Tight frames and related geometric problems

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Abstract. A tight frame is the orthogonal projection of some orthonormal basis of $\mathbb{R}^n$ onto $\mathbb{R}^k$. We show that a set of vectors is a tight frame if and only if the set of all cross products of these vectors is a tight frame. We reformulate a range of problems on the volume of projections (or sections) of regular polytopes in terms of tight frames and write a first-order necessary condition for local extrema of these problems. As applications, we prove new results for the problem of maximization of the volume of zonotopes.

1 Introduction

A *tight frame* in $\mathbb{R}^k$ is a set of vectors that is the orthogonal projection of some orthonormal basis of $\mathbb{R}^n$ onto $\mathbb{R}^k$. Or, equivalently, it is a set of vectors $\{v_1, \ldots, v_n\}$ that satisfy identity

$$(1.1) \qquad \sum_1^n v_i \otimes v_i = I_k,$$

where I_k is the identity operator in $\mathbb{R}^k$.

Tight frames appear and are used naturally in different branches of mathematics: from quantum mechanics and approximation theory (see [1] for details) to classical problems of convex analysis, starting with the well-known John condition [11] for the ellipsoid of maximal volume in a convex body.

We study properties of tight frames from the algebraic point of view. For this purpose, we use exterior algebra for studying properties of the exterior powers of the projection operators in Section 3. The necessary definitions on exterior algebra are included in Section 2.1 for completeness. Among other results on projection operators, we prove the following theorem.

Theorem 1.1 *A set of vectors $\{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ is a tight frame if and only if the set of cross products $[v_{i_1}, \ldots, v_{i_{k-1}}]$ for all $(k-1)$-tuples $\{i_1, \ldots, i_{k-1}\} \in \binom{[n]}{k-1}$ is a tight frame.*

There are standard geometric objects that can be described either in terms of Grassmannian $\mathrm{Gr}(n, k)$ of k-dimensional subspace of $\mathbb{R}^n$ or in terms of tight frames. Let us give some examples.

Let $\{v_1, \ldots, v_n\}$ be the orthogonal projection of the standard basis onto k-dimensional subspace $H \subset \mathbb{R}^n$. Consider the following classes of convex polytopes:

(1)   Projections of the cross-polytope. Let $\diamond^n = \{x \in \mathbb{R}^n \mid |x_1| + \cdots + |x_n| \le 1\}$ be the standard cross-polytope. Then, the projection of $\diamond^n$ onto H is the absolute convex hull of the projection of the standard basis of $\mathbb{R}^n$, that is $\mathrm{co}\{\pm v_1, \ldots, \pm v_n\}$.
(2)   Sections of the cube. Let $\square^n = [-1, 1]^n$ be the standard n-dimensional cube. The section $\square^n \cap H$ is the intersection of strips $\{x \in H : |\langle x, v_i \rangle| \le 1\}$.
(3)   Projections of the cube. Then, the projection of $\square^n$ onto H is the Minkowski sum of the segments $[-v_i, v_i]$.

Identifying H with $\mathbb{R}^k$, we identify the set $\{v_1, \ldots, v_n\}$ with a tight frame. Clearly, the volumes of these polytopes can be considered as functions on the set of tight frames.

Of course, much more types of convex bodies can be described in both ways. We list these three examples, since they can be considered as a standard position of special types of centrally symmetric polytopes. For example, any convex centrally symmetric polytope is an affine image of a central section of the cube or a projection of the cross-polytope of some dimension. The problems of finding the minimal and maximal volumes of these types of polytopes are well-studied (see [2, 3, 12, 14]). However, there are still a lot of open questions. For example, the tight upper bound on the volume of a projection of cross-polytope $\diamond^n$ onto a k-dimensional subspace H of $\mathbb{R}^n$ is unknown. The reasonable conjecture is that the upper bound is $\mathrm{vol}_k \diamond^k$, and it can be considered as a dual to Vaaler's theorem [18] that states that $\mathrm{vol}_k(\square^n \cap H_k) \ge \mathrm{vol}_k \square^k$.

In Section 4, we show how to reformulate problems about extrema of the volume of sections and projections of regular polytopes in the language of tight frames. For example, one can apply our results for listed above types of polytopes. A similar to ours approach was used by Filliman in [5–8]. However, Filliman preferred to consider the volume functions as the functions on the Grassmannian. Instead of this, we consider functions on the set of tight frames in $\mathbb{R}^k$; extend the domain of such functions to the set of the ordered sets of n vectors of $\mathbb{R}^k$ in a natural way; and give in Lemma 4.2 a necessary and sufficient condition for local extremum of such functions. Thus, we avoid working in the Grassmannians. The only thing we need the Grassmannians for is to prove a first-order approximation formula for a perturbation of a tight frame. As a consequence of Theorem 1.1 and the Cauchy–Binet formula, we obtain the following theorem.

Theorem 1.2   *Let $\{v_1, \ldots, v_n\}$ be a tight frame in $\mathbb{R}^k$, τ be a scalar, and $\{x_i\}_1^n$ be an arbitrary set of vectors of $\mathbb{R}^k$. Then,*

$$\det \sum_1^n (v_i + \tau x_i) \otimes (v_i + \tau x_i) = 1 + 2\tau \sum_{i=1}^n \langle x_i, v_i \rangle + o(\tau).$$

We illustrate our technique on the problem of maximizing function

(1.2) $F(H) = \mathrm{vol}_k([0, 1]^n | H), \quad \dim H = k \le n,$

where $[0,1]^n|H$ is the projection of the cube $[0,1]^n$ onto H. The projections of the cube $[0,1]^n$ may be considered as a special position of Minkowski sum of finitely many linear segments, so-called *zonotopes*.

Using the results of Section 4, we give a complete system of the first-order necessary conditions of a maximizer of (1.2) in Lemma 5.5. It yields the following geometric result.

Corollary 1.3   Let H be a local maximizer of (1.2) and $Q = [0,1]^n|H$. Let v_i denote the orthogonal projection of the standard basis vector e_i onto H, for $i \in [n]$, and $Q|v_i^\perp$ be the projection of Q onto the orthogonal complement of the line $\{\lambda v_i \mid \lambda \in \mathbb{R}\}$ in H. Then,

$$\mathrm{vol}_{k-1}\left(Q|v_i^\perp\right) = |v_i|\,\mathrm{vol}_k\, Q \quad \text{for all} \quad i \in [n].$$

Also, we obtain some properties of the maximizers.

Theorem 1.4   Let the maximum of (1.2) be attained at H. Let v_i denote the projection of the standard basis vector e_i onto H, for $i \in [n]$. Then, for any $i, j \in [n]$, the inequality

$$|v_i|^2 \geq (\sqrt{2}-1)|v_j|^2$$

holds.

As a consequence of Theorem 1.4 and McMullen's symmetric formula [13], which is

(1.3) $$F(H) = F(H^\perp),$$

where $H^\perp$ is the orthogonal complement of H, we prove the following corollary.

Corollary 1.5   Fix $q \in \mathbb{N}$. Let H_{n-q} be a maximizer of (1.2) for $k = n - q, n \geq q$. By M_n and m_n, we denote the maximum and the minimum length of the projections of the standard basis vectors of $\mathbb{R}^n$ onto H_{n-q}, respectively. Then,

$$\frac{m_n}{M_n} \to 1 \qquad as \qquad n \to \infty.$$

2   Definitions and preliminaries

We use C^n to denote an n-dimensional cube $\{x : 0 \leq x[i] \leq 1\}$ in $\mathbb{R}^n$. Here and throughout the paper, $x[i]$ stands for the ith coordinate of a vector x. As usual, $\{e_i\}_1^n$ is the standard orthonormal basis of $\mathbb{R}^n$. We use $\langle p, x \rangle$ to denote the *value of a linear functional p at a vector x*.

For a convex body $K \subset \mathbb{R}^n$ and a k-dimensional subspace H of $\mathbb{R}^n$, we denote by $K \cap H$ and $K|H$ the section of K by H and the orthogonal projection of K onto H, respectively. For a k-dimensional subspace H of $\mathbb{R}^n$ and a convex body $K \subset H$, we denote by $\mathrm{vol}_k\, K$ the k-dimensional volume of K. We use P^H to denote the orthogonal projection onto H.

For a positive integer n, we refer to the set $\{1, 2, \ldots, n\}$ as $[n]$. The set of all ℓ-element subsets (or ℓ-*tuple*) of a set $M \subset [n]$ is denoted by $\binom{M}{\ell}$. For two ℓ-tuples $I \in \binom{[a]}{\ell}, J \in \binom{[b]}{\ell}$, we will use $M_{\{I,J\}}$ to denote the determinant of the corresponding ℓ minor of the $a \times b$ matrix M. For the sake of convenience, we will write M_I whenever $I = J$. We use $M_{[i,j]}$ to denote the entry in the ith row and the jth column of a matrix M.

For the sake of convenience, we denote by $d_S(\{M\})$ the determinant of $\{v_i \; : \; i \in M\}$ for an ordered set of vectors $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ and a set M of k indexes from $[n]$ (the indexes may repeat).

Recall that a *zonotope* in a vector space is the Minkowski sum of finitely many line segments, i.e., $\sum_{i=1}^{n} [a_i, b_i]$, where $\sum$ stands for the Minkowski sum, and $[a, b]$ means the line segment between points a and b.

Definition 2.1 We will say that an ordered n-tuple of vectors $\{v_1, \ldots, v_n\} \subset H$ forms *a tight frame* in a vector space H if

$$(2.1) \qquad \left(\sum_{1}^{n} v_i \otimes v_i \right)\Bigg|_H = I_H,$$

where I_H is the identity operator in H and $A|_H$ is the restriction of an operator A onto H.

We use $\Omega(n, k)$ to denote the set of all tight frames with n vectors in $\mathbb{R}^k$.

2.1 Exterior algebra

For the sake of completeness and clarity, all needed definitions from exterior algebra are given here in a brief and noncanonical way. We assume that the equivalence of our definitions to the usual ones is quite obvious. As a proper introduction to multilinear algebra, we refer to Greub's book [9].

Let H be a finite-dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. We define the vector space $\Lambda^\ell(H)$ as the space of the multilinear skew-symmetric functions on ℓ vectors of H with the natural linear structure. The vectors of $\Lambda^\ell(H)$ are called ℓ-forms. As we consider only linear spaces with inner products, we assume that a space H and its dual H^* coincide. This allows us to simplify the following definitions.

For a sequence of ℓ vectors $\{x_1, \ldots, x_\ell\} \subset H$, we define an ℓ-form $x_1 \wedge \cdots \wedge x_\ell$ by its evaluation on vectors $\{y_1, \ldots, y_\ell\} \subset H$ given by

$$x_1 \wedge \cdots \wedge x_\ell(y_1, \ldots, y_\ell) = \det M, \qquad \text{where} \qquad M_{[ij]} = \langle x_i, y_j \rangle, \quad i, j \in [\ell].$$

By the properties of the determinant, $x_1 \wedge \cdots \wedge x_\ell$ is a multilinear skew-symmetric function of ℓ vectors of H. For the sake of convenience, for given n vectors $\{x_i\}_1^n$ and ℓ-tuple $L = \{i_1, \ldots, i_\ell\} \in \binom{[n]}{\ell}$, we denote by x_L the ℓ-form $x_{i_1} \wedge \cdots \wedge x_{i_\ell}$.

As usual, we use the set of ℓ-forms $e_{i_1} \wedge \cdots \wedge e_{i_\ell}$, where $\{i_1, \ldots, i_\ell\} \in \binom{[n]}{\ell}$, as the standard basis of $\Lambda^\ell(\mathbb{R}^n)$. We use the lexicographical order on ℓ-tuples $\{i_1, \ldots, i_\ell\} \in \binom{[n]}{\ell}$ to assign a number to $e_{i_1} \wedge \cdots \wedge e_{i_\ell}$ in this basis.

Recall that for a vector space H, an ℓ-form $w \in \Lambda^\ell(H)$ is said to be *decomposable* if it can be represented in the form $x_1 \wedge \cdots \wedge x_\ell$ for $\{x_i\}_1^\ell \subset H$. Every ℓ-form is a linear combination of some decomposable ℓ-forms (e.g., of the forms from the standard basis), but not all ℓ-forms are decomposable, e.g., $e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2(\mathbb{R}^4)$ is not decomposable. A line $l \subset \Lambda^\ell(\mathbb{R}^n)$ has a decomposable directional vector iff l is "generated" by some ℓ-dimensional subspace $H_\ell \subset \mathbb{R}^n$.

By linearity, it suffices to define a linear operator (or an inner product) on $\Lambda^\ell(\mathbb{R}^n)$ only for decomposable ℓ-forms, since the decomposable forms span $\Lambda^\ell(\mathbb{R}^n)$.

Recall that the exterior ℓ-power of an operator A on $\mathbb{R}^n$ is a linear operator on $\Lambda^\ell(\mathbb{R}^n)$, which is defined on decomposable forms by

$$\wedge^\ell A(x_1 \wedge \cdots \wedge x_\ell) = Ax_1 \wedge \cdots \wedge Ax_\ell.$$

Recall that the inner product of two decomposable ℓ-forms $a = a_1 \wedge \cdots \wedge a_\ell$ and $b = b_1 \wedge \cdots \wedge b_\ell$ is defined by

$$\langle a, b \rangle = \langle a_1 \wedge \cdots \wedge a_\ell, b_1 \wedge \cdots \wedge b_\ell \rangle = a_1 \wedge \cdots \wedge a_\ell(b_1, \ldots, b_\ell).$$

In this way, once we fix an inner product on $\mathbb{R}^n$, we fix the inner product on $\Lambda^\ell(\mathbb{R}^n)$. Then, for $\ell \in [n]$, we can define a special isometry $\star : \Lambda^\ell(\mathbb{R}^n) \to \Lambda^{n-\ell}(\mathbb{R}^n)$, the so-called *Hodge star operator*, by the following equation:

$$a \wedge \star(b) = \langle a, b \rangle \, e_1 \wedge \cdots \wedge e_n,$$

where a, b are ℓ-forms.

Using this, one can show the following. For given $a = a_1 \wedge \cdots \wedge a_\ell \in \Lambda^\ell(\mathbb{R}^n)$ and $b = b_1 \wedge \cdots \wedge b_{n-\ell} \in \Lambda^{n-\ell}(\mathbb{R}^n)$, we have

$$(2.2) \qquad \langle a, \star(b) \rangle = (-1)^{\ell(n-\ell)} \det\{a_1, \ldots, a_\ell, b_1, \ldots, b_{n-\ell}\}.$$

For a k-dimensional subspace $H_k \subset \mathbb{R}^n$, we use $\wedge^\ell H_k$ to denote the linear hull of forms $x_1 \wedge \cdots \wedge x_\ell$ in $\Lambda^\ell(\mathbb{R}^n)$ such that $\{x_1, \ldots, x_\ell\} \subset H_k$. Since H_k inherits the inner product of $\mathbb{R}^n$, the space $\wedge^\ell H_k$ inherits the inner product of $\Lambda^\ell(\mathbb{R}^n)$ and thus can be identified with the space $\Lambda^\ell(H_k)$ in a tautological way. This allows us to use the Hodge star operator for the spaces $\Lambda^\ell(H_k)$ and $\Lambda^{(k-\ell)}(H_k)$ for $\ell \in [k]$. Also, identity (2.2) can be rewritten in the following form. For given $a = a_1 \wedge \cdots \wedge a_\ell \in \Lambda^\ell(H_k)$ and $b = b_1 \wedge \cdots \wedge b_{k-\ell} \in \Lambda^{k-\ell}(H_k)$, we get

$$(2.3) \qquad \langle a, \star(b) \rangle = (-1)^{\ell(k-\ell)} \det\{a_1, \ldots, a_\ell, b_1, \ldots, b_{k-\ell}\},$$

where we understand the determinant as the determinant of k-vectors in a k-dimensional space H_k.

The next result is a straightforward consequence of the definitions. We show that the outer product and the tensor product commute for the exterior powers of operators in our case. Usually, definitions of the outer product involve some kind of (anti)symmetrization, which can give an additional factor when we exchange

the $\wedge$-product and $\otimes$-product. To avoid misunderstandings, we prove the following statement.

Lemma 2.1 *Let* $A = \sum\limits_{i=1}^{t} v_i \otimes v_i$ *for some integer* t *and vectors* $\{v_i\}_1^t \subset \mathbb{R}^n$. *Then,* $\wedge^\ell A = \sum\limits_{L \in \binom{[t]}{\ell}} v_L \otimes v_L$.

Proof   We can assume that $\ell \leq \dim \operatorname{Lin}\{v_1, \ldots, v_t\} \leq t$, otherwise $\wedge^\ell A = 0 = \sum\limits_{L \in \binom{[n]}{\ell}} v_L \otimes v_L$. It suffices to prove the identity only for the decomposable ℓ-forms.

Fix $\{x_1, \ldots, x_\ell\} \subset \mathbb{R}^n$. We have

$$\wedge^\ell A(x_1 \wedge \cdots \wedge x_\ell) = Ax_1 \wedge \cdots \wedge Ax_\ell = \sum_{i=1}^{t} \langle v_i, x_1 \rangle v_i \wedge \cdots \wedge \sum_{i=1}^{t} \langle v_i, x_\ell \rangle v_i.$$

By linearity and by skew symmetry, we expand the last identity, and the coefficient at $v_L = v_{i_1} \wedge \cdots \wedge v_{i_\ell}$ is

$$\sum_\sigma \operatorname{sgn} \sigma \prod_{j=1}^{\ell} \langle v_{i_j}, x_{\sigma(j)} \rangle,$$

where the summation is taken over all permutations of $[\ell]$. This is the determinant of the matrix $M_{ij} = \langle v_{i_j}, x_i \rangle$, where $i, j \in [\ell]$. By the definition of the inner product, this determinant is just $\langle v_L, (x_1 \wedge \cdots \wedge x_\ell) \rangle$, that is,

$$\wedge^\ell A(x_{i_1} \wedge \cdots \wedge x_{i_\ell}) = \sum_{L \in \binom{[t]}{\ell}} \langle v_L, (x_1 \wedge \cdots \wedge x_\ell) \rangle v_L = \sum_{L \in \binom{[t]}{\ell}} v_L \otimes v_L (x_1 \wedge \cdots \wedge x_\ell).$$

This completes the proof.   ∎

Identifying $\wedge^{k-1} H_k$ with $\Lambda^{k-1}(H_k)$, we identify H_k with $\wedge^{k-1} H_k$ using the Hodge star operator. Now, we can define the *cross product* of $k - 1$ vectors $\{x_1, \ldots, x_{k-1}\}$ by

$$[x_1, \ldots, x_{k-1}] = \star(x_1 \wedge \cdots \wedge x_{k-1}).$$

Or, in other words, by linearity of the determinant, the cross product $x = [x_1, \ldots, x_{k-1}]$ of $k - 1$ vectors $\{x_1, \ldots, x_{k-1}\}$ in a k-dimensional space H_k with the fixed inner product $\langle \cdot, \cdot \rangle$ is the vector defined by

$$\langle x, y \rangle = \det(x_1, \ldots, x_{k-1}, y) \quad \text{for all} \quad y \in H_k.$$

3 Properties of the exterior power of a projection operator

As a tight frame is a projection of an orthonormal basis, it is natural that its properties are connected with the properties of a projection operator. Moreover, sometimes it is useful to consider a lifting of a tight frame up to an orthonormal basis.

In the following trivial lemma, we understand $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of vectors, whose last $n - k$ coordinates are zero. For convenience, we will consider $\{w_i\}_1^n \subset \mathbb{R}^k \subset \mathbb{R}^n$ to be k-dimensional vectors.

Lemma 3.1   *The following assertions are equivalent:*

(1)   *a set of vectors $\{w_1, \ldots, w_n\} \subset \mathbb{R}^k$ is a tight frame;*
(2)   *there exists an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ such that w_i is the orthogonal projection of f_i onto $\mathbb{R}^k$, for any $i \in [n]$;*
(3) $\mathrm{Lin}\{w_1, \ldots, w_n\} = \mathbb{R}^k$ *and the Gram matrix Γ of vectors $\{w_1, \ldots, w_n\} \subset \mathbb{R}^k$ is the matrix of the projection operator from $\mathbb{R}^n$ onto the linear hull of the rows of matrix $M = (w_1, \ldots, w_n)$.*
(4)   *the $k \times n$ matrix $M = (w_1, \ldots, w_n)$ is a submatrix of an orthogonal matrix of order n.*

The main observation is that P is the Gram matrix of vectors $\{v_1, \ldots, v_n\} \subset H_k$. Since $\langle Pe_i, e_j \rangle = \langle P^2 e_i, e_j \rangle = \langle Pe_i, Pe_j \rangle = \langle v_i, v_j \rangle$, we have that for fixed $\ell \in [k]$ and ℓ-tuple $L \subset \binom{[n]}{\ell}$, the corresponding $\ell \times \ell$ submatrix of P is the Gram matrix of vectors $\{v_i\}_{i \in L}$, and the determinant of this Gram matrix is P_L. It is well known that for ℓ vectors $\{w_i\}_1^\ell \subset \mathbb{R}^\ell$, their squared determinant is equal to the determinant of their Gram matrix.

Lemma 3.2   *Let P be the orthogonal projection from $\mathbb{R}^n$ onto a k-dimensional subspace H_k. Then, for $\wedge^\ell P$, where $1 \le \ell \le k$, we have*

(1) $\wedge^\ell P$ *is an $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix such that*

$$(3.1) \qquad \wedge^\ell P_{[I,J]} = P_{\{I,J\}} \quad for \quad I, J \in \binom{[n]}{\ell};$$

(2) $\wedge^\ell P : \Lambda^\ell(\mathbb{R}^n) \to \Lambda^\ell(\mathbb{R}^n)$ *is the orthogonal projection onto $\wedge^\ell H_k$.*

Proof   (1) By definition, $\wedge^\ell P$ is an $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix. Identity (3.1) is the consequence of the definition of the inner product in $\Lambda^\ell(\mathbb{R}^n)$.

(2) For any decomposable ℓ-form $x_1 \wedge \cdots \wedge x_\ell$, we have

$$\wedge^\ell P(x_1 \wedge \cdots \wedge x_\ell) = Px_1 \wedge \cdots \wedge Px_\ell \in \wedge^\ell H_k.$$

By linearity, we have that $\wedge^\ell Px \in \wedge^\ell H_k$ for an arbitrary ℓ-form x. For $x \in \wedge^\ell H_k$, we see that $\wedge^\ell Px = x$. Thus, we showed that $\mathrm{Im} \wedge^\ell P = \wedge^\ell H_k$ and $\left(\wedge^\ell P\right)^2 = \wedge^\ell P$. By this and since $P^2 = P$, we have that $\wedge^\ell P$ is symmetric. This completes the proof.   ∎

In the following Lemma, we understand $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of vectors, whose last $n - k$ coordinates are zero. This embedding induces a natural embedding $\Lambda^\ell(\mathbb{R}^k) \subset \Lambda^\ell(\mathbb{R}^n)$ for $\ell \in [n]$.

Theorem 3.3   *The following assertions are equivalent for $\{v_1, \ldots, v_n\} \subset \mathbb{R}^k$:*

(1)   *there exists an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ such that v_i is the orthogonal projection of f_i onto $\mathbb{R}^k$, for any $i \in [n]$;*
(2) $P_k = \sum_1^n v_i \otimes v_i$, *where P_k is the projector from $\mathbb{R}^n$ onto $\mathbb{R}^k$;*

(3) *for any fixed $\ell \in [k-1]$, there exists an orthonormal basis $\{f_L\}$ of $\Lambda^\ell(\mathbb{R}^n)$ such that v_L is the orthogonal projection of f_L onto $\Lambda^\ell(\mathbb{R}^k)$, for all $L \in \binom{[n]}{\ell}$;*

(4) *for any fixed $\ell \in [k-1]$, the following identity is true:*

$$\Lambda^\ell P_k = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L,$$

where $\Lambda^\ell P_k$ is the projector from $\Lambda^\ell(\mathbb{R}^n)$ onto $\Lambda^\ell(\mathbb{R}^k)$.

Proof   By the equivalence (1) $\Leftrightarrow$ (2) in Lemma 3.1 and since I_k is the restriction of P_k onto $\mathbb{R}^k$, we have that (1) $\Leftrightarrow$ (2).

Identifying f_i and e_i for $i \in [n]$, we identify $\mathbb{R}^k$ with some subspace $H_k \subset \mathbb{R}^n$, $\Lambda^\ell(\mathbb{R}^k) \subset \Lambda^\ell(\mathbb{R}^n)$ with $\wedge^\ell(H^k) \subset \Lambda^\ell(\mathbb{R}^n)$, and P_k with P.

Lemma 3.2 says that $\Lambda^\ell P_k$ is exactly $\wedge^\ell P_k$. Thus, identifying f_i and e_i for $i \in [n]$, we identify $\Lambda^\ell P_k$ with $\wedge^\ell P$. Again, since (1) $\Leftrightarrow$ (2) in Lemma 3.1, we get that (3) $\Leftrightarrow$ (4).

By assertion (2) in Lemma 3.2, we have that (2) $\Rightarrow$ (4).

Hence, we must show that the implication (2) $\Leftarrow$ (4) holds to complete the proof. Let $\{v_1, \ldots, v_n\} \subset H_k$ such that $\wedge^\ell P = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L$ for a fixed $\ell \in [k-1]$.

Let us prove that $P = \sum_1^n v_i \otimes v_i$ to complete the proof. Assume the contrary, i.e., $P \neq \sum_1^n v_i \otimes v_i$, and define $A = \sum_1^n v_i \otimes v_i$. By Lemma 2.1, we have that $\wedge^l P = \wedge^l A$. Since the restriction of A on H_k is a positive semi-definite operator, the restriction of A has an orthonormal basis of eigenvectors in H_k. Let us denote these eigenvectors by $x_1, \ldots, x_k$ and the corresponding eigenvalues by $\lambda_1, \ldots, \lambda_k$. Since $\{x_1, \ldots, x_k\} \subset H_k$ and $\wedge^\ell P = \wedge^\ell A$ is the projection onto $\wedge^\ell H_k$, we have that

$$0 \neq x_{i_1} \wedge \cdots \wedge x_{i_\ell} = Px_{i_1} \wedge \cdots \wedge Px_{i_\ell} = \wedge^\ell P(x_{i_1} \wedge \cdots \wedge x_{i_\ell})$$

$$= \wedge^\ell A(x_{i_1} \wedge \cdots \wedge x_{i_\ell}) = Ax_{i_1} \wedge \cdots \wedge Ax_{i_\ell} = \left(\prod_{j=1}^\ell \lambda_{i_j}\right) \cdot x_{i_1} \wedge \cdots \wedge x_{i_\ell}$$

for any ℓ-tuple $\{i_1, \ldots, i_\ell\} \subset \binom{[n]}{\ell}$. Therefore, $\left(\prod_{j=1}^\ell \lambda_{i_j}\right) = 1$ for any ℓ-tuple $\{i_1, \ldots, i_\ell\} \subset \binom{[n]}{\ell}$. Obviously, since $0 < \ell < k$, it implies that all eigenvalues are one. Hence, the restriction of A onto H_k is the identity operator in H_k. This means that $A = P$. ∎

The implication (2) $\Leftarrow$ (4) of Theorem 3.3 still holds for $\ell = k$. By the same arguments as in the proof, we have that (4) implies $\det\left(\sum_1^n v_i \otimes v_i\right)\Big|_{\mathbb{R}^k} = 1$. Thus, we have the following corollary.

Corollary 3.4   *The following assertions are equivalent for $\{v_1, \ldots, v_n\} \subset \mathbb{R}^k \subset \mathbb{R}^n$:*

(1) $\det \left(\sum\limits_{1}^{n} v_i \otimes v_i \right) \Big|_{\mathbb{R}^k} = 1.$

(2)   *the following identity is true:*

$$\Lambda^k P_k = \sum_{L \in \binom{[n]}{k}} v_L \otimes v_L.$$

Proof of Theorem 1.1   By the definition of the cross product, Theorem 1.1 is the equivalence (1) $\Leftrightarrow$ (3) of Theorem 3.3 with $\ell = k - 1$.   ∎

By identity $(\wedge^\ell P)^2 = \wedge^\ell P$, we have

$$P_{\{I, J\}} = \sum_{L \in \binom{[n]}{\ell}} P_{\{I, L\}} P_{\{L, J\}}$$

for a fixed $\ell \in [k]$ and two ℓ-tuples $I, J \subset \binom{[n]}{\ell}$. But Theorem 3.3 gives us another identity, which connects the squared ℓ-dimensional volume (that is, P_I) with the sum of the squared k-dimensional volumes.

Lemma 3.5   *Let $\{v_1, \ldots, v_n\}$ is a tight frame in H_k. Then, the following identity holds:*

$$P_I = \sum_{Q \in \binom{[n]}{k-\ell}; |Q \cap I| = \varnothing} P_{I \cup Q} = \sum_{T \in \binom{[n]}{k} | I \subset T} P_T$$

for a fixed $\ell \in [k]$ and an ℓ-tuple $I \in \binom{[n]}{\ell}$.

Proof   Now, we identify $\wedge^p H_k$ with $\Lambda^p(H_k)$ for a fixed integer $p \in [k]$. Then, the Hodge star operator maps a p-form $w \in \Lambda^p(H_k)$ to a $(k-p)$-form $v \in \Lambda^{k-p}(H_k)$. By Theorem 3.3, we know that the set of $(k-\ell)$-forms v_Q, where $Q \in \binom{[n]}{k-\ell}$, is a tight frame in $\wedge^{(k-\ell)} H_k \equiv \Lambda^{(k-\ell)}(H_k)$. Using the Hodge star operator, we get that the set of ℓ-forms $\star(v_Q)$, where $Q \in \binom{[n]}{k-\ell}$, is a tight frame in $\wedge^\ell H_k \equiv \Lambda^\ell(H_k)$. By the definition of the inner product of two ℓ-forms, we have $P_I = \langle v_I, v_I \rangle$. Now, we can expand this using properties of the ℓ-forms $\star(v_Q)$, where $Q \in \binom{[n]}{k-\ell}$. So,

$$(3.2) \qquad P_I = \langle v_I, v_I \rangle = \sum_{Q \in \binom{[n]}{k-\ell}} \langle v_I, \star(v_Q) \rangle \langle \star(v_Q), v_I \rangle.$$

By identity (2.3), $\langle v_I, \star(v_Q) \rangle$ is the determinant of k vectors $(v_i)_{i \in I}, (v_q)_{q \in Q}$ of H_k. Thus, $\langle v_I, \star(v_Q) \rangle = 0$ whenever $I \cap Q \neq \varnothing$, and $\langle v_I, \star(v_Q) \rangle^2 = P_{I \cup Q}$ whenever $I \cap Q = \varnothing$. Combining these formulas with (3.2), we obtain

$$P_I = \sum_{Q \in \binom{[n]}{k-\ell}; Q \cap I = \varnothing} \langle v_I, \star(v_Q) \rangle \langle \star(v_Q), v_I \rangle = \sum_{Q \in \binom{[n]}{k-\ell}; |Q \cap I| = \varnothing} P_{I \cup Q}. \qquad ∎$$

3.1 Lagrange's identity and McMullen's formula

McMullen's formula (1.3) follows from Shepard's formula [17] for the volume of a $\sum_{i=1}^{n}[0, b_i] \subset \mathbb{R}^k$

$$(3.3) \qquad \mathrm{vol}_k \left(\sum_{i=1}^{n}[0, b_i] \right) = \sum_{\{i_1, \dots, i_k\} \in \binom{[n]}{k}} |\det\{b_{i_1}, \dots, b_{i_k}\}|,$$

and identity $|d_S(\{L\})| = |d_S(\{[n]\setminus L\})|$ for a tight frame S and a subset L of $[n]$. By assertion (4) of Lemma 3.1, this identity is equivalent to identity $|U_L| = |U_{[n]\setminus L}|$ for an orthogonal matrix U of rank n and any $L \subset [n]$, which is known as Lagrange's identity. However, it is natural to prove it in terms of properties of a projection operator.

Lemma 3.6 *Let H_{n-k} be the orthogonal complement of H_k in $\mathbb{R}^n$. Denote by P and $P^\perp$ the projections onto H_k and H_{n-k}, respectively. Then,*

$$\star \left(\wedge^k P[x] \right) = \left(\wedge^{(n-k)} P^\perp \right) [\star x]$$

for any $x \in \Lambda^k(\mathbb{R}^n)$.

Proof We use π to denote a unit directional vector of the line $\wedge^k H_k$ generated by H_k in $\Lambda^k(\mathbb{R}^n)$. Clearly, $\star \pi$ is the unit directional vector (with a proper orientation) of the line $\wedge^{(n-k)} H_{n-k}$ generated by H_{n-k} in $\Lambda^{n-k}(\mathbb{R}^n)$. For an arbitrary $x \in \Lambda^k(\mathbb{R}^n)$ and $y \in \Lambda^{n-k}(\mathbb{R}^n)$, we have $\wedge^k P[x] = c_1 \pi \in \wedge^k H_k$ and $\left(\wedge^{(n-k)} P^\perp \right)[y] = c_2(\star \pi) \in \wedge^{(n-k)} H_{n-k}$ for some scalars c_1, c_2. From this and by linearity of operators $\star \left(\wedge^k P \right)$ and $\left(\wedge^{(n-k)} P^\perp \right) \star$, we obtain that $\star \left(\wedge^k P \right)[x] = c \left(\wedge^{(n-k)} P^\perp \right)[\star x]$ for some scalar c. From the chain

$$\star \left(\wedge^k P \right) \pi = \star \pi = \left(\wedge^{(n-k)} P^\perp \right)[\star \pi],$$

we conclude that $c = 1$. This completes the proof. ∎

So, Lagrange's identity takes the following form with a simple geometric meaning.

Corollary 3.7 *In the notation of Lemma 3.6, we have*

$$(3.4) \qquad\qquad\qquad P_L = P^\perp_{[n]\setminus L}$$

for $L \in \binom{[n]}{k}$.

4 The tight frames and related geometric problems

4.1 Equivalent formulation

Tight frame $\{P^H e_1, \dots, P^H e_n\}$, where H is a k-dimensional subspace of $\mathbb{R}^n$ and P^H is the orthogonal projection onto H, can be considered as the ordered set of n vectors of $\mathbb{R}^k$. That allows us to avoid working in Grassmannian $\mathrm{Gr}(n, k)$. However, there is an ambiguity in correspondence $H \to \{P^H e_1, \dots, P^H e_n\}$. Any choice of orthonormal basis of H gives its own tight frame in $\mathbb{R}^k$, all of them are isometric but different from

each other. Let us formalize our technique and explain how one can deal with this ambiguity.

We endow $\Omega(n, k)$ with the metric

$$\mathrm{dist}(\{v_1, \ldots, v_n\}, \{w_1, \ldots, w_n\}) = \sqrt{\sum_1^n |v_i - w_i|^2}.$$

The orthogonal group $\mathrm{O}(k)$ acts on $\Omega(n, k)$ in a usual way

$$U\{v_1, \ldots, v_n\} = \{Uv_1, \ldots, Uv_n\} \quad \text{for} \quad U \in \mathrm{O}(k).$$

The equivalence classes of this action are called *O-classes*, and the O-class of a tight frame S is denoted as $[S]$. We say that a function $\Psi : \Omega(n, k) \to \mathbb{R}$ is *O-invariant* if it is constant on each O-class.

There is a one-to-one correspondence between $\mathrm{Gr}(n, k)$ and $\frac{\Omega(n,k)}{\mathrm{O}(k)}$:

- the map $\alpha : \mathrm{Gr}(n, k) \to \frac{\Omega(n,k)}{\mathrm{O}(k)}$ is defined by

$$\alpha(H) = \left[P^H e_1, \ldots, P^H e_n \right] \quad \text{for} \quad H \in \mathrm{Gr}(n, k).$$

- the map $\beta : \frac{\Omega(n,k)}{\mathrm{O}(k)} \to \mathrm{Gr}(n, k)$ is defined as follows. Let $\{v_1, \ldots, v_n\} \in \Omega(n, k)$. By Lemma 3.1, the Gram matrix P of $\{v_1, \ldots, v_n\}$ is the matrix of projection onto a k-dimensional subspace H. Then, $\beta(\left[\{v_1, \ldots, v_n\}\right]) = H$. Clearly, an orthogonal transformation does not change the Gram matrix. Therefore, β is defined correctly.

By assertion (4) of Lemma 3.1, the sets of vectors $\{v_1, \ldots, v_n\}$ and $\{Pe_1, \ldots, Pe_n\}$ are isometric. Thus, α and β are the inverse functions of each other.

Consequently, there is a one-to-one correspondence between the space of functions $\mathrm{Gr}(n, k) \to \mathbb{R}$ and the space of O-invariant functions $\Omega(n, k) \to \mathbb{R}$. For any function $\Psi : \mathrm{Gr}(n, k) \to \mathbb{R}$, we define its frame-function $\gamma(\Psi) : \Omega(k, n) \to \mathbb{R}$ by

$$\gamma(\Psi)(S) = \Psi(\beta([S])).$$

Clearly, the frame-function is O-invariant. Thus, by the above argument, Ψ and $\gamma(\Psi)$ have the same global extrema. Let us show that the local extrema of any continuous function $\mathrm{Gr}(n, k) \to \mathbb{R}$ and its frame-functions are the same up to factorization by $\mathrm{O}(k)$. Of course, we need to specify a topology or metric on $\mathrm{Gr}(n, k)$. We consider the natural one as a topology of a homogeneous space $(\mathrm{Gr}(n, k) = \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k)))$, which is the metric topology generated by metric

$$\mathrm{Dist}(H, H') = \left\| P^H - P^{H'} \right\|,$$

where $\|\cdot\|$ denotes the operator norm.

Lemma 4.1 *Let $\Psi : \mathrm{Gr}(n, k) \to \mathbb{R}$ be a continuous function. Then, the following assertions are equivalent:*

(1) $S \in \Omega(n, k)$ *is a local extremum of $\gamma(\Psi)$.*
(2) $[S] \subset \Omega(n, k)$ *in the identity. At the moment, it is written $[S] \in$.*
(3) $\beta([S])$ *is a local extremum of Ψ.*

Proof   The equivalence of the first two assertions is trivial as $\gamma(\Psi)$ is O-invariant. Let us show that they are equivalent to the third one.

We define a metric on $\frac{\Omega(n,k)}{O(k)}$ as follows:

(4.1) $$\rho\left(O_1, O_2\right) = \min\left\{\text{dist}(S_1, S_2): \ S_1 \in O_1, \ S_2 \in O_2\right\}.$$

Obviously, the minimum is attained. Let us check the triangle inequality for arbitrary O-classes O_1, O_2, and O_3. Since $O(k)$ acts transitively on each O-class, there is $S_2 \in O_2$ such that $\rho\left(O_1, O_2\right) = \text{dist}(S_1, S_2)$ for any $S_1 \in O_1$. We assume that the minimum in (4.1) is attained at S_1 and S_2 for O_1 and O_2, and at S_2 and S_3 for O_2 and O_3. Then,

$$\rho\left(O_1, O_2\right) + \rho\left(O_2, O_3\right) = \text{dist}(S_1, S_2) + \text{dist}(S_2, S_3) \geq \text{dist}(S_1, S_3) \geq \rho\left(O_1, O_3\right).$$

Clearly, the tight frames of $[S]$ are the local extrema of $\gamma(\Psi)$ iff $[S]$ is a local extremum of the function $\Psi_O : \frac{\Omega(n,k)}{O(k)} \to \mathbb{R}$ defined by $\Psi_O\left([S]\right) = \gamma(\Psi(S))$. To complete the proof, it suffices to show that $\text{Gr}(n, k)$ and $\frac{\Omega(n,k)}{O(k)}$ are homeomorphic. This follows from the observation that the operator norm is equivalent to the Hilbert–Schmidt operator norm $\left(\left\|P^H\right\|_{HS} = \sqrt{\sum_1^n |P^H e_i|^2}\right)$, and the latter is equivalent to metric ρ on $\frac{\Omega(n,k)}{O(k)}$. ∎

4.2 Perturbation of frames

Our main idea of finding local extrema of an O-invariant function Ψ is to transform a given tight frame S to a new one S'. However, it is not convenient to restrict ourselves to the set of tight frames. For example, consider function $\Psi_1 : \Omega(n, k) \to \mathbb{R}_+$ given by $\Psi_1\left(\{v_1, \dots, v_n\}\right) = \text{vol}_k \text{co}\{\pm v_1, \dots, \pm v_n\}$ (that is, the frame-function of the volume of a projection of the cross-polytope). It is clear what happens with the convex hull when we move one or several vectors of the tight frame, but we may destroy the tight frame condition with such a transformation. It is not a problem as we can easily extend the domain.

Definition 4.1   We will say that the ordered set $S = \{v_1, \dots, v_n\}$ of n vectors of $\mathbb{R}^k$ is a frame if vectors of S span $\mathbb{R}^k$. We consider the following objects related to frames:

(1) the operator $\sum_{i \in [n]} v_i \otimes v_i$. We use A_S to denote this operator and the matrix of this operator in the standard basis.

(2) The operator $B_S = A_S^{-\frac{1}{2}}$.

(3) The subspace $H_k^S \subset \mathbb{R}^n$, which is the linear hull of the rows of the $k \times n$ matrix $(v_1, \dots, v_n)$ and the projection operator P^S from $\mathbb{R}^n$ onto H_k^S. We use the same notation P^S for the matrix of this operator in the standard basis.

For a frame $S = \{v_1, \dots, v_n\}$ and a linear transformation L, we denote $\{Lv_1, \dots, Lv_n\}$ by LS. The operator B_S is well-defined as the condition $\text{Lin } S = \mathbb{R}^k$

implies that A_S is a positive definite operator. Clearly, for any frame S, the frame $B_S S$ is a tight frame:

$$\sum_{i=1}^{n} B_S v_i \otimes B_S v_i = B_S \left(\sum_{i=1}^{n} v_i \otimes v_i \right) B_S^T = B_S A_S B_S = I_k.$$

It is easy to see that the metric $\mathrm{dist}(\cdot, \cdot)$ on $\Omega(n, k)$ extends to the set of all frames. Also, a small enough perturbation of a frame S yields a small perturbation of B_S.

Typically, O-invariant functions can be extended to the set of all frames in a tautological way, for example, the extensions of O-invariant functions for the objects listed above might be:

(1) $\Psi_1(\{v_1, \ldots, v_n\}) = \mathrm{vol}_k \, \mathrm{co}\{\pm v_1, \ldots, \pm v_n\}$;

(2) $\Psi_2(\{v_1, \ldots, v_n\}) = \mathrm{vol}_k \left(\bigcap_1^n \{x \in \mathbb{R}^k : |\langle x, v_i \rangle| \le 1\} \right)$;

(3) $\Psi_3(\{v_1, \ldots, v_n\}) = \mathrm{vol}_k \left(\sum_1^n [-v_i, v_i] \right)$.

We prefer to extend the functions as follows:

$$(4.2) \qquad \Psi(\tilde{S}) = \frac{\Psi(B_{\tilde{S}} \tilde{S})}{\det B_{\tilde{S}}} \quad \text{for a frame } \tilde{S}.$$

One can see that Ψ_1 and Ψ_3 satisfy this condition, and $\overline{\Psi}_2 = 1/\Psi_2$ satisfies it as well. It looks as a natural extension at least for problems related to the volume of projections or sections of bodies because of the positive homogeneity of the volume $\mathrm{vol}_k L(K) = |\det L| \, \mathrm{vol}_k K$, where K is a convex set and L is a linear transformation.

In order to obtain properties of extremizers, we consider a composition of two operations:

$$(4.3) \qquad S \xrightarrow{\ \mathbf{T}\ } \tilde{S} \xrightarrow{\ B_{\tilde{S}}\ } S',$$

where $\mathbf{T}$ is a map from a subset of $\Omega(n, k)$ to the set of frames in $\mathbb{R}^k$ and $B_{\tilde{S}}$ just maps $\tilde{S} = T(S)$ to a new tight frame S'.

Using our setting, it is easy to write a necessary and sufficient condition for a local extremum of an O-invariant function.

Lemma 4.2 *A tight frame $S = \{v_1, \ldots, v_n\}$ is a local maximum (or minimum) of a non-negative O-invariant function Ψ iff the following inequality holds for any frame $\tilde{S}$ of some open neighborhood $U(S)$ of S in the set of frames:*

$$(4.4) \qquad \frac{\Psi(\tilde{S})}{\Psi(S)} \le \sqrt{\det A_{\tilde{S}}} \quad (or \ge).$$

In addition, S is a global maximum (or minimum) iff inequality (4.4) holds for all frames $\tilde{S}$.

Proof As mentioned above, $B_{\tilde{S}} \tilde{S}$ is a tight frame and, by continuity, it belongs to a small enough neighborhood of S in $\Omega(n, k)$ if $\tilde{S} \in U(S)$. Using these observations

and the definition of B_S, we have

$$\frac{\Psi(\tilde{S})}{\Psi(S)} = \frac{1}{\det B_{\tilde{S}}} \frac{\Psi(B_{\tilde{S}}\tilde{S})}{\Psi(S)} \le \frac{1}{\det B_{\tilde{S}}} = \sqrt{\det A_{\tilde{S}}}.$$

The equivalence for a global extremum is trivial. ∎

Choosing a proper simple operation **T** (e.g., scaling one or several vectors, moving one vector to the origin, and mapping one vector to another), we may understand the geometric meaning of the left-hand side of (4.4). On the other hand, the determinant in the right-hand side of (4.4) can be calculated directly. In particular, the first-order approximation of the determinant is obtained in Theorem 1.2.

4.3 Computation of the determinants

Proof of Theorem 1.2   First of all, by the Cauchy–Binet formula, we have

$$(4.5) \qquad \det\left(\sum_1^n v_i \otimes v_i\right) = \sum_{I \in \binom{[n]}{k}} P_I^S.$$

By linearity and identity $P_I^S = (d_S(\{I\}))^2$, it suffices to prove the theorem for the case when only one vector of the set $\{x_i\}_1^n$ is nonzero. Thus, the theorem follows from the following lemma. ∎

Lemma 4.3   *Let $S = \{v_1, \ldots, v_n\}$ be a tight frame and S' be a frame obtained from S by substitution $v_i \to v_i + tx$, where $t \in \mathbb{R}, x \in \mathbb{R}^k$. Then,*

$$\sqrt{\det A_{S'}} = 1 + t \langle v_i, x \rangle + o(t).$$

That is, $\sqrt{\det A_{S'}}$ is a differentiable function of t at $t = 0$, and the derivative equals $\langle v_i, x \rangle$.

Proof   Given a substitution $v_i \to v_i + tx$, we modify the minor P_I^S iff $i \in I$. By the properties of minors of P^S and definition of the Hodge star operator, we have that $P_I^S = (\langle v_i, \star(v_{I \setminus i}) \rangle)^2$ if $i \in I$. After the substitution, we get

$$\left(\langle v_i + tx, \star(v_{I \setminus i}) \rangle\right)^2 = \left(\langle v_i, \star(v_{I \setminus i}) \rangle\right)^2 + 2t \langle v_i, \star(v_{I \setminus i}) \rangle \langle \star(v_{I \setminus i}), x \rangle + o(t).$$

Hence,

$$\det A_{S'} = \sum_{I \in \binom{[n]\setminus i}{k}} P_I^S + \sum_{I \in \binom{[n]}{k}, i \in I} P_I^S + 2t \sum_{I \in \binom{[n]}{k}, i \in I} \langle v_i, \star(v_{I \setminus i}) \rangle \langle \star(v_{I \setminus i}), x \rangle + o(t).$$

If $i \in J$ for $J \in \binom{[n]}{k-1}$, then we have $\langle v_i, \star(v_J) \rangle = 0$. Therefore, we can rewrite the last identity

$$\det A_{S'} = \sum_{I \in \binom{[n]}{k}} P_I^S + 2t \sum_{J \in \binom{[n]}{k-1}} \langle v_i, \star(v_J) \rangle \langle \star(v_J), x \rangle + o(t).$$

By Theorem 1.1, we know that the vectors $(\star(v_J))_{J \in \binom{[n]}{k-1}}$ form a tight frame in H_k. Therefore,

$$\sum_{J \in \binom{[n]}{k-1}} \langle v_i, \star(v_J) \rangle \langle \star(v_J), x \rangle = \langle v_i, x \rangle.$$

Using this and by (4.5), we obtain

$$\sqrt{\det A_{S'}} = \sqrt{\det A_S + 2t \langle v_i, x \rangle + o(t)} = 1 + t \langle v_i, x \rangle + o(t). \qquad \blacksquare$$

Theorem 1.2 gives the first-order approximation of the left-hand side in inequality (4.4). It implies the first-order necessary condition for a differentiable O-invariant function Ψ.

Corollary 4.4 *Let Ψ be an O-invariant function, $S = \{v_1, \ldots, v_n\}$ be a local extremum of Ψ on $\Omega(n, k)$, and Ψ is differentiable at S. Then,*

$$\Psi(\tilde{S}) = \Psi(S) \left(1 + \tau \sum_{i=1}^{n} \langle x_i, v_i \rangle \right) + o(\tau)$$

for an arbitrary $\tilde{S} = \{v_1 + \tau x_1, \ldots, v_n + \tau x_n\}$.

By the dimension argument, Corollary 4.4 gives a complete system of the first-order necessary conditions. As a small enough perturbation of a frame is still a frame, we see that the set of frames looks like $\mathbb{R}^{nk}$ locally, and we can speak about local convexity of O-invariant functions. By a basic fact from subdifferential calculus, inequality (4.4) implies the differentiability of a locally convex function Ψ at its local maximum.

Corollary 4.5 *Let Ψ be an O-invariant function, $S = \{v_1, \ldots, v_n\}$ be a local extremum of Ψ on $\Omega(n, k)$, and Ψ is locally convex at S. Then, Ψ is a differentiable function at S, and it satisfies the identity of Corollary 4.4.*

Actually, the local convexity of Ψ is a rather typical situation when we consider the volume of the projection of a polytope in $\mathbb{R}^n$ onto a k-dimensional subspace, as the latter is piecewise linear on the corresponding Grassmannian by Theorem 1 in [7]. Another example is the so-called *linear parameter systems* (see [15]).

Here is another observation. One can notice that $\operatorname{tr} A_S = \sum_{1}^{n} |v_i|^2$ for a tight frame $S = \{v_1, \ldots, v_n\}$. Hence, $\operatorname{tr} A_S = k$ for a tight frame S. Let $\Omega'(n, k)$ denote the class of frames $S = \{v_1, \ldots, v_n\}$ such that $\sum_{1}^{n} |v_i|^2 = k (= \operatorname{tr} A_S)$. The same arguments as in Lemma 4.2 imply the following corollary.

Corollary 4.6 *Let Ψ be an O-invariant function. Then,*

$$\max_{S \in \Omega'(n,k)} \Psi(S) = \max_{S \in \Omega(n,k)} \Psi(S).$$

Proof   Since $\Omega(n,k) \subset \Omega'(n,k)$, it is enough to show that for any $S' \in \Omega'(n,k), S' \notin \Omega(n,k)$, there exists a tight frame S such that $\Psi(S') < \Psi(S)$. We put $S = B_{S'}S'$. Then, by Lemma 4.2, it is enough to prove that $\det A_{S'} < 1$. Considering $A_{S'}$ in the basis of its eigenvectors (in which it is a diagonal operator) and using the inequality of arithmetic and geometric means, we obtain

$$\det A_{S'} \leq \left(\frac{\operatorname{tr} A_{S'}}{k} \right)^k = 1,$$

where equality is attained iff all eigenvalues of $A_{S'}$ are ones. This means, the equality is attained iff $S' \in \Omega(n,k)$. This completes the proof. ∎

Remark 4.7   Corollary 4.6 was proved in [5, Theorem 1] with the same idea but different notation for the projections of regular polytopes, particularly, for the mentioned functions Ψ_1 and Ψ_3. However, Corollary 4.6 gives a nice property of function $1/\Psi_2$ as well.

4.4   Lifting of frames

Vectors $\{v_1, \dots, v_n\}$ of a tight frame $S \in \Omega(n,k)$ have a nice isometric embedding in $\wedge^{k-1}(H^S)$. Fix $i \in [n]$ and let $d_S(i)$ be the vector in $\Lambda^{k-1}(\mathbb{R}^n)$ such that its Lth coordinate in the standard basis of $\Lambda^{k-1}(\mathbb{R}^n)$ is $d_S(\{i,L\})$.

Lemma 4.8   *Let $S = \{v_1, \dots, v_n\}$ be an arbitrary tight frame. Then, the vectors $d_S(i), i \in [n]$, belong to $\wedge^{k-1}(H^S) \subset \Lambda^{k-1}(\mathbb{R}^n)$. Moreover, they form a tight frame in $\wedge^{k-1}(H^S)$, and the following identity holds:*

(4.6)
$$\langle d_S(i), d_S(j) \rangle = \langle v_i, v_j \rangle$$

for all $i, j \in [n]$.

Proof   Let $a_j, j \in [k]$, be the rows of the $k \times n$ matrix $M^S = (v_1, \dots, v_n)$. By the definition of H^S, we know that $a_j \in H^S_k, j \in [k]$, and that they form an orthonormal system in H^S. Hence, the $(k-1)$-forms $(a_{[k]\setminus j})_{j=1}^k$ are an orthonormal basis of $\wedge^{k-1}(H^S)$. Consider the $(k-1)$-forms $b_i = \sum_{j=1}^k (-1)^{j+1} v_i[j] \cdot a_{[k]\setminus j}$, where $i \in [n]$.
Then, by the definition of the inner product in $\Lambda^{k-1}(\mathbb{R}^n)$ and the Laplace expansion of the determinant, we have

$$b_i[L] = \langle b_i, e_L \rangle = \sum_{j=1}^k (-1)^{j+1} v_i[j] \langle a_{[k]\setminus j}, \rangle e_L$$

$$= \sum_{j=1}^k (-1)^{j+1} v_i[j] M^S_{[k]\setminus j, L} = M^S_{\{i,L\},\{i,L\}} = d_S(i)[L]$$

for $L \in \binom{[n]}{k-1}, i \notin L$. We get $b_i[L] = d_S(i)[L] = 0$ if $i \in L$. Thus, $d_S(i) = b_i \in \Lambda^{k-1}(H^S)$ and, clearly, identity (4.6) holds. ∎

Lemma 3.1 allows us to describe all substitutions $w_i \to w_i'$, $i \in [n]$, which preserve the operator A_S for a frame $S = \{w_1, \dots, w_n\}$. But we need a more suitable geometric description. So, let $S = \{w_1, \dots, w_n\}$ be a frame and $\ell \in [k]$, $L = \{i_1, \dots, i_\ell\} \subset \binom{[n]}{\ell}$. Consider a substitution $w_i \to w_i'$, for $i \in L$, and $w_i' = w_i$, for $i \notin L$, which preserves A_S, and denote by $S' = \{w_1', \dots, w_n'\}$ the new frame.

In this notation, we get the following.

Lemma 4.9 *The substitution preserves A_S (i.e., $A_S = A_{S'}$) iff there exists an orthogonal matrix U of rank ℓ such that*

(4.7)
$$\begin{pmatrix} w_{i_1}', & \dots, & w_{i_\ell}' \end{pmatrix} = \begin{pmatrix} w_{i_1}, & \dots, & w_{i_\ell} \end{pmatrix} U.$$

Additionally, in case S is a tight frame, let $\{f_1, \dots, f_n\}$ be any orthonormal basis of $\mathbb{R}^n$ given by the assertion (3.1) of Lemma 3.1. Then, the substitution preserves A_S iff the vectors of S' are the projection of an orthonormal basis $\{f_1', \dots, f_n'\}$ of $\mathbb{R}^n$, which is obtained from $\{f_1, \dots, f_n\}$ by an orthogonal transformation of $\{f_i\}_{i \in I}$ in their linear hull.

Proof   The identity $A_{S'} = A_S$ holds if and only if

$$\sum_{i \in I} w_i \otimes w_i = \sum_{i \in I} w_i' \otimes w_i'.$$

Writing this in a matrix form, we get another equivalent statement that the Gram matrices of the rows of the matrices $\begin{pmatrix} w_{i_1}', & \dots, & w_{i_\ell}' \end{pmatrix}$ and $\begin{pmatrix} w_{i_1}, & \dots, & w_{i_\ell} \end{pmatrix}$ are the same. The latter is equivalent to the existence of an isometry of $\mathbb{R}^\ell$, which maps the rows of the first matrix to the rows of the second. This isometry defines an orthogonal matrix U, which satisfies the assumptions of the lemma.

In case of a tight frame, extending U as the orthogonal transformation of $\mathrm{Lin}\{f_i : i \in I\}$, we obtain the suitable $\{f_1', \dots, f_n'\}$.   ∎

Also, we can reformulate the second claim of Lemma 4.9 in the following equivalent way: given a block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that the rows of $\begin{pmatrix} A & B \end{pmatrix}$ are orthonormal and the columns of $\begin{pmatrix} A \\ C \end{pmatrix}$ are orthonormal, then D can be chosen in such a way that M will be an orthogonal matrix.

5 Zonotopes and their volume

5.1 Definitions and history

A *zonotope* is the *Minkowski sum* of several segments in $\mathbb{R}^k$. Every zonotope can be represented (up to an affine transformation) as a projection of a higher-dimensional cube $C^n = [0,1]^n$. For example, it follows from the definition of B_S. To get more information about zonotopes, we refer the reader to Zong's book [20, Chapter 2] and to [16].

Several different bounds for the maximum in (1.2) are known. For example, G. D. Chakerian and P. Filliman [4] prove

$$\mathrm{vol}_k(\mathrm{C}^n\,|H) \le \sqrt{\frac{n!}{(n-k)!k!}} \quad \text{and} \quad \mathrm{vol}_k(\mathrm{C}^n\,|H) \le \frac{\omega_{k-1}^k}{\omega_k^{k-1}}\left(\frac{n}{k}\right)^{k/2},$$

where w_i is the volume of the i-dimensional Euclidean unit ball. The right-hand side inequality is asymptotically tight, as was shown in [5], and the left-hand side inequality is tight in the cases $k = 1, n - 1$. The tight upper and lower bounds in the limit case for the volume of a zonotope in a specific position (even for so-called L_p-zonoids) were obtained in [12, Theorem 2].

Among different upper bounds, all maximizers of (1.2) were described in the cases $k = 1, 2, n - 2, n - 1$ and $k = 3, n = 6$ in [5]. It appears that projections of the standard basis vectors onto H_k have the same length whenever H is a maximizer of (1.2) for all cases mentioned above, which gives a rather reasonable conjecture.

Conjecture 5.1 *Let the maximum volume projection of $\mathrm{C}^n = [0,1]^n$ onto a k-dimensional subspace be attained on H. Then, the projections of the vectors of the standard basis onto H have the same length.*

We note that there are k-dimensional subspaces of $\mathbb{R}^n$ such that the projections of the vectors of the standard basis onto them have the same length. An explicit construction of such subspaces can be found in [19]. In [10, Lemma 2.4], the author describes all possible lengths of the projections of the vectors of the standard basis.

Summarizing the observations of Section 4, we have that the problem to find and to study maximizers of

$$(5.1) \qquad F_2(S) = \mathrm{vol}_k\left(\sum_1^n [0, v_i]\right), \qquad \text{where} \qquad \{v_1, \dots, v_n\} \in \Omega(n, k),$$

is equivalent to that of (1.2). We illustrate the developed technique and write the first-order necessary condition of local maximum for (5.1) together with some geometric consequences.

5.2 The first-order necessary condition

We start with showing that $F_2(S)$ is a differentiable function at its local maximum.

Lemma 5.2 *Let $S = \{v_1, \dots, v_n\}$ be a local maximizer of (5.1) and S' be a frame obtained from S by substitution $v_i \to v_i + tx$, where $t \in \mathbb{R}, x \in \mathbb{R}^k$. Then, $\frac{F_2(S')}{F_2(S)}$ is a differentiable function of t at $t = 0$, and the derivative equals $\langle v_i, x \rangle$.*

Proof Given a substitution $v_i \to v_i + tx$, we change $d_S(\{L\})$ for $L \in \binom{[n]}{k}$ in (3.3) iff $i \in L$. Since the determinant is a linear function of each vector, this means that $|d_{S'}(\{L\})|$ as function of t (even as function of $x' = tx$) is a convex function of t (or even of $x' = tx \in \mathbb{R}^k$). Therefore, $F_2(S')/F_2(S)$ as function of t (or $x' = tx$) is a convex function. The result follows from Corollary 4.5. ∎

Corollary 5.3 *Let $S = \{v_1, \ldots, v_n\}$ be a local maximizer of* (5.1). *Then, the vectors of S are in general position in $\mathbb{R}^k$, i.e., $d_S(\{L\}) \neq 0$ for each $L \in \binom{[n]}{k}$.*

Proof Assume the contrary that there is a k-tuple J such that $d_S(\{J\}) = 0$. As the rank of the vectors of S is k, this implies that there is an $L = \{i_1, \ldots, i_k\} \in \binom{[n]}{k}$ such that the vectors $\{v_{i_1}, \ldots, v_{i_{k-1}}\}$ are linearly independent and the vectors $\{v_{i_1}, \ldots, v_{i_{k-1}}, v_{i_k}\}$ are linearly dependent (i.e., $d_S(\{L\}) = 0$). Taking $x \neq 0$ in the orthogonal complement of $\operatorname{Lin}\{v_{i_1}, \ldots, v_{i_{k-1}}\}$ and obtaining S' by a substitution $v_{i_k} \to v_{i_k} + tx$, we get that $F_2(S')/F_2(S)$ as well as the absolute value of $d_{S'}(\{L\})$ is not differentiable at $t = 0$. This contradicts Lemma 5.2. ∎

Lemma 5.4 *The function $F_2(\cdot)$ is differentiable at a local maximizer of* (5.1).

Proof Let $S = \{v_1, \ldots, v_n\}$ be a local maximizer of (5.1) and S' be a frame obtained from S by substitution $v_i \to v_i + t_i x_i$, $i \in [n]$.

The function $|d_{S'}(\{L\})|$ as a function of $(t_1, \ldots, t_n)$ is the absolute value of a polynomial of $(t_1, \ldots, t_n)$. Hence, it is differentiable at a point $(t_1, \ldots, t_n)$ whenever $d_{S'}(\{L\}) \neq 0$. By Corollary 5.3, we have $d_S(\{L\}) \neq 0$ for all k-tuples. Therefore, the function

$$F_2(S') = \sum_{L \in \binom{[n]}{k}} |d_{S'}(\{L\})|$$

as a function of $(t_1, \ldots, t_n)$ is differentiable at the origin. ∎

Now, we show the geometric meaning of identities of Corollary 4.4 for the function F_2. We define a *sign function* $\sigma_S(i, L)$ by

$$\sigma_S(i, L) = \begin{cases} 1/F_2(S), & \text{if} \quad d(\{i, L\}) > 0; \\ -1/F_2(S), & \text{if} \quad d(\{i, L\}) < 0; \\ 0, & \text{if} \quad d(\{i, L\}) = 0, \end{cases}$$

for a frame S, $i \in [n]$ and $L \in \binom{[n]}{k-1}$.

In the same way as with the vectors $d_S(i)$, $i \in [n]$, we identify $\sigma_S(i)$ with a vector in $\Lambda^{k-1}(\mathbb{R}^n)$ such that its Lth coordinate in the standard basis of $\Lambda^{k-1}(\mathbb{R}^n)$ is $\sigma_S(i, L)$. As a direct consequence of Lemma 5.4, we obtain the following lemma.

Lemma 5.5 *Let $S = \{v_1, \ldots, v_n\}$ be a local maximizer of* (5.1). *Then,*

$$(5.2) \quad \wedge^{k-1} P^S(\sigma_S(i)) = d_S(i) \quad \text{and} \quad \langle \sigma_S(i), d_S(j) \rangle = \langle v_i, v_j \rangle, \quad i, j \in [n].$$

Proof By Lemma 4.8, we have that the vectors $(d_S(i))_{i=1}^n$ form a tight frame in $\wedge^{k-1} H_k^S$. Therefore, by Theorem 3.3, it is enough to show that the rightmost identity in (5.2) is true.

Fix $i, j \in [n]$. Let S' be a frame obtained from S by the substitution $v_i \to v_i + tv_j$. By Corollary 5.3, we have that $d_S(\{L\}) \neq 0$ for every k-tuple L. Therefore, $d_{S'}(\{L\}) \neq 0$ and $d_{S'}(\{L\})$ has the same sign as $d_S(\{L\})$ for all k-tuples L and for a small

enough t. Using the substitution $v_i \to v_i + t v_j$, we only change the determinants of type $d_S(\{i, J\})$, where $i \notin J \subset \binom{[n]}{k-1}$. Thus, by the properties of absolute value (for a small enough t), we get

$$\frac{F_2(S')}{F_2(S)} = \frac{1}{F_2(S)} \sum_{L \in \binom{[n]}{k}} |d_{S'}(\{L\})| = 1 + t \sum_{J \in \binom{[n]}{k-1}, i \notin J} \sigma_S(i, J) d_S(\{j, J\}).$$

Since $\sigma_S(i, J) = 0$ whenever $i \in J$, we have that the coefficient at t in the previous formula is

$$\sum_{J \in \binom{[n]}{k-1}} \sigma_S(i, J) d_S(\{j, J\}).$$

But this is the inner product $\langle \sigma_S(i), d_S(j) \rangle$ of the $(k-1)$-forms $\sigma_S(i)$ and $d_S(j)$ written in the standard basis of $\Lambda^{k-1}(\mathbb{R}^n)$. By Lemma 5.4 and Corollary 4.4, we have that $\langle \sigma_S(i), d_S(j) \rangle = \langle v_i, v_j \rangle$. ∎

5.3 Proofs of Corollary 1.3, Theorem 1.4, and Corollary 1.5

Corollary 1.3 is a direct consequence of Lemma 5.5 with a simple geometric meaning.

Proof of Corollary 1.3 By Corollary 5.3, $|v_i| \neq 0$ for all $i \in [n]$. Let $S' = \{v_1', \dots, v_n'\}$ be projections of the vectors $\{v_1, \dots, v_n\}$ onto $v_i^\perp$. Clearly, $v_i' = 0$. By Shepard's formula (3.3) and the properties of the determinant, we have

$$\mathrm{vol}_{k-1}\left(Q|v_i^\perp\right) = \sum_{J \in \binom{[n]}{k-1}} |d_{S'}(\{J\})| = \frac{1}{|v_i|} \sum_{J \in \binom{[n]}{k-1}} |d_S(\{i, J\})|.$$

By Lemma 5.5, the latter is $|v_i| \, \mathrm{vol}_k Q$. ∎

The idea of the proof of Theorem 1.4 is the following. We use Lemma 4.9 to rotate vectors by $\pi/4$ (i.e., $v_i \to \cos(\pi/4) v_i - \sin(\pi/4) v_j$, $v_j \to \sin(\pi/4) v_i + \cos(\pi/4) v_j$), and after some simple calculations along with Corollary 4.4, we get the inequality of Theorem 1.4.

Proof of Theorem 1.4 We prove the theorem for a maximizer $S = \{v_1, \dots, v_n\}$ of (5.1). Fix i and j in $[n]$. We assume that $|v_i|^2 < |v_j|^2$, otherwise there is nothing to prove. Using Lemma 4.9, we have that the substitution $v_i \to \cos(\pi/4) v_i - \sin(\pi/4) v_j$, $v_j \to \sin(\pi/4) v_i + \cos(\pi/4) v_j$ preserves A_S and the absolute value of $d_S(\{L\})$ for all $L \subset \binom{[n]}{k}$ such that $i, j \in L$. Let S' be the tight frame obtained by this substitution. From the choice of S, we have $F_2(S') \leq F_2(S)$. Expanding these volumes by formula (3.3) and reducing common determinants, we get

$$\frac{\sqrt{2}}{2} \sum_{J \in \binom{[n]}{k-1}, \, i,j \notin J} \left(|d_S(\{i, J\}) - d_S(\{j, J\})| + |d_S(\{i, J\}) + d_S(\{j, J\})| \right)$$

$$\leq \sum_{J \in \binom{[n]}{k-1}, \, i,j \notin J} \left(|d_S(\{i, J\})| + |d_S(\{j, J\})| \right).$$

By identity $|a + b| + |a - b| = 2 \max\{|a|, |b|\}$, we obtain that each summand in the left-hand side is at least $2 \max\{|d_S(\{i, J\})|, |d_S(\{i, J\})|\}$ and, consequently, is at least $2|d_S(\{j, J\})|$. Hence, we show that

$$(\sqrt{2} - 1) \sum_{J \in \binom{[n]}{k-1}, \, i, j \notin J} |d_S(\{j, J\})| \leq \sum_{J \in \binom{[n]}{k-1}, \, i, j \notin J} |d_S(\{i, J\})|.$$

Consider the one-to-one correspondence between the set of all $(k - 1)$-tuples L with $i \in L$ and $j \notin L$ and the set of all $(k - 1)$-tuples J with $i \notin J$ and $j \in J$ given by $L \to (L \backslash \{i\}) \cup \{j\}$. In this case, $|d_S(\{j, L\})| = |d_S(\{i, (L \backslash \{i\}) \cup \{j\}\})|$. Adding all such determinants to the last inequality, we obtain

$$\sum_{J \in \binom{[n]}{k-1}, \, i \notin J} |d_S(\{i, J\})|$$

$$\geq (\sqrt{2} - 1) \sum_{J \in \binom{[n]}{k-1}, \, i, j \notin J} |d_S(\{j, J\})| + \sum_{J \in \binom{[n]}{k-1}, \, i \in J, \, j \notin J} |d_S(\{j, J\})|$$

$$\geq (\sqrt{2} - 1) \left(\sum_{J \in \binom{[n]}{k-1}, \, i, j \notin J} |d_S(\{j, J\})| + \sum_{J \in \binom{[n]}{k-1}, \, i \in J, \, j \notin J} |d_S(\{j, J\})| \right)$$

$$= (\sqrt{2} - 1) \sum_{J \in \binom{[n]}{k-1}, \, j \notin J} |d_S(\{j, J\})|.$$

Finally, the sum in the left-hand side is exactly $F_2(S) \langle \sigma_S(i), d_S(i) \rangle$, and by Lemma 5.2, it is equal to $F_2(S)|v_i|^2$. Similarly, we have $(\sqrt{2} - 1)F_2(S)|v_j|^2$ in the right-hand side of the last inequality. Dividing by $F_2(S)$, we obtain $|v_i|^2 \geq (\sqrt{2} - 1)|v_j|^2$. ∎

As mentioned in the Introduction, Corollary 1.5 is a consequence of Theorem 1.4 and McMullen's symmetric formula (1.3).

Proof of Corollary 1.5   Let H_q be the orthogonal complement of H_{n-q} in $\mathbb{R}^n$. Let v_i and v_i' be the projections of the vector e_i onto H_{n-q} and H_q, respectively. Clearly, $|v_i|^2 + |v_i'|^2 = 1$. By Theorem 1.4, we conclude that $|v_i'|^2$ is at most $1/(\sqrt{2} - 1)$ and at least $(\sqrt{2} - 1)$ of the average squared length of the projections of the standard basis, which is q/n. Therefore,

$$1 \geq \frac{m_n}{M_n} \geq \sqrt{\frac{1 - \frac{1}{\sqrt{2}-1}\frac{q}{n}}{1 - (\sqrt{2} - 1)\frac{q}{n}}},$$

which tends to 1 as n tends to infinity. ∎

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