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MAZUR–ULAM PROPERTY OF THE SUM OF TWO STRICTLY CONVEX BANACH SPACES

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Abstract

In this article, we study the Mazur–Ulam property of the sum of two strictly convex Banach spaces. We give an equivalent form of the isometric extension problem and two equivalent conditions to decide whether all strictly convex Banach spaces admit the Mazur–Ulam property. We also find necessary and sufficient conditions under which the ℓ^1-sum and the ℓ^∞-sum of two strictly convex Banach spaces admit the Mazur–Ulam property.

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1. Introduction and preliminaries

In 1987, Tingley proposed the following problem in [12].

PROBLEM 1.1 (Isometric extension problem). Let E and F be real Banach spaces and let V_0 be a surjective isometry between the unit spheres $S_1(E)$ and $S_1(F)$. Is V_0 necessarily the restriction of a linear isometry on the whole space?

The isometric extension problem is only considered in real Banach spaces, since the answer is clearly negative in the complex case. If it has a positive answer, the local geometric properties of a mapping on the unit sphere will determine the properties of the mapping on the whole space. This problem is related to the well-known Mazur–Ulam theorem.

THEOREM 1.2 (Mazur–Ulam theorem). *Let E and F be real Banach spaces and let $V : E \to F$ be a surjective isometry. Then V is affine.*

A Banach space E is said to *admit the Mazur–Ulam property* if, for any Banach space F, any surjective isometry V_0 between the unit spheres $S_1(E)$ and $S_1(F)$ is the restriction of a linear isometry between E and F (see [1]). It is clear that the

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isometric extension problem just asks whether all Banach spaces admit the Mazur–Ulam property.

In the past decade, the isometric extension problem was mainly considered in various classical Banach spaces (see [4]). The problem has been solved affirmatively if E is a classical Banach space and F is a general Banach space. In other words, all the classical Banach spaces admit the Mazur–Ulam property. The isometric extension problem for the ℓ^1-sum of strictly convex Banach spaces was solved affirmatively (see [14]) and also for the ℓ^∞-sum of strictly convex Banach spaces (see [7]).

Recently, the isometric extension problem was considered in finite-dimensional polyhedral Banach spaces (see [9]) and somewhere-flat real Banach spaces (see [1]). 'Sharp corner points' on the unit ball of dual Banach spaces were applied to consider this problem in Gâteaux differentiable spaces (see [6]). The problem was also studied in $\mathbb{R}^2$ with symmetric absolute normalised norms [10, 11].

We state two lemmas which will be useful in this article.

Lemma 1.3 [8, Theorem 2]. *Let E, F be real Banach spaces and $V_0 : S_1(E) \to S_1(F)$ be a surjective isometry. Suppose that*

$$\|V_0(u) - \lambda V_0(v)\| \le \|u - \lambda v\| \quad \forall u, v \in S_1(E), \lambda \in \mathbb{R}^+.$$

Then V_0 can be extended to a linear isometry on the whole space.

Lemma 1.4 [2, Lemma 2.1]. *Let E and F be real Banach spaces and let E be strictly convex. Suppose that V_0 is a surjective mapping between $S_1(E)$ and $S_1(F)$ and*

$$\|V_0(u) - V_0(v)\| \le \|u - v\| \quad \forall u, v \in S_1(E).$$

Then $V_0(-u) = -V_0(u)$ for any $u \in S_1(E)$.

We consider the isometric extension problem between the sum of two strictly convex Banach spaces and a general Banach space. In Section 2, we give an equivalent form of the isometric extension problem and we give two equivalent conditions to decide whether all strictly convex Banach spaces admit the Mazur–Ulam property. In Section 3, we prove that a surjective isometry between the ℓ^1-sum of two strictly convex Banach spaces and a general Banach space has a linear isometric extension under a condition. In Section 4, we prove that a surjective isometry between the ℓ^∞-sum of two strictly convex Banach spaces and a general Banach space has a linear isometric extension under the same condition. In Section 5, we obtain necessary and sufficient conditions under which the ℓ^1-sum and the ℓ^∞-sum of two strictly convex Banach spaces admit the Mazur–Ulam property.

Before we start, we need some definitions and notation. In this article, all Banach spaces are over $\mathbb{R}$. Let E and F be Banach spaces and let V be a surjective mapping between them. We call V a *sphere isometry* if $\|V(u) - V(v)\| = \|u - v\|$ for any $u, v \in E$ with $\|u\| = \|v\|$. We say that V *preserves spheres* if $\|V(u)\| = \|u\|$ for any $u \in E$. For a subspace $E_0 \subseteq E$, we say that V *preserves the subspace* E_0 if $V(E_0) \subseteq F$ is also

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a subspace. We call V *positive (real) homogeneous* if $V(\lambda u) = \lambda V(u)$ for any $u \in E$ and $\lambda > 0$ ($\lambda \in \mathbb{R}$).

Let E be a Banach space. We denote the unit sphere and the unit ball respectively by

$$S_1(E) := \{u \in E : \|u\| = 1\}, \quad B_1(E) := \{u \in E : \|u\| \leq 1\}.$$

Let E_1 and E_2 be Banach spaces and let $E_1 \oplus E_2$ be their direct sum. We denote by $E_1 \oplus_{\ell^1} E_2$ and $E_1 \oplus_{\ell^\infty} E_2$ the vector space $E_1 \oplus E_2$ with the ℓ^1-norm and the ℓ^∞-norm, respectively. For $x \in E_1, y \in E_2$ and $u \in E_1 \oplus E_2$, we write $u := (u_1, u_2) \in E_1 \oplus E_2$ and

$$\hat{x} := (x, 0) \in E_1 \oplus E_2, \quad \hat{y} := (0, y) \in E_1 \oplus E_2.$$

For any $\lambda, \mu \in \mathbb{R}$, we denote by $\max\{\lambda, \mu\}$ the larger one and write

$$\operatorname{sgn}(\mu) := \begin{cases} \mu/|\mu| & \text{if } \mu \neq 0, \\ 0 & \text{if } \mu = 0. \end{cases}$$

2. Equivalent forms of the isometric extension problem

In this section, we give an equivalent form of the isometric extension problem. In particular, we show that all the strictly convex Banach spaces admit the Mazur–Ulam property if and only if any surjective real homogeneous sphere isometry between a strictly convex Banach space and a general Banach space is linear.

Theorem 2.1. *The following are equivalent.*

(i) *For Banach spaces E and F and a surjective mapping V between them, if V is a positive homogeneous sphere isometry and preserves the sphere, then V is linear.*

(ii) *All Banach spaces admit the Mazur–Ulam property.*

Proof. If (i) holds and V_0 is a surjective isometry between $S_1(E)$ and $S_1(F)$, we define $\tilde{V}_0$ between E and F as follows:

$$\tilde{V}_0(u) = \begin{cases} 0 & \text{if } u = 0, \\ \|u\| V_0(\|u\|^{-1} u) & \text{if } u \neq 0. \end{cases}$$

It is clear that $\tilde{V}_0$ is positive homogeneous and preserves the sphere. Moreover, $\tilde{V}_0$ is surjective since V_0 is surjective. Now we prove that $\tilde{V}_0$ is a sphere isometry. Take $u, v \in E$ with $\|u\| = \|v\|$. If we denote $\lambda := \|u\| = \|v\|$, then

$$\|\tilde{V}_0(u) - \tilde{V}_0(v)\| = \left\|\lambda V_0\left(\frac{u}{\lambda}\right) - \lambda V_0\left(\frac{v}{\lambda}\right)\right\| = \lambda \left\|\frac{u}{\lambda} - \frac{v}{\lambda}\right\| = \|u - v\|.$$

Therefore, $\tilde{V}_0$ is linear and thus an isometry on the whole space.

Conversely, suppose that all Banach spaces admit the Mazur–Ulam property. If V is a surjective positive homogeneous sphere isometry between Banach spaces E and F and preserves the sphere, define V_0 to be the restriction of V on $S_1(E)$. It is clear that V_0 is a surjective isometry between $S_1(E)$ and $S_1(F)$. Then V_0 has a linear isometric extension $\tilde{V}_0$ from E to F. Since V is positive homogeneous, we see that $\tilde{V}_0 = V$ and thus V is linear. □

Theorems 2.1 and 1.2 show why the isometric extension problem is a refinement of the Mazur–Ulam theorem. We consider other equivalent forms of this problem between a strictly convex Banach space and a general Banach space. In fact, it can be seen from Lemma 1.4 that (ii) implies (i) in Theorem 2.2.

THEOREM 2.2. *The following are equivalent.*

(i) *For any Banach spaces E and F and a surjective mapping V between them, if E is strictly convex and V is a positive homogeneous sphere isometry and preserves the sphere, then V is linear.*
(ii) *For any Banach spaces E and F and a surjective mapping V between them, if E is strictly convex and V is a real homogeneous sphere isometry, then V is linear.*
(iii) *All strictly convex Banach spaces admit the Mazur–Ulam property.*

PROOF. By similar methods to Theorem 2.1, we can prove that (i) is equivalent to (iii). Now we want to prove that (i) is equivalent to (ii).

Suppose that (i) holds and V is a surjective real homogeneous sphere isometry between Banach spaces E and F, where E is strictly convex. For any $u \in E$,

$$\|V(u)\| = \left\|V\left(\frac{u}{2}\right) - V\left(-\frac{u}{2}\right)\right\| = \left\|\frac{u}{2} - \left(-\frac{u}{2}\right)\right\| = \|u\|$$

and thus V preserves the sphere.

Conversely, suppose that (ii) holds and V is a surjective positive homogeneous sphere isometry between Banach spaces E and F and preserves the sphere, where E is strictly convex. For any $u \in E$, there exists $v \in E$ such that $V(v) = -V(u)$. Note that $\|u\| = \|V(u)\| = \|V(v)\| = \|v\|$ since V preserves the sphere. Then

$$\|u - v\| = \|V(u) - V(v)\| = \|2V(u)\| = \|2u\| = \|u\| + \|v\|$$

and so $u = -v$ since E is strictly convex. Since V is positive homogeneous, we see that V is real homogeneous. This completes the proof. □

3. Mazur–Ulam property of $E_1 \oplus_{\ell^1} E_2$

We first reproduce a lemma in [13] and give the proof.

LEMMA 3.1. *Let E and F be Banach spaces and let $V_0 : S_1(E) \to S_1(F)$ be a surjective isometry. Then*

$$\|u + v\| = 2 \Longleftrightarrow \|V_0(u) + V_0(v)\| = 2 \quad \forall u, v \in S_1(E).$$

PROOF. Note that V_0 is surjective. We only need to prove the '$\Longrightarrow$' part. By the Hahn–Banach theorem, there exists $f \in S(E^*)$ such that $f(u + v) = \|u + v\| = 2$. Then

$$2 = \|u + v\| = |f(u + v)| \le |f(u)| + |f(v)| \le 2$$

and thus

$$f(u) = f(v) = 1. \tag{3.1}$$

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For $n \in \mathbb{N}$, set $u_n = (1 - n^{-1})u + n^{-1}v$. By (3.1), we have $\{u_n\} \subseteq S_1(E)$. Let $n \in \mathbb{N}$ and $w \in S_1(E)$ and suppose that

$$\|u_n + w\| = 2. \tag{3.2}$$

By the Hahn–Banach theorem and a similar argument, there exists $f_{(n,w)} \in S(E^*)$ such that $f_{(n,w)}(u_n + w) = 2$, which implies that

$$f_{(n,w)}(w) = f_{(n,w)}(v) = f_{(n,w)}(u_n) = 1.$$

Therefore,

$$\|v + w\| = 2, \tag{3.3}$$

since $w = f_{(n,w)}(v + w) \le \|v + u\| \le 2$. Note that

$$\|u_n - V_0^{-1}(-V_0(u_n))\| = \|V_0(u_n) + V_0(u_n)\| = \|2V_0(u_n)\| = 2 \quad \forall n \in \mathbb{N}.$$

By a similar method to the one we used to deduce (3.3) from (3.2),

$$\|v - V^{-1}(-V(u_n))\| = 2 \quad \forall n \in \mathbb{N}$$

and thus

$$\|V_0(v) + V_0(u_n)\| = 2 \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ gives $\|V_0(v) + V_0(u)\| = 2$ and completes the proof. □

Now, we begin to consider the isometries between $S_1(E_1 \oplus_{\ell^1} E_2)$ and $S_1(F)$, where E_1 and E_2 are strictly convex. In the following result, we prove that any surjective isometry between $S_1(E_1 \oplus_{\ell^1} E_2)$ and $S_1(F)$ necessarily maps antipodal points to antipodal points.

Proposition 3.2. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$ is a surjective isometry. Then $V_0(-u) = -V_0(u)$ for any $u \in S_1(E_1 \oplus_{\ell^1} E_2)$.*

Proof. We first prove that $V_0(-\hat{x}) = -V_0(\hat{x})$ for any $x \in S(E_1)$. Since V_0 is surjective, there exists $u \in S_1(E_1 \oplus_{\ell^1} E_2)$ such that $V_0(u) = -V_0(\hat{x})$. Then

$$\|u_1 - x\| + \|u_2\| = \|u - \hat{x}\| = \|V_0(u) - V_0(\hat{x})\| = \|-2V_0(\hat{x})\| = 2$$

and thus $\|u_1 - x\| = \|u_1\| + \|x\|$. Since E_1 is strictly convex, $u_1 = -\|u_1\|x$. For any $y \in S(E_2)$,

$$\|V_0(\hat{y}) + V_0(u)\| = \|V_0(\hat{y}) - V_0(\hat{x})\| = \|\hat{y} - \hat{x}\| = 2.$$

By Lemma 3.1,

$$\|u_1\| + \|y + u_2\| = \|\hat{y} + u\| = 2$$

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and thus $\|y + u_2\| = \|y\| + \|u_2\|$. Since y is arbitrary, we have $u_2 = 0$ and thus $u = -\hat{x}$. Similarly, we can prove that $V_0(-\hat{y}) = -V_0(\hat{y})$ for any $y \in S(E_2)$.

Next we prove that $V_0(-u) = -V_0(u)$ for all $u \in S_1(E_1 \oplus_{\ell^1} E_2)$. We can assume $u_1, u_2 \neq 0$. Since V_0 is surjective, there is a $v \in S_1(E_1 \oplus_{\ell^1} E_2)$ such that $V_0(v) = -V_0(u)$. Then

$$\|u_1 - v_1\| + \|u_2 - v_2\| = \|u - v\| = \|V_0(u) - V_0(v)\| = \|2V_0(u)\| = 2$$

and thus

$$2 = \|u_1 - v_1\| + \|u_2 - v_2\| \leq \|u_1\| + \|v_1\| + \|u_2\| + \|v_2\| = \|u\| + \|v\| = 2.$$

It follows that $\|u_1 - v_1\| = \|u_1\| + \|v_1\|$. Since E_1 is strictly convex,

$$v_1 = -\frac{\|v_1\|}{\|u_1\|} u_1.$$

By the result of the previous part of this proof, for any $x \in S(E_1)$,

$$\|\hat{x} - u\| = \|V_0(\hat{x}) - V_0(u)\| = \|-V_0(-\hat{x}) + V_0(v)\| = \|\hat{x} + v\|.$$

Set $x = \|u_1\|^{-1} u_1$. Then

$$1 - \|u_1\| + \|u_2\| = \| \, \|u_1\|^{-1} u_1 + v_1\| + \|v_2\| = 1 - \|v_1\| + \|v_2\|,$$

since $v_1 = -(\|v_1\|/\|u_1\|)u_1$. Therefore, $\|u_1\| = \|v_1\|$ and $u_1 = -v_1$. We can prove that $u_2 = -v_2$ by a similar argument. This completes the proof. □

The following lemma is a special case of [5, Lemma 5].

Lemma 3.3. *Let F be a Banach space and $w_1, w_2 \in S_1(F)$. Suppose that $\|w_1 \pm w_2\| = 2$. Then*

$$\|\lambda w_1 + \mu w_2\| = |\lambda| + |\mu| \quad \forall \lambda, \mu \in \mathbb{R}.$$

Proof. Assume that $\lambda \neq 0$ and $\mu \neq 0$. Let $\theta_1 = \operatorname{sgn}(\lambda)$ and $\theta_2 = \operatorname{sgn}(\mu)$. By the Hahn–Banach theorem, there exists $f \in S_1(F^*)$ such that

$$\theta_1 f(w_1) + \theta_2 f(w_2) = f(\theta_1 w_1 + \theta_2 w_2) = \|\theta_1 w_1 + \theta_2 w_2\| = 2.$$

Since $|\theta_i f(w_i)| \leq 1$ for $i = 1, 2$, we see that $\theta_1 f(w_1) = \theta_2 f(w_2) = 1$ and so

$$\begin{aligned} |\lambda| + |\mu| &= |\lambda|\theta_1 f(w_1) + |\mu|\theta_2 f(w_2) = f(|\lambda|\theta_1 w_1 + |\mu|\theta_2 w_2) = f(\lambda w_1 + \mu w_2) \\ &\leq \|\lambda w_1 + \mu w_2\| \leq |\lambda| + |\mu|. \end{aligned}$$

The case $\lambda = 0$ or $\mu = 0$ is clear. This completes the proof. □

Proposition 3.4. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$ is a surjective isometry. If $x \in S_1(E_1), y \in S_1(E_2)$ and $\lambda, \mu \in \mathbb{R}$ with $|\lambda| + |\mu| = 1$, then $V_0(\lambda\hat{x} + \mu\hat{y}) = \lambda V_0(\hat{x}) + \mu V(\hat{y})$.*

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Proof. By Lemma 3.1, since V_0 is an isometry,

$$\|V_0(\hat{x}) \pm V_0(\hat{y})\| = \|\hat{x} \pm \hat{y}\| = 2.$$

By Lemma 3.3,

$$\|\lambda V_0(\hat{x}) + \mu V_0(\hat{y})\| = |\lambda| + |\mu| = 1.$$

Since V is surjective, there exists $u \in S_1(E_1 \oplus_{\ell^1} E_2)$ such that $V_0(u) = \lambda V_0(\hat{x}) + \mu V_0(\hat{y})$. Now we prove that $u = \lambda\hat{x} + \mu\hat{y}$.

The case $|\lambda| = 1$ is clear. After Proposition 3.2, we can assume that $0 < \lambda < 1$. Then

$$\begin{aligned}\|u_1 + x\| + \|u_2\| &= \|u + \hat{x}\| = \|V_0(u) + V_0(\hat{x})\| \\ &= \|(1+\lambda)V_0(\hat{x}) + \mu V_0(\hat{y})\| = 1 + \lambda + |\mu| = 2\end{aligned}$$

by Proposition 3.2 and Lemma 3.3. Therefore, $\|u_1 + x\| = \|u_1\| + \|x\|$. Since E_1 is strictly convex, we get $u_1 = \|u_1\|x$. It follows that

$$\begin{aligned}2 - 2\|u_1\| &= 1 - \|u_1\| + \|u_2\| = \|u_1 - x\| + \|u_2\| = \|u - \hat{x}\| = \|V_0(u) - V_0(\hat{x})\| \\ &= \|(\lambda - 1)V_0(\hat{x}) + \mu V_0(\hat{y})\| = 1 - \lambda + |\mu| = 2 - 2\lambda.\end{aligned}$$

Therefore, $\|u_1\| = \lambda$ and $u_1 = \lambda x$. Similarly, we can prove that $u_2 = \mu y$. This completes the proof. □

Remark 3.5. Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$ is a surjective isometry. For $i = 1, 2$, we can define $V_i : S_1(E_i) \to S_1(F)$ by

$$V_i(x) = V_0(\hat{x}) \quad \forall x \in S_1(E_i).$$

Then, by Proposition 3.4,

$$V_0(\lambda\hat{x} + \mu\hat{y}) = \lambda V_1(x) + \mu V_2(y)$$

for any $x \in S_1(E_1), y \in S_1(E_2)$ and $\lambda, \mu \in \mathbb{R}$ with $|\lambda| + |\mu| = 1$.

Proposition 3.6. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$ is a surjective isometry and that*

$$\mathbb{R} \cdot V_0(S_1(E_1)) \subseteq F, \quad \mathbb{R} \cdot V_0(S_1(E_2)) \subseteq F$$

are both subspaces. For $i = 1, 2$, the V_i defined in Remark 3.5 can be extended to a linear isometry on E_i.

Proof. Assume that $i = 1$. We can define $\tilde{V}_1 : E_1 \to F$ by

$$\tilde{V}_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ \|x\|V_1(\|x\|^{-1}x) & \text{if } x \neq 0. \end{cases}$$

It is clear that $\tilde{V}_1|_{S_1(E_1)} = V_1$. Now, we prove that

$$\|\tilde{V}_1(x_1) - \tilde{V}_1(x_2)\| \geq \|x_1 - x_2\| \quad \forall x_1, x_2 \in E_1. \tag{3.4}$$

From Proposition 3.2, $\tilde{V}_1(\lambda x) = \lambda \tilde{V}_1(x)$ for any $x \in E_1$ and $\lambda \in \mathbb{R}$. To prove (3.4), we only need to prove that it holds for any $x_1, x_2 \in B_1(E_1)$. For $y \in S_1(E_2)$, set $u := (x_1, (1 - \|x_1\|)y)$ and $v := (x_2, (1 - \|x_2\|)y)$. Note that $u, v \in S_1(E_1 \oplus_{\ell^1} E_2)$. From Proposition 3.4,

$$\begin{aligned}\|V_0(u) - V_0(v)\| &= \left\| \left(\|x_1\| V_1\Big(\frac{x_1}{\|x_1\|}\Big) - \|x_2\| V_1\Big(\frac{x_2}{\|x_2\|}\Big) \right) + (\|x_2\| - \|x_1\|) V_2(y) \right\| \\ &\le \left\| \|x_1\| V_1\Big(\frac{x_1}{\|x_1\|}\Big) - \|x_2\| V_1\Big(\frac{x_2}{\|x_2\|}\Big) \right\| + |\, \|x_2\| - \|x_1\| \,| \, \|V_2(y)\| \\ &= \|\tilde{V}_1(x_1) - \tilde{V}_1(x_2)\| + |\, \|x_2\| - \|x_1\| \,|\end{aligned}$$

and

$$\|u - v\| = \|x_1 - x_2\| + |\, \|x_2\| - \|x_1\| \,|.$$

This yields (3.4), since $\|V_0(u) - V_0(v)\| = \|u - v\|$.

Let $F_1 := \mathbb{R} \cdot V_0(S_1(E_1))$ be a subspace of F. Then V_1 can be seen as a surjective isometry between $S(E_1)$ and $S(F_1)$. Define $V_1^{-1} : S_1(F_1) \to S_1(E_1)$ as the inverse of V_1. By (3.4), for any $w_1, w_2 \in S(F_1)$ and $\lambda \in \mathbb{R}^+$,

$$\|V_1^{-1}(w_1) - \lambda V_1^{-1}(w_2)\| \le \|\tilde{V}_1(V_1^{-1}(w_1)) - \tilde{V}_1(\lambda V_1^{-1}(w_2))\| = \|w_1 - \lambda w_2\|.$$

Then V_1^{-1} has a linear isomeric extension W_1 from F_1 to E_1 by Lemma 1.3. It is clear that $W_1 \circ \tilde{V}_1(x) = x$ for any $x \in E_1$ and thus $\tilde{V}_1$ is also a linear isometry. We can prove the case $i = 2$ similarly. This completes the proof. □

THEOREM 3.7. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$ is a surjective isometry and*

$$\mathbb{R} \cdot V_0(S_1(E_1)) \subseteq F, \quad \mathbb{R} \cdot V_0(S_1(E_2)) \subseteq F$$

are both subspaces. Then V_0 can be extended to a linear isometry on the whole space.

PROOF. In Proposition 3.6, we have linear isometries $\tilde{V}_i : E_i \to F$ for $i = 1, 2$. Define

$$\tilde{V}_0 : E_1 \oplus_{\ell^1} E_2 \to F$$

by $\tilde{V}_0(u) = \tilde{V}_1(u_1) + \tilde{V}_2(u_2)$ for $u \in E_1 \oplus_{\ell^1} E_2$. By Proposition 3.4, $\tilde{V}_0|_{S_1(E_1 \oplus_{\ell^1} E_2)} = V_0$ and $\tilde{V}_0$ is linear. This completes the proof. □

4. Mazur–Ulam property of $E_1 \oplus_{\ell^\infty} E_2$

In this section, we begin to consider the isometries between $S_1(E_1 \oplus_{\ell^\infty} E_2)$ and $S_1(F)$, where E_1 and E_2 are strictly convex. In the following result, we prove that the surjective isometry between $S_1(E_1 \oplus_{\ell^\infty} E_2)$ and $S_1(F)$ necessarily maps antipodal points to antipodal points.

PROPOSITION 4.1. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^\infty} E_2) \to S_1(F)$ is an isometry and*

$$-V_0(S_1(E_1 \oplus_{\ell^\infty} E_2)) \subseteq V_0(S_1(E_1 \oplus_{\ell^\infty} E_2)).$$

Then $V_0(-u) = -V_0(u)$ for any $u \in S_1(E_1 \oplus_{\ell^\infty} E_2)$.

Proof. We first prove that $V_0(-\hat{x}) = -V_0(\hat{x})$ for any $x \in S_1(E_1)$. Note that

$$-V_0(S_1(E_1 \oplus_{\ell^\infty} E_2)) \subseteq V_0(S_1(E_1 \oplus_{\ell^\infty} E_2)).$$

There exists $u \in S_1(E_1 \oplus_{\ell^\infty} E_2)$ such that $V_0(u) = -V_0(\hat{x})$. Consequently,

$$\|u - \hat{x}\| = \|V_0(u) - V_0(\hat{x})\| = \|-2V_0(\hat{x})\| = 2$$

and thus $\|u_1 - x\| = 2$. Since E_1 is strictly convex, we see that $u_1 = -x$. For any $y \in S_1(E_2)$, there exists $v = (v_1, v_2) \in S_1(E_1 \oplus_{\ell^\infty} E_2)$ such that $V_0(v) = -V_0(\hat{y})$. By a similar argument, $v_2 = -y$. Consequently,

$$\|u - v\| = \|V_0(u) - V_0(v)\| = \|-V_0(\hat{x}) + V_0(\hat{y})\| = \|-\hat{x} + \hat{y}\| = 1$$

and thus $\|u_2 + y\| = \|u_2 - v_2\| \le 1$. Since y is arbitrary, we get $u_2 = 0$. Therefore, we have $u = -\hat{x}$ and $V_0(-\hat{x}) = -V_0(\hat{x})$. We can prove that $V_0(-\hat{y}) = -V_0(\hat{y})$ for any $y \in S_1(E_2)$ by a similar argument.

Now we prove that $V_0(-u) = -V_0(u)$ for any $u \in S_1(E_1 \oplus_{\ell^\infty} E_2)$. We can assume that $u_1, u_2 \neq 0$. Since

$$-V_0(S_1(E_1 \oplus_{\ell^\infty} E_2)) \subseteq V_0(S_1(E_1 \oplus_{\ell^\infty} E_2)),$$

there exists $v = (v_1, v_2) \in S_1(E_1 \oplus_{\ell^\infty} E_2)$ such that $V_0(v) = -V_0(u)$. It is clear that $v_1, v_2 \neq 0$. Otherwise, $u_1 = 0$ or $u_2 = 0$ by the result of the previous part of this proof. Then, for any $x \in S(E_1)$,

$$\|\hat{x} - u\| = \|V_0(\hat{x}) - V_0(u)\| = \|-V_0(-\hat{x}) + V_0(v)\| = \|\hat{x} + v\|.$$

Let $x = \|v_1\|^{-1} v_1$. Then

$$1 + \|v_1\| = \left\| \frac{v_1}{\|v_1\|} - u_1 \right\| \le 1 + \|u_1\| \tag{4.1}$$

and thus $\|v_1\| \le \|u_1\|$. Similarly, if we let $x = -u_1/\|u_1\|$,

$$1 + \|u_1\| = \left\| -\frac{u_1}{\|u_1\|} + v_1 \right\| \le 1 + \|v_1\|$$

and thus $\|u_1\| \le \|v_1\|$. Therefore, $\|u_1\| = \|v_1\|$. From (4.1), since E_1 is strictly convex,

$$1 + \|u_1\| = \left\| \frac{v_1}{\|v_1\|} - u_1 \right\|$$

and thus $u_1 = -v_1$. Similarly, we can prove that $u_2 = -v_2$ to complete the proof. □

The following lemma is a special case of [3, Lemma 2].

Lemma 4.2. *Let F be a Banach space and $w_1, w_2 \in S_1(F)$ with $\|w_1 \pm w_2\| = 1$. Then*

$$\|\lambda w_1 + \mu w_2\| = \max\{|\lambda|, |\mu|\} \quad \forall \lambda, \mu \in \mathbb{R}.$$

Proof. Assume that $\lambda \neq 0$ and $|\lambda| \ge |\mu|$. Since $\|w_1 \pm w_2\| = 1$, by the Hahn–Banach theorem, there exists $f \in S_1(F^*)$ such that $f(w_1) = \|w_1\| = 1$ and $f(w_2) = 0$. Then

$$\max\{|\lambda|, |\mu|\} = |\lambda| = |f(\lambda w_1 + \mu w_2)| \le \|\lambda w_1 + \mu w_2\|.$$

By the Hahn–Banach theorem again, there exists $g \in S_1(F^*)$ such that

$$\begin{aligned}\|\lambda w_1 + \mu w_2\| &= |g(\lambda w_1 + \mu w_2)| = |\lambda g(w_1) + \mu g(w_2)| \\ &\le \max\{|\lambda|, |\mu|\} \cdot \max\{|g(w_1 + w_2)|, |g(w_1 - w_2)|\} \\ &\le \max\{|\lambda|, |\mu|\} \cdot \max\{\|w_1 + w_2\|, w_1 - w_2\|\} = \max\{|\lambda|, |\mu|\}.\end{aligned}$$

We can similarly handle the case $|\lambda| < |\mu|$. This completes the proof. □

Proposition 4.3. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^\infty} E_2) \to S_1(F)$ is a surjective isometry. If $x \in S(E_1)$, $y \in S(E_2)$ and $\lambda, \mu \in \mathbb{R}$ with $\max\{|\lambda|, |\mu|\} = 1$, then $V_0(\lambda\hat{x} + \mu\hat{y}) = \lambda V_0(\hat{x}) + \mu V_0(\hat{y})$.*

Proof. Since V_0 is an isometry, by Proposition 4.1,

$$\|V_0(\hat{x}) \pm V_0(\hat{y})\| = \|\hat{x} \pm \hat{y}\| = 1.$$

By Lemma 3.1,

$$\|\lambda V_0(\hat{x}) + \mu V_0(\hat{y})\| = \max\{|\lambda|, |\mu|\} = 1.$$

Since V_0 is surjective, there is $u \in S_1(E_1 \oplus_{\ell^\infty} E_2)$ such that $V_0(u) = \lambda V_0(\hat{x}) + \mu V_0(\hat{y})$. Now we prove that $u = \lambda\hat{x} + \mu\hat{y}$.

The case $\lambda = 0$ or $\mu = 0$ is clear. After Proposition 4.1, we can assume that $\lambda = 1$ and $\mu \neq 0$. Then

$$\|u + \hat{x}\| = \|V_0(u) + V_0(\hat{x})\| = \|2V_0(\hat{x}) + \mu V_0(\hat{y})\| = 2$$

by Proposition 4.1 and Lemma 3.1. Therefore, we have $\|u_1 + x\| = 2$. Since E_1 is strictly convex, $u_1 = x$. It is clear that

$$\|u_2\| = \|u - \hat{x}\| = \|V_0(u) - V_0(\hat{x})\| = \|\mu V_0(\hat{y})\| = |\mu|.$$

Let $\theta = \operatorname{sgn}(\mu)$. It is clear that $|\mu + \theta| > 1$ and so

$$\|u + \theta\hat{y}\| = \|V_0(u) + \theta V_0(\hat{y})\| = \|V_0(\hat{x}) + (\mu + \theta)V_0(\hat{y})\| = |\mu + \theta|,$$

by Lemma 3.1, and

$$\|u_2 + \theta y\| = |\mu + \theta| = |\mu| + 1.$$

Since $\|u_2\| = |\mu|$ and E_2 is strictly convex, $u_2 = \theta|\mu|y = \mu y$. Then $u = \hat{x} + \mu\hat{y}$, which completes the proof. □

Remark 4.4. Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^\infty} E_2) \to S_1(F)$ is a surjective isometry. For $i = 1, 2$, define $V_i : S_1(E_i) \to S_1(F)$ by

$$V_i(x) = V_0(\hat{x}), \quad \forall x \in S_1(E_i).$$

Then, by Proposition 4.3,

$$V_0(\lambda\hat{x} + \mu\hat{y}) = \lambda V_1(x) + \mu V_2(y)$$

for any $x \in S_1(E_1), y \in S_1(E_2)$ and $\lambda, \mu \in \mathbb{R}$ with $\max\{|\lambda|, |\mu|\} = 1$.

PROPOSITION 4.5. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^\infty} E_2) \to S_1(F)$ is a surjective isometry and*

$$\mathbb{R} \cdot V_0(S_1(E_1)) \subseteq F, \quad \mathbb{R} \cdot V_0(S_1(E_2)) \subseteq F$$

are both subspaces. For $i = 1, 2$, the V_i defined in Remark 4.4 can be extended to a linear isometry on E_i.

PROOF. Assume that $i = 1$. We can define $\tilde{V}_1 : E_1 \to F$ by

$$\tilde{V}_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ \|x\| V_1(\|x\|^{-1} x) & \text{if } x \neq 0. \end{cases}$$

We first prove that $\tilde{V}_1$ is an isometry. From Proposition 4.1,

$$\tilde{V}_1(\lambda x) = \lambda \tilde{V}_1(x) \quad \forall \lambda \in \mathbb{R}$$

and $\tilde{V}_1|_{S_1(E_1)} = V_1$. Therefore, we only need to prove that $\|\tilde{V}_1(x_1) - \tilde{V}_1(x_2)\| = \|x_1 - x_2\|$ for any $x_1, x_2 \in B_1(E_1)$. In fact, for any $y \in S_1(E_2)$, write $u := (x_1, y)$ and $v := (x_2, y)$. Since $u, v \in S_1(E_1 \oplus_{\ell^\infty} E_2)$, by Proposition 4.3,

$$\begin{aligned} \|\tilde{V}_1(x_1) - \tilde{V}_1(x_2)\| &= \left\| \|x_1\| V_1\left(\frac{x_1}{\|x_1\|}\right) - \|x_2\| V_1\left(\frac{x_2}{\|x_2\|}\right) \right\| \\ &= \|V_0(u) - V_0(v)\| = \|u - v\| = \|x_1 - x_2\|. \end{aligned}$$

By hypothesis, $F_1 := \mathbb{R} \cdot V_0(S_1(E_1))$ is a subspace of F. So, $\tilde{V}_1$ can be seen as a surjective isometry between E_1 and F_1. By Theorem 1.2, $\tilde{V}_1$ is linear. The case $i = 2$ is similar. □

THEOREM 4.6. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V_0 : S_1(E_1 \oplus_{\ell^\infty} E_2) \to S_1(F)$ is a surjective isometry and*

$$\mathbb{R} \cdot V_0(S_1(E_1)) \subseteq F, \quad \mathbb{R} \cdot V_0(S_1(E_2)) \subseteq F$$

are both subspaces. Then V_0 can be extended to a linear isometry on the whole space.

PROOF. In Proposition 4.5, we have defined linear isometries $\tilde{V}_i : E_i \to F$ for $i = 1, 2$. Then we can define

$$\tilde{V}_0 : E_1 \oplus_{\ell^\infty} E_2 \to F$$

by $\tilde{V}_0(u) = \tilde{V}_1(u_1) + \tilde{V}_2(u_2)$ for any $u \in E_1 \oplus_{\ell^\infty} E_2$. From Proposition 4.5, it is clear that $\tilde{V}_0|_{S_1(E_1 \oplus_{\ell^\infty} E_2)} = V_0$ and $\tilde{V}_0$ is linear. This completes the proof. □

5. Main results

In this section, we prove the main results, which give necessary and sufficient conditions under which the ℓ^1-sum and the ℓ^∞-sum of two strictly convex Banach spaces admit the Mazur–Ulam property.

Theorem 5.1. *Let E_1 and E_2 be strictly convex Banach spaces and let F be a Banach space. Suppose that $V : E_1 \oplus_{\ell^1} E_2 \to F$ is a surjective positive homogeneous sphere isometry and preserves the sphere. If V preserves the subspaces E_1 and E_2, then V is linear. The same result holds if we replace ℓ^1 by ℓ^∞.*

Proof. We prove the assertion for ℓ^1. Let

$$V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$$

be the restriction of V to $S_1(E_1 \oplus_{\ell^1} E_2)$. Since V preserves the subspaces E_1 and E_2,

$$\mathbb{R} \cdot V_0(S_1(E_1)) = V(E_1) \subseteq F, \quad \mathbb{R} \cdot V_0(S_1(E_2)) = V(E_2) \subseteq F$$

are both subspaces. The desired result follows from Theorem 3.7. The proof of the second part is similar by means of Theorem 4.6. □

The following result is a straightforward deduction from Theorems 2.1 and 5.1.

Theorem 5.2. *Let E_1 and E_2 be strictly convex Banach spaces. The following assertions are equivalent.*

(i) *$E_1 \oplus_{\ell^1} E_2$ admits the Mazur–Ulam property.*

(ii) *Suppose that F is a Banach space and $V : E_1 \oplus_{\ell^1} E_2 \to F$ is a surjective positive homogeneous sphere isometry and preserves the sphere. Then V preserves the subspaces E_1 and E_2.*

(iii) *Suppose that F is a Banach space and $V_0 : S_1(E_1 \oplus_{\ell^1} E_2) \to S_1(F)$ is a surjective isometry. Then*

$$\mathbb{R} \cdot V_0(S_1(E_1)) \subseteq F, \quad \mathbb{R} \cdot V_0(S_1(E_2)) \subseteq F$$

are both subspaces.

Moreover, the assertions (i′), (ii′) and (iii′) obtained by replacing ℓ^1 by ℓ^∞ in (i), (ii) and (iii) are also equivalent.

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