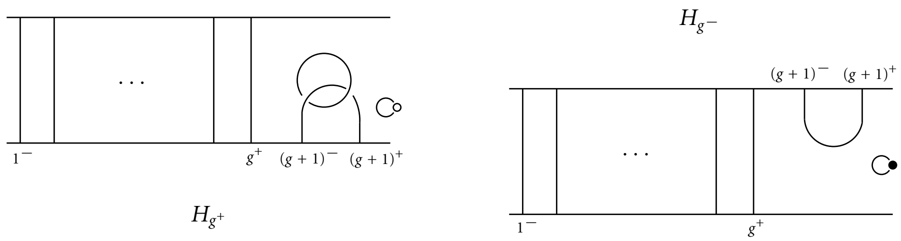

Figure 9: Tangles for Handle additions.

We included ± 1-framed circles to adjust the 2-framings of $\mathbf{H}_g^{\pm}$. A 0-framed circle around a strand has the effect of inserting $\lambda = S^{+}(1) = \frac{1}{i}\rho$. In this normalization we find with $\rho = i\Pi\lambda$ and (33) that

$$\mathcal{V}_{\mathcal{N}}(\mathbf{H}_g^{+}) \colon \alpha \mapsto \alpha \otimes \rho \quad \forall \alpha \in \mathcal{N}_0^{\otimes g}. \tag{56}$$

Similarly, we obtain from (34) that

$$\mathcal{V}_{\mathcal{N}}(\mathbf{H}_g^{-}) \colon \alpha \otimes x \mapsto \mu_0(x)\alpha \quad \forall \alpha \in \mathcal{N}_0^{\otimes g}, x \in \mathcal{N}_0, \tag{57}$$

where μ_0 is as in (43). We note the following:

Lemma 9 *The generators in (55), (56), and (57) intertwine the* $\mathrm{SL}(2,\mathbb{R})$*-action on* $\mathcal{N}_0^{\otimes g}$.

Proof The fact that $\mathbb{A}$ and $\mathbb{D}$ commute with the $\mathrm{SL}(2,\mathbb{R})$-action follows from invariance of ν and $\mathcal{M}$. From (50) we see that $\mathbb{S}$ is scalar on the non-invariant part, and thus commutes as well. Finally, ρ and μ_0 are clearly invariant. ■

For $g \geq 0$ set $\chi_g := S_g \circ \cdots \circ S_1$, $h_g^{+} := \mathbf{H}_{g-1}^{+} \circ \cdots \circ \mathbf{H}_0^{+}$, and $h_g^{-} := \mathbf{H}_0^{-} \circ \cdots \circ \mathbf{H}_{g-1}^{-}$. We define a standard closure of a 2-framed 3-cobordism as the closed 3-manifold

$$\langle M \rangle := h_{g_2}^{-} \circ \chi_{g_2} \circ M \circ \chi_{g_1}^{-1} \circ h_{g_1}^{+} \cup D^3. \tag{58}$$

If M is represented by a tangle T we obtain, similarly, a link $\langle T\rangle$. We introduce the following function from the class of 2-framed cobordisms into $\mathbb{Z}/2$:

$$\varphi(M) := \beta_1(\langle M\rangle) + \operatorname{sign}(\langle T\rangle) \bmod 2, \tag{59}$$

where β_j denotes the j-th Betti number. We further denote by $\mathcal{C}ob_3^{22\,\mathrm{fr},*} \subset \mathcal{C}ob_3^{2\,\mathrm{fr},*}$ the subset of all cobordisms M with $\varphi(M) = 0$, which we will call *evenly* 2-framed.

Lemma 10

(1) $\varphi(M) = |\langle T\rangle| \bmod 2$, *where* $|\langle T\rangle| := \#$ *components of* $\langle T\rangle$.
(2) $\varphi(M) = \#$ *components of T not connected to the bottom line.*
(3) $\mathcal{V}_{\mathcal{N}}(M)$ *is real if* $\varphi(M) = 0$ *and imaginary for* $\varphi(M) = 1$.
(4) $\mathcal{C}ob_3^{22\,\mathrm{fr},*}$ *is a subcategory.*

Proof Let W be the 4-manifold given by adding 2-handles to D^4 along $\langle T\rangle \subset S^3$ so that $\langle M\rangle = \partial W$, and let L_T be the linking matrix of $\langle T\rangle$. We have $\beta_2(W) = |\langle T\rangle| = d_+ + d_- + d_0$, where d_+, d_-, and d_0 are the number of eigenvalues of L_T that are > 0, < 0, and $= 0$ respectively. From the exact sequence $0 \to H_2(\langle M\rangle) \to H_2(W) \xrightarrow{L_T} H^2(W) \to H_1(\langle M\rangle) \to 0$ we find that $\beta_1(\langle M\rangle) = d_0$, which implies (1) using $\operatorname{sign}(W) = d_+ - d_-$. (2) follows immediately from the respective tangle compositions.

The possible components not connected to the bottom line are strands connecting point pairs at the top line or closed components. From the rules (32) through (35) we see that these are just the types of components that involve an evaluation against $\mu = i\mu_0$. All other parts of the Hennings procedure involve only real maps. Finally, (4) follows from counting tangle components under composition. ∎

Proposition 6 *The Hennings procedure yields a relative, 2-framed,* $\mathrm{SL}(2,\mathbb{R})$*-equivariant, half-projective TQFT*

$$\mathcal{V}_{\mathcal{N}}\colon \mathcal{C}ob_3^{2\,\mathrm{fr},\bullet} \longrightarrow \mathrm{SL}(2,\mathbb{R})\text{-}\mathrm{mod}_{\mathbb{C}},$$

which is $\mathbb{Z}/4$*-projective on* $\mathcal{C}ob_3^\bullet$. *We have a restriction*

$$\mathcal{V}_{\mathcal{N}}^{(2)}\colon \mathcal{C}ob_3^{22\,\mathrm{fr},\bullet} \longrightarrow \mathrm{SL}(2,\mathbb{R})\text{-}\mathrm{mod}_{\mathbb{R}},$$

which is $\mathbb{Z}/2$*-projective on* $\mathcal{C}ob_3^\bullet$.

Proof From Lemma 9 we know that the generators of Γ_g are represented $\mathrm{SL}(2,\mathbb{R})$-equivariantly, hence also Γ_g itself. The decomposition in (10) and equivariance of the maps in (56) and (57) implies the same for general cobordisms. That this TQFT is half-projective follows from the fact that $\mathcal{N}$ is non-semisimple, or, equivalently, that $\mathcal{V}_{\mathcal{N}}(S^1 \times S^2) = \mu(1) = \varepsilon(\lambda) = 0$, see [20]. The projective phase of the TQFT is determined by the value $\mu(v) = i$ on the 1-framed circle.

Lemma 10 (3) implies that $\mathcal{V}_{\mathcal{N}}^{(2)}$ maps into the *real* $\mathrm{SL}(2,\mathbb{R})$-equivariant maps and modules. This reduces the ambiguity of multiplication with i to a sign ambiguity. ■

An important point of view in the TQFT constructions in [25] is the existence of a categorical Hopf algebra, which can be understood as the TQFT image of a topological Hopf algebra given as an object in $\mathcal{C}ob_3^{\bullet}$.

To be more precise, in [51] and [19] $\mathcal{C}ob_3^{\bullet}$ is described as a braided tensor category, and it is found that the object $\Sigma_{1,1} \in \mathcal{C}ob_3^{\bullet}$ is naturally identified as a braided Hopf algebra in this category in the sense of [33] and [32]. Particularly, $\Sigma_{2,1}$ is identified with $\Sigma_{1,1} \otimes \Sigma_{1,1}$ since the tensor product on $\mathcal{C}ob_3^{\bullet}$ is defined by sewing two surfaces together along a pair of pants. The multiplication and comultiplication are thus given by elementary cobordisms $\mathbf{M}\colon \Sigma_{2,1} \to \Sigma_{1,1}$ and $\boldsymbol{\Delta}\colon \Sigma_{1,1} \to \Sigma_{2,1}$. Their tangle diagrams are worked out explicitly in [3], and depicted in Figure 10 with minor modifications in the conventions:

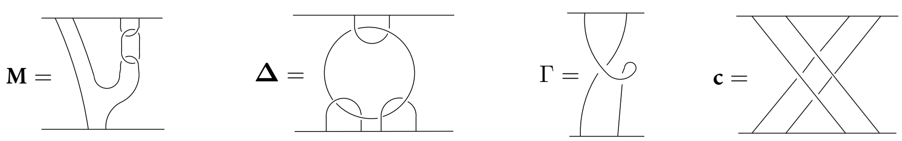

Figure 10: Tangles for multiplications.

Here $\mathbf{c}\colon \Sigma_{2,1} \to \Sigma_{2,1}$ is the braid isomorphism. The braided antipode is given by the tangle $\Gamma = (S^+)^2$, with S^+ as in Figure 6.

Lemma 11 *The cobordisms $\mathbf{M}$ and $\boldsymbol{\Delta}$ have the following Heegaard decompositions.*

$$\mathbf{M} = \mathbf{H}_2^- \circ \mathbf{I}_{D_1 \circ S_2} \quad \textit{and} \quad \boldsymbol{\Delta} = \mathbf{I}_{S_1 \circ D_1^{-1} \circ S_1^{-1} \circ S_2^{-1}} \circ \mathbf{H}_2^+.$$

Proof Verification by composition of the associated tangles. ■

The explicit formulas for the linear maps associated to the generators of the mapping class group and the handle attachments in Section 7 allow us now to compute the braided Hopf algebra structure induced on $\mathcal{N}_0 = \mathcal{V}_{\mathcal{N}}(\Sigma_{1,1})$. We write $M_0 := \mathcal{V}_{\mathcal{N}}(\mathbf{M})$, $\Delta_0 := \mathcal{V}_{\mathcal{N}}(\boldsymbol{\Delta})$, $S_0 := \mathcal{V}_{\mathcal{N}}(S_1^2)$, and $c_0 := \mathcal{V}_{\mathcal{N}}(\mathbf{c})$ for the braided multiplication, comultiplication, antipode and braid isomorphism respectively.

Lemma 12 *The induced braided Hopf algebra structure on $\mathcal{N}_0$ is the canonical $\mathbb{Z}/2$-graded Hopf algebra with:*

$$M_0(x \otimes y) = xy \quad c_0(x \otimes y) = (-1)^{d(x)d(y)} y \otimes x \quad \forall x, y \in \mathcal{N}_0$$
$$\textit{and} \quad \Delta_0(w) = w \otimes 1 + 1 \otimes w \quad \Gamma_0(w) = -w \quad \forall w \in \mathbb{E}.$$

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In particular, $\mathcal{N}_0$ is commutative and cocommutative in the graded and braided sense, $\mathcal{N}_0 \cong \mathcal{N}_0^*$ is self dual, $\mathrm{SL}(2,\mathbb{R})$ *still acts by Hopf automorphisms on $\mathcal{N}_0$, and S_0 is an involutory homomorphism on $\mathcal{N}_0$.*

Proof For $\mathbf{M}$ and $\boldsymbol{\Delta}$ insert the morphism associated to the generators in Lemma 11. The braid isomorphism is given via the Hennings rules by acting with the operator $\mathrm{ad} \otimes \mathrm{ad}(\mathcal{R})$ on $\mathcal{N}_0^{\otimes 2}$ and then permuting the factors. It is easy to see that $\mathrm{ad} \otimes \mathrm{ad}(\mathcal{Z})$ acts on $x \otimes y$ by multiplying $(-1)^{d(x)d(y)}$, where $d(x)$ is the $\mathbb{Z}/2$-degree of x in $\mathcal{N}_0$. Moreover, we we know that the adjoint action of $\mathcal{N}_0$ on itself is trivial so that the term $\theta \otimes K\bar{\theta}$ in the second factor of $\mathcal{R}$ in (45) does not contribute. ∎

8 Skein Theory for $\mathcal{V}_{\mathcal{N}}$

The skein theory of the Hennings calculus over $\mathcal{N}$ is mostly a consequence of the form $v = 1 + \rho$ of the ribbon element as in (46). In the Hennings procedure we substitute a strand with decoration $\frac{1}{i}\rho$ by a dotted strand (with possibly more decorations) as shown on the left of Figure 11. Observe from (47) that

$$\mathcal{M}^{\pm 1}(1 \otimes \rho) = (1 \otimes \rho) \quad \text{and} \quad \mathcal{M}^{\pm 1}(\rho \otimes 1) = (\rho \otimes 1).$$

This means that for a dotted strand we do not have to distinguish between over and undercrossing with other strands as indicated on the right of Figure 11. As a result such a strand can be disentangled from the rest of the diagram.

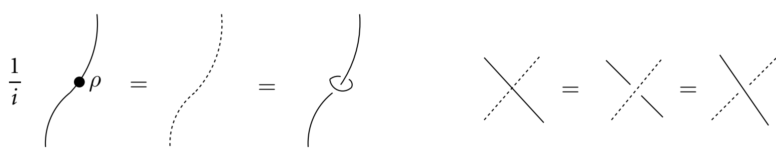


Figure 11: Transparent ρ-decorated strand.

The next additional ingredient in the calculus are symbols for 1-handles. They are used in the bridged link calculus as described in [21] and [25]. We indicate a pair of 1-surgery balls by pairs of coupons. The defining relation is the modification move depicted on the left of Figure 12. The move indicated on the right of Figure 12 and its reflections is a standard consequence of the boundary move from (24).

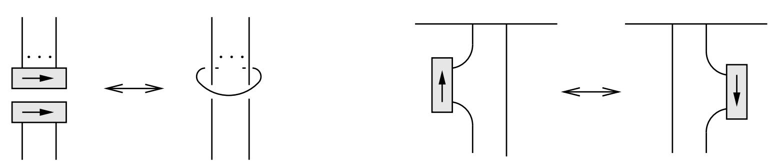


Figure 12: Coupons for 1-handles.

Since $v^k = 1 + k\rho$ for $k \in \mathbb{Z}$ we find that the framing of any component can be changed at the expense of introducing dotted lines. This translates to the diagrams in Figure 13.

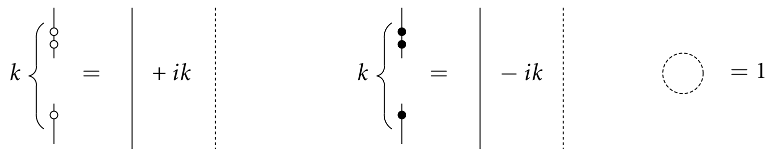


Figure 13: Framing shift.

The skein relation is now obtained by applying Figure 13 to the Fenn–Rourke move as in Figure 14, see also [36].

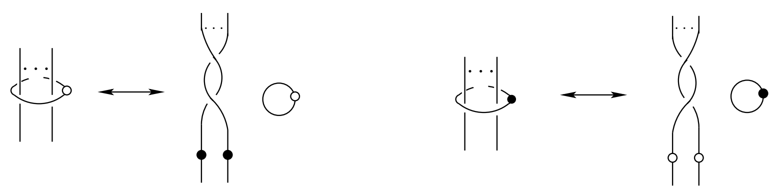
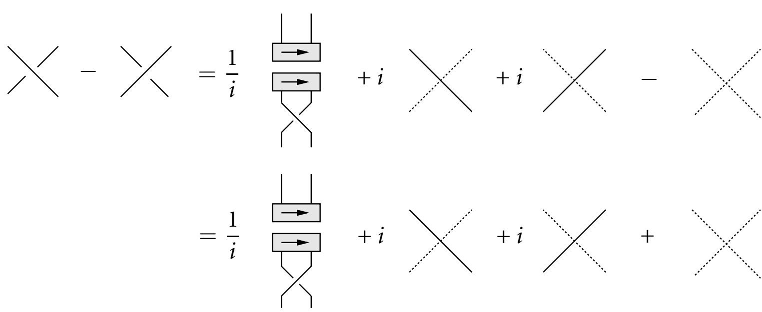
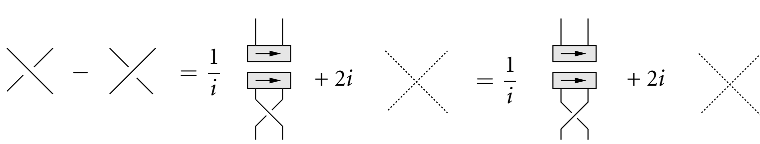


Figure 14: Fenn–Rourke move.

Lemma 13 *For two strands belonging to two different components of a tangle diagram we have the relation*



For strands belonging to the same component of the tangle the relation is



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At this point it is convenient to extend the tangle presentations to general diagrams, dropping the condition that a strand starting at a point j^- has to end at a point j^+ (or the corresponding condition for through strands). From such a general tangle diagram we can get to an admissible one by applying boundary moves (24) at all intervals $[j^-, j^+]$. (This is in fact the original definition used in [21].) We shall allow the occurrence of coupons but restrict ourselves to the cases where exactly two strands enter (or exit) a coupon as in Lemma 13.

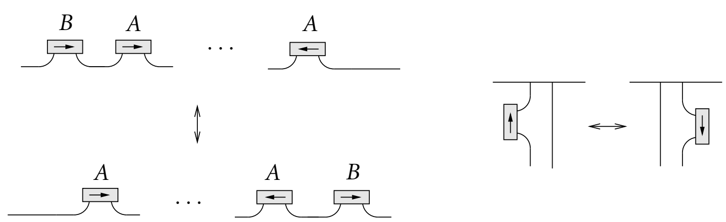
We also introduce two notions of components: The first is that of a *diagram component* $\mathcal{X}$ of a generalized tangle diagram. It is given by a concatenation of curve segments, coupons that have two strands going in on one side, and intervals $[j^-, j^+]$ connecting a strand ending in j^- with the one ending in j^+.

The second is a *strand component*, which is also a collection of curves that can be joined in two ways. As before curves that end in two sides of the same interval $[j^-, j^+]$ belong to the same strand component, as well as curves exiting and entering a coupon pair that would be connected under application of Figure 12.

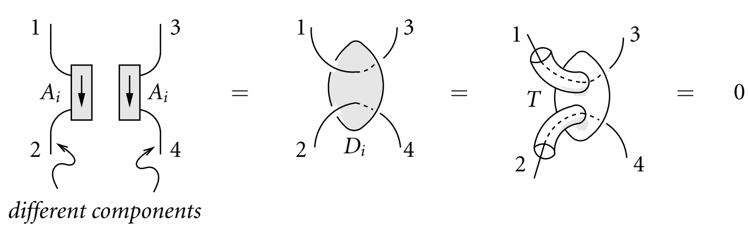
We have the following rules for manipulating the coupons:

Lemma 14 *In the following equivalences the labels $A, B, \ldots$ indicate which coupons form a pair.*

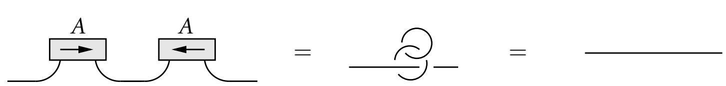
(1) *1-handles can be slid over other 1-handles, through a boundary interval, and hence anywhere along a strand component.*

(60)


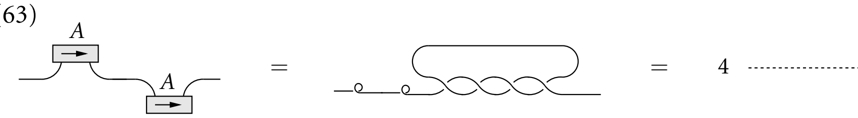
(2) *If in a diagram the coupons of a pair belong to different diagram components the entire diagram does not contribute, i.e., is evaluated as zero. Hence only diagrams contribute in which the diagram components coincide with strand components.*

(61)


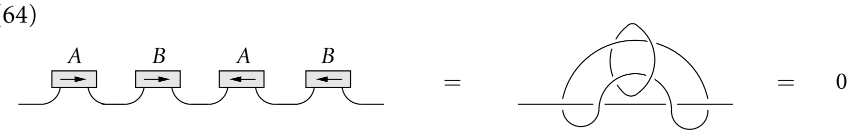
(3) *Direct 1-handle cancellation: If coupons with the same label are adjacent on the same side of a strand they can be canceled:*

(62)


(4) *Opposite 1-handle cancellation: If coupons with the same label are adjacent on opposite sides of a strand the strand is replaced by a dotted strand and the evaluation gains a factor of 4.*

(63)


(5) *If a generalized tangle diagram contains a coupon configuration as indicated the entire diagram is evaluated as zero.*

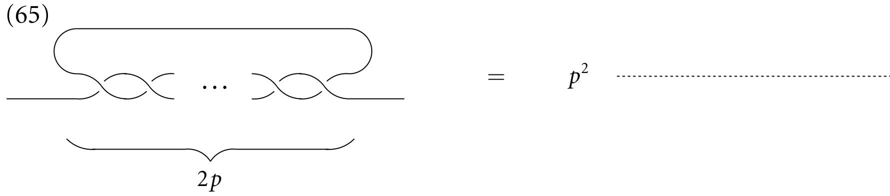
(64)


Proof The slide of B over the pair A in (60) translates to a simple isotopy if we apply the move in Figure 12 to the A-pair. Similarly, the slide through a boundary interval is given by an isotopy conjugated by a σ-move as in (24).

For (2) let $\mathcal{X}$ be a diagram component that contains coupons $A_1, \ldots, A_n$ whose partners lie on different diagram components. Performing boundary moves we can make $\mathcal{X}$ a true inner component. Furthermore, we can eliminate the other coupons on $\mathcal{X}$ that occur in pairs by undoing the modification from Figure 12. The component $\mathcal{X}$ is now a closed curve interrupted only by coupons $A_1, \ldots, A_n$. We undo the modification also for these and the corresponding annuli added in the move bound discs that we denote by $D_1, \ldots, D_n$. Note that the arcs of $\mathcal{X}$ all end in only one side of a disc D_j since the strands emerging from the other side belong to a different component. We can thus surger the discs along the arcs, as shown in (61), so that we obtain a torus T with n holes $\partial T = \partial D_1 \sqcup \cdots \sqcup \partial D_n$ which misses all other parts of the tangle. After surgery along the annuli the torus T can be capped off so that we have found a non-separating surface inside the represented cobordism. Since we are dealing with a non-semisimple TQFT this implies that the associated linear map is zero.

The direct cancellation in (62) follows by applying Figure 12. In the resulting configuration in the middle of (62) the Hopf link can be slid off and removed.

The opposite cancellation in (63) and the remodification from Figure 12 give the tangle in the middle. Now consider in general a straight strand that is entangled with an annulus with $2p$ positive crossings as in (65).

(65)


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Using the formula in (48) we find by applying the Hennings procedure and evaluating the elements on the annulus against the integral that the resulting element on the open strand is

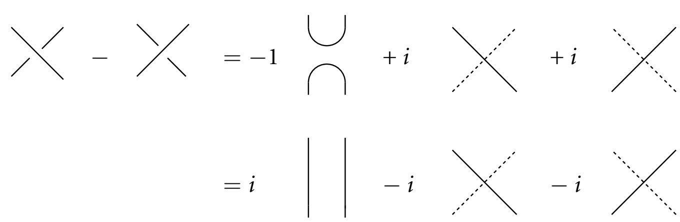
$$\mu \otimes id(\mathcal{M}^p) = \frac{p^2}{i}\rho$$

which with Figure 11 implies the claim.

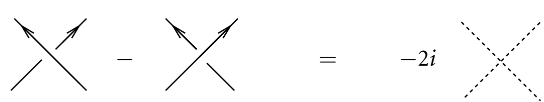
Finally, we also reexpress the coupons in (64) by a tangle. As before, non-semi-simplicity of the TQFT implies that a diagram containing such a subdiagram is always zero. For example, the 0-framed annulus clearly bounds a surface disjoint from the rest of the link so that the cobordism contains a non-separating surface. ∎

We now combine the previous two lemmas in the following skein relations without coupons.

Theorem 7 *For generalized tangle diagrams we have the following skein relations:*
For crossings of strands of different components:

(66)


For crossing of strands of the same component we need to introduce an orientation on the component.

(67)


Proof The proof is given by moving the coupons in the skein relations of Lemma 13 through the components using Lemma 14. ∎

Note that relation (67) implies the relation for the Kauffman polynomial for $z = \frac{1}{2}$. However, the framing relations are quite different.

Let $\widehat{\mathcal{B}}_g$ be the group of tangles in $2g$ strands generated by the braidings $\mathbf{c}$ of double strands and the braided antipodes Γ as in Figure 10 acting in different positions. It is thus the image of the abelian extension $B_g \ltimes (\mathbb{Z}/2)^g$ of the braid group.

Moreover, let us introduce a few elementary generalized tangles $M_k\colon k \to 0$,

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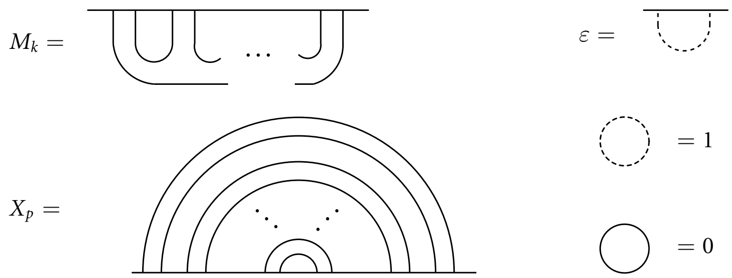
$\varepsilon \colon 1 \to 0$ and $X_n \colon 0 \to 2n$ as depicted below.

(68)



Theorem 8 *Every tangle $T \colon G \to 0$ with $2G$ starting (top) points and* no *endpoints can be resolved via the skein relations in Theorem 7 into a combination of tangles of the form*

$$T = (M_{k_1} \otimes \cdots \otimes M_{k_r} \otimes \epsilon^{\otimes N}) \circ B,$$

with $B \in \widehat{\mathcal{B}}_G$ and $\sum_{i=1}^{r} k_i = G - N$.

Proof We consider generalized tangles without coupons. We proceed by induction on the number m of connected components of T. We only count components that involve solid lines; those with dotted lines reduce to a collection of ε-diagrams at the intervals belonging to that component or closed dotted circles that do not contribute. Suppose now T has only one component, which we equip with some orientation. Applying Γ's to the intervals we can arrange it that the strands enter an interval $[j^-, j^+]$ at the left point j^- and leave at the right one j^+. Furthermore, we can find a permutation of intervals so that the strand exiting j^+ enters at $(j+1)^-$, except for G^+, which is connected to 1^-. Hence, by multiplying an element of $\widehat{\mathcal{B}}_G$ to T we can assume that the endpoints of the intervals are connected to each other by strands as they are for M_G.

Next we note that the skein relation (67) from Theorem 7 does not change this connectivity property for the solid lines and any diagram with dotted lines collapses to ε-diagrams.

For diagrams where equally labeled coupons are on the same components there are three planar moves that allow us to manipulate the arrangement of coupons. They are the 1-handle slide and the 1-handle cancellation depicted below, and the boundary flip as in Figure 12. In fact it is easy to see that we have the skein relation $T = M_G + iw(T)\varepsilon^{\otimes G}$, where $w(T)$ is the generalization of the writhe number of the diagram as defined, for example, in [30]. In case $G = 0$ the diagram M_0 is a closed solid circle which therefore makes the entire diagram zero.

Assume now T has m components and the claim is true for all diagrams with $m-1$ components. Pick one component C and apply an element of $\widehat{\mathcal{B}}_G$ such that the intervals included in this component are all to the left of the other intervals. Note that the set of intervals that belongs to C may also be empty. Next apply the skein relations (66) from Theorem 7 to untangle C from the other components. In each step of changing crossings of a strand of C with the strand of another component D, we can choose the relation for which the tangle that belongs to the first local diagram on

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the right side of the equation has one component less, since C and D are connected. The other diagrams on the right side also have one less component since we do not count dotted lines. Hence, by induction, the error of changing a crossing between C and another component can be resolved into elementary diagrams as claimed. After C is untangled we have expressed T, modulo elementary diagrams, in the form $C \otimes T'$ (juxtaposition) where T' has $m-1$ components. Again each factor can be resolved independently by induction, and hence the whole diagram, since $\otimes$-products of elementary diagrams are again elementary. ∎

Next note that every tangle $R\colon g_1 \to g_2$ is in fact of the form

$$R = (T \otimes id_{g_2}) \circ (id_{g_1} \otimes X_{g_2}) \tag{69}$$

for some $T\colon g_1 + g_2 \to 0$. Thus, in order to evaluate a general tangle diagram it suffices by Theorem 8 to specify the evaluations of the elementary tangles in (68). To this end we define the tensor

$$A = \frac{1}{i} S \otimes 1 \Delta(\rho) = \frac{1}{i}(\rho \otimes 1 + 1 \otimes \rho - \bar{\theta} \otimes \theta + \theta \otimes \bar{\theta}) \in \mathcal{N}_0^{\otimes 2}. \tag{70}$$

Corollary 9 *Every diagram can be resolved into a sum of composites of diagrams in (68). The linear maps associated to them are*

$$\mathcal{V}_{\mathcal{N}}(X_1)\colon \mathbb{C} \to \mathcal{N}_0^{\otimes 2} : 1 \mapsto A = \sum_{\nu} x_\nu \otimes y_\nu \tag{71}$$

$$\begin{aligned} \mathcal{V}_{\mathcal{N}}(X_n) &= \left(1^{\otimes(n-1)} \otimes \mathcal{V}_{\mathcal{N}}(X_1) \otimes 1^{\otimes(n-1)}\right) \circ \mathcal{V}_{\mathcal{N}}(X_{n-1})\colon \mathbb{C} \to \mathcal{N}_0^{\otimes 2n} \\ &: 1 \mapsto A_{\{n\}} = \sum_{\nu_1,\ldots,\nu_n} x_{\nu_1} \otimes x_{\nu_2} \otimes \cdots \otimes x_{\nu_n} \otimes y_{\nu_n} \otimes \cdots \otimes y_{\nu_2} \otimes y_{\nu_1} \end{aligned} \tag{72}$$

$$\mathcal{V}_{\mathcal{N}}(M_n)\colon \mathcal{N}_0^{\otimes n} \to \mathbb{C} : a_1 \otimes \cdots \otimes a_n \mapsto \mu(a_1 \cdot \cdots \cdot a_n). \tag{73}$$

Dotted circles can be removed and diagrams with solid circles do not contribute.

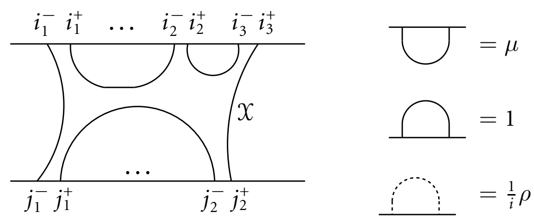
Proof The formulas follow easily from the pictures in Figure 10 to which we assigned linear maps in Lemma 12. Particularly, we find that the upside down reflection of the multiplication tangle **M** is mapped to the S-conjugate coproduct

$$\widetilde{\Delta} = i\mathbb{S}^{-1} \otimes \mathbb{S}^{-1} \Delta_0 \mathbb{S}\colon \mathcal{N}_0 \otimes \mathcal{N}_0 \to \mathcal{N}_0. \tag{74}$$

The tangle X_1 is obtained by capping this off with an arc at the top, which corresponds to the insertion of the unit. Hence, $A = \widetilde{\Delta}(1)$. The diagrams M_p are easily identified as composites $M^p = (M \otimes 1^{\otimes(p-1)}) \circ M^{p-1}$ capped off with an arc at the bottom, which is hence assigned to the p-fold multiplication followed by an evaluation against the integral $\mu \in \mathcal{N}^*$. ∎

Let us consider a few examples. One useful case is when the braid $B \in \widehat{\mathcal{B}}_n$ can be chosen trivially. Hence the contribution to the linear map for a tangle $R\colon g_1 \longrightarrow g_2$ is given by a union of planar diagrams as depicted in (75):

(75)



Define the map

$$C_p^q = \widetilde{\Delta}^{q-1} \circ M_0^{p-1}\colon \mathcal{N}_0^{\otimes p} \longrightarrow \mathcal{N}_0^{\otimes q}, \tag{76}$$

where the exponents denote the usual multiple products and coproducts. The linear map associated to a planar diagram is now the tensor product of maps associated to the individual components of the diagram. For example, if we want to evaluate the linear map on a homogeneous vector $x_1 \otimes \cdots \otimes x_{g_1}$ and the diagram has a component with solid lines as in (75) containing top intervals $[i_1^-, i_1^+], \ldots, [i_p^-, i_p^+]$ and bottom intervals $[j_1^-, j_1^+], \ldots, [j_q^-, j_q^+]$, we compute the vector $C_p^q(x_{i_1} \otimes \cdots \otimes x_{i_p}) \in \mathcal{N}_0^{\otimes q}$ and insert the entries in order into the positions $j_1, \ldots, j_q$ in $\mathcal{N}_0^{\otimes g_2}$.

With these rules the computation of the maps associated to the generators of the mapping class group are readily carried out. For example we can evaluate the diagram for the S-transformation from Figure 4. We resolve the rightmost crossing by taking the skein relation in the first row in Proposition 7 but with every diagram rotated clockwise by $\frac{\pi}{2}$. The result is

$$\mathbb{S} = id - \rho \otimes \mu_0 - 1 \otimes \epsilon - 1 \otimes \mu_0 + \rho \otimes \epsilon.$$

This yields exactly the formula from (53).

As another example we may consider the C_1 waist cycle in Σ_2. The diagram consists of four parallel strands with a 1-framed annulus around the second and third. We apply Figure 13 and then Figure 12 to this annulus. The resulting coupons can be canceled. We find

$$\mathcal{V}_{\mathcal{N}}(\mathbf{I}_{C_1}) = id - iC_1^1.$$

This implies the formula for the D-transformation from (54).

Finally, let us show how to use the skein calculus to find the precise formula for the invariant of a 2-framed closed 3-manifold presented by a link $\mathcal{L} \subset S^3$. It is basically given by the order of the first integral homology. More precisely, let

$$\eta(M) := \begin{cases} |H_1(M, \mathbb{Z})| & \text{for } \beta_1(M) = 0 \\ 0 & \text{for } \beta_1(M) > 0. \end{cases} \tag{77}$$

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Lemma 15 *For a given framed link $\mathcal{L} \subset S^3$ and η as in (77) we have*

$$\mathcal{V}_{\mathcal{N}}(M_{\mathcal{L}}) = i^{|\mathcal{L}|}\det(\mathcal{L}\cdot\mathcal{L}) = \pm i^{|\mathcal{L}|}\eta(M).$$

Proof By 2-handle slides we can move $\mathcal{L}$ into a link $\mathcal{L}^\delta$ so that the linking form $\mathcal{L}^\delta\cdot\mathcal{L}^\delta$ is diagonal and equivalent to the original one $\mathcal{L}\cdot\mathcal{L}$. Suppose f_j is the framing number of the j-th component $\mathcal{L}^\delta_j$. From Figure 13 we see that

$$\mathcal{V}_{\mathcal{N}}(\mathcal{L}^\delta) = \mathcal{V}_{\mathcal{N}}(\mathcal{L}^{\delta,-f_j}) + i f_j \mathcal{V}_{\mathcal{N}}(\mathcal{L}^\delta - \mathcal{L}^\delta_j).$$

Here, $\mathcal{L}^{\delta,-f_j}$ is the link in which the framing of the j-th component is shifted to zero. As a result the manifold represented by this link has non-trivial rational homology. Since $\mathcal{V}_{\mathcal{N}}$ is a non-semisimple theory this implies that $\mathcal{V}_{\mathcal{N}}(\mathcal{L}^{\delta,-f_j}) = 0$. Iterating the above identity we find $\mathcal{V}_{\mathcal{N}}(\mathcal{L}^\delta) = \prod_{j=1}^{|\mathcal{L}|}(i f_j)\mathcal{V}_{\mathcal{N}}(\varnothing)$. Clearly, $\prod_{j=1}^{|\mathcal{L}|}(f_j)$ is the determinant of the linking form of $\mathcal{L}^\delta$ and hence also the one of $\mathcal{L}$. ∎

9 Equivalence of $\mathcal{V}^{(2)}_{\mathcal{N}}$ and $\mathcal{V}^{\mathrm{FN}}$

In this section we compare the two topological quantum field theories $\mathcal{V}^{\mathrm{FN}}$ described in Section 3 and $\mathcal{V}^{(2)}_{\mathcal{N}}$ constructed in Section 7. We already found a number of general properties that are shared by both theories:

By Lemma 2 and Proposition 6 both theories are $\mathbb{Z}/2$-projective on $\mathcal{C}ob_3^\bullet$ and non-semisimple, fulfilling the property of Lemma 1. The $\mathbb{Z}/2$-projectivity is due to ambiguities of even 2-framings in the case of $\mathcal{V}^{(2)}_{\mathcal{N}}$ and ambiguities of orientations in the case of $\mathcal{V}^{\mathrm{FN}}$. The non-semisimple half-projective property results in the case of $\mathcal{V}^{\mathrm{FN}}$ from representation varieties that are transversely disjoint, and in the case of $\mathcal{V}^{(2)}_{\mathcal{N}}$ from the nilpotency of the integral $\lambda \in \mathcal{N}$. Further common features are the dimensions of vector spaces ($= 4^g$), actions of $\mathrm{SL}(2,\mathbb{R})$, see Sections 7 and 10, and the fact that $\mathcal{J}_g$ lies in the kernel of the mapping class group representations.

We construct now an explicit isomorphism between $\mathcal{V}^{\mathrm{FN}}$ and $\mathcal{V}^{(2)}_{\mathcal{N}}$. Let $\mathcal{Q} = \bigwedge^*\langle a,b\rangle$ be the exterior algebra over $\mathbb{R}^2$ with basis $a,b \in \mathbb{R}^2$. We obtain a canonical isomorphism, which is defined on monomial elements as follows:

$$i_*\colon \mathcal{Q}^{\otimes g} \xrightarrow{\sim} \bigwedge\nolimits^* H_1(\Sigma_g) : q_1\otimes\cdots\otimes q_g \mapsto i_1(q_1)\wedge\cdots\wedge i_g(q_g), \tag{78}$$

where $i_j\colon \mathcal{Q} \xrightarrow{\sim} \bigwedge^*\langle [a_j],[b_j]\rangle$ is the canonical map sending a and b to $[a_j]$ and $[b_j]$ respectively. Next, we define an isomorphism between $\mathcal{Q}$ and $\mathcal{N}_0$, seen as linear spaces, by the following assignment of basis vectors:

$$\phi\colon \mathcal{N}_0 \xrightarrow{\sim} \mathcal{Q} \qquad \text{with} \qquad \begin{matrix} \phi(1) = b, & \phi(\bar\theta\theta) = a, \\ \phi(\theta) = a\wedge b, & \phi(\bar\theta) = 1. \end{matrix} \tag{79}$$

Note that this map has odd $\mathbb{Z}/2$-degree and is, in particular, not an algebra homomorphism. From (79) we infer directly the following identities:

$$\phi(\theta x) = -\phi(x)\wedge a, \quad \phi(x\theta) = a\wedge\phi(x), \tag{80}$$

$$\phi(\mathbb{A}x) = [A_1]\phi(x), \quad \phi(\mathbb{S}x) = [S_1]\phi(x). \tag{81}$$

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Here, $\mathbb{A}$ and $\mathbb{S}$ are as in (55), and $[A_1]$ and $[S_1]$ are the maps on $H_1(\Sigma_1)$ as in (19) and (21).

Moreover, let us introduce a sign-operator $(-1)^{\Lambda}$ on $\mathcal{Q}^{\otimes g}$ defined on monomials by

(82) $$(-1)^{\Lambda_g}(q_1 \otimes \cdots \otimes q_g) = (-1)^{\lambda_g(d_1,\ldots,d_g)} q_1 \otimes \cdots \otimes q_g.$$

The function λ_N is defined in the N-fold product of $\mathbb{Z}/2$'s as follows:

(83) $$\lambda_N \colon (\mathbb{Z}/2)^N \to \mathbb{Z}/2 \quad \text{with} \quad \lambda_N(d_1,\ldots,d_N) = \sum_{i<j} d_i(1-d_j),$$

where $d_j = \deg(q_j) \bmod 2$. Consider now the following isomorphism of vector spaces:

(84) $$\xi_g := i_* \circ (-1)^{\Lambda_g} \circ \phi^{\otimes g} : \mathcal{N}_0^{\otimes g} \xrightarrow{\sim} \bigwedge\nolimits^* H_1.$$

Given a linear map, $F \colon \mathcal{N}^{\otimes g_1} \to \mathcal{N}^{\otimes g_2}$, we write $(F)^{\xi} := \xi_{g_2} \circ F \circ \xi_{g_1}^{-1}$ for the respective map on homology. Moreover, we denote by $\mathbf{L}_x^{(k)}$ the operator on $\mathcal{N}^{\otimes g}$ that multiplies the k-th factor in the tensor product by x from the left, and by $\mathbf{R}_x^{(k)}$ the respective operator for multiplication from the right. We compute:

(85) $$\begin{aligned} (\mathbf{L}_\theta^{(k)})^{\xi}(\alpha \wedge u_k \wedge \beta) &= (-1)^{g-k+s+1} \alpha \wedge a_k \wedge u_k \wedge \beta, \\ \text{and} \quad (\mathbf{R}_\theta^{(k)})^{\xi}(\alpha \wedge u_k \wedge \beta) &= (-1)^{g-k+s} \alpha \wedge u_k \wedge a_k \wedge \beta, \end{aligned}$$

where $s = \sum_{j=1}^{g} d_j$ is the total degree of $\alpha \wedge u_k \wedge \beta$, $\alpha \in \bigwedge^* \langle a_1, \ldots, b_{k-1} \rangle$, and $\beta \in \bigwedge^* \langle a_{k+1}, \ldots, b_g \rangle$.

Lemma 16 *For every standard generator $G \in \{A_j, D_j, S_j\}$, we have*

$$\big(\mathcal{V}_{\mathcal{N}}(\mathbf{I}_G)\big)^{\xi} = \bigwedge\nolimits^* [G],$$

where $[G]$ denotes as before the action on homology.

Proof For the A_j and S_j this follows readily from (81), and the fact that $[A_j]$ and $[S_j]$ do not change the degrees d_j and hence commute with $(-1)^{\Lambda_g}$.

The operator in (54) decomposes into $\mathbb{D} = \mathbb{D}^0 + \mathbb{D}^1$, where $\mathbb{D}^0 = id - \mathbf{R}_\rho \otimes \mathbf{L}_\rho$ and $\mathbb{D}^1 = \mathbf{R}_\theta \otimes \mathbf{L}_{\bar\theta} - \mathbf{R}_{\bar\theta} \otimes \mathbf{L}_\theta$. Now $\mathbb{D}^0$ does not change the $\mathbb{Z}/2$-degree of both factors, and $\mathbb{D}^1$ flips the degree of both factors. One readily verifies that

$$\lambda_g(\ldots, 1-d_j, 1-d_{j+1}, \ldots) - \lambda_g(\ldots, d_j, d_{j+1}, \ldots) = d_j + d_{j+1} \quad \bmod 2$$

so that

$$\begin{aligned} \mathcal{V}_{\mathcal{N}}(\mathbf{I}_{D_j})^{\xi} &= \big(\mathcal{V}^0_{\mathcal{N}}(\mathbf{I}_{D_j})\big)^{\zeta} + (-1)^{d_j+d_{j+1}} \big(\mathcal{V}^1_{\mathcal{N}}(\mathbf{I}_{D_j})\big)^{\zeta} \\ &= \big(I^{\otimes j-1} \otimes (\mathbb{D}^0)^{\phi^{\otimes 2}} \otimes I^{\otimes g-j-1}\big)^{i_*} \\ &\quad + (-1)^{d_j+d_{j+1}} \big(I^{\otimes j-1} \otimes (\mathbb{D}^1)^{\phi^{\otimes 2}} \otimes I^{\otimes g-j-1}\big)^{i_*}. \end{aligned}$$

Here, $\zeta_g = i_* \circ \phi^{\otimes g}$ and $\mathcal{V}^i_{\mathcal{N}}(\mathbf{I}_{D_j})$ is the operator with $\mathbb{D}^i$ in j-th position. Since $\zeta_g = \zeta_1^{\otimes g}$, the ζ-conjugate maps only act on the generators $\{a_j, b_j, a_{j+1}, b_{j+1}\}$; the action is the same for all positions j. Observe that also $[D_j]$ acts only on the homology generators $\{a_j, b_j, a_{j+1}, b_{j+1}\}$. It is, therefore, enough to prove the relation for $g = 2$ and $\mathcal{V}_{\mathcal{N}}(\mathbf{I}_{D_1}) = \mathbb{D}$.

Now, from (54) it is obvious that $\mathcal{V}_{\mathcal{N}}(\mathbf{I}_{D_j})$ commutes with $\mathbf{L}_\theta^{(j)}$ and $\mathbf{R}_\theta^{(j+1)}$. Moreover, it is easy to see that $\bigwedge^*[D_j]$, as given in (20), commutes with $(\mathbf{L}_\theta^{(j)})^\xi$ and $(\mathbf{R}_\theta^{(j+1)})^\xi$ from (85). Specifically, we use that $\bigwedge^*[D_j]$ does not change the total degree, and acts trivially on a_j and a_{j+1}. It thus suffices to check

$$\bigwedge^2[D_1] \circ \zeta_2(x_1 \otimes x_2) = \zeta_2 \circ \mathbb{D}^0(x_1 \otimes x_2) + (-1)^{d_1+d_2}\zeta_2 \circ \mathbb{D}^1(x_1 \otimes x_2) \tag{86}$$

with $d_i = \deg\big(\phi(x_i)\big)$, and only for $x_i \in \{1, \bar{\theta}\}$. For example, for $x_1 = x_2 = 1$, with $d_1 + d_2 = 0$, we find from (54) and (20) that

$$\begin{aligned}
\zeta_2 \circ \mathbb{D}(1 \otimes 1) &= \zeta_2(1 \otimes 1 + \theta \otimes \bar{\theta} - \bar{\theta} \otimes \theta - \rho \otimes \rho) \\
&= b_1 \wedge b_2 + a_1 \wedge b_1 - a_2 \wedge b_2 - a_1 \wedge a_2 \\
&= (b_1 - a_2) \wedge (b_2 - a_1) = \bigwedge^2[D_1](b_1 \wedge b_2) = \bigwedge^2[D_1]\big(\zeta_2(1 \otimes 1)\big).
\end{aligned}$$

We also compute, for the case $x_1 = \bar{\theta}$ and $x_2 = 1$, with $d_1 + d_2 = 1$:

$$\begin{aligned}
\zeta_2 \circ (\mathbb{D}^0 - \mathbb{D}^1)(\bar{\theta} \otimes 1) &= \zeta_2(\bar{\theta} \otimes 1 - \bar{\theta}\theta \otimes \bar{\theta}) = b_2 - a_1 \\
&= \bigwedge^2[D_1](b_2) = \bigwedge^2[D_1]\big(\zeta_2(\bar{\theta} \otimes 1)\big).
\end{aligned}$$

The other two cases follow similarly. ∎

As the $\{A_j, D_j, S_j\}$ generate Γ_g we conclude from Lemma 16 and (13) that $\big(\mathcal{V}_{\mathcal{N}}(\mathbf{I}_\psi)\big)^\xi = \mathcal{V}^{\mathrm{FN}}(\mathbf{I}_\psi)$ for all $\psi \in \Gamma_g$. Let us also consider the maps associated by both functors to the handle additions $\mathbf{H}_g^\pm$. We note that

$$\lambda_{g+1}(d_1, \ldots, d_g, 1) = \lambda_g(d_1, \ldots, d_g)$$

so that we find from (56), (15) and (79) that $\big(\mathcal{V}_{\mathcal{N}}(\mathbf{H}_g^+)\big)^\xi = \mathcal{V}^{\mathrm{FN}}(\mathbf{H}_g^+)$. Similarly, (57), (16) and (43) imply $\big(\mathcal{V}_{\mathcal{N}}(\mathbf{H}_g^-)\big)^\xi = \mathcal{V}^{\mathrm{FN}}(\mathbf{H}_g^-)$. Using the Heegaard decomposition (10) we finally infer equivalence:

Proposition 10 *The maps ξ_g defined in (84) give rise to an isomorphism*

$$\xi \colon \mathcal{V}_{\mathcal{N}} \xrightarrow{\bullet\cong} \mathcal{V}^{\mathrm{FN}}$$

of relative, non-semisimple, $\mathbb{Z}/2$-projective functors from $\mathcal{C}ob_3^\bullet$ to $\mathrm{Vect}(\mathbb{K})$.

10 Hard–Lefschetz Decomposition and Invariants

The tangent bundle over the moduli space $J(\Sigma_g)$ is trivial with fiber $H^*(\Sigma_g,\mathbb{R})$ so that its cohomology ring is naturally $\bigwedge^* H_1(\Sigma_g,\mathbb{R})$. There is an almost complex structure on $J(\Sigma_g)$ given by a map J with $\mathrm{J}^2 = -1$ in the cohomology. It is given by $\mathrm{J}.[a_j] = -[b_j]$ and $\mathrm{J}.[b_j] = [a_j]$. With the Kähler form $\omega_g \in H^2\big(J(\Sigma_g)\big)$ defined in (23) it is also a Kähler manifold. The dual Kähler metric provides us with a Hodge star $\star\colon \bigwedge^j H_1(\Sigma_g) \to \bigwedge^{2g-j} H_1(\Sigma_g)$ for a given volume form $\Omega \in \bigwedge^{2g} H_1(\Sigma_g)$ by the equation $\alpha \wedge \star\beta = \langle \alpha, \beta\rangle \Omega$. Specifically, for the $2g$ generators $\{[a_1],\dots,[b_g]\}$ of $H_1(\Sigma_g)$ with volume form $\Omega = [a_1]\wedge\cdots\wedge[b_g]$, the Hodge star is given by $\star(a_1^{1-\epsilon_1}\wedge \cdots \wedge b_g^{1-\epsilon_{2g}}) = (-1)^{\lambda_{2g}(\epsilon_1,\dots,\epsilon_{2g})} a_1^{\epsilon_1}\wedge\cdots\wedge b_g^{\epsilon_{2g}}$, where λ_{2g} is as in (83).

As a Kähler manifold, $H^*\big(J(\Sigma_g)\big)$ admits an $\mathrm{SL}(2,\mathbb{R})$-action, see for example [13], given for the standard generators $E, F, H \in \mathfrak{sl}_2(\mathbb{R})$ by

$$(87)\qquad H\alpha := (j-g)\alpha \quad \forall \alpha \in \bigwedge^j H_1(\Sigma_g), \quad E\alpha := \alpha \wedge \omega_g, \quad F := \star \circ E \circ \star^{-1}.$$

Lemma 17 *The functor $\mathcal{V}^{\mathrm{FN}}$ is $\mathrm{SL}(2,\mathbb{R})$-equivariant with respect to the action in (87).*

Proof Commutation with H follows from counting degrees. Since ω_g is invariant under the $\mathrm{Sp}(2g,\mathbb{R})$-action, E commutes with the maps in (13), and since $\omega_g \wedge [a_{g+1}] = [a_{g+1}] \wedge \omega_{g+1}$, also with the ones in (15) and (16). Finally, as all maps $\mathcal{V}^{\mathrm{FN}}(M)$ are isometries with respect to $\langle\, .\,,.\,\rangle$ they also commute with F. ■

In order to finish the proof of Theorem 1 we still need to show that the ξ_g are $\mathrm{SL}(2,\mathbb{R})$-equivariant as well. The fact that H commutes with ξ_g is again a matter of counting degrees. We have $E = \sum (E_1^{(i)})^{i_*}$, where $E_1^{(i)}$ acts on the i-th factor of $\mathcal{Q}^{\otimes g}$ by $q \mapsto E_1(q) = q \wedge a \wedge b$. Since E does not change degrees we find that $E^{\xi} = \sum (E^{(i)})^{\phi^{(i)}}$, where $(E^{(i)})^{\phi^{(i)}}$ acts on the i-th factor by E_1^{ϕ}. We find $E_1^{\phi}(\bar\theta) = \theta$, and $E_1^{\phi}(1) = E_1^{\phi}(\theta) = E_1^{\phi}(\bar\theta\theta) = 0$, which yields precisely the desired action of E on $\mathcal{N}_0$. The conjugate action of $\star$ on $\mathcal{N}_0^g$ is as follows:

$$(88)\qquad \star^{\xi}\colon x_1 \otimes\cdots\otimes x_g \mapsto (-1)^{\sum_{i<j} d_i d_j} (\star x_1) \otimes \cdots \otimes (\star x_g) \quad \forall x_j \in \mathcal{N}_0,$$

where $\star\theta = \bar\theta$, $\star\bar\theta = \theta$, $\star\bar\theta\theta = 1$, and $\star 1 = -\bar\theta\theta$. From this we see that F^{ξ} acts on each factor by $F_1^{\phi}(\theta) = \bar\theta$, and $F_1^{\phi}(1) = F_1^{\phi}(\bar\theta) = F_1^{\phi}(\bar\theta\theta) = 0$, as required.

With Lemma 17 and equivariance of ξ_g we have thus completed the proof of Theorem 1. Henceforth, we will use the simpler notation $\mathcal{V} = \mathcal{V}^{\mathrm{FN}} = \mathcal{V}_{\mathcal{N}}$. ■

The $\mathrm{SL}(2,\mathbb{R})$-action implies a Hard–Lefschetz decomposition [13] as follows:

$$(89)\qquad H^*\big(J(\Sigma_g)\big) \cong \bigoplus_{j=1}^{g+1} V_j \otimes W_{g,j}.$$

Here, V_j is the irreducible $\mathfrak{sl}_2$-module with $\dim(V_j) = j$, and

$$(90)\qquad W_{g,j} := \{u \in \bigwedge^{g-j+1} H_1(\Sigma_g) : F.u = 0\}$$

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is the space of Ω-*harmonic* vectors of degree $(g-j+1)$, or, equivalently, the space of $\mathfrak{sl}_2$-highest weight vectors of weight $(j-1)$. On each of these spaces we have an action of the mapping class groups from (13) factoring through $\mathrm{Sp}(2g,\mathbb{R})$.

Theorem 11 **([12, Theorem 5.1.8])** *Each $W_{g,j}$ is an irreducible $\mathrm{Sp}(2g,\mathbb{R})$-module with fundamental highest weight ϖ_{g-j+1} and dimension*

$$\dim(W_{g,j}) = \binom{2g}{g-j+1} - \binom{2g}{g-j-1}.$$

In particular, the pair of subgroups

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{Sp}(2g,\mathbb{R}) \subset \mathrm{GL}\Big(H^*\big(J(\Sigma_g)\big)\Big)$$

forms a Howe pair, that is, the two subgroups are exact commutants of each other.

The fundamental weights are given as in [12] by $\varpi_k = \epsilon_1 + \cdots + \epsilon_k$ with ϵ_j as in (22).

In the decomposition into irreducible TQFT's the one for $j = 1$ associated to the trivial $\mathrm{SL}(2,\mathbb{C})$ representation plays a special role for invariants of closed manifolds.

For any invariant, τ, of closed 3-manifolds there is a standard "reconstruction" of TQFT vector spaces as follows. We take the formal $\mathbb{K}$-linear span $\mathfrak{C}_g^+$ of cobordisms $M\colon \varnothing \to \Sigma_g$ and $\mathfrak{C}_g^-$ of cobordisms $N\colon \Sigma_g \to \varnothing$. We obtain a pairing $\mathfrak{C}_g^- \times \mathfrak{C}_g^+ \to \mathbb{K}\colon (N,M) \to \tau(N \circ M)$. If $\mathfrak{N}_g^+ \subset \mathfrak{C}_g^+$ is the null space of this pairing we define $\mathcal{V}^{\tau\text{-rec}}(\Sigma_g) = \mathfrak{C}_g^+/\mathfrak{N}_g^+$. For generic τ these vector spaces are infinite dimensional. The exception is when τ stems from a TQFT. In this case $\mathcal{V}^{\tau\text{-rec}}(\Sigma_g)^* = \mathfrak{C}_g^-/\mathfrak{N}_g^-$, and the linear map $\mathcal{V}^{\tau\text{-rec}}(P)$ associated to a cobordism P is reconstructed from its matrix elements $\tau(N \circ P \circ M)$. This construction, which basically imitates the GNS construction of operator algebras, is folklore since the emergence of TQFT's and appears, for example, in [46].

Theorem 12

(1) *The TQFT functor from Theorem 1 decomposes into a direct sum*

$$\mathcal{V} = \bigoplus \mathbb{R}^j \otimes \mathcal{V}^{(j)} = \mathcal{V}^{(1)} \oplus \mathbb{R}^2 \otimes \mathcal{V}^{(2)} \oplus \mathbb{R}^3 \otimes \mathcal{V}^{(3)} \cdots$$

of irreducible TQFT's with multiplicities.

(2) *The associated vector space for each TQFT is $\mathcal{V}^{(j)}(\Sigma_g) = W_{g,j}$ so that $\mathcal{V}^{(j)}(\Sigma_g) = 0$ whenever $j > g+1$. In particular, for any closed 3-manifold M and $j > 1$ we have $\mathcal{V}^{(j)}(M) = 0$ so that $\mathcal{V}(M) = \mathcal{V}^{(1)}(M)$.*

(3) *The vector spaces associated to the invariant $\pm\eta$ from (77) are finite dimensional. The reconstructed $\mathbb{Z}/2$-projective TQFT is $\mathcal{V}^{\eta\text{-rec}} = \mathcal{V}^{(1)}$ with dimensions $\dim\big(\mathcal{V}^{\eta\text{-rec}}(\Sigma_g)\big) = \dim(W_{g,1}) = \frac{2}{g+2}\binom{2g+1}{g}$.*

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Proof The fact that the TQFT's decompose in the prescribed manner follows from the $SL(2,\mathbb{R})$-covariance. Irreducibility of each $\mathcal{V}^{(j)}$, meaning there are no proper sub-TQFT's, results from the fact that each $Sp(2g,\mathbb{Z})$ representation is irreducible so that in a sub-TQFT the vector spaces for each g are either $\mathcal{V}^{(j)}(\Sigma_g)$ or 0. Since the handle maps yield non-zero maps between these vector spaces if one space is non-zero none of them can be. The reconstructed TQFT must be a quotient TQFT of $\mathcal{V}^{(1)}$, which is, however, irreducible. Hence, they are equal. ■

Let us finally give an alternative proof of Lemma 15 using the language in which the Frohman–Nicas invariant is constructed.

We present M by a Heegaard splitting $M_\psi = h_g^- \circ \mathbf{I}_\psi \circ h_g^+$, as defined in (10) and (58). The invariant is given as the matrix coefficient of $\bigwedge^g[\psi]$ for the basis vector $\mathcal{V}(h_g^+) = [a_1] \wedge [a_2] \wedge \cdots \wedge [a_g]$. If we denote by $[\psi]_{aa}$ the $g \times g$-block of $[\psi]$ acting on the Lagrangian subspace spanned by the $[a_i]$'s this number is just $\det([\psi]_{aa})$. At the same time, the Mayer–Vietoris sequence for M_ψ shows that $[\psi]_{aa}$ is a presentation matrix for the group $H_1(M_\psi,\mathbb{Z})$ so that the order of $H_1(M_\psi,\mathbb{Z})$ is, indeed, given by $\pm\det([\psi]_{aa})$. ■

11 Alexander–Conway Calculus for 3-Manifolds

Let M be a 3-manifold with an epimorphism $\varphi\colon H_1(M,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}$. We recall the definition of the *(reduced) Alexander polynomial* $\Delta_\varphi(M)$, as it is given in the case of knot and link complements for example in [4].

Let $\widetilde{M} \to M$ be the cyclic cover associated to φ and view $H_1(\widetilde{M})$ as a $\mathbb{Z}[t,t^{-1}]$-module with t acting by Decktransformation. Let $E_1 \subset \mathbb{Z}[t,t^{-1}]$ be the first elementary ideal generated by the $n \times n$ minors of an $n \times m$ presentation matrix $A(t)$ of $H_1(\widetilde{M})$. Then $\Delta_\varphi(M)$ is the generator of the smallest principal idea containing E_1, or, equivalently, the g.c.d. of the $n \times n$ minors of a presentation matrix. Particularly, if $A(t)$ is a square matrix $\Delta_\varphi(M) = \det\big(A(t)\big)$ and if $n > m$, *i.e.*, there are more rows than columns, $\Delta_\varphi(M) = 0$.

Another important invariant of a 3-manifold is its Reidemeister Torsion, which is obtained as the torsion of a chain complex over $\mathbb{Q}[t,t^{-1}]$ obtained from a cell decomposition of $\widetilde{M}$. The Alexander polynomial turns out to be almost the same as the Reidemeister Torsion of a 3-manifold. The relation described in the next theorem was first proven for homology circles by Milnor and in the general case by Turaev.

Theorem 13 **([38], [45])** *Let M be a compact, oriented 3-manifold, $\varphi\colon H_1(M) \to \mathbb{Z}$ an epimorphism as above, $r_\varphi(M)$ its Reidemeister Torsion, and $\Delta_\varphi(M)$ its Alexander polynomial.*

(1) *If $\partial M \neq \varnothing$ then $r_\varphi(M) = \frac{1}{(t-1)}\Delta_\varphi(M)$.*
(2) *If $\partial M = \varnothing$ then $r_\varphi(M) = \frac{1}{(t-1)^2}\Delta_\varphi(M)$.*

For a 3-manifold given by surgery along a framed link we will now give a procedure to compute the Alexander polynomial (and thus also Reidemeister Torsion).

Let $\mathcal{Z} \sqcup \mathcal{L} \subset S^3$ be a framed link consisting of a framed link $\mathcal{L}$ and a curve $\mathcal{Z}$ which has trivial linking number of all components of $\mathcal{L}$, *i.e.*, with $\mathcal{L} \cdot \mathcal{Z} = 0$. We denote by

$M^{\bullet}_{\mathcal{Z},\mathcal{L}}$ the manifold obtained by cutting out a tubular neighborhood of $\mathcal{Z}$ and doing surgery along $\mathcal{L}$. Hence, $\partial M^{\bullet}_{\mathcal{Z},\mathcal{L}} = S^1 \times S^1$, with canonical meridian and longitude (given by 0-framing). Also let $M_{\mathcal{Z},\mathcal{L}}$ be the closed manifold obtained by doing 0-surgery along $\mathcal{Z}$ so that $M_{\mathcal{Z},\mathcal{L}} = M^{\bullet}_{\mathcal{Z},\mathcal{L}} \cup D^2 \times S^1$. The special component $\mathcal{Z}$ defines an epimorphism $\varphi_{\mathcal{Z}} \colon H_1(M^{(\bullet)}) \to \mathbb{Z}$, for example via intersection numbers with a Seifert surface. We write $\Delta_{\mathcal{Z},\mathcal{L}} = \Delta_{\varphi_{\mathcal{Z}}}(M_{\mathcal{Z},\mathcal{L}}) = \Delta_{\varphi_{\mathcal{Z}}}(M^{\bullet}_{\mathcal{Z},\mathcal{L}})$ for the associated reduced Alexander polynomial, which is the same in both cases.

Consider a general Seifert surface $\Sigma^{\bullet} \subset S^3$ with $\partial \Sigma^{\bullet} = \mathcal{Z}$ and $\Sigma^{\bullet} \cap \mathcal{L} = \varnothing$. By removing a neighborhood of the surface we obtain a relative cobordism $C^{\bullet}_{\Sigma} = M^{\bullet}_{\mathcal{Z},\mathcal{L}} - \Sigma^{\bullet} \times (-\epsilon, \epsilon)$ from $\Sigma^{\bullet}$ to itself. Similarly, $C_{\Sigma} = M_{\mathcal{Z},\mathcal{L}} - \Sigma \times (-\epsilon, \epsilon)$, where Σ is the closed, capped-off surface $\Sigma^{\bullet} \cup D^2$. The cobordism C_{Σ} is obtained from $C^{\bullet}_{\Sigma}$ by gluing in a full cylinder $D^2 \times [0, 1]$.

Denote by $\psi^{(\bullet)}_{\pm} \colon \Sigma^{\bullet}_{\pm} \hookrightarrow C_{\Sigma}$ the inclusion maps of the bounding surfaces, and by

$$A_{\pm} = H_1(\psi^{(\bullet)}_{\pm}) \colon H_1(\Sigma) \to H_1(C^{(\bullet)}_{\Sigma}) \to H_1^{\mathrm{free}}(C^{(\bullet)}_{\Sigma}),$$

the maps on the free part of homology, where the free part is $G^{\mathrm{free}} = \frac{G}{\mathrm{Tors}(G)}$. As $H_1(\widetilde{M}) \cong H_1^{\mathrm{free}}(\widetilde{M}) \oplus \mathrm{Tors}\big(H_1(M)\big) \otimes \mathbb{Z}[t, t^{-1}]$, we will consider the first elementary ideal for the free part, which differs only by a factor of $\big|\mathrm{Tors}\big(H_1(M)\big)\big|$.

Suppose first that C does not have interior homology. This means the $A_{\pm}$ can be presented as square matrices, and $A_+ - tA_-$ is a presentation matrix. Consequently $\Delta_{\mathcal{Z},\mathcal{L}} = \pm t^p \det(A_+ - tA_-)$. By some linear algebra [8] this is the same as the Lefschetz polynomial

$$\det(A_+ - tA_-) = \sum_{k=0}^{2g} (-t)^{2g-k} \,\mathrm{trace}\Big(\big(\bigwedge^k A_+\big) \circ \star^{-1} \circ \big(\bigwedge^{2g-k} A_-^*\big) \circ \star \Big).$$

In [8] it is also shown that the expression inside the trace is the same as $\mathcal{V}^{\mathrm{FN}}(C^{\bullet}_{\Sigma})_k$ or $\mathcal{V}^{\mathrm{FN}}(C_{\Sigma})_k$ depending on context. Hence, we have (multiplying by a unit $(-t)^{-g}$) that

$$\Delta_{\mathcal{Z},\mathcal{L}} = \sum_{k=0}^{2g} (-t)^{g-k} \,\mathrm{trace}\big(\mathcal{V}^{\mathrm{FN}}(C_{\Sigma})_k\big) \tag{91}$$

$$= \mathrm{trace}\big((-t)^{-H} \mathcal{V}^{\mathrm{FN}}(C_{\Sigma})\big) \tag{92}$$

$$= \sum_{j=1} [j]_{-t} \,\mathrm{trace}\big(\mathcal{V}^{(j)}(C_{\Sigma})\big) = \sum_{j=1} [j]_{-t} \Delta^{(j)}_{\mathcal{Z},\mathcal{L}}, \tag{93}$$

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. In (92) we used the generator H of the $\mathrm{SL}(2,\mathbb{R})$-Lefschetz action. Formula (93) is a consequence of the Hard–Lefschetz decomposition from (89). We call the invariant $\Delta^{(j)}_{\mathcal{Z},\mathcal{L}}$ the *j-th Alexander Character* of the Alexander polynomial.

In case C does have interior rational homology, the dimension of $H_1^{\mathrm{free}}(C^{(\bullet)}_{\Sigma})$ is bigger than $H_1(\Sigma)$ so that $H_1(\widetilde{M})$ has $\mathbb{Z}[t, t^{-1}]$ as a direct summand. Consequently, the Alexander polynomial vanishes. At the same time $\mathcal{V}^{\mathrm{FN}}(C_\Sigma)$ is zero since it is a non-semisimple TQFT. Hence, (93) holds for all cases.

Suppose that in our presentation $\mathcal{Z} \subset S^3$ is the unknot. In this case we can isotop the diagram $\mathcal{L} \sqcup \mathcal{Z} \subset S^3$ into the form shown on the left side of Figure 15. Specifically, we arrange it that the strands of one link component alternate orientations as we go from left to right. By application of the connecting annulus moves, see for example [21], we can modify the link further such that the resulting tangle $\mathcal{T}$ in the indicated box is admissible without through pairs as described in the beginning of Section 5 or, again, [21]. There is a canonical Seifert surface $\Sigma_{\mathcal{T}}$ associated to a diagram as in Figure 15 obtained by surgering the disc bounded by $\mathcal{Z}$ along the framed components of $\mathcal{L}$ emerging at the bottom side. By construction $\mathcal{T}$ is then a tangle presentation of $C_{\Sigma_{\mathcal{T}}}$.

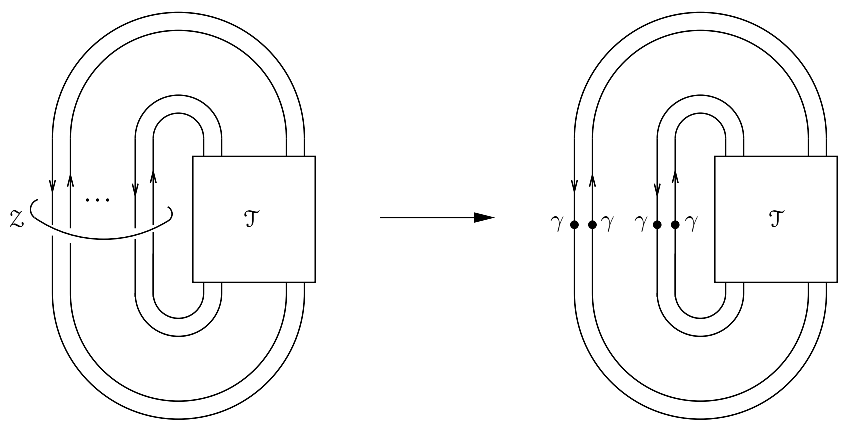


Figure 15: Standard presentation.

For the evaluation of this diagram it is convenient to introduce an extension of $\mathcal{N}$ over $\mathbb{Z}[t, t^{-1}]$, given by $\mathbb{Z}[\gamma^{\pm 1}] \ltimes \mathcal{N}$. The extra generator γ is group-like with $S(\gamma) = \gamma^{-1}$ and it acts on $\mathcal{N}$ by $\gamma x \gamma^{-1} = t^H x = t^{\deg(x)} x$ for $x \in \mathcal{N}$ and $\deg(x)$ the degree for homogenous elements.

In order to evaluate the diagram we apply the Hennings substitutions for crossing (29) and rules (30) through (32) to the $\mathcal{T}$ part to obtain a combination of $\mathcal{N}$-decorated arcs as in (33) and (34). Furthermore, we remove the circle $\mathcal{Z}$ at the expense of introducing a γ-decoration on each strand. The Hennings procedure is continued with the extended algebra over $\mathbb{Z}[t, t^{-1}]$. It is easy to see that the elements that have to be evaluated against the integral all lie in $\mathbb{Z}[t, t^{-1}] \otimes \mathcal{N}$ and that μ is cyclic also with respect to γ. Hence, the evaluation is well defined.

Lemma 18 *The evaluation procedure for a diagram as in Figure 15 yields the Alexander polynomial.*

Proof The standard evaluation of $\mathcal{T}$ yields a sum of diagrams with top and bottom arcs, where the j-th bottom arc is decorated by b_j and the j-th top arc by c_j as in (33) and (34). Hence, $\mathcal{V}_{\mathcal{N}}(C_\Sigma)$ is the sum over all diagrams of linear maps $\bigotimes_j^g \Big(b_j \otimes \mu\big(S(\_)c_j\big)\Big)$. The extended evaluation yields closed curves, each of which is decorated with four elements b_j, c_j, γ, and γ^{-1}. Using the antipodal sliding rule from (32) we collect them at one side of a circle so that the evaluation becomes

$$\begin{aligned}\mu\big(S^{-1}(b_j)\gamma c_j \gamma^{-1}\big) &= (-1)^{\deg(b_j)} t^{\deg(c_j)} \mu\big(S(b_j)c_j\big)\\ &= (-t)^{-\deg(b_j)} \operatorname{trace}\Big(b_j \otimes \mu\big(S(\_)c_j\big)\Big).\end{aligned}$$

Note here that $S^2(b_j) = (-1)^{\deg(b_j)}$ and that the evaluation is non-zero only if $\deg(c_j) + \deg(b_j) = 0$. The sum (over all decorations) of the products (over j) of these individual traces is thus just the trace of $(-t)^{-H}\mathcal{V}_{\mathcal{N}}(C_\Sigma)$. Since this is (up to sign) identical with $(-t)^{-H}\mathcal{V}^{\mathrm{FN}}(C_\Sigma)$, it follows from (92) that the evaluation gives the Alexander polynomial. ■

The evaluation of a standard diagram can be described also more explicitly without the use of the $\mathbb{Z}[\gamma]$ extension. Let $\mathcal{T}^\#: 2g \to 0$ be the diagram consisting of the tangle $\mathcal{T}: g \to g$ and the lower arcs. That is, $\mathcal{T} = (1^g \otimes \mathcal{T}^\#) \circ (X_g \otimes 1^g)$ and $\mathcal{T}^\# = (X_g^\dagger) \circ (1^g \otimes \mathcal{T})$, where $X_g^\dagger$ is the upside-down reflection of X_g. We define $A^\gamma \in \mathcal{N}_0^{\otimes 2} \otimes \mathbb{Z}[t, t^{-1}]$ as

$$A^\gamma = (\gamma \otimes 1)A(\gamma^{-1} \otimes 1) = \frac{1}{i}(\rho \otimes 1 + 1 \otimes \rho - t^{-1}\bar{\theta} \otimes \theta + t\theta \otimes \bar{\theta}). \tag{94}$$

Moreover, we define $A^\gamma_{\{g\}} \in \mathcal{N}_0^{\otimes 2g} \otimes \mathbb{Z}[t, t^{-1}]$ from A^γ as $A_{\{g\}}$ in (72) is defined from A in (70) and (71), or, equivalently, by

$$A^\gamma_g = (\gamma^{\otimes g} \otimes 1^{\otimes g}) \circ A_{\{g\}} \circ \big((\gamma^{-1})^{\otimes g} \otimes 1^{\otimes g}\big).$$

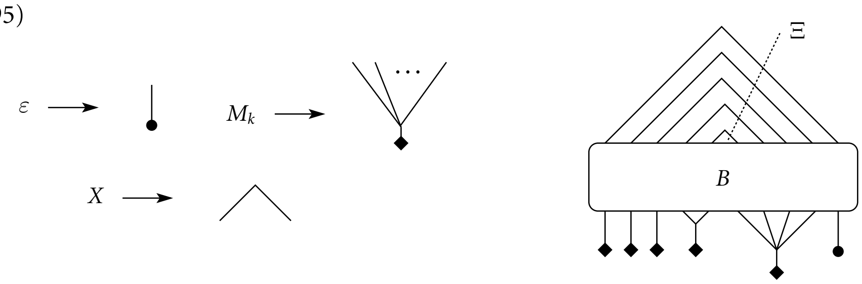
This tensor is assigned to the upper arcs and the γ elements in the standard diagram. Hence, by the extended Hennings evaluation procedure the Alexander polynomial is given by the composition

$$\Delta_{\mathcal{Z},\mathcal{L}} = \mathcal{V}^{\mathrm{FN}}(\mathcal{T}^\#)(A^\gamma_g),$$

where we think of $\mathcal{V}^{\mathrm{FN}}(\mathcal{T}^\#): \mathcal{N}_0^{\otimes 2g} \to \mathbb{C}$ as being naturally extended to a $\mathbb{Z}[t, t^{-1}]$-map from $\mathcal{N}_0^{\otimes 2g} \otimes \mathbb{Z}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}]$.

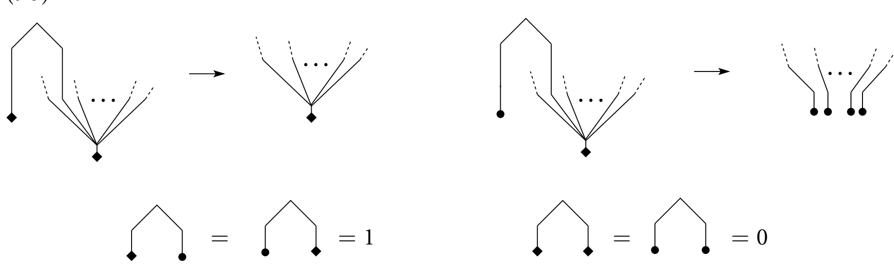
For further evaluation we use Theorem 8 to write $\mathcal{V}^{\mathrm{FN}}(\mathcal{T}^\#) = \sum_\nu \mathcal{V}^{\mathrm{FN}}(E_\nu)$ as a combination of elementary tangles $E_\nu = (M_{k_1} \otimes \cdots \otimes M_{k_r} \otimes \epsilon^{\otimes N}) \circ B$, so that the Alexander polynomial is the sum of polynomials $E_\nu(A^\gamma_g)$. For the computation of these elementary polynomials it is convenient to use the following graphical notation. As shown in (95) we indicate the morphism M_k by a tree with k incoming branches. The morphism X_1 is drawn as an arc and X_g as g concentric arcs.

(95)



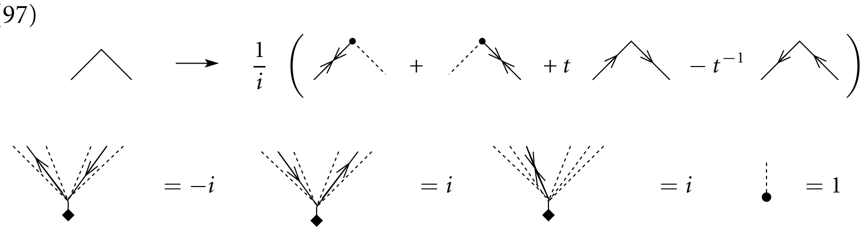
For $E = (M_1^{\otimes 3} \otimes M_2 \otimes M_4 \otimes \epsilon) \circ B$ we obtain the composite shown on the right of (95). Using relations $(\mu \otimes 1)A^\gamma = (1 \otimes \mu)A^\gamma = 1$, $(\varepsilon \otimes 1)A^\gamma = (1 \otimes \varepsilon)A^\gamma = \frac{1}{i}\rho$, and $\mu(x\frac{1}{i}\rho) = \varepsilon(x)$, we find the graphical relations depicted in (96).

(96)



Now, to each of the arcs the tensor A^γ is associated, containing the four terms $\rho \otimes 1$, $1 \otimes \rho$, $\bar{\theta} \otimes \theta$, and $\theta \otimes \bar{\theta}$ with coefficients of the form $\pm i t^m$. We represent the elementary polynomial thus as a sum over all combinations of these terms, *i.e.*, 4^g terms for $A^\gamma_{\{g\}}$. We indicate a combination in a diagram by drawing a line with a down arrow for $\bar{\theta}$, a line with an up arrow for θ, a line with arrows for ρ and a dashed line for 1. Hence, (94) becomes the first line in (97).

(97)



The tensors associated to the M_k are non-zero only in two cases. Namely, if one element is θ, another $\bar{\theta}$ and all others 1, or if one element is ρ and all others 1. In diagrams we obtain the evaluation rules as depicted. All other configurations are evaluated to zero.

For an elementary diagram let $N_x(= g)$ be the number of arcs at the top, N_0 the number of ε's, and N_k the number of M_k's at the bottom of the diagram for $k \geq 1$.

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Let us also call an elementary diagram *reduced* if $N_0 = N_1 = 0$. We can now give the recipe for evaluating elementary diagrams:

Proposition 14

(1) *We have the relations*

$$2N_x = N_0 + \sum_{k\geq 1} kN_k, \quad \textit{and} \quad N_x = \sum_{k\geq 1} N_k.$$

(2) *Every elementary diagram is zero or equivalent to a reduced one by application of the moves in (96).*

(3) *A reduced diagram is non-zero only of $N_j = 0$ for $j \geq 3$. That is, if the diagram is of the form $D = M_2^{\otimes g} \circ B \circ X_g$.*

(4) *A contributing reduced diagram $D = P_1 \sqcup \cdots \sqcup P_n$ is the union of closed paths P_j, and the polynomial $\Delta_D = \prod_j \Delta_{P_j}$ assigned to D is the product of the polynomials assigned to the the components P_j.*

(5) *The polynomial associated to a connected component is*

$$\Delta_P = 2 - (-1)^b (t^p + t^{-p}),$$

where p is the algebraic intersection number of the closed path P with a radial line segment Ξ as in (95), and b is the total number of half twists (or antipode insertions) in B.

Proof (1) In a diagram as in (95) the number of strands entering from the top is $2N_x$, two for each arc, and the number of strands entering from the bottom is $N_0 + \sum_{k\geq 1} kN_k$. Obviously, both numbers have to be equal. For an admissible configuration of a contributing diagram we can also call weighted edges, where the dashed ones are weighted 0, the ones with one arrow as 1, and those with double arrows as 2. The top part of the diagram shows that the total weight has to be $2N_x$ since every admissible arc has weight 2. Also every tree has weight 2 and the ϵ's have weight 0 so that the total weight must also be given by $\sum_{k\geq 1} 2N_k$.

(2) This is clear since every non-reduced one allows the application of a move that reduces the number of edges.

(3) If we subtract twice the second identity in (1) from the first we find $0 = N_0 - N_1 + N_3 + 2N_4 + 3N_5 + \cdots$. In the reduced case with $N_0 = N_1 = 0$, this implies $0 = N_3 = N_4 = N_5 = \cdots$, since these are all non-negative integers.

(4) Any graph where all vertices have valency 2 is the union of closed paths. Since we have a symmetric commutativity constraint we can untangle components from each other and move them apart. The evaluation of disjoint unions of diagrams is given by their products.

(5) There are four configurations that contribute to Δ_P for a closed path. Two of them are given by dashed lines alternating with double arrow lines. This corresponds to paring factors $\frac{1}{i}\rho$ with integrals μ in two different ways, each evaluated as 1. Thus these two cases contribute the 2 in the expression. The other two configurations are

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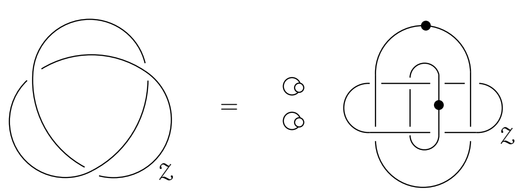
given by two orientations of P with single arrows everywhere. For one given orientation we get from (97) a factor $\frac{1}{i}t$ if P crosses Ξ left to right and a factor $\frac{1}{i}(-t^{-1})$ if P crosses right to left. Thus the arcs yield a tensor $\pm(\frac{1}{i})^g t^b(x_1 \otimes \cdots \otimes x_{2g})$, where each x_i is either θ or $\bar{\theta}$. Application of B yields a tensor $\pm(\frac{1}{i})^g t^b(y_1 \otimes \cdots \otimes y_g)$ where each y_j is either $\theta \otimes \bar{\theta}$ or $\bar{\theta} \otimes \theta$ depending on which way the path runs through the M_2 piece. The pairwise multiplication thus yields the tensor $\pm t^b(\frac{1}{i}\rho)^{\otimes g}$ and evaluation against μ yields the factor $\pm t^b$. For the opposite orientation the tensor for the arcs is obtained by exchanging t for t^{-1} and multiplying by a factor $(-1)^g$. The factor picked up by application of B is unchanged, and in the evaluation against the μ we pick up a factor $(-1)^g$ because the orders of θ and $\bar{\theta}$ are exchanged, canceling the one from the top. Hence the contribution for the opposite orientation is the same with t and t^{-1} exchanged. Thus $\Delta_P = 2 \pm (t^b + t^{-b})$. The sign can be determined by evaluating the polynomial at $t = 1$. This is identical with the usual Hennings invariant of the 3-manifold given by surgery along a link associated to the connected diagram P as follows.

First choose over and under crossing for P pushing it slightly outside the plane of projection into a knot P^*. This knot is thickened to a band $N(P^*)$, which is parallel to the plane of projection except for half twists that are introduced at the points where $B \subset P$ has antipodes inserted.

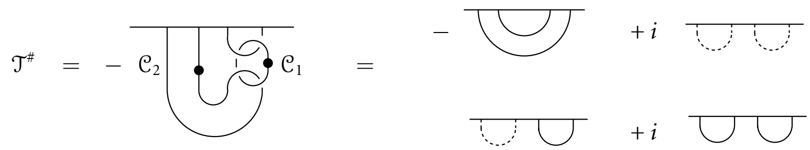
Consider the link $\partial N(P^*)$ given by the edges of the band. Generically this link consists of parallel strands that double cross as in Figure 10 at simple crossings of P^* and has Γ-diagram also as in Figure 10 for every half twist. We further modify this link at some generic point in the band by replacing the parallel strands by a configuration with a connecting annulus as in the σ-Move of (24). We obtain a two component link $\mathcal{L}_P = \mathcal{A}_P \sqcup \mathcal{C}_P$, where $\mathcal{A}_P$ is the 0-framed annulus. The other part $\mathcal{C}_P$ bounds the disc obtained by removing the small piece from the band where we applied the σ-Move, and thus carries a natural framing. We have by construction that $\Delta_P(1) = \pm\eta(M_{\mathcal{L}_P})$ with η as in (77). For self-intersection numbers we clearly have $\mathcal{A}_P \cdot \mathcal{A}_P = 0$ and $\mathcal{C}_P \cdot \mathcal{C}_P = 0$. For an even number of twists in the band $N(P^*)$ we obtain also $\mathcal{A}_P \cdot \mathcal{C}_P = 0$ and for an odd number of twists we have $\mathcal{A}_P \cdot \mathcal{C}_P = \pm 2$. Hence $\eta(M_{\mathcal{L}_P}) = 0$ in the first case and $\eta(M_{\mathcal{L}_P}) = 4$ in the second. ■

Note that the form of the Δ_P implies again the symmetry $\Delta(t) = \Delta(t^{-1})$ of the Alexander polynomial. In order to instill some confidence in our procedure let us recalculate the familiar formula for the left-handed trefoil in this setting. Using the Fenn–Rourke move from Figure 14 we present the trefoil as an unknotted curve $\mathcal{Z}$ in a surgery diagram of Borromean rings as in (98).

(98) 

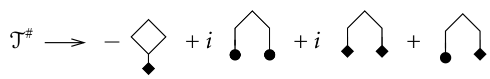
The standard form is obtained by moving $\mathcal{C}_1$ to the right off $\mathcal{Z}$ and letting $\mathcal{C}_2$ follow at the ends. The tangle $\mathcal{T}^\#$ is then as depicted on the left of (99) below. Using the framing moves from Figure 13 we expand it into elementary diagrams as on the right of (99).

(99)



The translation into Hopf algebra diagrams and subsequently polynomials is indicated next in (100).

(100)



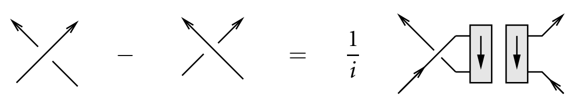
$$\longrightarrow\ -(2-t-t^{-1})+0+0+1$$

Thus the polynomial comes out to be $t+t^{-1}-1$ as it had to be. The same calculation carries through if we change the framings f_j of the components $\mathcal{C}_j$ in (98). The difference is the sign of the first summand, that is $\Delta_{\mathcal{Z}} = f_1 f_2(t+t^{-1}-2)+1$. Thus, if we flip both framings we obtain the right-handed trefoil with the same polynomial. If we flip only one framing so that $f_1 = -f_2$, we obtain one of two figure-eight knots with polynomial $-t - t^{-1} + 3$. Many other Alexander polynomials with multiple twists, as for example (p,q,r)-pretzel knots, can be computed quite conveniently in this fashion using Fenn–Rourke moves and the nilpotency of the ribbon element $v^k = 1 + k\rho$. Thus, our method proves to be quite useful in the calculation of the Alexander Polynomial for knots although its primary application is the generalization to 3-manifolds.

We describe next a more systematic way to unknot the special strand $\mathcal{Z}$ in a general diagram more akin the traditional skein theory. The additional relations that allow us to put any diagram $\mathcal{L} \sqcup \mathcal{Z}$ into a standard form are as follows.

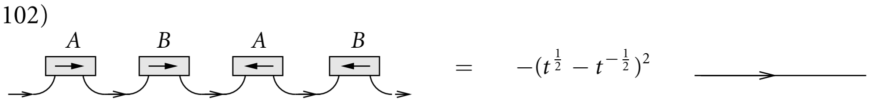
Proposition 15 *We have the following two skein relations for the special strand $\mathcal{Z}$*

(101)



and

(102)



as well as the slide and cancellations moves analogous to (60), and a vanishing property as in (61).

These equivalences allow us to express the Alexander polynomial of any diagram $\mathcal{Z} \sqcup \mathcal{L} \subset S^3$ as a combination of the evaluations of diagrams in standard form.

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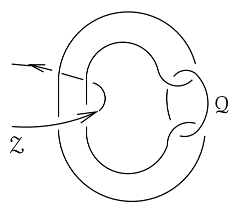
Proof As before we change a self-crossing of $\mathcal{Z}$ by sliding a 1-framed annulus $\mathcal{A}$ over the crossing. Note that we do not have to keep track of the framing of $\mathcal{Z}$ as it is unchanged and by convention zero. Using the orientation of $\mathcal{Z}$ we can do this such that the linking numbers of $\mathcal{Z}$ and $\mathcal{A}$ remain zero. It is easy to see that we can bring a diagram into the standard position as in Figure 15 without ever sliding a strand over the new component $\mathcal{A}$. The evaluation is obtained as the weighted trace over the linear map associated by $\mathcal{V}_\mathcal{N}$ to the cobordism represented by the tangle, which contains $\mathcal{A}$. Inserting the relation from Figure 13 we see that this linear map, and hence the associated polynomial, is the combination of the one for which $\mathcal{A}$ has been removed and the one for which the framing of $\mathcal{A}$ has been shifted by one. In both cases the unknotting procedure can be reversed so that we obtain the original pictures with $\mathcal{A}$ removed or its framing shifted by one. The situation in which $\mathcal{A}$ is removed corresponds to the opposite crossing. In the other contribution we have a 0-framed annulus around the crossing which can be rewritten as an index-1 surgery represented by a pair of coupons. This yields (101).

The coupon combination in (102) can be reexpressed by a tangle as in (64); it can be isotoped into the position shown in (103).

(103)



The extra tangle piece $\mathcal{Q}$ maps to the identity on a torus block. More precisely, $\mathcal{V}_\mathcal{N}(\mathcal{Q} \sqcup \mathcal{T}) = id_{\mathcal{N}_0} \otimes \mathcal{V}_\mathcal{N}(\mathcal{T})$. The weighted traces thus differ by a factor $\mathrm{trace}_{\mathcal{N}_0}\big((-t)^{-H}\big) = -t + 2 - t^{-1} = -(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$. ∎

For ordinary link and knot complements there are well known skein relations that uniquely characterize the Alexander–Conway polynomial of the knot, see for example [4, Chapter 12.C].

Corollary 16 *For ordinary knot complements (that is if $\mathcal{L} = \varnothing$) the relations in Proposition 15 reduce to the ordinary Alexander–Conway skein relations.*

Proof It is clear that with Proposition 15 we can resolve every diagram into disjoint circles in the plane with coupons on them in exactly the same way as for the Alexander–Conway polynomial. The difference is that wherever we pick up a factor $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ from the smoothening in the traditional calculus we obtain a factor $\frac{1}{t}$ and a pair of coupons in our case, but all other numbers are the same.

Suppose now after resolving the crossings we have more than one circle. Since the strand $\mathcal{Z}$ has to run though all of these components we must have coupons that are paired but on different circles. By (61) of Lemma 14 it follows that such a configuration must vanish. In the Alexander–Conway calculus we also have the rule that the link invariant for the unlinked union of an unknot with a non-trivial link is zero.

Hence we only need to compare the contributions that come from single circles. If in the process of applying the skein relations we carried out N smoothenings of crossings the circle will carry $2N$ coupons.

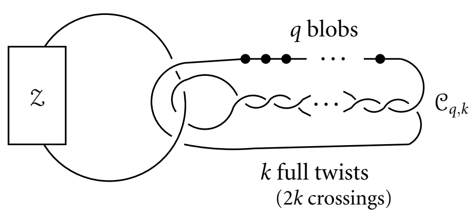
Next we claim that it is not possible to slide two paired coupons in adjacent position. To this end note that the coupons in the resolution of Proposition 15 stay all on one side of the special strand, *i.e.*, in the depicted orientation of $\mathcal{Z}$ the coupons are always on the left of $\mathcal{Z}$. Thus, if they become adjacent we would have a situation as in (62) of Lemma 14. This is not possible since then $\mathcal{Z}$ would have at least two components. Thus the number $2N$ of coupons will remain the same under handle slides.

We next observe that a circle with edges that are labeled in pairs and subject to handle slides also occurs in the classification of compact, oriented surfaces via their triangulations as in [34, Chapter 1]. It is shown there that any such configuration is, under application of handle slides and cancellation moves as in (62), equivalent to a sequence of blocks as in (102). As before we may assume that all coupons lie on one side of the circle. In fact, as $\mathcal{Z}$ is connected we see from [34] that we can move to the configuration in standard block form without the use of cancellations.

Thus, we have $\frac{N}{2}$ 4-coupon (torus) blocks as in (102) contributing a factor of $\left(-(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^2\right)^{N/2}=(i)^N(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^N$. Recall that in each resolution we also had a factor $\frac{1}{i}$, so that the total factor for the circle is just $(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^N$ and N is the number of smoothenings. But $(t^{\frac{1}{2}}-t^{-\frac{1}{2}})$ is precisely the factor assigned to each smoothening by the usual Alexander–Conway calculus. ∎

Although we now have a systematic procedure for computing the Alexander polynomial of a 3-manifold, it is often efficient to use the skein relations leading up to it directly. We illustrate this by computing $\Delta_{\mathcal{C}_{k,l},\mathcal{Z}}$, where $\mathcal{C}_{k,l}$ is the component depicted in (104):

(104)



The two middle strands are twisted with each other k times generating $2k$ crossings, and we have q full circles on the upper strand indicating shifts in the framing by -1. The definition for $k<0$ or $q<0$ is given by choosing the opposite twistings.

Lemma 19 *The Alexander Polynomial of $M_{\mathcal{C}_{k,l},\mathcal{Z}}$ is given by the ordinary Alexander polynomial of the knot as follows:*

$$\Delta_{\mathcal{C}_{k,l},\mathcal{Z}}=i\left(k(t+t^{-1})-q\right)\Delta_{\mathcal{Z}}.$$

Proof We combine every twist with two circles so that we have k twist configurations as in Figure 14 and $l = q - 2k$ remaining circles. Applying the Fenn–Rourke move to each of these, we obtain a configuration in which we have a parallel instead of twisted pair of strands in the middle, surrounded by k annuli with an empty circle on them. In addition, we have k separate annuli with full circles. Denote by $\Delta_{k,l}$ the associated Alexander Polynomial. For $k > 0$ we choose one of the first annuli and apply the framing shift relation (13) to the empty circle on it. In the second contribution we omit the dotted line so that we obtain the same configuration with one less annulus around the double strands. The factor i in (13) is canceled against one of the separate annuli with a full circle so that the second contribution is exactly $\Delta_{l,k-1}$. In the first contribution we have a 0-framed annulus which, by Figure 12, can be turned into a pair of coupons. The other $k - 1$ coupons can thus be slid off and canceled against $k - 1$ annuli with full circles. Moreover, the remaining l full circles on the upper strand can be removed, since inserting a dotted line leaves two isolated coupons, which yields zero. The resulting configuration is the knot $\mathcal{Z}$ with a tangle piece $\mathcal{Q}$ as in (103), contributing an extra factor $-(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$, and an extra annulus with full circle with a factor $-i$. We thus obtain the recursion relation $P_{k,l} = i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 + P_{k-1,l}$ so that $P_{k,l} = ik(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 \Delta_{\mathcal{Z}} + P_{0,l}$. But the configuration for $k = 0$ is the separate union of $\mathcal{Z}$ and an annulus with l full circles. The latter yields a factor $-il$ so that $P_{k,l} = i\big(k(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 - l\big)\,\Delta_{\mathcal{Z}}$, which computes to the desired formula. ∎

12 Lefschetz Compatible Hopf Algebra Structures on $H^*\big(J(\Sigma)\big)$

It is easy to see that the natural ring structure on the cohomology $H^*\big(J(\Sigma)\big) \cong \bigwedge^* H_1(\Sigma)$ is not compatible with the $\mathrm{SL}(2,\mathbb{R})$ Lefschetz action as described in Section 10. For example $E(x \wedge y) = x \wedge y \wedge \omega$ but $(Ex) \wedge y + x \wedge (Ey) = 2x \wedge y \wedge \omega$. The isomorphism with $\mathcal{N}_0^{\otimes g}$ however induces another multiplication structure compatible with the $\mathrm{SL}(2,\mathbb{R})$ action. In this section we will describe it explicitly.

The $\mathbb{Z}/2$-graded Hopf algebra structure on $\mathcal{N}_0$ given in Lemma 12 extends to a $\mathbb{Z}/2$-graded Hopf algebra structure $\mathcal{H}_{\mathcal{N}}$ on $\mathcal{N}_0^{\otimes g}$ with

$$(x_1 \otimes \cdots \otimes x_g)(y_1 \otimes \cdots \otimes y_g) = (-1)^{\sum_{i<j} d(x_j)d(y_i)} x_1 y_1 \otimes \cdots \otimes x_g y_g.$$

The formula for $\boldsymbol{\Delta}$ is the dual analog. The precise form of $\mathcal{H}_{\mathcal{N}}$ is given as follows:

Lemma 20 *For a choice of basis of $\mathbb{R}^g$ there is a natural isomorphism of Hopf algebras*

$$\varrho\colon \bigwedge^*(\mathbb{E} \otimes \mathbb{R}^g) \xrightarrow{\sim} \mathcal{N}_0^{\otimes g}$$

so that $\mathrm{Aut}(\mathcal{N}_0^{\otimes g}, \mathcal{H}_{\mathcal{N}}) \cong \mathrm{GL}(\mathbb{E} \otimes \mathbb{R}^g)$.

Proof Let $\{e_j\}$ be a basis of $\mathbb{R}^g$. The generating set of primitive vectors of $\bigwedge^*(\mathbb{E} \otimes \mathbb{R}^g)$ is given by $\mathbb{E} \otimes \mathbb{R}^g$. On this subspace we set $\varrho(w \otimes e_j) = 1 \otimes \cdots \otimes 1 \otimes w \otimes 1 \otimes \cdots \otimes 1$, with w in j-th position. We easily see that the vectors in $\varrho(\mathbb{E} \otimes \mathbb{R}^g)$ form again a generating set of anticommuting, primitive vectors of $\mathcal{N}_0^{\otimes g}$, so that ϱ extends to a Hopf algebra epimorphism. Equality of dimensions thus implies that ϱ is an isomorphism. ∎

The canonical $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{N}_0^{\otimes g}$ is still compatible with $\mathcal{H}_{\mathcal{N}}$ since it preserves the degrees and factors. Under the isomorphism in Lemma 20 it is readily identified as the $\mathrm{SL}(2,\mathbb{R})$-action on the $\mathbb{E}$-factor. The remaining action on the $\mathbb{R}^g$-part can be understood geometrically. Specifically, $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $\mathcal{N}_0^{\otimes g}$, since the $\mathcal{V}$-representation of the mapping class group factors through the symplectic group with representation $\mathcal{V}^{\mathrm{Sp}}\colon \mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{GL}(\mathcal{N}_0^{\otimes g})\colon [\psi] \mapsto \mathcal{V}^{\mathrm{Sp}}([\psi]) := \mathcal{V}(\mathbf{I}_\psi)$. For a given decomposition into Lagrangian subspaces we denote the standard inclusion

$$\kappa\colon\ \mathrm{SL}(g,\mathbb{Z}) \hookrightarrow \mathrm{GL}(g,\mathbb{Z}) \hookrightarrow \mathrm{Sp}(2g,\mathbb{Z})\colon A \mapsto \kappa(A) := A \oplus (A^{-1})^T. \tag{105}$$

Lemma 21 *The action of* $\mathrm{SL}(g,\mathbb{Z})$ *on* $\mathcal{N}_0^{\otimes g}$ *induced by* $\mathcal{V}^{\mathrm{Sp}}\circ\kappa$ *is compatible with* $\mathcal{H}_{\mathcal{N}}$*, and under the isomorphism* ϱ *from Lemma 20 it is identical with the* $\mathrm{SL}(g,\mathbb{Z})$*-action on* $\mathbb{R}^g$ *for the given basis. In particular, it commutes with the* $\mathrm{SL}(2,\mathbb{R})$*-action so that we have the following natural inclusion of the Howe pairs:*

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(g,\mathbb{Z}) \subset \mathrm{GL}(\mathbb{E}\otimes\mathbb{R}^g) = \mathrm{Aut}(\mathcal{N}_0^{\otimes g},\mathcal{H}_{\mathcal{N}}).$$

Proof Consider the elements $P_j := S_j \circ D_j^{-1} \circ S_j^{-1}$ and $Q_j := S_{j+1} \circ D_j^{-1} \circ S_{j+1}^{-1}$ of $\Gamma_{g,1}$. From (20) and (21) we compute the homological action as $[R_j] = \kappa(I_g + E_{j+1,j})$ and $[Q_j] = \kappa(I_g + E_{j,j+1})$, with conventions again as in [12]. The matrices $I_g + E_{j+1,j}$ and $I_g + E_{j,j+1}$ generate $\mathrm{SL}(g,\mathbb{Z})$, and hence $[P_j]$ and $[Q_j]$ generate $\kappa\big(\mathrm{SL}(g,\mathbb{Z})\big) \subset \mathrm{Sp}(2g,\mathbb{Z})$. The actions of $\mathcal{V}(\mathbf{I}_{P_j})$ and $\mathcal{V}(\mathbf{I}_{Q_j})$ on $\mathcal{N}_0^{\otimes g}$ are given by placing the maps $\mathbb{P} := (\mathbb{S}\otimes 1)\mathbb{D}^{-1}(\mathbb{S}^{-1}\otimes 1)$ and $\mathbb{Q} := (1\otimes\mathbb{S})\mathbb{D}^{-1}(1\otimes\mathbb{S}^{-1})$ in the j-th and $j+1$-st tensor positions. In order to show that the actions of P_j and Q_j on $\mathcal{N}_0^{\otimes g}$ yield Hopf algebra automorphisms it thus suffices to prove this for the maps $\mathbb{P}$ and $\mathbb{Q}$ in the case $g = 2$. From the tangle presentations we find identities $\mathbf{I}_{Q_1} = (\mathbf{M}\otimes 1)\circ(1\otimes\mathbf{\Delta})$ and $\mathbf{I}_{P_1} = (1\otimes\mathbf{M})\circ(\mathbf{\Delta}\otimes 1)$. It follows that $\mathbb{P}(x\otimes y) = \Delta_0(x)(1\otimes y)$ and $\mathbb{Q}(x\otimes y) = (x\otimes 1)\Delta_0(y)$. The fact that these are Hopf automorphisms on $\mathcal{N}_0\otimes\mathcal{N}_0$ can be verified by direct computations. For the multiplication this amounts to verification of equations such as $\Delta(w)1\otimes v = -1\otimes v\Delta(w)$, $\forall v, w\in\mathbb{E}$, and for the comultiplication we use the fact that $\mathcal{N}_0$ is self dual.

From the above identities we have that $\mathcal{V}(\mathbf{I}_{Q_1}) = (M_0\otimes 1)\circ(1\otimes\Delta_0)$, so that $\mathcal{V}(\mathbf{I}_{Q_j})$ is given on a monomial by taking the coproduct of the element in the $(j+1)$-st position, multiplying the first factor of that to the element in the j-th position and placing the second factor into the $(j+1)$-st position. We readily infer for every $w\in\mathbb{E}$ that $\mathcal{V}(\mathbf{I}_{Q_j})\big(\varrho(w\otimes e_k)\big) = \varrho(w\otimes e_k + \delta_{j+1,k} w\otimes e_j) = \varrho\big(w\otimes(I_g + E_{j+1,j})e_k\big)$. The analogous relation holds for $[P_j]$ so that

$$\mathcal{V}^{\mathrm{Sp}}\big(\kappa(A)\big)(w\otimes x) = w\otimes(Ax) \quad \forall A\in\mathrm{SL}(g,\mathbb{Z}).$$

This is precisely the claim made in Lemma 21. ∎

The structure $\mathcal{H}_{\mathcal{N}}$ is mapped by the isomorphism ξ_g from (84) to a $\mathbb{Z}/2$-graded Hopf algebra structure $\mathcal{H}_\Lambda$ on $H^*\big(J(\Sigma_g)\big)$. *A priori* the isomorphism ξ_g and thus

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also $\mathcal{H}_\Lambda$ depend on the choice of a basis of $H_1(\Sigma_g)$. However, the $\mathrm{SL}(g,\mathbb{Z})$-invariance determined in Lemma 21 translates to the $\mathrm{SL}(g,\mathbb{Z})$-invariance of $\mathcal{H}_\Lambda$, where $\kappa\big(\mathrm{SL}(g,\mathbb{Z})\big) \subset \mathrm{Sp}(2g,\mathbb{Z})$ acts in the canonical way on $H^*\big(J(\Sigma_g)\big)$. Hence, $\mathcal{H}_\Lambda$ only depends on the oriented subspaces $\Lambda = \langle [a_1],\dots,[a_g]\rangle \subset H_1(\Sigma_g,\mathbb{Z})$ and $\Lambda^* = \langle [b_1],\dots,[b_g]\rangle \subset H_1(\Sigma_g,\mathbb{Z})$, but not the specific choice of basis within them. The orientations can be given by volume forms $\omega_\Lambda := [a_1]\wedge\cdots\wedge[a_g]$ and $\omega_{\Lambda^*} := [b_1]\wedge\cdots\wedge[b_g]$. The primitive elements $\varrho(\theta\otimes e_j)$ and $\varrho(\bar\theta\otimes e_j)$ of $\mathcal{N}_g^{\otimes g}$ are mapped by ξ_g to

$$\pm[a_j]\wedge\omega_{\Lambda^*} \in \bigwedge^{g+1} H_1(\Sigma_g) \quad\text{and}\quad \pm\, i^*_{z_j}(\omega_{\Lambda^*}) \bigwedge^{g-1} H_1(\Sigma_g) \tag{106}$$

respectively, where $[a_j] \in H_1(\Sigma_g)$ and $z_j \in H^1(\Sigma_g)$, with $z_j([b_j]) = 1$ and $z_j([x]) = 0$ on all other basis vectors. We also have $\xi_g(1) = \omega_{\Lambda^*}$ and $\xi_g(\rho^{\otimes g}) = \omega_\Lambda$.

This completes the proof of Theorem 3. ■

In the remainder of this section we give a more explicit description of the structure $\mathcal{H}_\Lambda$ on $H^*\big(J(\Sigma_g)\big)$, and relate it to an involution, τ, on $H^*\big(J(\Sigma_g)\big)$, which acts as identity on the Λ-factor and, modulo signs, as a Hodge star on the opposite Λ^*-factor.

The product $\diamond$ on $\Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_\Lambda\Big)$ is given on a genus one block, $\bigwedge^*\langle [a],[b]\rangle$, as follows:

Table for $u\diamond t := \phi\big(\phi^{-1}(u)\phi^{-1}(t)\big)$

(107)

$u \backslash t$	1	$[a]$	$[b]$	$[a]\wedge[b]$
1	0	0	1	$[a]$
$[a]$	0	0	a	0
$[b]$	1	$[a]$	$[b]$	$[a]\wedge[b]$
$[a]\wedge[b]$	$-[a]$	0	$[a]\wedge[b]$	0

It extends to $\bigwedge^* H_1(\Sigma_g)$ via the formula

$$(u_1\wedge\cdots\wedge u_g)\diamond(t_1\wedge\cdots\wedge t_g) = (-1)^{\sum_{i<j} d_i l_j}(u_1\diamond t_1)\wedge\cdots\wedge(u_g\diamond t_g), \tag{108}$$

where $u_i, t_i \in \bigwedge^*\langle [a_i],[b_i]\rangle$, $d_i = 1-\deg(u_i)$ and $l_j = 1-\deg(t_j)$. In particular, we have $u\diamond t = (-1)^{dl} t\diamond u$, with $d = \sum_i d_i = g - \deg(u)$ and $l = \sum_i l_i = g-\deg(t)$, which reflects the $\mathbb{Z}/2$-commutativity of $H^*\big(J(\Sigma_g)\big)$.

The product structure and another proof of Lemma 21 can be also found from an involution, τ, defined as follows:

Every cohomology class $x \in H^*\big(J(\Sigma_g)\big)$ is uniquely written as $x = \alpha\wedge\beta$, where $\alpha \in \bigwedge^*\Lambda$ and $\beta \in \bigwedge^*\Lambda^*$. For x in this form the map τ is uniquely determined by the relations

$$\tau(\alpha\wedge\beta) = \alpha\wedge\tau(\beta) \quad\text{and}\quad \tau(b_1^{\epsilon_1}\wedge\cdots\wedge b_g^{\epsilon_g}) = b_1^{1-\epsilon_1}\wedge\cdots\wedge b_g^{1-\epsilon_g}. \tag{109}$$

From the formulas in (107) and (108) we find that $\tau^2 = 1$, and

$$\tau(u\diamond t) = \tau(t)\wedge\tau(u), \tag{110}$$

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and that τ maps $\bigwedge^* \Lambda$ as well as $\bigwedge^* \Lambda^*$ to itself. It is clear from (109) and (110) that $\mathrm{SL}(g, \mathbb{Z})$-variance of $\diamond$ on $H^*\big(J(\Sigma_g)\big)$ is equivalent to $\mathrm{SL}(g, \mathbb{Z})$-variance of $\diamond$ on $\bigwedge^* \Lambda^*$. Now, for any $A \in \mathrm{SL}(\Lambda^*)$ the following identity holds:

$$\tau \circ \Big(\bigwedge^* A\Big) \circ \tau = \bigwedge^* \iota(A), \tag{111}$$

where ι is the involution on $\mathrm{SL}(\Lambda^*)$ defined by

$$\iota(A) := D \circ (A^{-1})^T \circ D, \quad \text{with } D[b_j] = (-1)^j [b_j].$$

This can be proven either by considering again generators of $\mathrm{SL}(\Lambda^*)$, or by applying the generalized Leibniz formula for the expansion of the determinant of a $g \times g$-matrix into products of determinants of $k \times k$ and $(g-k) \times (g-k)$-submatrices. See also Lemma 5.2 in [9]. Now (110) together with (111) implies that $\diamond$ depends only on the decomposition $H_1(\Sigma_g, \mathbb{Z}) = \Lambda \oplus \Lambda^*$.

In summary, we have the following isomorphism of $\mathbb{Z}/2$-graded Hopf algebras:

$$\tau' := \bigwedge^* D \circ \tau \colon \Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_\Lambda\Big) \xrightarrow{\sim} \Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_{\mathrm{ext}}\Big),$$

The Howe pair $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(g, \mathbb{R}) \subset \mathrm{GL}\big(H_1(\Sigma_g)\big) = \mathrm{Aut}\Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_{\mathrm{ext}}\Big)$, with $H_1(\Sigma_g) = \mathbb{E} \otimes \Lambda$, is conjugated by τ' to the pair $\mathrm{SL}(2, \mathbb{R})_{\text{Lefsch.}} \times \kappa\big(\mathrm{SL}(g, \mathbb{R})\big) \subset \mathrm{Aut}\Big(H^*\big(J(\Sigma_g)\big), \mathcal{H}_\Lambda\Big)$.

13 More Examples of Homological TQFT's and Open Questions

A Relations to Gauge Theories and the TQFT-Ring Ω Generated by $\mathcal{V}$

We begin by collecting the ingredients that imply Theorem 4. The first identity (2) has already been computed in (93).

The invariants I^{DC} and I_d^{SW} are obtained by Donaldson in [5] from TQFT's $\mathcal{V}^{\mathrm{DC}}$ and $\mathcal{V}_d^{\mathrm{SW}}$ respectively. For both TQFT's the vector spaces associated to a surface Σ are the homologies of natural moduli spaces. In the case of $\mathcal{V}^{\mathrm{DC}}$ this is the moduli space $\mathcal{M}(\Sigma)$ of flat connections on a non-trivial $\mathrm{SO}(3)$ bundle. For $\mathcal{V}_d^{\mathrm{SW}}$ the moduli space of solutions to certain vortex equations is considered, which is in turn identified with the symmetric products of the surface. The action of the mapping class group on the resulting homologies also factors through the symplectic groups (with the familiar $\mathbb{F}_2$-ambiguity). Donaldson thus derives the following isomorphisms between $\mathrm{Sp}(2g, \mathbb{Z})$-modules.

$$\begin{aligned} \mathcal{V}^{\mathrm{DC}}(\Sigma_g) &= H_*\big(\mathcal{M}(\Sigma_g)\big) \cong \bigoplus_{j=0}^{g} \mathbb{Q}^{j^2} \otimes \bigwedge^{g-j} H_1(\Sigma_g) \quad \text{and} \\ \mathcal{V}_d^{\mathrm{SW}}(\Sigma_g) &= H_*\big(\mathrm{Sym}^k(\Sigma_g)\big) \cong \bigoplus_{j=1}^{g-d} \mathbb{Q}^{j} \otimes \bigwedge^{g-d-j} H_1(\Sigma_g), \end{aligned} \tag{112}$$

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where $k = g - 1 - d$ is the degree of the holomorphic line bundle of which the vortex solutions are sections. The $\mathrm{Sp}(2g, \mathbb{Z})$-representations can be further identified with the irreducible parts, which in our notation takes the form

$$\bigwedge^{g-j} H_1(\Sigma_g) = \mathcal{V}^{(j+1)}(\Sigma) \oplus \mathcal{V}^{(j+3)}(\Sigma) \oplus \mathcal{V}^{(j+5)}(\Sigma) \oplus \cdots . \tag{113}$$

Inserting (113) into the isomorphisms in (112) we find that the $\mathcal{V}^{\mathrm{DC}}(\Sigma_g)$ and $\mathcal{V}^{\mathrm{SW}}(\Sigma_g)$ are direct sums of the $\mathcal{V}^{(j)}(\Sigma_g)$ with g-independent multiplicities given by precisely the non-negative coefficients in (3) and (4). In Chapter 5 of [5] Donaldson exploits this fact to show that the decomposition thus extends to the entire TQFT's, meaning that cobordisms act trivially on the mulitiplicity spaces and have block-wise actions on the $\mathcal{V}^{(j)}$ components equivalent to those in the $\mathcal{V}^{\mathrm{FN}}$ case. In summary, we have the following isomorphisms *of TQFT's*:

$$\begin{aligned} \mathcal{V}^{\mathrm{DC}} &\cong \bigoplus_{j\geq 2} \mathbb{Q}^{\binom{j+1}{3}} \otimes \mathcal{V}^{(j)} \\ \mathcal{V}^{\mathrm{SW}}_d &\cong \bigoplus_{j\geq d+2} \mathbb{Q}^{[\![(\frac{j-d}{2})^2]\!]} \otimes \mathcal{V}^{(j)}. \end{aligned} \tag{114}$$

Identities (3) and (4) are now immediate. For the last equation (5) in Theorem 4 we refer to [24]. ■

In an effort to find new knot invariants Frohman and Nicas generalized their approach in [9] to higher rank Lie algebras. They construct a TQFT $\mathcal{V}_k^{\mathrm{PSU}(n)}$, whose vector spaces are given as intersection homology groups of certain restricted moduli spaces of $\mathrm{PSU}(n)$-representations. Further they derived from these via similar trace formulas invariants $\lambda_{n,k}$ depending on the rank n and weight k. In [7] Frohman finds a recursive procedure to compute the invariants $\lambda_{n,k}$ and shows that they are determined by the polynomial expressions in the coefficients of the Alexander polynomial. Consequently, they are also polynomial in the Alexander Characters, so that we can write

$$\lambda_{n,k} = R_{n,k}(\Delta^{(1)}, \Delta^{(2)}, \dots), \tag{115}$$

with $R_{n,k} \in \mathbb{Z}[x_1, x_2, \dots]$. A general closed formula and some integrality issues for the $R_{n,k}$ are still unresolved though, see also [2]. This relation in (115) is more general than those expressed in Theorem 4 as it is no longer linear.

More precisely, define the space of invariants $\mathbf{\Omega}^{[0]} = \{n_1\Delta^{(1)} + n_2\Delta^{(2)} + \cdots \mid n_i \in \mathbb{Z}^{+,0}\}$. Then it is clear that any invariant that descends from a TQFT that is homological must be in $\mathbf{\Omega}^{[0]}$, where the $n_i \geq 0$ are the multiplicities of the irreducible summands. Thus $I^{\mathrm{DC}}, I_d^{\mathrm{SW}} \in \mathbf{\Omega}^{[0]}$, but we also have $\lambda_L \notin \mathbf{\Omega}^{[0]}$ since some of the coefficients are negative. λ_L is, nevertheless, related to the quantum TQFT's, but the derivations use slightly more subtle p-modular interpretations, see [24].

Similar to sums we can derive the invariant given by the product of two Alexander Characters, say $\Delta^{(i)} \cdot \Delta^{(j)}$, from the tensor product of the corresponding TQFT's, namely $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$. Thus, if the coefficients of all the $R_{n,k}$ were non-negative integers we could easily produce a homological TQFT by taking corresponding direct sums and tensor products of the $\mathcal{V}^{(i)}$ in order to reproduce $\lambda_{n,k}$. This invariant, indeed, descends from the TQFT $\mathcal{V}_k^{PSU(n)}$, however, the coefficients of the $R_{n,k}$ can be negative. The point to observe here is that, for example, $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$ is generally not an irreducible TQFT and can be decomposed.

Denote by $\mathcal{V}^{(\vec{\lambda})}$ the irreducible TQFT's obtained as summands of quotients of multiple tensor products of the $\mathcal{V}^{(i)}$. The superscript label, $\vec{\lambda} \in \vec{\Lambda}$, may be roughly thought of as a semi-infinite branching path for $\mathrm{Sp}(2) \subset \mathrm{Sp}(4) \subset \mathrm{Sp}(6) \subset \cdots$. The space of TQFT's

$$\mathfrak{Q}^{[+]} = \Big\{ \bigoplus_{\vec{\lambda}\in\vec{\Lambda}} \mathbb{Q}^{n_{\vec{\lambda}}} \otimes \mathcal{V}^{(\vec{\lambda})} \;\Big|\; n_{\vec{\lambda}} \in \mathbb{N} \cup \{0\} \Big\} \tag{116}$$

thus has a natural ring structure with operations $\oplus$ and $\otimes$ and can be thought of as a type of Grothendieck K_0-ring for a homological subquotient of $\mathcal{C}ob_3$. We denote the corresponding set of higher Alexander Characters abusively in the same way, since it possesses the same ring structure under usual addition and multiplication. Clearly, $\mathfrak{Q}^{[0]} \subset \mathfrak{Q}^{[+]}$. The following conjecture together with an understanding of the ring structure of $\mathfrak{Q}^{[+]}$ should shed light on the general structure of the polynomials $R_{n,k}$.

Conjecture 17

$$\mathcal{V}_k^{\mathrm{PSU}(n)} \in \mathfrak{Q}^{[+]}.$$

B Homology TQFT's from the Reshetikhin–Turaev Theory

Recall that the TQFT $\mathcal{V}^{(j)}$ is in fact a functor to the category of free $\mathbb{Z}$-modules rather than just the category of vector spaces over $\mathbb{Q}$. Now for any prime $p \geq 3$, by taking all lattices modulo p, this in turn maps to the category of vector spaces over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The resulting TQFT $\mathcal{V}_p^{(j)}$ over $\mathbb{F}_p$ is now no longer irreducible, but it has a *unique* irreducible subquotient, which we denote by $\overline{\overline{\mathcal{V}}}_p^{(j)}$, see [22].

Another way of generating TQFT's over $\mathbb{F}_p$ is to consider the Reshetikhin–Turaev Theory for quantum-SO(3) at a primitive p-th root of unity ζ_p. As shown in [11] this can be regarded as a TQFT over the ring of cyclotomic integers $\mathbb{Z}[\zeta_p]$. The TQFT obtained from the ring reduction $\mathbb{Z}[\zeta_p] \twoheadrightarrow \mathbb{F}_p \colon \zeta_p \mapsto 1$ is denoted $\mathcal{V}_p^{\mathrm{RT}}$. The example $p = 5$, which is in some sense a fundamental case, is analyzed in [23]. We obtain an exact, but non-split sequence of TQFT's as follows:

$$0 \to \overline{\overline{\mathcal{V}}}_5^{(4)} \longrightarrow \mathcal{V}_5^{\mathrm{RT}} \longrightarrow \overline{\overline{\mathcal{V}}}_5^{(1)} \to 0. \tag{117}$$

As an extension of the mapping class group Γ_g (117) involves a Johnson–Morita subquotient of Γ_g. The precise modular structure of the $\mathcal{V}_p^{(j)}$ and $\overline{\overline{\mathcal{V}}}_p^{(j)}$ TQFT's is unraveled in [22]. There we find resolutions of the $\overline{\overline{\mathcal{V}}}_p^{(j)}$ in terms of the $\mathcal{V}_p^{(j)}$, which lead to important identities between the p-modular versions of the invariants from Theorem 4 and the Reshetikhin–Turaev Invariants.

It is easy to see that the irreducible factors of $\mathcal{V}_p^{\mathrm{RT}}$ for $p \geq 7$ can no longer be reductions of the $\mathcal{V}^{(j)}$. There is, however, evidence that suggests that the irreducible factors are reductions of summands in the symmetric powers of the fundamental ones. That is, TQFT's of the form

$$\mathcal{V}^{\vec{\lambda}} \subseteq S^{\frac{p-3}{2}} \mathcal{V}^{\mathrm{FN}} \in \mathbf{Q}^{[+]}. \tag{118}$$

This is closely related to the conjecture that the Lescop invariant for a closed 3-manifold M with $\beta_1(M) \geq 1$ relates to the Reshetikhin–Turaev Invariant as follows.

$$\mathcal{V}_{\zeta_p}^{\mathrm{RT}}(M) = C_p \cdot \big((\zeta_p - 1)\lambda_L(M)\big)^{\frac{p-3}{2}} + \mathcal{O}\big((\zeta_p - 1)^{\frac{p-1}{2}}\big). \tag{119}$$

This has been verified for $p = 5$ in [24].

C Relation of Reshetikhin–Turaev and Hennings Theory

Given a quasitriangular Hopf algebra, $\mathcal{A}$, we have described in Section 5 a procedure to construct a topological quantum field theory, $\mathcal{V}_{\mathcal{A}}^{H}$. In [43] and [46], Reshetikhin and Turaev give another procedure to construct a TQFT, $\mathcal{V}_{\mathcal{S}}^{\mathrm{RT}}$, from a *semisimple* modular category, $\mathcal{S}$. A more general construction in [25] allows us to construct a TQFT, $\mathcal{V}_{\mathcal{C}}^{\mathrm{KL}}$, also for modular categories, $\mathcal{C}$, that are not semisimple, and we show in [19] that $\mathcal{V}_{\mathcal{A}}^{H} = \mathcal{V}_{\mathcal{A}\text{-mod}}^{\mathrm{KL}}$ and $\mathcal{V}_{\mathcal{S}}^{\mathrm{RT}} = \mathcal{V}_{\mathcal{S}}^{\mathrm{KL}}$ for semisimple $\mathcal{S}$. For a non-semisimple, quasitriangular algebra, $\mathcal{A}$, the semisimple category used in [43] and [46] is given as the semisimple trace-quotient $\mathcal{S}(\mathcal{A}) = \overline{\mathcal{A}\text{-mod}}$ of the representation category of $\mathcal{A}$. The relation between $\mathcal{V}_{\mathcal{A}}^{H}$ and $\mathcal{V}_{\mathcal{S}(\mathcal{A})}^{\mathrm{RT}}$ is generally unknown. We make the following conjecture in the case of quantum $\mathfrak{sl}_2$.

Conjecture 18 *Let $\mathcal{A} = U_q(\mathfrak{sl}_2)^{\mathrm{red}}$, with q an odd p-th root of unity, and relations $E^p = F^p = 0$ and $K^{2p} = 1$ for the standard generators. Then there is a monomorphic, natural transformation*

$$\mathcal{V}^{\mathrm{FN}} \otimes \mathcal{V}_{\mathcal{S}(\mathcal{A})}^{\mathrm{RT}} \hookrightarrow \mathcal{V}_{\mathcal{A}}^{H}. \tag{120}$$

In the genus one case we have shown in [18] and [19] that the mapping class group representations and invariants of lens spaces of both theories in (120) are in fact equal. The above inclusion of TQFT functors can also be phrased in the form $\mathcal{V}_{\mathcal{C}^{\#}}^{\mathrm{KL}} \hookrightarrow \mathcal{V}_{\mathcal{C}}^{\mathrm{KL}}$, where $\mathcal{C} := U_q(\mathfrak{sl}_2)^{\mathrm{red}}$-mod and $\mathcal{C}^{\#} := (\mathcal{N}\text{-mod}) \otimes \overline{\mathcal{C}}$. The categories $\mathcal{C}$ and $\mathcal{C}^{\#}$ are in fact rather similar as linear abelian categories. From [20] it follows that there an isomorphism of *abelian* categories

$$\mathcal{H}\colon \mathcal{C}^{\#} \oplus 2 \cdot \mathrm{Vect}(\mathbb{C}) \xrightarrow{\cong} \mathcal{C}, \tag{121}$$

where the two extra $\mathrm{Vect}(\mathbb{C})$'s account for the two p-dimensional, irreducible Steinberg modules. This, however, is *not* a monoidal functor. Instead we have a natural set of monomorphisms of the form $\mathcal{H}(X) \otimes \mathcal{H}(Y) \hookrightarrow \mathcal{H}(X \otimes Y)$. As a result the braidings, integrals, and coends that enter in a crucial way the construction of the TQFT's [25] can no longer be naïvely identified. Strategies of proof would include a basis of $\mathcal{A}$ as worked out in [18] and the use of the special central, nilpotent element Q defined in [19].

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