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Existence results for semilinear problems in the two-dimensional hyperbolic space involving critical growth

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We consider semilinear elliptic problems on two-dimensional hyperbolic space. A model problem of our study is

$$-\Delta_{g_{\mathbb{B}^2}} u = f(x,t), \quad u \in H^1(\mathbb{B}^2),$$

where $H^1(\mathbb{B}^2)$ denotes the Sobolev space on the disc model of the hyperbolic space and $f(x,t)$ denotes the function of critical growth in dimension 2. We first establish the Palais–Smale (PS) condition for the functional corresponding to the above equation, and using the PS condition we obtain existence of solutions. In addition, using a concentration argument, we also explore existence of infinitely many sign-changing solutions.

Keywords: semilinear elliptic problem; hyperbolic space; critical growth; Moser–Trudinger inequality

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1. Introduction

In this paper we are concerned with the existence and multiplicity of solutions of the problem

$$-\Delta_{g_{\mathbb{B}^N}} u = f(x,u), \quad u \in H^1(\mathbb{B}^N), \tag{1.1}$$

where $H^1(\mathbb{B}^N)$ denotes the Sobolev space on the disc model of the hyperbolic space $\mathbb{B}^N$ endowed with the Poincaré metric $g_{\mathbb{B}^N}$, $\Delta_{g_{\mathbb{B}^N}}$ denotes the Laplace–Beltrami operator on $\mathbb{B}^N$, and $f\colon \mathbb{B}^N \times \mathbb{R} \to \mathbb{R}$ is a C^1 function with $f(x,-t) = -f(x,t)$.

Equation (1.1) has been the subject of intensive research in the past few years after its connection with various geometrical problems was discovered. For example, (1.1) with $f(x,t) = \lambda t + |t|^{p-2}t$, $2 < p \leqslant 2N/(N-2)$ when $N \geqslant 3$ and $2 < p < \infty$

when $N = 2$, arises in the study of the Grushin operator [9], the Hardy–Sobolev–Maz'ya equation [14, 15, 23], and in the prescription of Webster curvature on the Heisenberg group. In this case, great attention has been devoted to the study of positive solutions. More precisely, existence, uniqueness, regularity, symmetry and non-degeneracy properties of positive solutions have been thoroughly investigated in [12, 20, 23].

In the seminal paper [23], with the above choice of f and p subcritical, it was shown that the problem always admits a positive solution. The solutions were also shown to be unique up to hyperbolic isometries except in the case of dimension 2. However, when $N \geqslant 3$ and $p = 2N/(N-2)$, i.e. the critical case, the study of existence of solutions becomes more interesting due to the lack of compactness of the Sobolev embedding in the hyperbolic space. It has been shown that (1.1) admits a positive solution provided that

$$\frac{N(N-2)}{4} < \lambda \leqslant \left(\frac{N-1}{2}\right)^2.$$

This is in contrast with the Euclidean case, where a positive solution exists if and only if $\lambda = 0$, it is unique up to translations and dilations, and is explicitly known.

The next step is to characterize all sign-changing solutions. Existence of sign-changing solutions has been investigated in [11, 12]. Furthermore, the extension to general manifolds was also discussed in [10]. The results in [10] hold for quite general nonlinearities f and non-energy solutions are also dealt with. However, in the critical case in which $p = 2N/(N-2)$, the problem becomes more delicate and has been thoroughly studied in [19]. One of the important results obtained in [19] is the existence of infinitely many sign-changing radial solutions for $N \geqslant 7$. The question remains open for $N \leqslant 6$.

In this paper we are interested in problem (1.1) when $N = 2$ and the nonlinearity is 'critical'. Criticality comes from the critical Sobolev embedding or, more precisely, the Moser–Trudinger (MT) inequality (see [27]).

First let us recall the MT inequality on the hyperbolic space. Recently, Mancini and Sandeep [24] and Adimurthi and Tintarev [3] proved that MT holds true in the hyperbolic space. In fact they proved the following theorem.

Theorem (Mancini and Sandeep [24]). *Let $\mathbb{D}$ be the unit open disc in $\mathbb{R}^2$, endowed with a conformal metric $h = \rho g_{\mathrm{E}}$, where g_{E} denotes the Euclidean metric and $\rho \in C^2(\mathbb{D})$, $\rho > 0$. Then*

$$\sup_{\substack{u \in C_0^\infty(\mathbb{D}),\\ \int_{\mathbb{D}} |\nabla_h u|^2 \leqslant 1}} \int_{\mathbb{D}} (\mathrm{e}^{4\pi u^2} - 1)\,\mathrm{d}v_h < \infty, \tag{1.2}$$

holds true if and only if $h \leqslant c g_{\mathbb{B}^2}$ for some positive constant c.

Inequality (1.2) is sharp in the sense that the 'critical' constant 4π cannot be improved. We refer the reader to [8, 25, 26] for the MT inequality in the higher-dimensional hyperbolic space. However, the existence of extremals of the MT inequality is still an *open question*. In this direction some partial results were obtained by Manicini *et al*. [26]. They showed the existence of extremals for a modified MT inequality. In particular, they proved that

$$\tilde{S} := \sup_{\|u\|_{\mathcal{H}} \leqslant 1} \int_{\mathbb{B}^2} (\mathrm{e}^{4\pi u^2} - 1 - 4\pi u^2) \, \mathrm{d}v_{g_{\mathbb{B}^2}}$$

is finite and attained or, in other words, that the corresponding Euler–Lagrange equation

$$-\Delta_{g_{\mathbb{B}^2}} u - \tfrac{1}{4} u = \beta u (\mathrm{e}^{4\pi u^2} - 1), \quad \beta = \frac{1}{\int_{\mathbb{B}^2} u^2 (\mathrm{e}^{4\pi u^2} - 1) \, \mathrm{d}v_{g_{\mathbb{B}^2}}} \tag{1.3}$$

admits a positive (radial) solution in $\mathcal{H}$, where $\mathcal{H}$ denotes closure of $C_0^\infty(\mathbb{B}^2)$ with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{B}^2} [|\nabla_{g_{\mathbb{B}^2}} u|^2 - \tfrac{1}{4}|u|^2] \, \mathrm{d}v_{g_{\mathbb{B}^2}}.$$

Now, it is important to remark that the solution of (1.3) u satisfies

$$|u(x)| \geqslant C(1 - |x|^2)^{1/2},$$

and hence is not an element of $H^1(\mathbb{B}^2)$.

Motivated by the Euler–Lagrange equation (1.3) satisfied by the MT inequality, we plan to address the question of existence of solutions to problem (1.1) in dimension 2 and involving exponential nonlinearity. In particular, we are interested in the existence of positive solutions, sign-changing solutions, and their multiplicity when $N = 2$ and

$$f(x, t) = h(x, t)(\mathrm{e}^{\lambda t^2} - 1)$$

is a function of critical growth (see definition 1.1). Hence, from here onwards we shall consider the following problem:

$$-\Delta_{g_{\mathbb{B}^2}} u = h(x, u)(\mathrm{e}^{\lambda u^2} - 1), \quad u \in H^1(\mathbb{B}^2). \tag{1.4}$$

In the Euclidean setting, i.e. when (1.4) is posed on $\Omega \subset \mathbb{R}^2$, a bounded domain, many important existence results were obtained; see, for example, [1, 4, 7, 13, 17, 18, 31, 32]. Adimurthi [1] proved existence of non-trivial solutions and also established the Palais–Smale (PS) condition for the functional corresponding to (1.4). Thereafter the focus had been on the existence of sign-changing solutions. Adimurthi and Yadava [4] obtained existence of sign-changing solutions when $\sup_{x \in \bar{\Omega}} f'(x, 0) < \mu_1(\Omega)$, where $\mu_1(\Omega)$ denotes the first eigenvalue of a Dirichlet boundary-value problem involving the Euclidean Laplacian. In addition, they also proved, when Ω is a Euclidean ball, that (1.4) admits infinitely many radial sign-changing solutions. Also, in the critical case Adimurthi *et al*. [6] obtained non-existence results under some suitable conditions for the Euclidean setting. However, a complete study of the borderline between existence and non-existence was provided by Adimurthi and Prashanth [2]. All these results use the variational approach in order to tackle existence results. The key step in using such a theory is the verification of conditions that allow the use of the PS condition. Recently, Guozhen and Nguyen [29] obtained existence of solutions of (1.4) without assuming the Ambrosetti–Rabinowitz condition.

Before going further, we first introduce the definition of a critical growth function. In view of the MT embedding in the Euclidean setting, the notion of functions of critical growth was first introduced by Adimurthi [1]. However, in the same spirit we intend to generalize the concept to the hyperbolic setting. The recent development of the MT inequality in the hyperbolic space [24] enables us to define the following class of critical growth functions.

Definition 1.1. Let $h\colon \mathbb{B}^2 \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and let $\lambda > 0$. The function $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2} - 1)$ is said to be a function of critical growth on $\mathbb{B}^2$ if $f(x,t) > 0$ for $t > 0$, if $f(x,-t) = -f(x,t)$, and if it satisfies the following growth conditions.

There exists a constant $M_1 > 0$ such that, for every $\varepsilon > 0$ and for all $(x,t) \in \mathbb{B}^2 \times (0,\infty)$, the following hold:

(C1) $h(\cdot,\cdot) \in L^\infty(\mathbb{B}^2 \times [-L, L])$ for all $L > 0$, and

$$\sup_{x\in\mathbb{B}^2} h(x,t) = O(t^a) \quad \text{near } t = 0 \text{ for some } a > 0;$$

(C2) $f'(x,t) > \dfrac{f(x,t)}{t}$, where $f'(x,t) = \dfrac{\partial f}{\partial t}(x,t)$;

(C3) $F(x,t) \leqslant M_1(g(x) + f(x,t))$, where

$$F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s \quad \text{and} \quad g \in L^1(\mathbb{B}^2, \mathrm{d}v_{g_{\mathbb{B}^2}});$$

(C4) for any compact set $K \subset \mathbb{B}^2$, it holds that

$$\lim_{t\to\infty} \inf_{x\in K} h(x,t)\mathrm{e}^{\varepsilon t^2} = \infty \quad \text{and} \quad \lim_{t\to\infty} \sup_{x\in\mathbb{B}^2} h(x,t)\mathrm{e}^{-\varepsilon t^2} = 0.$$

For examples of functions of critical growth, we refer the reader to § 2. Moreover, the class of critical growth functions defined above does not depend on the choice of the origin, that is, the assumption of radiality can be posed with respect to an arbitrary point in the hyperbolic space, and changing this point to the origin by a Möbius transformation will not change the assumptions in definition 1.1.

Now we will briefly discuss some of the hurdles we may encounter in dealing with the problem in the hyperbolic space. First of all, we have to deal with the infinite volume case, which makes the problem very different from the bounded one. Secondly, one of the major difficulties comes from the lack of compactness. The lack of compactness can occur due to the concentration phenomenon as well as through the vanishing of mass in the sense of the concentration-compactness of Lions (see [21]). In the Euclidean case, by dilating a given sequence we can assume that all the functions involved have a fixed positive mass in a given ball, and hence we can overcome the vanishing of the mass, but in the case of the hyperbolic space $\mathbb{B}^2$ this is not possible as the conformal group of $\mathbb{B}^2$ is the same as the isometry group. We will overcome this difficulty by using the growth estimates near ∞.

To the best of our knowledge, this is the first paper that deals with the critical growth function in the two-dimensional hyperbolic space. We establish the PS condition for the functional corresponding to (1.4) (see theorem 4.1), which leads us to the following existence theorem.

Theorem 1.2. *Let f be a function of critical growth. Furthermore, assume that f is radial and that for any $K \subset \mathbb{B}^2$ compact there holds*

$$\lim_{t\to\infty} \inf_{x\in K} h(x,t)t = \infty. \tag{1.5}$$

Then (1.4) admits a positive solution.

Remark. If we write (1.4) in the Euclidean local coordinate, then theorem 1.2 tells us that

$$\Delta u = \left(\frac{2}{1-|x|^2}\right)^2 h(|x|,u)(\mathrm{e}^{\lambda u^2} - 1) \tag{1.6}$$

has a radial solution in $H_0^1(\mathbb{B}^2)$ under the assumption that $h(x,u) = O(u^a)$ for some $a > 0$ near $u = 0$. This allows us to consider the quadratic singularity (or integrability) at the boundary, i.e. that of order $1/(1-|x|^2)^2$.

Remark. Theorem 1.2 is also true for f non-radial with some assumption on the growth of f. Please see theorem B.1 for further details.

Also, using variational methods and a concentration argument we obtain the following result.

Theorem 1.3. *Let f be a function of critical growth, radial, and given any $N > 0$ and compact set $K \subset \mathbb{B}^2$, there exists $t_{N,K} > 0$ such that*

$$\inf_{x\in K} h(x,t)t \geqslant \mathrm{e}^{Nt} \quad \forall t \geqslant t_{N,K} \tag{1.7}$$

holds. Then (1.4) has a radial sign-changing solution.

Remark. In theorem 1.3, condition (1.7) is necessary in order to get a radial sign-changing solution. If we consider $f(x,t) = (1-|x|^2)^2 t\mathrm{e}^{t^2+|t|^a}$, $0 < a \leqslant 1$, then by conformal invariance, (1.4) does not admit any radial sign-changing solution (see [5]).

Once we obtain existence of a radial sign-changing solution, we can go further and investigate their multiplicity. The main idea is the following: for a positive integer k, one can divide $\mathbb{B}^2$ into k annuli and, considering functions satisfying certain conditions on each annuli, one can get existence of solution(s) having k nodes. More precisely, we have the following theorem.

Theorem 1.4. *Let f be a function of critical growth, radial, and satisfying condition (1.7). Then (1.4) has infinitely many radial sign-changing solutions.*

Remark. Theorem 1.4 gives an affirmative answer to the question of existence of infinitely many sign-changing radial solutions for problem (1.1) in dimension 2.

We also give existence of non-radial solutions to the above problem. Please see theorem B.1 for a discussion and the proof of existence of non-radial solutions.

The paper is organized as follows. We divide the paper into seven sections. The preliminaries and some technical frameworks are discussed in §§ 2 and 3. The PS condition and several convergence results are devoted to § 4. The results of § 4 are used to prove the main existence theorems (theorems 1.2–1.4) in § 5. In appendix A we give a sketch of the proof of lemma 5.2. The final section, appendix B, is devoted to the existence of non-radial solutions.

2. Notation and functional analytic preliminaries

In this section we introduce some of the notation and definitions used in this paper and also recall some of the embeddings related to the Sobolev space in the hyperbolic space. We also obtain estimates for radial functions.

We will denote by $\mathbb{B}^2$ the disc model of the hyperbolic space, i.e. the unit disc equipped with the Riemannian metric

$$g_{\mathbb{B}^2} := \sum_{i=1}^{2} \frac{2}{(1-|x|^2)^2}\,\mathrm{d}x_i^2.$$

To simplify our notation we will denote $g_{\mathbb{B}^2}$ by g.

The corresponding volume element is given by

$$\mathrm{d}v_g = \frac{2}{(1-|x|^2)^2}\,\mathrm{d}x,$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^2$. The hyperbolic gradient ∇_g and the hyperbolic Laplacian Δ_g are given by

$$\nabla_g = \left(\frac{1-|x|^2}{2}\right)^2 \nabla, \qquad \Delta_g = \left(\frac{1-|x|^2}{2}\right)^2 \Delta.$$

Notation (Sobolev space). We will denote by $H^1(\mathbb{B}^2)$ the Sobolev space on the disc model of the hyperbolic space $\mathbb{B}^2$.

Throughout this paper we denote the norm of $H^1(\mathbb{B}^2)$ by

$$\|u\| := \left(\int_{\mathbb{B}^2} |\nabla_g u|^2\,\mathrm{d}v_g\right)^{1/2}.$$

2.1. A sharp Poincaré–Sobolev inequality (see [23])

For $N \geqslant 3$ and $p \in (1,(N+2)/(N-2)]$ there exists an optimal constant $S_{N,p} > 0$ such that

$$S_{N,p}\left(\int_{\mathbb{B}^N} |u|^{p+1}\,\mathrm{d}v_{\mathbb{B}^N}\right)^{2/(p+1)} \leqslant \int_{\mathbb{B}^N}\left[|\nabla_{\mathbb{B}^N} u|^2 - \frac{(N-1)^2}{4}u^2\right]\mathrm{d}v_{\mathbb{B}^N} \tag{2.1}$$

for every $u \in C_0^\infty(\mathbb{B}^N)$. If $N = 2$, any $p > 1$ is allowed.

A basic fact is that the bottom of the spectrum of $-\Delta_g$ on $\mathbb{B}^2$ is

$$\frac{1}{4} = \inf_{u \in H^1(\mathbb{B}^2)\setminus\{0\}} \frac{\int_{\mathbb{B}^2} |\nabla_g u|^2\,\mathrm{d}v_g}{\int_{\mathbb{B}^2} |u|^2\,\mathrm{d}v_g}. \tag{2.2}$$

Also, from conformal invariance we have the following lemma.

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Lemma 2.1. *If $u \in H^1(\mathbb{B}^2)$, then*

$$\int_{\mathbb{B}^2} |\nabla_g u|^2 \,\mathrm{d}v_g = \int_{\mathbb{B}^2} |\nabla u|^2 \,\mathrm{d}x, \tag{2.3}$$

where ∇ denotes the Euclidean gradient on $\mathbb{R}^2$.

Proof. In local coordinates we have

$$\begin{aligned}\int_{\mathbb{B}^2} |\nabla_g u|^2 \,\mathrm{d}v_g &= \int_{\mathbb{B}^2} \left(\frac{1-|x|^2}{2}\right)^2 |\nabla u|^2 \left(\frac{2}{1-|x|^2}\right)^2 \mathrm{d}x \\ &= \int_{\mathbb{B}^2} |\nabla u|^2 \,\mathrm{d}x. \end{aligned} \tag{2.4}$$

□

Let $H^1_R(\mathbb{B}^2)$ denote the subspace

$$H^1_R(\mathbb{B}^2) := \{u \in H^1_R(\mathbb{B}^2) \colon u \text{ is radial}\}.$$

Since the hyperbolic sphere with centre $0 \in \mathbb{B}^2$ is also a Euclidean sphere with centre $0 \in \mathbb{B}^2$ (see [33]), $H^1_R(\mathbb{B}^2)$ can also be seen as the subspace consisting of hyperbolic radial functions.

Proposition 2.2. *Let $u \in H^1_R(\mathbb{B}^2)$. Then*

$$|u(x)| \leqslant \frac{\|u\|}{(4\pi)^{1/2}} \frac{(1-|x|^2)^{1/2}}{|x|^{1/2}}. \tag{2.5}$$

Proof. Since $u \in H^1_R(\mathbb{B}^2)$, we have $u(x) = u(|x|)$ by denoting the radial function by u. For u radial, in hyperbolic polar coordinates $|x| = \tanh \frac{1}{2}t$ and we have

$$\int_{\mathbb{B}^2} |\nabla_g u|^2 \,\mathrm{d}v_g = \omega_2 \int_0^\infty \sinh t |u'(t)|^2 \,\mathrm{d}t < \infty.$$

Thus, for $u \in H^1_R(\mathbb{B}^2)$ and $t < \tau$,

$$\begin{aligned}|u(\tau) - u(t)| = \left| \int_t^\tau u'(s) \,\mathrm{d}s \right| &\leqslant \left(\int_0^\infty (\sinh s)|u'(s)|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_t^\infty \frac{\mathrm{d}s}{\sinh s} \right)^{1/2} \\ &\leqslant \|u\|_{H^1} \left(\frac{1}{2\pi \sinh t} \right)^{1/2}. \end{aligned} \tag{2.6}$$

Since

$$\int_{\mathbb{B}^2} u^2 \,\mathrm{d}v_g = \omega_2 \int_0^\infty u^2 \sinh t \,\mathrm{d}t < \infty,$$

this implies that $\liminf_{\tau \to \infty} u(\tau) = 0$, and we obtain

$$|u(t)| \leqslant \|u\|_{H^1} \left(\frac{1}{2\pi \sinh t} \right)^{1/2}. \tag{2.7}$$

Now, substituting $t = 2\tanh^{-1}(|x|)$,

$$\begin{aligned}\sinh t = \frac{e^t - e^{-t}}{2} &= \frac{e^{2\tanh^{-1}(|x|)} - e^{-2\tanh^{-1}(|x|)}}{2} \\ &= \left(\exp\left(2\log\left(\frac{1+|x|}{1-|x|}\right)\right) - 1\right)\left(2\exp\left(\log\left(\frac{1+|x|}{1-|x|}\right)\right)\right)^{-1} \\ &= \frac{2|x|}{(1-|x|^2)}, \qquad (2.8)\end{aligned}$$

and hence, substituting (2.8) into (2.7), we get

$$|u(x)| \leqslant \frac{\|u\|}{(4\pi)^{1/2}} \frac{(1-|x|^2)^{1/2}}{|x|^{1/2}}.$$

This completes the proof of the proposition. □

Remark. The above proposition is redundant. Instead one can use the standard estimate

$$|u(r)| \leqslant \frac{1}{(2\pi)^{1/2}} \sqrt{\log\frac{1}{r}}\, \|\nabla u\|_2$$

on the ball (see [34]), which is sharper than (2.5) as $r := |x| \to 1$. However, for the sake of notational brevity we use estimate (2.5), which does not weaken the results we obtain in this paper.

2.2. Compactness lemma

Next we shall prove the compactness lemma of Lions [22] in the hyperbolic setting. The main ingredient of the proof is using a suitable covering of hyperbolic space with a Möbius transformation developed by Adimurthi and Tintarev [3]. Adopting their approach we prove the following lemma.

Lemma 2.3 (hyperbolic version of Lions's lemma). *Let $\{u_k : \|u_k\| = 1\}$ be a sequence in $H^1(\mathbb{B}^2)$ converging weakly to a non-zero function u. Then for every $p < (1 - \|u\|^2)^{-1}$,*

$$\sup_k \int_{\mathbb{B}^2} (e^{4\pi p u_k^2} - 1)\, dv_g < \infty. \qquad (2.9)$$

Proof. Let us fix an open set U in $\mathbb{B}^2$ such that $\bar{U} \subset \mathbb{B}^2$ and define

$$\|u\|_U^2 = \int_U |\nabla u|^2\, dx + \int_U u^2 \left(\frac{2}{1-|x|^2}\right)^2 dx. \qquad (2.10)$$

Then, following [3], we can conclude that there exists a number $q > 0$ such that for all $u \in H^1(\mathbb{B}^2)$ with $\|u\|_U < 1$ there holds

$$\int_U (e^{qu^2} - 1)\, dv_g \leqslant C \frac{\|u\|_U^2}{1 - \|u\|_U^2}. \qquad (2.11)$$

Let us fix a u_k. Let $\{\phi_i\}_i$ be a countable family of Möbius transforms such that $\{\phi_i(U)\}_i$ covers $\mathbb{B}^2$, having finite multiplicity, say R_0. Then define

$$S_k := \left\{ i \colon \|u_k \circ \phi_i\|_U^2 > \frac{q}{8\pi p} \right\}. \tag{2.12}$$

Proceeding as in [3], we can show that number of elements in S_k is less than $40\pi p R_0/q + 1$ and

$$\sum_{i \notin S_k} \int_{\phi_i(U)} (\mathrm{e}^{4\pi p u_k^2} - 1)\, \mathrm{d}v_g \leqslant C, \tag{2.13}$$

where C is independent of u_k. Also we have

$$\begin{aligned} \sum_{i \in S_k} \int_{\phi_i(U)} (\mathrm{e}^{4\pi p u_k^2} - 1)\, \mathrm{d}v_g &\leqslant C \sum_{i \in S_k} \int_{\mathbb{B}^2} (\mathrm{e}^{4\pi p (u \circ \phi_i)_k^2} - 1) \\ &\leqslant C\left(\frac{40\pi p R_0}{q} + 1 \right), \end{aligned} \tag{2.14}$$

by $\|v \circ \phi_i\| = \|v\|$ for all $v \in H^1(\mathbb{B}^2)$, and the Euclidean version of Lions's lemma [22]. Therefore, from (2.13) and (2.14) we get (2.9). □

Finally, we end this section with some examples of functions having critical growth and the definition of a Moser function.

Examples (functions of critical growth).

(i) $f(x,t) = t(\mathrm{e}^{\lambda t^2} - 1)$ is an example of a function of critical growth. This example suggests that we can allow the singularity at the boundary of the ball of order $1/(1 - |x|^2)^2$.

(ii) Let $h(x,t) \in C^1(\mathbb{B}^2 \times (0,\infty))$ be a positive function satisfying (C1), (C4) and $h'(x,t) \geqslant h(x,t)/t$. Then $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2} - 1)$ is a function of critical growth.

Proof. One can easily show that $f'(x,t) > f(x,t)/t$. It remains to show that f satisfies (C3).

For $t \leqslant 1/\sqrt{\lambda}$ we have, from the definition of $F(x,t)$,

$$F(x,t) = \int_0^t f(x,s)\, \mathrm{d}s \leqslant t f(x,t) \leqslant \frac{1}{\sqrt{\lambda}} f(x,t).$$

For $t > 1/\sqrt{\lambda}$ we have

$$\begin{aligned} F(x,t) &= \int_0^t h(x,s)(\mathrm{e}^{\lambda s^2} - 1)\, \mathrm{d}s \\ &= \frac{1}{2\lambda} \int_0^t \frac{h(x,s)}{s} \frac{\mathrm{d}}{\mathrm{d}s} (\mathrm{e}^{\lambda s^2} - \lambda s^2)\, \mathrm{d}s \\ &= \frac{1}{2\lambda} \int_0^t \frac{1}{s} \left[\frac{h(x,s)}{s} - h'(x,s) \right] (\mathrm{e}^{\lambda s^2} - \lambda s^2)\, \mathrm{d}s + \frac{1}{2\lambda} \frac{h(x,t)}{t} (\mathrm{e}^{\lambda t^2} - \lambda t^2). \end{aligned}$$

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Therefore, using $h'(x,t) \geqslant h(x,t)/t$, we get

$$F(x,t) \leqslant Cf(x,t).$$

This proves that f satisfies (C3). □

Definition (Moser function). For $0 < l < R_0 < 1$, $m_{l,R_0}(x)$ is the Moser function defined by

$$m_{l,R_0}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log(R_0/l))^{1/2} & \text{if } 0 < |x| < l, \\ \dfrac{\log(R/|x|)}{(\log(R_0/l))^{1/2}} & \text{if } l < |x| < R_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{\mathbb{B}^2} |\nabla_g m_{l,R_0}|^2_{\mathbb{B}^2} \, \mathrm{d}v_g = \int_{\mathbb{B}^2} |\nabla m_{l,R_0}|^2 \, \mathrm{d}x = 1.$$

3. Variational framework

We use variational methods in order to prove the main theorems. Taking advantage of the MT inequality and the radial estimate (2.5), we shall derive a variational principle for (1.4) in the Sobolev space $H^1_R(\mathbb{B}^2)$. The solutions of (1.4) are the critical points of the energy functional given by

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{B}^2} |\nabla_g u|^2 \, \mathrm{d}v_g - \int_{\mathbb{B}^2} F(x,u) \, \mathrm{d}v_g. \tag{3.1}$$

Indeed, by proposition 3.1 and lemma 4.2, J_λ is a well-defined C^1 functional on $H^1_R(\mathbb{B}^2)$. Assuming f to be radial in its first variable, it is enough to find critical points of J_λ on $H^1_R(\mathbb{B}^2)$ by the principle of symmetric criticality [30]. Hence, from now on we shall denote $f(x,t) := g(|x|,t)$ by f.

Proposition 3.1. *If $u \in H^1_R(\mathbb{B}^2)$, then*

$$\int_{\mathbb{B}^2} F(x,u) \, \mathrm{d}v_g < \infty. \tag{3.2}$$

Proof. Without loss of generality we can assume that $u \geqslant 0$. By (C2) we have for all $t > 0$,

$$F(x,t) \leqslant \tfrac{1}{2} t f(x,t). \tag{3.3}$$

Hence, using radial estimate (2.5) and (3.3), we have

$$\int_{\mathbb{B}^2} F(x,u) \, \mathrm{d}v_g < \frac{1}{2} \int_{\mathbb{B}^2} u f(x,u) \, \mathrm{d}v_g$$
$$= \int_{\mathbb{B}^2 \cap \{|x| > 1/2\}} u f(x,u) \, \mathrm{d}v_g + \int_{\mathbb{B}^2 \cap \{|x| < 1/2\}} u f(x,u) \, \mathrm{d}v_g. \tag{3.4}$$

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Consider the first integral of (3.4):

$$\begin{aligned}\int_{\mathbb{B}^2\cap\{|x|>1/2\}} uf(x,u)\,\mathrm{d}v_g &= \int_{\mathbb{B}^2\cap\{|x|>1/2\}} uh(x,u)(\mathrm{e}^{\lambda u^2}-1)\,\mathrm{d}v_g \\ &= \int_{\mathbb{B}^2\cap\{|x|>1/2\}} uh(x,u)\frac{(\mathrm{e}^{\lambda u^2}-1)}{(1-|x|^2)^2}\,\mathrm{d}x \\ &\leqslant C\|u\|^{3/2}\int_{\mathbb{B}^2\cap\{|x|>1/2\}}(1-|x|^2)^{-1/2}\,\mathrm{d}x<\infty. \end{aligned} \tag{3.5}$$

The second integral of (3.4) is finite by using the Euclidean version of the MT inequality (1.2). Hence, this proves the proposition. □

Before going further, we need some notation and definitions. Let f be a function of critical growth on $\mathbb{B}^2$. Define

$$\begin{gathered}\mathcal{M}=\left\{u\in H^1_R(\mathbb{B}^2)\setminus\{0\}\colon \|u\|^2=\int_{\mathbb{B}^2} f(x,u)u\,\mathrm{d}v_g\right\},\\ \mathcal{M}_1=\{u\in\mathcal{M}\colon u^{\pm}\in\mathcal{M}\},\\ I_\lambda(u)=\frac{1}{2}\int_{\mathbb{B}^2} f(x,u)u\,\mathrm{d}v_g-\int_{\mathbb{B}^2}F(x,u)\,\mathrm{d}v_g,\\ \tfrac{1}{2}\eta(f)^2=\inf_{u\in\mathcal{M}}J_\lambda(u),\qquad \tfrac{1}{2}\eta_1(f)^2=\inf_{u\in\mathcal{M}_1}I_\lambda(u).\end{gathered}$$

We show the existence of solutions of (1.4) by minimizing the functional J_λ over $\mathcal{M}$. However, the main difficulty lies in the validity of the PS condition. The next section is devoted to the study of the PS condition.

4. Palais–Smale condition and some convergence results

In this section we study the PS condition of the problem

$$\left.\begin{aligned}-\Delta_g u=f(x,u)\quad\text{in }\mathbb{B}^2,\\ u\in H^1(\mathbb{B}^2),\end{aligned}\right\} \tag{4.1}$$

where $f(x,u)$ denotes the function of critical growth. We say that $u_k\in H^1(\mathbb{B}^2)$ is a PS sequence for J_λ at a level c if $J_\lambda(u_k)\to c$ and $J'_\lambda(u_k)\to 0$ in $H^{-1}(\mathbb{B}^2)$. We show that if we restrict J_λ to $H^1_R(\mathbb{B}^2)$, then J_λ satisfies the $(\mathrm{PS})_c$ condition for all $c\in(0,2\pi/\lambda)$. To be precise, we state the following theorem.

Theorem 4.1. *Let $f(x,t)=h(x,t)(\mathrm{e}^{\lambda t^2}-1)$ be a function of critical growth on $\mathbb{B}^2$ and let $J_\lambda\colon H^1_R(\mathbb{B}^2)\to\mathbb{R}$ be defined as in (3.1). Then*

(i) *J_λ satisfies the PS condition on $(0,2\pi/\lambda)$;*

(ii) *moreover, if h satisfies*

$$\varlimsup_{t\to\infty}\inf_{x\in K}h(x,t)t=\infty\quad\textit{for any compact subset } K \textit{ of } \mathbb{B}^2, \tag{4.2}$$

then

$$0<\eta(f)^2<\frac{4\pi}{\lambda}.$$

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Theorem 4.1 will play a crucial role in the study of existence of solutions. The main difficulties in studying the PS condition come from the concentration phenomenon and through vanishing of mass. However, vanishing can be handled by using the radial estimate proved in § 2 (lemma 2.5). Keeping this in mind, we plan to address some important propositions involving convergence of critical growth functions. The propositions and lemmas needed in the proof of theorem 4.1 are collected below.

Lemma 4.2. *Let $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2} - 1)$ be a function of critical growth. Then we have the following.*

(i) *$f(x,u) \in L^p(\mathbb{B}^2, \mathrm{d}v_g)$ for all $p \in [1,\infty)$ and $u \in H^1(\mathbb{B}^2)$.*

(ii) *$I_\lambda(u) \geqslant 0$ for all u and $I_\lambda(u) = 0$ if and only if $u \equiv 0$. Moreover, given $\varepsilon > 0$, there exists a constant $C_0(\varepsilon) > 0$ such that for all $u \in H^1_R(\mathbb{B}^2)$,*

$$\int_{\mathbb{B}^2} f(x,u)u \,\mathrm{d}v_g \leqslant C_0(\varepsilon)(1 + I_\lambda(u)) + \varepsilon\|u\|^2. \tag{4.3}$$

Proof. (i) By (C4), for a given $\varepsilon > 0$ there exists an $N_0 > 0$ such that for all $t \geqslant N_0$ we have

$$f(x,t) \leqslant C(\mathrm{e}^{(\lambda+\varepsilon)t^2} - 1).$$

For $p \in [1,\infty)$, using the inequality $(\mathrm{e}^t - 1)^p \leqslant (\mathrm{e}^{pt} - 1)$ for $t \geqslant 0$ and the hyperbolic version of the MT inequality (1.2), we have

$$\begin{aligned}\int_{\mathbb{B}^2} |f(x,u)|^p \,\mathrm{d}v_g &\leqslant \int_{\{|u|>N_0\}} |f(x,u)|^p \,\mathrm{d}v_g + \int_{\{|u|\leqslant N_0\}} |f(x,u)|^p \,\mathrm{d}v_g \\ &\leqslant C\int_{\mathbb{B}^2} (\mathrm{e}^{p(\lambda+\varepsilon)u^2} - 1)\,\mathrm{d}v_g + C\int_{\mathbb{B}^2} (\mathrm{e}^{\lambda p u^2} - 1)\,\mathrm{d}v_g, \\ &< +\infty.\end{aligned}$$

(ii) By (C2), $f(x,t)t - 2F(x,t) \geqslant 0$ and equal to 0 if and only if $t = 0$; hence, this implies that

$$I_\lambda(u) \geqslant 0 \text{ and } I_\lambda(u) = 0 \iff u \equiv 0.$$

For the second part it is enough to prove the inequality for all $u \in H^1_R(\mathbb{B}^2)$ with $u \geqslant 0$. Fix $\varepsilon > 0$. By (C3), $F(x,t) \leqslant M_1(g(x) + f(x,t))$ for some positive function $g \in L^1(\mathbb{B}^2, \mathrm{d}v_g)$. Then

$$\begin{aligned}2I_\lambda(u) &= \int_{\mathbb{B}^2} [f(x,u)u - 2F(x,u)]\,\mathrm{d}v_g \\ &\geqslant \int_{\mathbb{B}^2} [f(x,u)u - 2M_1(g(x) + f(x,t))]\,\mathrm{d}v_g \\ &= \int_{\mathbb{B}^2} f(x,u)(u - 2M_1)\,\mathrm{d}v_g - 2M_1\int_{\mathbb{B}^2} g(x)\,\mathrm{d}v_g \\ &\geqslant \int_{\mathbb{B}^2} f(x,u)(u - 2M_1)\,\mathrm{d}v_g - C.\end{aligned} \tag{4.4}$$

Observing that $u - 2M_1 \geqslant \frac{1}{2}u$ on $\{u \geqslant 4M_1\}$, we have

$$\begin{aligned}\int_{\mathbb{B}^2} f(x,u)(u-2M_1)\,\mathrm{d}v_g &= \int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g + \int_{\mathbb{B}^2\cap\{u>4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g \\ &\geqslant \int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g + C\int_{\mathbb{B}^2\cap\{u>4M_1\}} f(x,u)u\,\mathrm{d}v_g. \qquad (4.5)\end{aligned}$$

Therefore, from (4.4) and (4.5) we have

$$\begin{aligned}\int_{\mathbb{B}^2\cap\{u>4M_1\}} f(x,u)u\,\mathrm{d}v_g &\leqslant C\left|\int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g\right| + C\int_{\mathbb{B}^2} f(x,u)(u-2M_1)\,\mathrm{d}v_g \\ &\leqslant C\left|\int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g\right| + C(1+I_\lambda(u)). \qquad (4.6)\end{aligned}$$

Next we estimate

$$\left|\int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g\right|.$$

Let $\delta > 0$ be a small number, depending on ε, whose smallness will be decided later. By the radial estimate (2.5), there exists a compact set K_0 such that the set $\{u > \delta\}$ is contained in K_0 for every $u \in H^1_R(\mathbb{B}^2)$. We can write

$$\begin{aligned}\int_{\{u\leqslant 4M_1\}} f(x,u)\,\mathrm{d}v_g &= \int_{\{\delta<u\leqslant 4M_1\}} h(x,u)(\mathrm{e}^{\lambda u^2}-1)\,\mathrm{d}v_g + \int_{\{u\leqslant\delta\}} h(x,u)(\mathrm{e}^{\lambda u^2}-1)\,\mathrm{d}v_g \\ &\leqslant C + \int_{\{u\leqslant\delta\}} h(x,u)(\mathrm{e}^{\lambda u^2}-1)\,\mathrm{d}v_g. \qquad (4.7)\end{aligned}$$

Now, by (C1) we can estimate the last integral in (4.7) as

$$\begin{aligned}\int_{\{u\leqslant\delta\}} h(x,u)(\mathrm{e}^{\lambda u^2}-1)\,\mathrm{d}v_g &\leqslant C\int_{\{u\leqslant\delta\}} u^a(\mathrm{e}^{\lambda u^2}-1)\,\mathrm{d}v_g \\ &\leqslant C\delta^a\int_{\mathbb{B}^2} u^2\,\mathrm{d}v_g \\ &\leqslant C\delta^a\|u\|^2, \qquad (4.8)\end{aligned}$$

where the constant C in (4.8) does not depend on u. Now, choosing $6M_1C\delta^a < \varepsilon/2$ we get

$$\left|\int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)(u-2M_1)\,\mathrm{d}v_g\right| \leqslant C + \frac{\varepsilon}{2}\|u\|^2. \qquad (4.9)$$

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Similarly, it follows that

$$\int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)u\,\mathrm{d}v_g \leqslant C + \frac{\varepsilon}{2}\|u\|^2. \tag{4.10}$$

Hence, from (4.6), (4.9) and (4.10) we get

$$\begin{aligned}\int_{\mathbb{B}^2} f(x,u)u\,\mathrm{d}v_g &= \int_{\mathbb{B}^2\cap\{u\leqslant 4M_1\}} f(x,u)u\,\mathrm{d}v_g + \int_{\mathbb{B}^2\cap\{u>4M_1\}} f(x,u)u\,\mathrm{d}v_g \\ &\leqslant C_0(1+I_\lambda(u)) + \varepsilon\|u\|^2.\end{aligned}$$

□

LEMMA 4.3. *Let $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2}-1)$ be a function of critical growth. Then*

$$\tilde{c}^2 := \sup\left\{c^2 \colon \sup_{\substack{u\in H^1_R(\mathbb{B}^2),\\ \|u\|\leqslant 1}} \int_{\mathbb{B}^2} f(x,cu)u\,\mathrm{d}v_g < +\infty\right\} = \frac{4\pi}{\lambda}.$$

Proof. Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. By (C4), there exist constants $t_1, t_2, C_1(\varepsilon), C_2(\varepsilon) > 0$ such that

$$f(x,t)t \leqslant C_1(\varepsilon)(\mathrm{e}^{\lambda(1+\varepsilon)t^2}-1) \quad \text{for all } t \geqslant t_1, \tag{4.11}$$

$$f(x,t)t \geqslant C_2(\varepsilon)(\mathrm{e}^{\lambda(1-\varepsilon)t^2}-1) \quad \text{for all } t \geqslant t_2 \text{ and } |x| \leqslant \alpha. \tag{4.12}$$

Now assume $c > 0$ to be such that

$$\sup_{\substack{u\in H^1_R(\mathbb{B}^2),\\ \|u\|\leqslant 1}} \int_{\mathbb{B}^2} f(x,cu)u\,\mathrm{d}v_g < +\infty.$$

Then, using (4.12),

$$\begin{aligned}\int_{\mathbb{B}^2} f(x,cu)u\,\mathrm{d}v_g &= \frac{1}{c}\int_{\mathbb{B}^2} f(x,cu)(cu)\,\mathrm{d}v_g \\ &\geqslant \frac{1}{c}\int_{\{|x|\leqslant\alpha\}\cap\{u\geqslant t_2/c\}} f(x,cu)(cu)\,\mathrm{d}v_g \\ &\geqslant \frac{C_2(\varepsilon)}{c}\int_{\{|x|\leqslant\alpha\}\cap\{u\geqslant t_2/c\}} [\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2}-1]\,\mathrm{d}v_g \end{aligned} \tag{4.13}$$

and

$$\int_{\{|x|\leqslant\alpha\}\cap\{u\leqslant t_2/c\}} [\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2}-1]\,\mathrm{d}v_g \leqslant C(\alpha,t_2,c). \tag{4.14}$$

Therefore, (4.13) and (4.14) together give

$$\int_{\{|x|\leqslant\alpha\}} (\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2}-1)\,\mathrm{d}v_g \leqslant C(\alpha,\varepsilon,c)\int_{\mathbb{B}^2} f(x,cu)u\,\mathrm{d}v_g + C(\alpha,t_2,c). \tag{4.15}$$

Define

$$\tilde{C} = \frac{1}{\sqrt{4\pi}}\left[\frac{1-\alpha^2}{\alpha}\right]^{1/2}.$$

Now, by the radial estimate we have that for all $u \in H^1_R(\mathbb{B}^2)$ with $\|u\| \leqslant 1$,

$$|u(x)| \leqslant \tilde{C} \quad \text{whenever } |x| > \alpha.$$

Therefore, we have

$$\begin{aligned}\int_{\{|x|>\alpha\}} (\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2} - 1)\,\mathrm{d}v_g &\leqslant \int_{\{|u|\leqslant\tilde{C}\}} (\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2} - 1)\,\mathrm{d}v_g \\ &\leqslant C\int_{\{|u|\leqslant\tilde{C}\}} u^2\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2}\,\mathrm{d}v_g \\ &\leqslant C\int_{\mathbb{B}^2} u^2\,\mathrm{d}v_g \\ &\leqslant C\|u\|^2 \\ &\leqslant C. \end{aligned} \tag{4.16}$$

Taking into account (4.15) and (4.16), we obtain

$$\sup_{\substack{u\in H^1_R(\mathbb{B}^2),\\ \|u\|\leqslant 1}} \int_{\mathbb{B}^2} (\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2} - 1)\,\mathrm{d}v_g < +\infty. \tag{4.17}$$

Now, using the hyperbolic version of the MT inequality (1.2), we have $(1-\varepsilon)c^2 \leqslant 4\pi/\lambda$. Since $\varepsilon > 0$ was arbitrary, we deduce that $\tilde{c}^2 \leqslant 4\pi/\lambda$.

Now, suppose that $\tilde{c}^2 < 4\pi/\lambda$. Choose $\varepsilon > 0$ such that $(1+\varepsilon)^3\tilde{c}^2 < 4\pi/\lambda$. Then for all $u \in H^1_R(\mathbb{B}^2)$ with $\|u\| \leqslant 1$ we have

$$\begin{aligned}\int_{\mathbb{B}^2} f(x,(1+\varepsilon)\tilde{c}u)u\,\mathrm{d}v_g &\leqslant C\int_{\{|u|>t_1/(1+\varepsilon)\tilde{c}\}} f(x,(1+\varepsilon)\tilde{c}u)(1+\varepsilon)\tilde{c}u\,\mathrm{d}v_g \\ &\quad + C\int_{\{|u|\leqslant t_1/(1+\varepsilon)\tilde{c}\}} f(x,(1+\varepsilon)\tilde{c}u)(1+\varepsilon)\tilde{c}u\,\mathrm{d}v_g \\ &\leqslant C\int_{\mathbb{B}^2} (\mathrm{e}^{\lambda(1+\varepsilon)^3\tilde{c}^2u^2} - 1)\,\mathrm{d}v_g \\ &\quad + C\int_{\{|u|\leqslant t_1/(1+\varepsilon)\tilde{c}\}} (\mathrm{e}^{\lambda(1+\varepsilon)^2\tilde{c}^2u^2} - 1)\,\mathrm{d}v_g \\ &\leqslant C + C\int_{\{|u|\leqslant t_1/(1+\varepsilon)\tilde{c}\}} u^2\mathrm{e}^{\lambda(1+\varepsilon)^2\tilde{c}^2u^2}\,\mathrm{d}v_g \\ &\leqslant C + C\int_{\mathbb{B}^2} u^2\,\mathrm{d}v_g \\ &\leqslant C + C\|u\|^2. \end{aligned} \tag{4.18}$$

As a consequence, we derive that

$$\sup_{\substack{u\in H^1_R(\mathbb{B}^2),\\ \|u\|\leqslant 1}} \int_{\mathbb{B}^2} f(x,(1+\varepsilon)\tilde{c}u)u\,\mathrm{d}v_g < +\infty,$$

which contradicts the definition of $\tilde{c}$. So we must have $\tilde{c}^2 = 4\pi/\lambda$. □

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Proposition 4.4. *Let $\{u_k\}$ be a sequence in $H^1_R(\mathbb{B}^2)$ such that u_k converges weakly to some u in the space $H^1_R(\mathbb{B}^2)$ and assume that*

$$\sup_k \int_{\mathbb{B}^2} f(x,u_k)u_k \,\mathrm{d}v_g < +\infty.$$

Then we have the following convergence results:

(i) $\displaystyle\lim_{k\to\infty}\int_{\{|x|<\alpha\}} f(x,|u_k|)\,\mathrm{d}v_g = \int_{\{|x|<\alpha\}} f(x,|u|)\,\mathrm{d}v_g$ *for any $\alpha<1$,*

(ii) $\displaystyle\lim_{k\to\infty}\int_{\mathbb{B}^2} F(x,u_k)\,\mathrm{d}v_g = \int_{\mathbb{B}^2} F(x,u)\,\mathrm{d}v_g.$

Proof. (i) Fix $\alpha>0$. Then we have

$$\int_{\{|x|<\alpha\}\cap\{|u_k|>N\}} f(x,|u_k|)\,\mathrm{d}v_g \leqslant \frac{1}{N}\int_{\mathbb{B}^2} f(x,|u_k|)|u_k|\,\mathrm{d}v_g \leqslant \frac{C}{N}.$$

Therefore,

$$\begin{aligned}\int_{\{|x|<\alpha\}} f(x,|u_k|)\,\mathrm{d}v_g &= \int_{\{|x|<\alpha\}\cap\{|u_k|\leqslant N\}} f(x,|u_k|)\,\mathrm{d}v_g \\ &\quad + \int_{\{|x|<\alpha\}\cap\{|u_k|>N\}} f(x,|u_k|)\,\mathrm{d}v_g \\ &= \int_{\{|x|<\alpha\}\cap\{|u_k|\leqslant N\}} f(x,|u_k|)\,\mathrm{d}v_g + O\left(\frac{1}{N}\right).\end{aligned}$$

Hence, using the dominated convergence theorem, letting $k\to\infty$ followed by $N\to\infty$ we have

$$\lim_{k\to\infty}\int_{\{|x|<\alpha\}} f(x,|u_k|)\,\mathrm{d}v_g = \int_{\{|x|<\alpha\}} f(x,|u|)\,\mathrm{d}v_g. \tag{4.19}$$

(ii) Fix some $\alpha\in(0,1)$ close to 1. Since $u_k \rightharpoonup u$ in $H^1_R(\mathbb{B}^2)$ we have

$$\sup_k \|u_k\| \leqslant C < +\infty.$$

So by radial estimate (2.5),

$$\sup_k |u_k(x)| \leqslant C(1-|x|^2)^{1/2} \quad \text{for } |x|>\alpha$$

so that

$$F(x,u_k) \leqslant C(1-|x|^2)^{3/2} \quad \text{for } |x|>\alpha. \tag{4.20}$$

Since $(1-|x|^2)^{3/2}\in L^1(\{|x|>\alpha\},\mathrm{d}v_g)$, by the dominated convergence theorem we get

$$\lim_{k\to+\infty}\int_{\{|x|>\alpha\}} F(x,u_k)\,\mathrm{d}v_g = \int_{\{|x|>\alpha\}} F(x,u)\,\mathrm{d}v_g. \tag{4.21}$$

For $\{|x|<\alpha\}$, we can use (C3) and (4.19) to conclude that

$$\lim_{k\to+\infty}\int_{\{|x|<\alpha\}} F(x,u_k)\,\mathrm{d}v_g = \int_{\{|x|<\alpha\}} F(x,u)\,\mathrm{d}v_g. \tag{4.22}$$

So (4.21) and (4.22) together prove the lemma. □

Proposition 4.5. *Let $\{u_k\}$ and $\{v_k\}$ be bounded sequences in $H^1_R(\mathbb{B}^2)$ converging weakly to u and v, respectively. Furthermore, assume that*

$$\sup_k \|u_k\|^2 < \frac{4\pi}{\lambda}.$$

Then, for every $l \geqslant 2$,

$$\lim_{k\to+\infty} \int_{\mathbb{B}^2} \frac{f(x,u_k)}{u_k} v_k^l \, \mathrm{d}v_g = \int_{\mathbb{B}^2} \frac{f(x,u)}{u} v^l \, \mathrm{d}v_g. \tag{4.23}$$

Proof. Fix $\delta > 0$. Since v_k converges weakly to v, we have $\sup_k \|v_k\| < +\infty$. Let $\tilde{C}^2 > 4\pi/\lambda$ be such that

$$\sup_k \|v_k\| \leqslant \tilde{C}.$$

Define

$$\alpha = \frac{1}{\tilde{C}} \left[\sqrt{\tilde{C}^2 + \left(\frac{2\pi\delta^2}{\tilde{C}} \right)^2} - \frac{2\pi\delta^2}{\tilde{C}} \right].$$

Then, by the radial estimate, it holds that

$$\sup_k |u_k(x)| \leqslant \delta \quad \text{whenever } |x| > \alpha \tag{4.24}$$

and

$$\sup_k |v_k(x)| \leqslant \frac{\tilde{C}}{\sqrt{4\pi\alpha}} (1-|x|^2)^{1/2} \quad \text{whenever } |x| > \alpha. \tag{4.25}$$

Now, since $\sup_k \|u_k\|^2 < 4\pi/\lambda$, we can choose $\varepsilon > 0$ sufficiently small and $p > 1$ such that

$$(\lambda+\varepsilon)p\|u_k\|^2 < 4\pi \quad \text{for all } k. \tag{4.26}$$

By (C4), there exists $N_0 > 0$ such that for all $t \geqslant N_0$,

$$h(x,t) \leqslant C\mathrm{e}^{\varepsilon t^2} \quad \text{for all } x.$$

Therefore, for all $N \geqslant N_0$, it holds that

$$\begin{aligned} \int_{\{|u_k|>N\}} |f(x,u_k)|^p \, \mathrm{d}v_g &= \int_{\{|u_k|>N\}} |h(x,u_k)|^p (\mathrm{e}^{\lambda u_k^2} - 1)^p \, \mathrm{d}v_g \\ &\leqslant C \int_{\{|u_k|>N\}} \mathrm{e}^{\varepsilon p u_k^2} (\mathrm{e}^{\lambda p u_k^2} - 1) \, \mathrm{d}v_g \\ &\leqslant C \int_{\{|u_k|>N\}} (\mathrm{e}^{(\lambda+\varepsilon) p u_k^2} - 1) \, \mathrm{d}v_g \\ &\leqslant C \int_{\mathbb{B}^2} (\mathrm{e}^{(\lambda+\varepsilon)p\|u_k\|^2 (u_k/\|u_k\|)^2} - 1) \, \mathrm{d}v_g \\ &\leqslant C_1. \end{aligned} \tag{4.27}$$

Let q be the conjugate exponent of p. Then

$$\sup_k \int_{\mathbb{B}^2} |v_k|^{lq} \, \mathrm{d}v_g \leqslant C \sup_k \|v_k\|^{lq} \leqslant C(l,q). \tag{4.28}$$

By (4.27), (4.28) and Hölder's inequality, we have

$$\int_{\{|u_k|>N\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g = O\left(\frac{1}{N}\right). \tag{4.29}$$

Now, using (4.24) and (4.25), we get

$$\begin{aligned}
\left|\int_{\{|u_k|\leqslant\delta\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g\right| &\leqslant \int_{\{|u_k|\leqslant\delta\}} |h(x,u_k)| \frac{(\mathrm{e}^{\lambda u_k^2}-1)}{|u_k|} |v_k^l| \,\mathrm{d}v_g \\
&\leqslant \int_{\{|u_k|\leqslant\delta\}} |h(x,u_k)|\,|u_k| \frac{(\mathrm{e}^{\lambda u_k^2}-1)}{|u_k^2|} |v_k^l| \,\mathrm{d}v_g \\
&\leqslant C \int_{\{|u_k|\leqslant\delta\}} |u_k|\,|v_k^l| \,\mathrm{d}v_g \\
&\leqslant C \int_{\{|x|>\alpha\}} (1-|x|^2)^{(1+l)/2} \,\mathrm{d}v_g + O(\delta) \\
&= \circ(1) \quad \text{as } \delta \to 0.
\end{aligned} \tag{4.30}$$

From (4.29) and (4.30) we get

$$\int_{\mathbb{B}^2} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g = \int_{\{\delta\leqslant|u_k|\leqslant N\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g + O\left(\frac{1}{N}\right) + O(\delta). \tag{4.31}$$

Now the proof follows by the dominated convergence theorem, and thereafter by letting $N \to \infty$, $\delta \to 0$. □

Remark. By taking $v_k = u_k$ and $l = 2$, we see that if $\sup_k \|u_k\|^2 < 4\pi/\lambda$, then

$$\int_{\mathbb{B}^2} f(x,u_k)u_k \,\mathrm{d}v_g \to \int_{\mathbb{B}^2} f(x,u)u \,\mathrm{d}v_g.$$

In general it is difficult to prove that $\sup_k \|u_k\|^2 < 4\pi/\lambda$ from the functional itself. However, we need this compactness criterion in order to get the existence of a minimizer on $\mathcal{M}$, and hence a solution of (1.4).

Next we will investigate under which circumstances we can pass to the limit without the condition mentioned in the above remark. We use the hyperbolic version of Lions's lemma (lemma 2.3) in order to give an affirmative answer on passing to the limit.

Proposition 4.6. *Let $\{u_k\}$ be a sequence converging weakly to a non-zero function u in $H_R^1(\mathbb{B}^2)$ and assume that*

(i) *there exists $c \in (0, 2\pi/\lambda)$ such that $J_\lambda(u_k) \to c$,*

(ii) $\|u\|^2 \geqslant \int_{\mathbb{B}^2} f(x,u)u \,\mathrm{d}v_g$,

(iii) $\sup_k \int_{\mathbb{B}^2} f(x,u_k)u_k \,\mathrm{d}v_g < +\infty$.

Then

$$\lim_{k\to\infty}\int_{\mathbb{B}^2} f(x,u_k)u_k\,\mathrm{d}v_g = \int_{\mathbb{B}^2} f(x,u)u\,\mathrm{d}v_g.$$

Proof. Arguing as in the proof of [1, lemma 3.3] and using the hyperbolic version of Lions's lemma (lemma 2.3), we get

$$\sup_k \int_{\mathbb{B}^2} (\mathrm{e}^{(1+\varepsilon)\lambda u_k^2} - 1)\,\mathrm{d}v_g < +\infty.$$

By (C4), we can assume that

$$M_2 = \sup h(x,t)t\mathrm{e}^{-\lambda\varepsilon t^2/2} < +\infty,$$

so that

$$\begin{aligned}\int_{\{|u_k|>N\}} f(x,u_k)u_k\,\mathrm{d}v_g &= \int_{\{|u_k|>N\}} h(x,u_k)u_k(\mathrm{e}^{\lambda u_k^2}-1)\,\mathrm{d}v_g \\ &\leqslant \int_{\{|u_k|>N\}} (h(x,u_k)u_k\mathrm{e}^{-\lambda\varepsilon u_k^2})(\mathrm{e}^{(1+\varepsilon)\lambda u_k^2}-1)\,\mathrm{d}v_g \\ &\leqslant M_2\mathrm{e}^{-\lambda\varepsilon N^2/2}\int_{\mathbb{B}^2}(\mathrm{e}^{(1+\varepsilon)\lambda u_k^2}-1)\,\mathrm{d}v_g \\ &\leqslant C\mathrm{e}^{-\lambda\varepsilon N^2/2} \qquad (4.32)\end{aligned}$$

holds.

Fix $\delta > 0$ and let $\tilde{C}$ be such that $\sup_k \|u_k\| \leqslant \tilde{C}$ and let α depending on $\tilde{C}$ be as before. Then $\alpha = 1 - O(\delta)$ as $\delta \to 0$, and there holds

$$|u_k(x)| \leqslant C(1-|x|^2)^{1/2} \quad \text{whenever } |x| > \alpha.$$

By using the above, we have

$$\begin{aligned}\int_{\{|u_k|\leqslant\delta\}} f(x,u_k)u_k\,\mathrm{d}v_g &\leqslant \int_{\{|u_k|\leqslant\delta\}} h(x,u_k)u_k(\mathrm{e}^{\lambda u_k^2}-1)\,\mathrm{d}v_g \\ &\leqslant C\int_{\{|x|>\alpha\}} |u_k|^3\,\mathrm{d}v_g + O(\delta) \\ &\leqslant C\int_{\{|x|>\alpha\}} (1-|x|^2)^{3/2}\,\mathrm{d}v_g + O(\delta) \\ &\leqslant C(1-\alpha^2)^{1/2} + O(\delta) \\ &= O(\delta). \qquad (4.33)\end{aligned}$$

Thus, from (4.32) and (4.33) we get

$$\int_{\mathbb{B}^2} f(x,u_k)u_k\,\mathrm{d}v_g = \int_{\{\delta\leqslant|u_k|\leqslant N\}} f(x,u_k)u_k\,\mathrm{d}v_g + O(\mathrm{e}^{-\lambda\varepsilon N^2/2}) + O(\delta).$$

So the lemma follows by letting $k \to \infty$ and then letting $N \to \infty$, $\delta \to 0$ successively. □

Lemma 4.7. *Let* $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2} - 1)$ *be a function of critical growth. Fix* $0 < R_0 < 1$ *and* $0 < l_0 < R_0$. *Define*

$$h_{0,l_0}(t) := \inf_{x \in B_{l_0}(0)} h(x,t) \quad \text{and} \quad M_0 := \sup_{t \geqslant 0} h_{0,l_0}(t)t,$$

and

$$k_0 = \begin{cases} \dfrac{2}{M_0(R_0^2 - l_0^2)} & \text{if } M_0 < +\infty, \\ 0 & \text{if } M_0 = +\infty. \end{cases}$$

Let $a \geqslant 0$ *be such that*

$$\sup_{\|u\| \leqslant 1} \int_{\mathbb{B}^2} f(x, au)u \, \mathrm{d}v_g \leqslant a.$$

If $k_0/\lambda < 1$, *then* $a^2 < 4\pi/\lambda$.

Proof. From lemma 4.3, we have $a^2 \leqslant 4\pi/\lambda$. Suppose, if possible, that $a^2 = 4\pi/\lambda$.

Let m_{l,R_0} be the Moser function defined in § 2. Then m_{l,R_0} is constant in $\{|x| < l\}$. Let $t = am_{l,R_0}$ when $|x| < l$, and note the following trivial inequality:

$$1 \leqslant \frac{1}{(1 - |x|^2)^2} \leqslant \frac{1}{(1 - l^2)^2} \quad \text{for } 0 < |x| < l.$$

Then we have

$$\begin{aligned} a^2 &\geqslant \int_{\mathbb{B}^2} f(x, am_{l,R_0})(am_{l,R_0}) \, \mathrm{d}v_g \\ &\geqslant \int_{\{|x|<l\}} f(x,t)t \, \mathrm{d}v_g \\ &\geqslant \int_{\{|x|<l\}} h(x,t)(\mathrm{e}^{\lambda t^2} - 1)t \, \mathrm{d}x \\ &\geqslant 2\pi h_{0,l_0}(t)t(\mathrm{e}^{\lambda t^2} - 1)l^2. \end{aligned} \tag{4.34}$$

Now, by our assumption that $a^2 = 4\pi/\lambda$, using $\mathrm{e}^{\lambda t^2} = R_0^2/l^2$ and (4.34) we get

$$\frac{4\pi}{\lambda} \geqslant 2\pi h_{0,l_0}(t)t(R_0^2 - l^2) \geqslant 2\pi h_{0,l_0}(t)t(R_0^2 - l_0^2) \geqslant 2\pi M_0(R_0^2 - l_0^2).$$

This gives $\lambda \leqslant 2/M_0(R_0^2 - l_0^2) = k_0$, which is a contradiction. Hence, we must have $a^2 < 4\pi/\lambda$. □

Now we can prove theorem 4.1.

Proof of theorem 4.1. Let $\{u_k\}$ be a sequence in $H_R^1(\mathbb{B}^2)$ such that

$$\begin{aligned} \lim_{k \to +\infty} J_\lambda(u_k) &= c, \\ \lim_{k \to +\infty} J'_\lambda(u_k) &= 0 \end{aligned} \tag{4.35}$$

for some $c \in (0, 2\pi/\lambda)$.

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CLAIM. *$\{u_k\}$ is a bounded sequence in $H^1_R(\mathbb{B}^2)$.*

Proof of claim. As $J_\lambda(u_k) \to c$ and $J'_\lambda(u_k) \to 0$, we have

$$J_\lambda(u_k) \leqslant M_0 \quad \text{and} \quad \langle J'_\lambda(u_k), u_k\rangle \leqslant M_0(1 + \|u_k\|) \quad \text{for all } k,$$

where $M_0 > 0$ is a constant. Also,

$$J_\lambda(u_k) - \tfrac{1}{2}\langle J'_\lambda(u_k), u_k\rangle = I_\lambda(u_k), \tag{4.36}$$

which gives $I_\lambda(u_k) \leqslant M_1(1 + \|u_k\|)$. Hence, from lemma 4.2(ii) we have, for $\varepsilon > 0$ small,

$$\int_{\mathbb{B}^2} f(x, u_k)u_k \,\mathrm{d}v_g \leqslant M_2(\varepsilon)(1 + \|u_k\|) + \varepsilon\|u_k\|^2. \tag{4.37}$$

Therefore, from (C2) it follows that

$$\int_{\mathbb{B}^2} F(x, u_k) \,\mathrm{d}v_g \leqslant C_0(\varepsilon)(1 + \|u_k\|) + \tilde{C}\varepsilon\|u_k\|^2.$$

Now using the boundedness of $J_\lambda(u_k)$, and choosing ε such that $1 - \tilde{C}\varepsilon > 0$, we have

$$(1 - \tilde{C}\varepsilon)\|u_k\|^2 \leqslant C_0(1 + \|u_k\|),$$

which proves that the $\|u_k\|$s are bounded. This proves the claim. □

We also infer from (4.37) that

$$\sup_k \int_{\mathbb{B}^2} f(x, u_k)u_k \,\mathrm{d}v_g < +\infty. \tag{4.38}$$

By extracting a subsequence (if necessary) we may assume that u_k converges to a function $u \in H^1_R(\mathbb{B}^2)$ weakly. Now we shall consider two cases.

CASE 1 ($c \leqslant 0$). Using (4.36) and lemma 4.2(ii) we have

$$\begin{aligned} 0 \leqslant I_\lambda(u) &\leqslant \liminf_{k\to\infty} I_\lambda(u_k) \\ &= \liminf_{k\to\infty} \{J_\lambda(u_k) - \tfrac{1}{2}\langle J'_\lambda(u_k), u_k\rangle\} \\ &= c. \end{aligned}$$

It follows that no PS sequence exists if $c < 0$. If $c = 0$, then from proposition 4.4(ii) we have

$$\lim_{k\to\infty} \|u_k\|^2 = 2 \lim_{k\to\infty} \left\{ J_\lambda(u_k) + \int_{\mathbb{B}^2} F(x, u_k) \,\mathrm{d}v_g \right\} = 0,$$

and hence u_k converges strongly to 0 in $H^1_R(\mathbb{B}^2)$.

Case 2 ($c \in (0, 2\pi/\lambda)$). First we shall show that $u \not\equiv 0$. Suppose, if possible, that $u \equiv 0$. From (4.38) and proposition 4.4(ii) we have

$$\begin{aligned}\lim_{k\to+\infty} \|u_k\|^2 &= 2 \lim_{k\to+\infty} \left\{ J_\lambda(u_k) + \int_{\mathbb{B}^2} F(x, u_k)\, \mathrm{d}v_g \right\} \\ &= 2c \\ &< \frac{4\pi}{\lambda}.\end{aligned}$$

It follows that u_k satisfies the hypothesis of proposition 4.5 with $v_k = u_k$, $l = 2$, and hence we have

$$\lim_{k\to+\infty} \int_{\mathbb{B}^2} f(x, u_k) u_k \, \mathrm{d}v_g = \int_{\mathbb{B}^2} f(x, u) u \, \mathrm{d}v_g = 0.$$

This gives

$$\lim_{k\to+\infty} I_\lambda(u_k) = \lim_{k\to+\infty} \left\{ \frac{1}{2} \int_{\mathbb{B}^2} f(x, u_k) u_k \, \mathrm{d}v_g - \int_{\mathbb{B}^2} F(x, u_k)\, \mathrm{d}v_g \right\} = 0.$$

But from (4.36) we get

$$c = \lim_{k\to+\infty} J_\lambda(u_k) = \lim_{k\to+\infty} \{ I_\lambda(u_k) + \tfrac{1}{2} \langle J'_\lambda(u_k), u_k \rangle \} = 0,$$

which is a contradiction. Hence, we must have $u \not\equiv 0$. By the definition of $J'_\lambda(u)$ and a standard density argument it follows that

$$\|u\|^2 = \int_{\mathbb{B}^2} f(x, u) u \, \mathrm{d}v_g. \tag{4.39}$$

Now, since u_k and u satisfy all the hypotheses of proposition 4.6, we have

$$\lim_{k\to+\infty} \int_{\mathbb{B}^2} f(x, u_k) u_k \, \mathrm{d}v_g = \int_{\mathbb{B}^2} f(x, u) u \, \mathrm{d}v_g,$$

and hence by lower semi-continuity of the norm we obtain

$$\begin{aligned}\|u\|^2 &\leqslant \liminf_{k\to+\infty} \|u_k\|^2 \\ &= 2 \liminf_{k\to+\infty} \left\{ J_\lambda(u_k) + \int_{\mathbb{B}^2} F(x, u_k)\, \mathrm{d}v_g \right\} \\ &= 2 \liminf_{k\to+\infty} \left\{ I_\lambda(u_k) + \tfrac{1}{2} \langle J'_\lambda(u_k), u_k \rangle + \int_{\mathbb{B}^2} F(x, u_k)\, \mathrm{d}v_g \right\} \\ &= \liminf_{k\to+\infty} \left\{ \int_{\mathbb{B}^2} f(x, u_k) u_k \, \mathrm{d}v_g + \langle J'_\lambda(u_k), u_k \rangle \right\} \\ &= \int_{\mathbb{B}^2} f(x, u) u \, \mathrm{d}v_g \\ &= \|u\|^2.\end{aligned}$$

This implies that $u_k \to u$ strongly in $H^1_R(\mathbb{B}^2)$, and thus completes the proof of part (i).

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Proof of part (ii). The proof goes along the same lines as in [1] with obvious modifications, so we will briefly sketch the proof here.

STEP 1 ($\eta(f) > 0$). If possible, we assume that $\eta(f) = 0$ and let $\{u_k\}$ be a sequence in $\mathcal{M}$ such that $J_\lambda(u_k) = I_\lambda(u_k)$ converges to 0. Then, from lemma 4.2(ii) and proceeding as before, we can assume that

$$\sup_k \|u_k\| < +\infty,$$
$$\sup_k \int_{\mathbb{B}^2} f(x, u_k)u_k \, \mathrm{d}v_g < +\infty.$$

By extracting a subsequence and using Fatou's lemma and proposition 4.4 we can conclude that

$$u_k \to 0 \quad \text{strongly in } H^1_R(\mathbb{B}^2).$$

In contrast, considering $v_k = u_k/\|u_k\|$, we have that v_k converges weakly to v. Then, by proposition 4.5 and observing that $u_k \in \mathcal{M}$, we get

$$\begin{aligned} 1 &= \lim_{k\to\infty} \int_{\mathbb{B}^2} \frac{f(x, u_k)}{u_k} v_k^2 \, \mathrm{d}v_g \\ &= \int_{\mathbb{B}^2} f'(x, 0)v^2 \, \mathrm{d}v_g \\ &= 0. \end{aligned}$$

This contradiction proves that $\eta(f) > 0$.

Now, for the second part we need the following claim.

CLAIM. *For every $u \in H^1_R(\mathbb{B}^2)\backslash\{0\}$, there exists a constant $\gamma(u) > 0$ such that $\gamma(u)u \in \mathcal{M}$. In addition, if one assumes that*

$$\|u\|^2 \leqslant \int_{\mathbb{B}^2} f(x, u)u \, \mathrm{d}v_g,$$

then $\gamma(u) \leqslant 1$, and $\gamma(u) = 1$ if and only if $u \in \mathcal{M}$.

Considering

$$\psi(\gamma) = \frac{1}{\gamma} \int_{\mathbb{B}^2} f(x, \gamma u)u \, \mathrm{d}v_g \quad \text{for } \gamma > 0,$$

one observes that

$$\lim_{\gamma\to 0} \psi(\gamma) = \int_{\mathbb{B}^2} f'(x, 0)u^2 \, \mathrm{d}v_g = 0 < \|u\|^2,$$
$$\lim_{\gamma\to\infty} \psi(\gamma) = \infty.$$

So the first part of the claim follows by continuity of ψ. Since, by (C2), we have that $(f(x, tu)/t)u$ is an increasing function of t, the second part of the claim follows.

STEP 2 ($\eta(f)^2 < 4\pi/\lambda$). In view of (4.2) and lemma 4.7, it is enough to prove that

$$\sup_{\|u\|\leqslant 1} \int_{\mathbb{B}^2} f(x, \eta(f)u)u \, \mathrm{d}v_g \leqslant \eta(f).$$

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Let $u \in H^1_R(\mathbb{B}^2)$ with $\|u\| = 1$. By the above claim there exists $\gamma(u) > 0$ such that $\gamma(u)u \in \mathcal{M}$. Then

$$\frac{\eta(f)^2}{2} \leqslant J_\lambda(\gamma u) \leqslant \frac{\gamma^2}{2},$$

that is, $\eta(f) \leqslant \gamma$, and hence by monotonicity of $(f(x,tu)/t)u$ with respect to t we have

$$\int_{\mathbb{B}^2} \frac{f(x,\eta(f)u)}{\eta(f)} u \, \mathrm{d}v_g \leqslant \int_{\mathbb{B}^2} \frac{f(x,\gamma u)}{\gamma} u \, \mathrm{d}v_g = 1,$$

and this completes the proof. □

5. Proof of main theorems

In this section we prove the existence of solutions for (1.4). First we state the following abstract result.

Lemma 5.1. *Let f be a function of critical growth on $\mathbb{B}^2$.*

(1) *Let $u_0 \in \mathcal{M}_1$ be such that $J'_\lambda(u_0) \not\equiv 0$. Then*

$$J_\lambda(u_0) > \inf\{J_\lambda(u) \colon u \in \mathcal{M}_1\}.$$

(2) *Let u_1 and u_2 be two non-negative linearly independent functions in $H^1_R(\mathbb{B}^2)$. Then there exist $p, q \in \mathbb{R}$ such that $pu_1 + qu_2 \in \mathcal{M}_1$.*

The proof of lemma 5.1 follows from the result of Cerami *et al*. (see [16]) with obvious modifications.

Remark. Lemma 5.1(1) holds for functions in $\mathcal{M}$ as well.

Proof of theorem 1.2. As $J_\lambda(u) = J_\lambda(|u|)$, it is enough to prove that the minimum is attained on $\mathcal{M}$ for some non-zero function (thanks to the above remark and the principle of symmetric criticality). Hence, we only need to show that there exists $u \in \mathcal{M}$ with $u \not\equiv 0$ such that

$$J_\lambda(u) = \frac{\eta(f)^2}{2}.$$

Also, by theorem 4.1(ii), we know that

$$0 < \eta(f)^2 < \frac{4\pi}{\lambda}.$$

Let $\{u_k\}$ be a minimizing sequence. Since $J_\lambda = I_\lambda$ on $\mathcal{M}$, we have that (from lemma 4.2(ii))

$$\int_{\mathbb{B}^2} f(x,u_k)u_k \, \mathrm{d}v_g \leqslant C(1 + I_\lambda(u_k)) + \varepsilon \|u_k\|^2 \tag{5.1}$$

and, arguing as before, we get

$$\sup_k \|u_k\| < +\infty, \tag{5.2}$$

$$\sup_k \int_{\mathbb{B}^2} f(x,u_k)u_k \, \mathrm{d}v_g < +\infty. \tag{5.3}$$

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By extracting a subsequence we can assume that u_k converges to u weakly in $H^1_R(\mathbb{B}^2)$.

We claim that $u \not\equiv 0$ and $u \in \mathcal{M}$. Indeed, if possible, we assume that $u \equiv 0$. By (5.3) and proposition 4.4(ii), we conclude that

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} F(x, u_k)\,\mathrm{d}v_g = 0.$$

This gives

$$\begin{aligned}\lim_{k\to\infty} \|u_k\|^2 &= 2 \lim_{k\to\infty} \left\{ J_\lambda(u_k) + \int_{\mathbb{B}^2} F(x, u_k)\,\mathrm{d}v_g \right\} \\ &= \eta(f)^2 \in \left(0, \frac{4\pi}{\lambda}\right). \end{aligned} \tag{5.4}$$

Also, (5.4) enables us to use proposition 4.5 with $v_k = u_k$ and $l = 2$ to conclude that

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} f(x, u_k)u_k\,\mathrm{d}v_g = 0,$$

which is not possible, otherwise this would give

$$\eta(f)^2 = 2 \lim_{k\to\infty} J_\lambda(u_k) = 2 \lim_{k\to\infty} I_\lambda(u_k) = 0.$$

Hence, we must have $u \not\equiv 0$. Now it remains to show that $u \in \mathcal{M}$.

First assume that

$$\|u\|^2 > \int_{\mathbb{B}^2} f(x, u)u\,\mathrm{d}v_g.$$

This together with (5.3) and proposition 4.6 gives

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} f(x, u_k)u_k\,\mathrm{d}v_g = \int_{\mathbb{B}^2} f(x, u)u\,\mathrm{d}v_g.$$

Then lower semi-continuity of the norm implies that

$$\begin{aligned}\|u\|^2 &\leqslant \liminf_{k\to\infty} \|u_k\|^2 \\ &= \liminf_{k\to\infty} \int_{\mathbb{B}^2} f(x, u_k)u_k\,\mathrm{d}v_g \\ &= \int_{\mathbb{B}^2} f(x, u)u\,\mathrm{d}v_g.\end{aligned}$$

But this contradicts our initial assumption. Hence, we must have

$$\|u\|^2 \leqslant \int_{\mathbb{B}^2} f(x, u)u\,\mathrm{d}v_g.$$

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It follows from the proof of theorem 4.1(ii) that there exists $0 < \gamma \leqslant 1$ such that $\gamma u \in \mathcal{M}$. Then by monotonicity of $(f(x,tu)u)/t$ we have

$$\begin{aligned}\frac{\eta(f)^2}{2} \leqslant J_\lambda(\gamma u) &= I_\lambda(\gamma u)\\ &\leqslant I_\lambda(u)\\ &\leqslant \liminf_{k\to\infty} I_\lambda(u_k)\\ &= \liminf_{k\to\infty} J_\lambda(u_k) = \frac{\eta(f)^2}{2},\end{aligned}$$

and then, again using theorem 4.1(ii), we conclude that $\gamma = 1$ and $J_\lambda(u) = \eta(f)^2/2$. This completes the proof. □

Our next job is to investigate existence of a sign-changing solution, the proof of which will heavily depend on the following concentration lemma. The proof of the concentration lemma follows along the same lines as in [4, lemma 3.1] with some modifications (see appendix A).

Here we state the lemma.

Lemma 5.2. *Let $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2} - 1)$ be a function of critical growth on $\mathbb{B}^2$ and let V be the one-dimensional subspace defined by $\{pu_0 \colon p \in \mathbb{R}\}$ of $H^1_R(\mathbb{B}^2)$. Let $h_{0,\beta}(t) = \inf\{h(x,t); x \in \overline{B(0,\beta)}\}$ and $C(V) = \sup\{J_\lambda(u) \colon u \in V\}$. Assume that for every $N > 0$ there exists $t_N > 0$ such that*

$$h_{0,\beta}(t)t \geqslant \mathrm{e}^{Nt} \quad \forall t \geqslant t_N.$$

Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\sup_{u\in V,\ t\in\mathbb{R}} J_\lambda(u + tm_{\varepsilon,\beta}) < C(V) + \frac{2\pi}{\lambda}, \tag{5.5}$$

where $m_{\varepsilon,\beta}$ is the Moser function.

Now we can prove theorem 1.3.

Proof of theorem 1.3. From lemma 5.1, it is sufficient to show that the infimum of J_λ is achieved on $\mathcal{M}_1$. We first make the following claim.

Claim 1. $0 < \dfrac{\eta_1(f)^2}{2} < \dfrac{\eta(f)^2}{2} + \dfrac{2\pi}{\lambda}$.

By definition it is clear that $\eta_1(f)^2 \geqslant \eta(f)^2$. By theorem 1.2, let $u_0 \in \mathcal{M}$ be such that

$$\sup_{\alpha\in\mathbb{R}} J_\lambda(\alpha u_0) = J_\lambda(u_0) = \frac{\eta(f)^2}{2} > 0; \tag{5.6}$$

hence, this gives $\eta_1(f) > 0$. From lemma 5.1(2), for any $n_0 > 0$,

$$\frac{\eta_1(f)^2}{2} \leqslant \sup_{p,q\in\mathbb{R}} J_\lambda(pu_0 + qm_{n_0,\beta}), \tag{5.7}$$

where $m_{n_0,\beta}$ is the Moser function. Again from (5.6) and by considering $V = \{pu_0, p \in \mathbb{R}\}$ in lemma 5.2, there exists $n_1 > 0$ such that for $0 < n_0 < n_1$,

$$\sup_{p,q\in\mathbb{R}} J_\lambda(pu_0 + qm_{n_0,\beta}) < \frac{\eta(f)^2}{2} + \frac{2\pi}{b}. \tag{5.8}$$

Hence, claim 1 follows from (5.7) and (5.8).

Let u_k be in $\mathcal{M}_1$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \frac{\eta_1(f)^2}{2}.$$

Since $J_\lambda = I_\lambda$ on $\mathcal{M}_1$, from lemma 4.2(ii) we obtain

$$\sup_k \|u_k\| < \infty, \qquad \sup_k \int_{\mathbb{B}^2} f(x, u_k)u_k \, \mathrm{d}v_g < \infty. \tag{5.9}$$

Therefore, we can extract a subsequence of $\{u_k\}$ such that

$$u_k^{\pm} \to u_0^{\pm} \quad \text{weakly}.$$

From (5.9) and proposition 4.4, we get

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} F(x, u_k^{\pm}) \, \mathrm{d}v_g = \int_{\mathbb{B}^2} F(x, u_0^{\pm}) \, \mathrm{d}v_g. \tag{5.10}$$

In accordance with claim 1, we can choose $\varepsilon > 0$, $m_0 > 0$ such that for all $k \geqslant m_0$,

$$\eta_1(f)^2 \leqslant 2J_\lambda(u_k) \leqslant \eta(f)^2 + \frac{4\pi}{\lambda} - \varepsilon.$$

This, together with $J_\lambda(u_k^{\pm}) \geqslant \eta(f)^2/2$, gives

$$J_\lambda(u_k^{\pm}) \leqslant \frac{2\pi}{\lambda} - \frac{\varepsilon}{2}. \tag{5.11}$$

Claim 2. $u_0^{\pm} \not\equiv 0$ and $\|u_0^{\pm}\|^2 \leqslant \int_{\mathbb{B}^2} f(x, u_0^{\pm})u_0^{\pm} \, \mathrm{d}v_g$.

We shall only prove this for u_0^+. A similar proof holds for u_0^- as well. Suppose that $u_0^+ \equiv 0$. Then, from (5.10) and (5.11), we have

$$\limsup_{k\to\infty} \|u_k^+\|^2 = 2\limsup_{k\to\infty} \left(J_\lambda(u_k^+) + \int_{\mathbb{B}^2} F(x, u_k^+) \, \mathrm{d}v_g \right) \leqslant \frac{4\pi}{\lambda} - \varepsilon.$$

Therefore, from proposition 4.5,

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} f(x, u_k^+)u_k^+ \, \mathrm{d}v_g = 0. \tag{5.12}$$

Since $u_k^+ \in \mathcal{M}$, we get from (5.12) that $\lim_{k\to\infty} \|u_k^+\| = 0$. Together with $\eta(f) > 0$, this gives a contradiction. This proves that $u_0^+ \not\equiv 0$. Now suppose that

$$\|u_0^+\|^2 > \int_{\mathbb{B}^2} f(x, u_0^+)u_0^+ \, \mathrm{d}v_g. \tag{5.13}$$

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Then $\{u_k^+, u_0^+\}$ satisfies all the hypotheses of proposition 4.6, and hence

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} f(x, u_k^+)u_k^+ \,\mathrm{d}v_g = \int_{\mathbb{B}^2} f(x, u_0^+)u_0^+ \,\mathrm{d}v_g.$$

Therefore, we have

$$\|u_0^+\|^2 \leqslant \liminf_{k\to\infty} \|u_k^+\|^2 = \liminf_{k\to\infty} \int_{\mathbb{B}^2} f(x, u_k^+)u_k^+ \,\mathrm{d}v_g = \int_{\mathbb{B}^2} f(x, u_0^+)u_0^+ \,\mathrm{d}v_g,$$

which contradicts (5.13) and hence proves claim 2.

Thanks to claim 2, the property

$$\|u_0^\pm\|^2 \leqslant \int_{\mathbb{B}^2} f(x, u_0^\pm)u_0^\pm \,\mathrm{d}v_g$$

enables us to choose $0 < r_1 \leqslant 1$, $0 < r_2 \leqslant 1$, such that

$$v = r_1 u_0^+ - r_2 u_0^- \in \mathcal{M}_1.$$

Also, we have

$$\begin{aligned}
\frac{\eta_1(f)^2}{2} \leqslant J_\lambda(v) \leqslant I_\lambda(v) &= I_\lambda(r_1 u_0^+) + I_\lambda(r_2 u_0^-) \\
&\leqslant I_\lambda(u_0^+) + I_\lambda(u_0^-) \\
&\leqslant \liminf_{k\to\infty} I_\lambda(u_k) \\
&= \lim_{k\to\infty} J_\lambda(u_k) \\
&= \frac{\eta_1(f)^2}{2}.
\end{aligned}$$

Hence, $r_1 = r_2 = 1$, which gives $u_0 \in \mathcal{M}_1$ and $J_\lambda(u_0) = \eta_1(f)^2/2$. This completes the proof of theorem 1.3. □

Proof of theorem 1.4. The proof of this theorem follows similar lines as that of [4, theorem 1.3] with a lemma similar to [28, lemma 3.1]. For the sake of brevity, we have omitted the detailed verification.

Appendix A.

In this section we shall try to give a sketch of the proof of lemma 5.2.

Proof of lemma 5.2. From the radial estimate 2.5, it is very clear that blow-up can occur only at the origin. Hence, we only need to analyse near the origin. Denote the one-dimensional vector space $\{pu_0 : p \in \mathbb{R}\}$ by V.

Let $u_l = p_l u_0 + t_l m_{l,\beta}$ be such that $t_l \geqslant 0$ and

$$J_\lambda(u_l) = \sup_{\alpha, t\in\mathbb{R}} J_\lambda(\alpha u_0 + t m_{l,\beta}).$$

Since $J'_\lambda(u_l) = 0$ on $\{\alpha u_0 + t m_{l,\beta} : \alpha, t \in \mathbb{R}\}$, we have

$$\|u_l\|^2 = \int_{\mathbb{B}^2} f(x, u_l)u_l \,\mathrm{d}v_g. \tag{A 1}$$

Now suppose that (5.5) is not true. Then there exists a sequence l_n such that $l_n \to 0$ as $n \to 0$ and for $v_n := \alpha_n u_0 = \alpha_{l_n} u_0$, $m_{n,\beta} = m_{l_n,\beta}$, $t_n = t_{l_n}$, $u_n = u_{l_n}$,

$$C(V) + \frac{2\pi}{\lambda} \leqslant J_\lambda(u_n). \tag{A 2}$$

Step 1. $\{v_n\}$ and t_n are bounded.

Suppose that this is not true. Then either

$$\lim_{n\to\infty} \frac{t_n}{\|v_n\|} > 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{t_n}{\|v_n\|} = 0.$$

In the first case, there exist a subsequence of $\{v_n, t_n\}$ and a constant $C > 0$ such that for large n,

$$\frac{t_n}{\|v_n\|} \geqslant C \quad \text{and} \quad t_n \to \infty \quad \text{as } n \to \infty. \tag{A 3}$$

As $\|m_{n,\beta}\| = 1$, we have from (A 3) that

$$\|u_n\|^2 = t_n^2 + 2t_n\langle v_n, m_{n,\beta}\rangle + \|v_n\|^2 \leqslant C_1 t_n^2, \tag{A 4}$$

where

$$C_1 = 1 + \frac{2}{C} + \frac{1}{C^2}.$$

Since $\|v_n\|/t_n$ is bounded and $v_n \in \{pu_0 \colon p \in \mathbb{R}\}$, we have that $|v_n|_\infty/t_n$ is bounded. Hence, for $x \in B(0, l_n)$ and for large n,

$$\begin{aligned} u_n(x) &= v_n(x) + t_n m_{n,\beta}(x) \\ &= t_n m_{n,\beta}(x)\left(1 + \frac{v_n(x)}{t_n}\frac{1}{m_{n,\beta}(x)}\right) \\ &\geqslant \tfrac{1}{2} t_n m_{n,\beta}(x). \end{aligned} \tag{A 5}$$

Hence, we have

$$\begin{aligned} C_1 t_n^2 \geqslant \|u_n\|^2 &= \int_{\mathbb{B}^2} f(x, u_n) u_n \, \mathrm{d}v_g \\ &\geqslant \int_{B(0,\beta)} h(x, u_n) u_n (\mathrm{e}^{\lambda u_n^2} - 1) \, \mathrm{d}v_g \\ &\geqslant \int_{B(0,\beta)} h_{0,\beta}(u_n) u_n (\mathrm{e}^{\lambda u_n^2} - 1) \, \mathrm{d}v_g \\ &\geqslant \int_{B(0,l_n)} h_{0,\beta}(u_n) u_n (\mathrm{e}^{\lambda u_n^2} - 1) \, \mathrm{d}v_g \\ &\geqslant C_2 (\mathrm{e}^{\lambda t_n^2 m_{n,\beta}^2(0)/8} - 1) l_n^2, \end{aligned} \tag{A 6}$$

where C_2 is a positive constant. This implies that

$$C_1 \geqslant C_2 \left(\exp\left(\frac{\lambda t_n^2}{16\pi} \log\frac{\beta}{l_n} - 2\log\frac{1}{l_n} - 2\log t_n \right) - l_n^2 \right) \to \infty$$

as $l_n \to 0$, which gives a contradiction, and hence the first case can not occur.

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In the second case, first note that $\|v_n\| \to \infty$. Let

$$z_n = \frac{v_n}{\|v_n\|}, \qquad \varepsilon_n = \frac{t_n^2}{\|v_n\|^2} + \frac{2t_n}{\|v_n\|}\langle z_n, m_{n,\beta}\rangle.$$

Then, up to a subsequence and using the fact that $z_n \in \{pu_0 : p \in \mathbb{R}\}$, we can assume that

$$\lim_{n\to\infty} z_n = z_0, \qquad z_0 \in \{pu_0 : p \in \mathbb{R}\} \setminus \{0\}, \qquad \lim_{n\to\infty} \varepsilon_n = 0. \tag{A 7}$$

Also,

$$\|u_n\|^2 = \|v_n\|^2 + 2t_n\langle v_n, m_{n,\beta}\rangle + t_n^2 \tag{A 8}$$
$$= \|v_n\|^2(1 + \varepsilon_n). \tag{A 9}$$

Hence,

$$\frac{u_n}{\|u_n\|} = \frac{1}{(1+\varepsilon_n)^{1/2}}\left(z_n + \frac{t_n}{\|v_n\|}m_{n,\beta}\right) \to z_0 \not\equiv 0 \quad \text{in } H^1(\mathbb{B}^2). \tag{A 10}$$

Now, using Fatou's lemma,

$$\infty = \int_{\mathbb{B}^2} \liminf_{n\to\infty} \frac{f(x,u_n)}{u_n}\left(\frac{u_n}{\|u_n\|}\right)^2 \mathrm{d}v_g$$
$$\leqslant \liminf_{n\to\infty} \frac{1}{\|u_n\|^2}\int_{\mathbb{B}^2} f(x,u_n)u_n \,\mathrm{d}v_g = 1, \tag{A 11}$$

which is a contradiction. Hence, this proves step 1.

Therefore, up to a subsequence we can assume that

$$\lim_{n\to\infty} v_n = v_0 \quad \text{in } V, \qquad \lim_{n\to\infty} t_n = t_0.$$

Also, $u_n \rightharpoonup v_0$ weakly in $H^1(\mathbb{B}^2)$ and for almost all x in $\mathbb{B}^2$.

Remark. $\lim_{n\to\infty} v_n = v_0$ in V implies there exist a sequence $\alpha_n \in \mathbb{R}$ such that $\alpha_n u_0 \to \alpha u_0$.

Using proposition 4.4, we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{B}^2} F(x,u_n)\,\mathrm{d}v_g = \int_{\mathbb{B}^2} F(x,v_0)\,\mathrm{d}v_g. \tag{A 12}$$

Now, letting $n \to \infty$ in (A 2) and using convergence results, we get

$$C(V) + \frac{2\pi}{\lambda} \leqslant J_\lambda(v_0) + \frac{t_0^2}{2} \leqslant C(V) + \frac{t_0^2}{2}. \tag{A 13}$$

Step 2. $t_0^2 = 4\pi/\lambda$ and $J_\lambda(v_0) = C(V)$.

From (A 13) we have $t_0^2 \geqslant 4\pi/\lambda$. Suppose that $t_0^2 > 4\pi/\lambda$. Then, arguing the same as in step 2 of [4, lemma 3.3] and using step 1, we can get, for $n \geqslant n_0$,

$$M = \sup_n \|u_n\|^2 \geqslant C_1[l_n^{-2((1+\varepsilon/4)(1-\varepsilon_n)-1)} - l_n^2] \tag{A 14}$$

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for some positive constant C_1. As $\varepsilon_n \to 0$, $l_n \to 0$, (A 14) gives a contradiction. Hence, $t_0^2 = 4\pi/\lambda$ and (A 13) gives $J_\lambda(v_0) = C(V)$.

Step 3. There exist positive constants n_0 and C_0 such that for all $n \geqslant n_0$,

$$C_0 + \log(M + \pi p_{n,\beta}(0) l_n^2) \geqslant \left(t_n^2 - \frac{4\pi}{\lambda}\right) m_{n,\beta}^2(0) - \frac{1}{\lambda}\varepsilon_n m_{n,\beta}(0) + \frac{1}{\lambda}\log p_{n,\beta}(0), \quad \text{(A 15)}$$

where

$$\varepsilon_n = 2\lambda t_n \sup_{x\in\mathbb{B}^2} |v_n(x)|$$

and

$$p_{n,\beta}(0) = \inf\{t h_{0,\beta}(t) \colon t \in [\tfrac{1}{2} t_n m_{n,\beta}(0), 2 t_n m_{n,\beta}(0)]\}.$$

A straightforward calculation gives

$$M \geqslant \pi p_{n,\beta}(0) l_n^2 (\mathrm{e}^{\lambda t_n^2 m_{n,\beta}^2(0) - \varepsilon_n m_{n,\beta}(0)} - 1), \quad \text{(A 16)}$$

and from (A 16), step 3 follows easily.

Step 4. There exists a constant $C_1 > 0$ such that for large n,

$$\left(\log \frac{\beta}{l_n}\right)^{1/2} \left(\frac{4\pi}{\lambda} - t_n^2\right) \leqslant \tilde{C}_1 |\Delta v_n|_{L^2_{\mathrm{loc}}} \leqslant C_1 \alpha_n. \quad \text{(A 17)}$$

Proof of step 4 follows by convexity of $t \to F(x,t)$ and elliptic regularity.

Finally, we are in a position to prove the final step. By hypothesis, given any $N > 0$ and a compact set $\overline{B(0,\beta)}$, there exists $t_{N,\beta} > 0$ such that $h_{0,\beta}(t)t \geqslant \mathrm{e}^{Nt}$ for all $t \geqslant t_{N,\beta}$. Since $m_{n,\beta}(0) \to \infty$, by (A 15) and (A 17) we obtain for large n,

$$\left[\frac{N t_n}{2\lambda} - \frac{\varepsilon_n}{\lambda} - \frac{\sqrt{2\pi} C_0}{(\log(\beta/l_n))^{1/2}} - \frac{\sqrt{2\pi}\log(M + \pi \mathrm{e}^{N t_n} l_n^2)}{(\log(\beta/l_n))^{1/2}}\right] \leqslant \frac{C_1}{\sqrt{2\pi}} \alpha_n \quad \text{(A 18)}$$

since ε_n, α_n are bounded and $t_n \to t_0 > 0$. From above, we get

$$\frac{N t_0}{2\lambda} \leqslant \tilde{C}_1$$

for some positive constant $\tilde{C}_1$. Since N is arbitrary, we get a contradiction. Hence, this proves the lemma. □

Appendix B.

This section is devoted to the existence of non-radial solutions. Typically, existence of non-radial solutions on the hyperbolic space is a difficult question due to the lack of compactness through vanishing (mentioned earlier). We have made an attempt to give an existence theorem for a non-radial solution in certain cases. We have eliminated concentration at ∞ by considering a suitable growth condition on the nonlinearity, or, in other words, by considering a penalty assumption that sets the asymptotic nonlinear part to zero.

Moreover, by invariance with respect to Möbius transformations, we are able to prove Lions's lemma (lemma 2.3) for $H^1(\mathbb{B}^N)$, which plays an important role in the subsequent proof. In this regard, we first modify the function of critical growth.

A model problem for our study is

$$-\Delta_g u = f(x,u), \quad x \in \mathbb{B}^2, \tag{B 1}$$

where $f(x,t)$ is a function of critical growth as defined in definition 1.1 and, in addition, satisfies some growth condition near ∞. To be precise, we assume that $f(x,t) = h(x,t)(\mathrm{e}^{\lambda t^2} - 1)$ satisfies the following conditions.

($\overline{\mathrm{C3}}$) We have

$$F(x,t) \leqslant C(g(x) + f(x,t)), \quad g \in L^1(\mathbb{B}^2, \mathrm{d}v_g) \cap L^p(\mathbb{B}^2, \mathrm{d}v_g) \text{ for some } p \in (1,2]. \tag{B 2}$$

(C5) There exists a $\delta > 0$ such that

$$\frac{h(x,t)}{(1-|x|^2)^\delta} \in L^\infty(\{|x| > \alpha\} \times [-N,N]) \quad \text{for all } N. \tag{B 3}$$

(C6) For every $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that

$$h(x,t) \geqslant (1-|x|^2)^l \mathrm{e}^{-\varepsilon t^2} \quad \text{for some } l > 0,\ |x| > \alpha(\varepsilon) \text{ and } t \text{ positive large.} \tag{B 4}$$

A prototypical example of such a function is $f(x,t) = (1-|x|^2)^l t(\mathrm{e}^{\lambda t^2} - 1)$ for some $l > 0$. Hence, unlike in the radial case, here we can allow a singularity of maximum order $1/(1-|x|^2)^{2-\varepsilon}$ at the boundary. We prove the following theorem.

Theorem B.1. *Let $f(x,t)$ be a function of critical growth satisfying* ($\overline{\mathrm{C3}}$), (C5) *and* (C6). *Furthermore, assume that*

$$\lim_{t\to\infty} \inf_{x\in K} h(x,t)t = \infty \tag{B 5}$$

for every compact set $K \subset \mathbb{B}^2$. Then (B 1) *has a positive solution.*

Remark. The function of critical growth $f(x,t)$ defined in the above theorem is not necessarily a radial function in its first variable. Thus, from invariance of $\Delta_{\mathbb{B}^N}$ under orthogonal transformations we infer that the solution thus obtained in theorem B.1 is non-radial if we assume that $f(x,t)$ is non-radial in its first variable.

It is easy to see that under the above assumptions we can estimate the growth of $F(x,u)$ and $f(x,u)$ near ∞. As a consequence, we can provide all the necessary tools to acquire existence of solutions of (B 1). For the sake of completeness, we will outline some of the steps. In the rest of the section, $f(x,t)$ stands for a function of critical growth satisfying ($\overline{\mathrm{C3}}$), (C5) and (C6), and I_λ, J_λ are defined as before (see § 3) corresponding to this f.

Now we will provide proofs of lemmas that significantly differ from the radial ones. We have used some refined arguments and have taken advantage of growth conditions to prove the following lemmas.

Lemma B.2. *Let $\{u_k\}$ be a sequence in $H^1(\mathbb{B}^2)$ converging weakly to a function u in $H^1(\mathbb{B}^2)$. Furthermore, assume that*

$$\sup_k \int_{\mathbb{B}^2} f(x,u_k)u_k \,\mathrm{d}v_g < +\infty.$$

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Then

$$\int_{\mathbb{B}^2} F(x, u_k)\, \mathrm{d}v_g \to \int_{\mathbb{B}^2} F(x, u)\, \mathrm{d}v_g. \tag{B 6}$$

Proof. It is enough to show that $\int_{\{|x|>\alpha\}} F(x, u_k)\, \mathrm{d}v_g$ can be made arbitrarily small by choosing α close to 1. Indeed

$$\begin{aligned}\int_{\{|x|>\alpha\}} F(x, u_k)\, \mathrm{d}v_g &\leqslant C \int_{\{|x|>\alpha\}\cap\{|u_k|\leqslant N\}} (1-|x|^2)^\delta (\mathrm{e}^{(\lambda+\varepsilon)u_k^2} - 1)\, \mathrm{d}v_g \\ &\quad + \int_{\{|x|>\alpha\}\cap\{|u_k|>N\}} F(x, u_k)\, \mathrm{d}v_g \\ &\leqslant C\mathrm{e}^{(\lambda+\varepsilon)N^2}(1-\alpha)^\delta \|u_k\|^2 + \frac{C}{N}\int_{\mathbb{B}^2} (g(x)u_k + f(x, u_k)u_k)\, \mathrm{d}v_g \\ &\leqslant C\mathrm{e}^{(\lambda+\varepsilon)N^2}(1-\alpha)^\delta + \frac{C}{N}.\end{aligned}$$

Here we used $g \in L^p(\mathbb{B}^2, \mathrm{d}v_g)$ for some $p \in (1, 2]$, and this completes the proof. □

Lemma B.3. *Given $\mu > 0$, there exists a constant $C(\mu) > 0$ such that*

$$\int_{\mathbb{B}^2} f(x, u)u\, \mathrm{d}v_g \leqslant C(\mu)(1 + I_\lambda(u)) + \mu\|u\|^2 \quad \text{for all } u \in H^1(\mathbb{B}^2). \tag{B 7}$$

Proof. We note that

$$\begin{aligned}\int_{\{u\leqslant 4M_1\}} f(x, u)\, \mathrm{d}v_g &\leqslant \int_{\{u\leqslant 4M_1\}\cap\{|x|>\alpha\}} f(x, u)\, \mathrm{d}v_g + C(\alpha) \\ &\leqslant C(1-\alpha)^\delta \int_{\mathbb{B}^2} u^2\, \mathrm{d}v_g + C(\alpha) \\ &\leqslant C(1-\alpha)^\delta \|u\|^2 + C(\alpha) \qquad \text{(B 8)} \\ &= \frac{\mu}{2}\|u\|^2 + C(\mu), \qquad \text{(B 9)}\end{aligned}$$

by choosing α close to 1. Therefore, proceeding as in lemma 4.2 and using (B 8), we get (B 7). □

Lemma B.4. *Let $f(x, t) = h(x, t)(\mathrm{e}^{\lambda t^2} - 1)$ be a function of critical growth. Then*

$$d^2 := \sup\left\{c^2 : \sup_{u\in H^1(\mathbb{B}^2), \|u\|\leqslant 1} \int_{\mathbb{B}^2} f(x, cu)u\, \mathrm{d}v_g < +\infty\right\} = \frac{4\pi}{\lambda}. \tag{B 10}$$

Proof. The proof goes along the same lines as before, with obvious modifications. We will only mention the steps that differ from the previous one. We see that

$$\begin{aligned}\int_{\{|x|>\alpha\}} f(x, cu)u\, \mathrm{d}v_g &\geqslant C \int_{\{|x|>\alpha\}\cap\{u\geqslant t_0\}} h(x, cu)(\mathrm{e}^{\lambda c^2u^2} - 1)\, \mathrm{d}v_g \\ &\geqslant C \int_{\{|x|>\alpha\}\cap\{u\geqslant t_0\}} (1-|x|^2)^{l-2}(\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2} - 1)\, \mathrm{d}x.\end{aligned}$$

From this and proceeding as in lemma 4.4, we conclude that

$$\sup_{\substack{u\in H^1(\mathbb{B}^2),\\ \|u\|^2\leqslant 1}} \int_{\mathbb{B}^2} (1-|x|^2)^{l-2}(\mathrm{e}^{\lambda(1-\varepsilon)c^2u^2}-1)\,\mathrm{d}x < +\infty,$$

and hence $\lambda d^2 \leqslant 4\pi$. The proof of the reverse inequality is similar to in lemma 4.4. □

LEMMA B.5. *Let $\{u_k\}$ and $\{v_k\}$ be bounded sequences in $H^1(\mathbb{B}^2)$ converging weakly to u and v, respectively. Furthermore, assume that $\sup_k \|u_k\|^2 < 4\pi/\lambda$. Then, for all $l \geqslant 2$,*

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g = \int_{\mathbb{B}^2} \frac{f(x,u)}{u} v^l \,\mathrm{d}v_g. \tag{B 11}$$

Proof. As before, we can show that

$$\int_{\{|u_k|>N\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g = O\left(\frac{1}{N}\right).$$

Now we estimate

$$\int_{\{|u_k|\leqslant N\}\cap\{|x|>\alpha\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g.$$

In fact we can show that

$$\int_{\{|u_k|\leqslant N\}\cap\{|x|>\alpha\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g \leqslant C(N)\mathrm{e}^{(\lambda+\varepsilon)N^2}(1-\alpha)^\delta,$$

so that

$$\int_{\mathbb{B}^2} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g = \int_{\{|u_k|\leqslant N\}\cap\{|x|\leqslant\alpha\}} \frac{f(x,u_k)}{u_k} v_k^l \,\mathrm{d}v_g + O\left(\frac{1}{N}\right) + C(N)\mathrm{e}^{(\lambda+\varepsilon)N^2}(1-\alpha)^\delta. \tag{B 12}$$

From (B 12) we can easily see that (B 11) holds. □

A similar argument gives the following lemma.

LEMMA B.6. *Let $\{u_k\}$ be a sequence in $H^1(\mathbb{B}^2)$ converging weakly to a non-zero function u and assume that*

(i) *there exists $c \in (0, 2\pi/\lambda)$ such that $J_\lambda(u_k) \to c$,*

(ii) $\|u\|^2 \geqslant \int_{\mathbb{B}^2} f(x,u)u \,\mathrm{d}v_g$,

(iii) $\sup_k \int_{\mathbb{B}^2} f(x,u_k)u_k \,\mathrm{d}v_g < +\infty$.

Then

$$\lim_{k\to\infty} \int_{\mathbb{B}^2} f(x,u_k)u_k \,\mathrm{d}v_g = \int_{\mathbb{B}^2} f(x,u)u \,\mathrm{d}v_g. \tag{B 13}$$

Proof of theorem B.1. We omit the proof because it goes along the same lines as that of theorem 1.2 (see §§ 4 and 5 for details).

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