J. Fluid Mech. (1997), *vol.* 339, *pp.* 261–274. Printed in the United Kingdom

© 1997 Cambridge University Press

On the inviscid stability of parallel bubbly flows

By LUCA D'AGOSTINO[1], FABRIZIO D'AURIA[1] AND CHRISTOPHER E. BRENNEN[2]

[1]Università degli studi di Pisa, Pisa 56125, Italy

[2]California Institute of Technology, Pasadena, CA 91125, USA

(Received 20 October 1995 and in revised form 9 January 1997)

This paper investigates the effects of bubble dynamics on the stability of parallel bubbly flows of low void fraction. The equations of motion for the bubbly mixture are linearized for small perturbations and the parallel flow assumption is used to obtain a modified Rayleigh equation governing the inviscid stability problem. This is then used for the stability analysis of two-dimensional shear layers, jets and wakes. Inertial effects associated with the bubble response and energy dissipation due to the viscosity of the liquid, the heat transfer between the two phases, and the liquid compressibility are included. Numerical solutions of the eigenvalue problems for the modified Rayleigh equation are obtained by means of a multiple shooting method. Depending on the characteristic velocities of the various flows, the void fraction, and the ambient pressure, the presence of air bubbles can induce significant departures from the classical stability results for a single-phase fluid.

1. Introduction

The stability of parallel single-phase flows (both incompressible and compressible) has been the object of intensive and well-documented research in a wide variety of configurations (Betchov & Criminale 1967; Schlichting 1968; Drazin & Reid 1981; White 1991) including shear layers, boundary layers, jets, wakes, and internal flows. The stability characteristics of two-dimensional shear layers, jets and wakes having different velocity profiles have been studied by Michalke (1965), Drazin & Howard (1966), Betchov & Criminale (1966), Sato (1960), Sato & Kuriki (1961), and Tatsumi & Kakutani (1958) among others. An excellent review of stability analyses of compressible flows is given by Mack (1987). The work of Blumen (1970), Blumen, Drazin & Billings (1975), Drazin & Davey (1977), and Lees & Reshotko (1962) is also relevant to the present context. The corresponding literature for two-phase flows is much more limited. Only during the last decade or so have investigations of the stability of two-phase flows begun to be made. Most of this recent work has focused on liquid–solid suspensions (Herbolzeimer 1983; Shaqfeh & Acrivos 1986; Yang *et al.* 1990).

The aim of the present paper is to investigate the inviscid stability of bubbly parallel flows by including the effects associated with the dynamic response of the bubbles. The reviews of Prosperetti (1982) and van Wijngaarden (1972) provide a good summary of bubble dynamics effects in other contexts. Even at very low void fractions, the dynamic properties of the liquid, such as the acoustic speed, are dramatically modified by the presence of bubbles dispersed in the liquid (d'Agostino & Brennen 1983, 1988, 1989). The interactions occurring between the mean flow

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

and the compliant, inertial and dissipative nature of the bubble dynamics can be rather complex (Brennen 1995; Plesset & Prosperetti 1977). Inertial effects in the bubble dynamics become particularly important when the unstable frequencies of the flow approach the natural frequency of oscillation of individual bubbles. Then the bubbly mixture no longer behaves as a compressible barotropic fluid and significant deviations from the classical compressible flow solution are to be expected (d'Agostino & Brennen 1988; d'Auria, d'Agostino & Brennen 1994). Seeking a more detailed understanding of the phenomena associated with the stability of parallel bubbly flows, the authors (d'Agostino, d'Auria & Brennen 1995; d'Auria, d'Agostino & Brennen 1995) have already reported on some substantial effects due the presence of a dispersed phase in an unbounded shear layer and a Bickley jet.

In the present paper the linearized perturbation equations for a bubbly mixture are applied to a study of the inviscid stability of two-dimensional shear layers and jets. The computed eigenvalues exhibit significant deviations from the single-phase incompressible flow solutions for the flow parameters typical of bubbly flows with pressures below atmospheric pressure.

2. Basic equations

The basic equations employed are identical to those used by d'Agostino & Brennen (1983, 1989) so only a brief review will be included here. Neglecting the mass of the bubbles and their relative motion with respect to the surrounding liquid (d'Agostino, Brennen & Acosta 1988), the continuity equation for the mixture is written as

$$\nabla\cdot\boldsymbol{u} = \frac{1}{1+\beta\tau}\frac{\mathrm{D}(\beta\tau)}{\mathrm{D}t} - \frac{1}{\rho a^2}\frac{\mathrm{D}p}{\mathrm{D}t} \tag{2.1}$$

where $\boldsymbol{u}$ is the flow velocity, p, ρ, and a are respectively the pressure, density and sound speed of the liquid, β is the bubble concentration per unit liquid volume, $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\nabla$ is the Lagrangian time derivative, and $\tau = 4\pi R^3/3$ is the volume of a bubble, assumed spherical with radius $R(\boldsymbol{x}, t)$. The void fraction, α, is assumed to be very small compared with unity, so that only terms of $O(\alpha)$ are retained. Neglecting relative motion between the two phases and assuming a uniform initial population, it follows that β is constant in the bubbly fluid. Moreover, in the absence of body forces and viscous effects in the large-scale flow, the momentum equation for the fluid motion becomes

$$\rho(1-\alpha)\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p. \tag{2.2}$$

The relation between the pressure and the bubble radius is determined by the Rayleigh–Plesset equation modified as indicated by Prosperetti (1984) to account for the effects of liquid compressibility:

$$\left(1-\frac{1}{a}\frac{\mathrm{D}R}{\mathrm{D}t}\right)R\frac{\mathrm{D}^2R}{\mathrm{D}^2t} + \frac{3}{2}\left(\frac{\mathrm{D}R^2}{\mathrm{D}t}\right)^2\left(1-\frac{1}{3a}\frac{\mathrm{D}R}{\mathrm{D}t}\right)$$
$$= \left(1+\frac{1}{a}\frac{\mathrm{D}R}{\mathrm{D}t}\right)\frac{p_R(t)+p(t+R/a)}{\rho} + \frac{R}{\rho a}\frac{\mathrm{d}p_R(t)}{\mathrm{d}t}. \tag{2.3}$$

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

In (2.3) $p_R(t)$ is the liquid pressure at the bubble surface. The bubble internal pressure, p_B, is assumed uniform and is related to $p_R(t)$ by

$$p_B(t) = p_R(t) + \frac{2S}{R} + 4\mu\frac{\mathrm{D}R}{\mathrm{D}t} \tag{2.4}$$

where S is the surface tension of the bubble surface and μ is the viscosity of the liquid. The closure of the problem requires that (2.1), (2.2), (2.3), and (2.4) be supplemented by the mechanical and thermal equations of state and by the energy conservation equations for the two phases with the relevant boundary conditions.

3. Linear stability equations

The equations governing the stability of this two-dimensional parallel flow are similar to those derived by d'Agostino *et al.* (1995), and d'Auria *et al.* (1995). A justification of the use of the parallel flow approximation for unbounded flows is beyond the scope of this paper: we note, however, that high Reynolds numbers are a necessary condition for this approximation to be meaningful (Drazin & Reid 1981). This condition is clearly satisfied in the inviscid formulation.

The continuity and momentum equations of the two phases are perturbed around their mean values by small linear fluctuations denoted by a hat accent:

$$u = U(y) + \hat{u}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \quad v = \hat{v}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

$$p = p_0 + \hat{p}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \quad R = R_0 + \hat{R}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

where, with standard notations, x, y, and u, v are the coordinates and velocity components in the streamwise and normal directions, while ω and k are the perturbation frequency and wavenumber, respectively. Then, in the limit of small void fraction, linearization of the continuity and momentum equations to the first perturbation order produces

$$\mathrm{i}k\hat{u} + v' = -\mathrm{i}\omega_L\frac{3\alpha}{R_0}\hat{R} + \mathrm{i}\omega_L\frac{1}{\rho a^2}\hat{p}, \tag{3.1}$$

$$\rho(1-\alpha)(-\mathrm{i}\omega_L\hat{u} + U'\hat{v}) = -\mathrm{i}k\hat{p}, \tag{3.2}$$

$$\rho(1-\alpha)\mathrm{i}\omega_L\hat{v} = \hat{p}', \tag{3.3}$$

where primes indicate differentiation with respect to the independent variable y. The quantity $\omega_L = \omega - kU$ is the Lagrangian frequency experienced by the bubbles in their motion with the mean flow.

To the same order of approximation, the bubble dynamic equations yield the following relation (Prosperetti 1984):

$$(-\omega_L^2 - \mathrm{i}\omega_L\lambda + \omega_B^2)\hat{R} = -\left(1 + \mathrm{i}\omega_L\frac{R_0}{a}\right)\frac{\hat{p}}{\rho R_0} \tag{3.4}$$

where $\omega_B(\omega_L)$ and $\lambda(\omega_L)$ are the natural frequency and damping coefficient of individual bubbles excited at frequency ω_L in an unbounded liquid. The three terms of the left-hand side respectively account for inertial, damping and compressibility effects associated with the bubble dynamic response. The physical origin and functional dependence of ω_B and λ are illustrated in more detail in d'Agostino & Brennen (1989), and Brennen (1995). Here we only mention that the damping coefficient, λ, is given by the sum of three terms accounting for the viscous, acoustical, and thermal dissipation.

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

The equivalent of a Rayleigh equation governing the inviscid stability of a bubbly flow is given by the following system, obtained by elimination of $\hat{R}$ and $\hat{p}$ from (3.1), (3.2), (3.3), and (3.4) to yield

$$\hat{u}' = \mathrm{i}k\hat{v} - \mathrm{i}\frac{U''}{\omega_L}\hat{v} - \mathrm{i}\frac{U'}{ka_M^2}(\mathrm{i}\omega_L\hat{u} - U'\hat{v}), \tag{3.5}$$

$$\hat{v}' = -\mathrm{i}k\hat{u} + \frac{\omega_L}{ka_M^2}(\mathrm{i}\omega_L\hat{u} - U'\hat{v}), \tag{3.6}$$

where $a_M = a_M(\omega_L)$ is the complex and dispersive (frequency-dependent) speed of propagation of a harmonic disturbance of angular frequency, ω_L, in the bubbly mixture. This is determined by the dispersion relation

$$\frac{1}{a_M^2} = \frac{\omega_{B0}^2}{a_{M0}^2}\frac{1 + \mathrm{i}\omega R_0/a}{\omega_B^2 - \omega^2 - \mathrm{i}\omega 2\lambda} + \frac{1-\alpha}{a^2} \tag{3.7}$$

where

$$\omega_{B0}^2 = \frac{3p_{B0}}{\rho R_0^2} - \frac{2S}{\rho R_0^3} \quad \text{and} \quad a_{M0}^2 = \frac{\omega_{B0}^2 R_0^2}{3\alpha(1-\alpha)} \tag{3.8}$$

are respectively the natural frequency of oscillation of a single bubble under isothermal conditions and the low-frequency sound speed in a free bubbly flow with incompressible liquid ($\omega_L \to 0$ and $a \to \infty$). Notice that by eliminating the velocity component $\hat{u}$ from (3.5) and (3.6) and setting $a_M \to \infty$ we recover the classical Rayleigh stability equation for a single phase fluid.

For the mathematical problem to be well posed, (3.5) and (3.6) must be supplemented by two appropriate boundary conditions on $\hat{u}$, $\hat{v}$ and their first derivatives for the specific flow configuration under consideration. As will be seen later, this leads to a linear second-order eigenvalue problem for the free parameters ω or k. As in the single-phase formulation, the set of admissible (generally complex) values of ω or k (the eigenvalues) is uniquely determined by the condition that the corresponding non-trivial solutions (the eigenfunctions) satisfy the boundary conditions. Any two of the real and imaginary parts of the complex frequency and wavenumber can be specified; the remaining parts are then determined. Spatially growing oscillations are studied by assigning a real frequency, ω, and solving for the complex wavenumber, $k = k_r + \mathrm{i}k_i$, which is the eigenvalue of the problem; k_i, the imaginary part of k, is the spatial attenuation rate of the perturbation, while $2\pi/k_r$ is its wavelength. A negative value of k_i implies therefore amplification of the perturbation. On the other hand, temporally growing oscillations may be studied by assigning a real value to k and solving for the complex ω. The two cases become identical at neutral stability.

4. Inviscid shear layer

We first consider the simple classical case of a two-dimensional inviscid bubbly free shear layer of thickness δ between two parallel single-phase streams of velocities $U_1 (y < 0)$ and $U_2 (y > 0)$. It is assumed that the unperturbed velocity profile can be approximated by the hyperbolic tangent profile:

$$U(y) = \frac{U_1 + U_2}{2} + \frac{U_2 - U_1}{2}\tanh\left(\frac{y}{\delta}\right).$$

Inside the shear layer the equations governing the perturbations must, in general, be integrated numerically. Outside the shear layer, where U, ω_L, and a_M are constant,

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

the perturbation equations reduce to

$$\hat{u}' = \mathrm{i}k\hat{v}' \quad \text{and} \quad \hat{v}' = -\mathrm{i}k\hat{u} + \mathrm{i}\frac{\omega_L^2}{ka_M^2}\hat{u}$$

or

$$\hat{v}'' + \left(-k^2 + \frac{\omega_L^2}{ka_M^2}\right)\hat{v}.$$

The above can be integrated in closed form to obtain

$$\hat{v} = A_{1,2}\mathrm{e}^{\pm y(k^2-\omega_L^2/a_M^2)^{1/2}}$$

$$\hat{u} = \pm A_{1,2}\frac{\mathrm{i}k}{(k^2-\omega_L^2/a_M^2)^{1/2}}\mathrm{e}^{\pm y(k^2-\omega_L^2/a_M)^{1/2}},$$

where $A_{1,2}$ are arbitrary complex constants (the principal branch of the square root is implied) and the appropriate sign is determined by requiring that the solution does not diverge as $y \to \pm\infty$. From a practical standpoint, the stability problem is first transformed into a boundary value problem by assigning zero derivatives to the eigenvalues with respect to y within the integration range. For the solution to be accurate, the integration must start and end far away from the shear layer upper and lower boundaries ($|y| \gg \delta$). The boundary value problem is then solved numerically by using a multiple shooting method (Stoer & Bulirsch 1980). The integration is carried out with a fourth-order Runge–Kutta method (extrapolated to the fifth order), with self-adaptive step-size in order to obtain the required accuracy. Finally, the eigenvalues are corrected using a multi-dimensional modified Newton–Raphson method, in order to improve the convergence of the algorithm. The code has been validated against the results reported for the single-phase flow by Betchov & Criminale (1967) and Michalke (1965) for both the spatially and temporally growing oscillations for the single-phase flow case.

In the case of spatial stability calculations, the computation starts with some tentative candidate for the complex eigenvalue, k. The arbitrary constant A_1 is chosen to give the simple initial conditions

$$\hat{u} = \frac{\mathrm{i}k}{(k^2-\omega_L^2/a_M^2)^{1/2}}$$

and

$$\hat{v} = 1$$

at $y = -n\delta$ ($n \gg 1$). The equations are integrated up to $y = n\delta$, where the computed values of $\hat{u}$ and $\hat{v}$ must be continuous with the upper asymptotic solution. Therefore, at $y = n\delta$, the condition

$$\hat{u} = -\frac{\mathrm{i}k}{(k^2-\omega_L^2/a_M^2)^{1/2}}\hat{v}$$

must be satisfied. This relation is used to iteratively correct the assumed complex eigenvalue and the process is repeated to convergence.

5. Inviscid jets and wakes

In this section we briefly describe results for the stability of inviscid jets and wakes. In order to do so we choose to study the Bickley jet (hyperbolic secant) profile since

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

it exemplifies the stability of a symmetric two-dimensional jet with a single velocity maximum, U_{max} (Drazin & Reid 1981). It is assumed that the unperturbed velocity profile can be approximated by the hyperbolic secant profile:

$$U(y) = U_{max} \cosh^{-2}\left(y/\delta\right), \quad -\infty < y < \infty.$$

It is well known (Betchov & Criminale 1966, 1967; Drazin & Howard 1966; Drazin & Reid 1981) that this profile, when subject to either spatial or temporal excitation, gives rise to two distinct modes of oscillation corresponding to symmetric ('sinuous') and antisymmetric ('varicose') eigenfunctions.

For a bubbly jet surrounded by a single-phase fluid the equations governing the perturbations have to be integrated numerically. The numerical procedure is identical to the one described in the previous section. The computations for a single-phase flow have been validated against the results reported by Betchov & Criminale (1966) and Drazin & Howard (1966), for both the spatially and temporally growing oscillations, and for the symmetric and antisymmetric modes of oscillation of the jet.

The basic physical trends observed in the analysis of jets can be extended to the case of wakes. In the case of a timewise stability analysis (where k is real and ω is complex) a constant mean velocity can be superposed on a jet flow leaving the eigenvalues unaltered, because the problem is invariant to a Galilean transformation. This means that the results obtained for jets are immediately applicable to wakes (Betchov & Criminale 1967). For the spacewise stability analysis different considerations are in order. The equivalence of the spatial and temporal analysis of wakes, proved by Mattingly (1968) (see Wazzan 1975), must be invoked. The extension of the results of spatial stability analysis of jets to spatially growing disturbances in wakes can then be argued.

6. Results and discussion

For illustrative purposes, we choose to study the spatial stability characteristics of parallel flows involving a mixture of air bubbles ($\gamma = 1.4$) and water ($\rho = 1485\,\mathrm{Kg\,m^{-3}}$, $\mu = 0.001\ \mathrm{N\,s\,m^{-2}}$, $S = 0.0728\,\mathrm{N\,m^{-1}}$). All quantities are expressed in dimensionless form (denoted by an asterisk) using the shear layer velocity difference, ΔU, or the jet maximum velocity, U_{max}, and the typical width of the flow (either the layer or the jet), δ, as reference velocity and length. The three non-dimensional parameters in the present analysis are the bubble natural frequency, ω^*_{B0}, the bubble radius, R^*, and the void fraction, α.

The effect of the bubbles on the stability characteristics of the hyperbolic tangent shear layer is illustrated in figure 1(a), while figures 1(b) and 1(c) are relevant to the Bickley jet, for the sinuous and varicose modes of oscillation, respectively. The graphs show that the presence of bubbles causes substantial modifications of the stability characteristics of the flow when the bubble natural frequency, ω^*_{B0}, roughly decreases below the value $\omega^*_{B0} \leqslant 20 \rightarrow 30$, depending on the velocity profile under consideration. It is readily seen that the bubbles have a stabilizing effect. This effect gets stronger as the bubble natural frequency, ω^*_{B0}, decreases. In the barotropic limit for $\omega^*_{B0} \gg \omega^*$ (i.e. $\omega^*_{B0} \geqslant 20 \rightarrow 30$) the effect of the presence of bubbles in the liquid has been found to be negligible: within the limit $\alpha \ll 1$, the void fraction (and therefore the compressibility of the bubbly mixture) has also been varied to see if it would affect the stability characteristics of flows with $\omega^*_{B0} \gg \omega^*$, but it did not (the results of these computations are not reported here). The most unstable frequency, that is the frequency corresponding to the maximum amplification

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

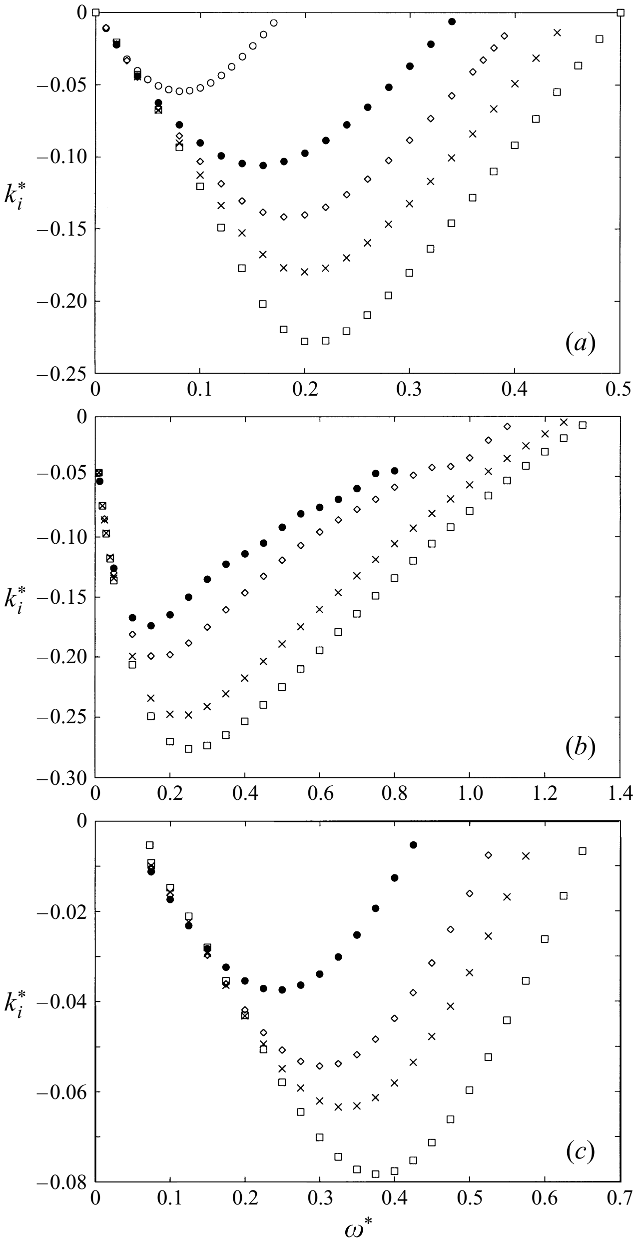

FIGURE 1. (*a*) The attenuation rate, k_i^*, of the hyperbolic tangent shear layer as a function of the excitation frequency, ω^*, for several values of the bubble resonance frequency: $\omega_{B0}^* = 21(\times)$, 15 ($\diamond$), 12 ($\bullet$), and 9 ($\circ$). In all cases $\alpha = 0.01$ and $R^* = 0.01$. The incompressible flow solution ($\alpha = 0$) is also shown for comparison ($\square$). (*b,c*) As (*a*) but for the Bickley jet: (*b*) $\omega_{B0}^* = 17.5(\times)$, 9.5 ($\diamond$), 7.5 ($\bullet$); (*c*) $\omega_{B0}^* = 30(\times)$, 22 ($\diamond$), 17 ($\bullet$).

rate (denoted here by $k_{i_{MIN}}^*$) also exhibits a shift towards smaller values (the most unstable frequency computed by Michalke (1965) for the single-phase shear layer is $\omega^* = 0.207$). Notice from the difference in the vertical scales in figures 1(*b*) and 1(*c*), that the symmetric (sinuous) mode is less stable than the antisymmetric

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

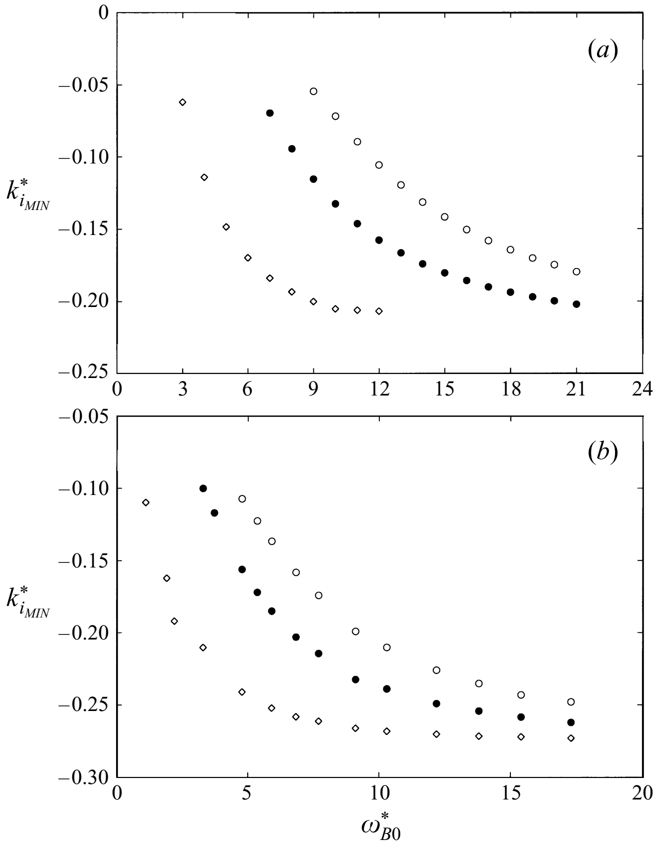

FIGURE 2. The most unstable attenuation rate, $k^*_{i_{MIN}}$, for (*a*) the hyperbolic tangent shear layer, and (*b*) symmetric oscillations of the Bickley jet, as a function of the bubble natural frequency, ω^*_{B0}, for three values of the void fraction: $\alpha = 0.001$ (⋄), 0.005 (•), and 0.01 (○). The bubble size is $R^* = 0.01$ for all cases.

(varicose) mode. In the following we shall therefore focus on the symmetric or sinuous mode (characterized by even eigenfunctions) of the jet, since it is the most unstable.

The effects of changing the bubble radius, R^*, and the void fraction, α, have also been investigated. The stabilizing effect due to an increase in the void fraction is illustrated in figures 2(*a*) and 2(*b*), for the shear layer and the jet, respectively. Here the maximum amplification rate (the minimum of k^*_i, denoted by $k^*_{i_{MIN}}$) is plotted as a function of the natural frequency of the bubble, ω^*_{B0}. The similar effect of a decrease of the bubble radius is shown in figures 3(*a*) and 3(*b*) for both flow configurations. It is worth noting that the ratio $3\alpha(1-\alpha)/R^{*2}$ coincides with the bubble cloud parameter $3\alpha(1-\alpha)/R_0^2$ originally identified by d'Agostino & Brennen (1983, 1988, 1989) in the dynamic analysis of bubbly flows. Hence, it can be stated in more general terms that the inviscid stability of parallel bubbly flows increases with $3\alpha(1-\alpha)/R_0^2$.

Figures 4(*a*) and 4(*b*) show how the most unstable frequency, ω^*_m, increases with the bubble natural frequency, ω^*_{B0}, for different values of the void fraction.

The effect of the bubble natural frequency, ω^*_{B0}, on the real phase velocity, $c^*_r = \omega^*/k^*_r$, is illustrated in figures 5(*a*) and 5(*b*). A comparison of the results for the shear layer and the jet reveals that the jet exhibits a larger shift in the most unstable

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

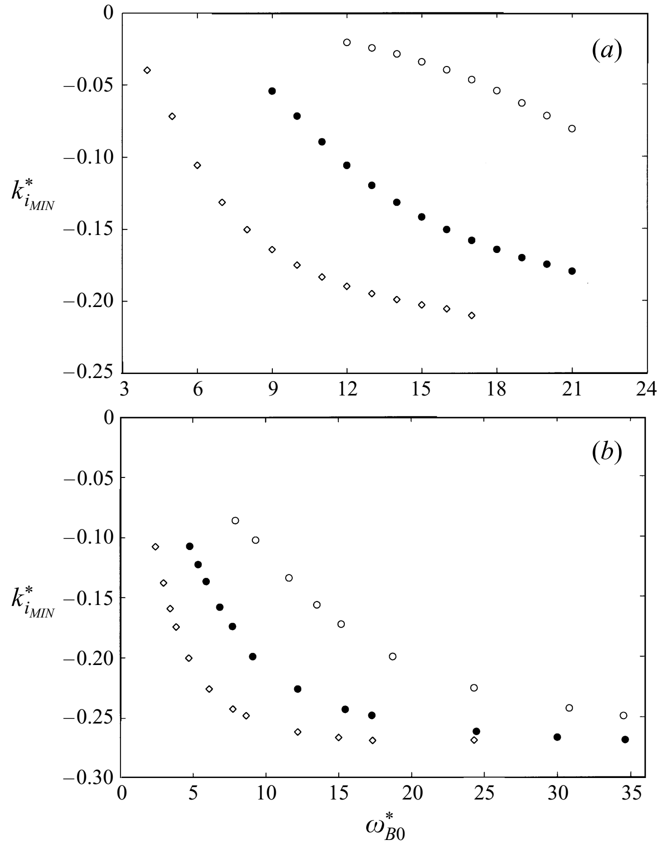

FIGURE 3. The most unstable attenuation rate, $k^*_{i_{MIN}}$, for (*a*) the hyperbolic tangent shear layer, and (*b*) symmetric oscillations of the Bickley jet, as a function of the bubble natural frequency, ω^*_{B0}, for three values of the bubble size: $R^*_0 = 0.005$ (◇), 0.01 (•), and 0.02 (○). The void fraction is $\alpha = 0.01$ for all cases.

frequency. On the other hand, the variation of c^*_r with ω^*_{B0} for the Bickley jet is not as significant as for the shear layer. However the qualitative effects of the bubbles are the same in both cases.

We should note that previous figures are relevant to those physical situations where the excitation frequency, ω^*, is considerably smaller than the bubble resonance frequency, ω^*_{B0}. Typical values of the flow parameters, such as those listed earlier, for gas bubbles in liquids suggest that this is the case in many practical applications, where the bulk pressure of the liquid is relatively close to atmospheric pressure and therefore the dominant contribution of the bubbles to the flow stability arises from their compressibility, while inertial and dissipation effects play a relatively minor role. However, we will show in a later publication that this may no longer be true in cavitating flows, where the reduced value of the liquid pressure and the possible occurrence of thermal cavitation can lead to significant inertial coupling and dissipation effects even at the relatively low frequencies imposed by the flow perturbation.

In order to explore the full range of possibilities, calculations were also carried out to examine the role played by bubble dynamics at or near resonance ($\omega^* \approx \omega^*_{B0}$). Typical results for the shear layer and jet are shown in figures 6(*a*) and 6(*b*). Notice

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

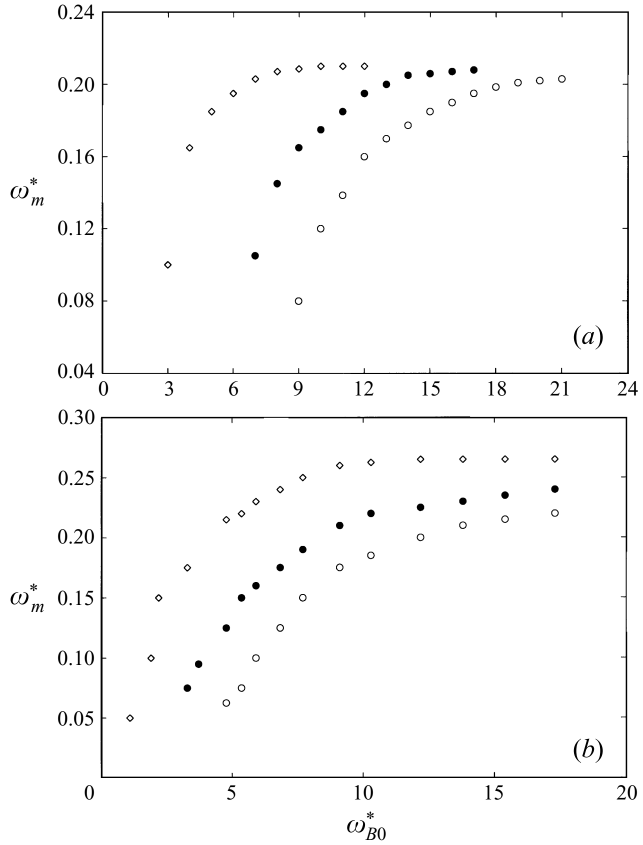

FIGURE 4. The frequency, ω_m^*, of the most unstable oscillation of (*a*) the hyperbolic tangent shear layer, and (*b*) the symmetric oscillations of the Bickley jet, as a function of the bubble natural frequency, ω_{B0}^*, for for three values of the void fraction: $\alpha = 0.001$ ($\diamond$), 0.005 ($\bullet$), and 0.01 ($\circ$). The bubble size is $R^* = 0.01$ for all cases.

that in both cases the attenuation rate peaks near resonance ($\omega^* \approx \omega_{B0}^*$), where the flow is significantly stabilized because the bubble response is larger and dissipates perturbation energy more effectively. As a consequence of this effect, the curves intersect each other and flows characterized by a lower bubble natural frequency can be less stable than others over some portion of the perturbation spectrum. More importantly, the most unstable frequency suddenly jumps to a higher value when the resonance frequency is reduced below the minimum of the attenuation curve in the absence of resonance, as illustrated by the curves for $\omega_{B0}^* = 0.38$ and 0.31 in figure 6(*b*). However, even the presence of significant resonance effects does not modify the general conclusion that the flow characterized by the highest bubble natural frequency is invariably the most unstable.

It is worth comparing the stabilizing effect due to the presence of uniformly dispersed bubbles in the liquid with the classical results of the linear stability theory of compressible flows. In his temporal stability analysis of symmetric inviscid shear layers in a perfect gas at uniform temperature, Blumen (1970) found that flow compressibility is a stabilizing feature, since he observed a decrease in the growth rates as the Mach number (in his case defined as $M = U/a$) increases. His results, obtained for $M \leqslant 1$, were later revised and expanded to all values of Mach number by

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

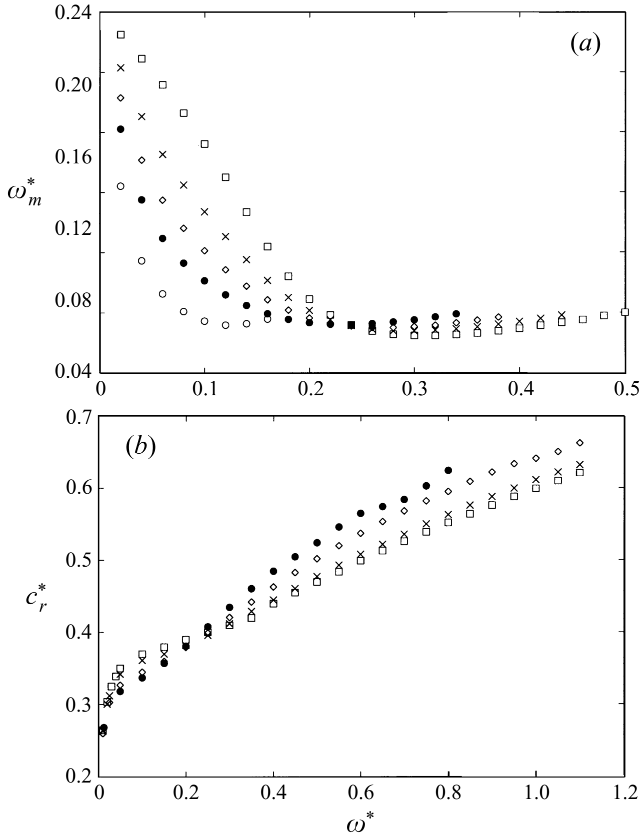

FIGURE 5. (*a*) The real phase velocity, c_r^*, of the hyperbolic tangent shear layer as a function of the excitation frequency, ω^*, for several values of the bubble resonance frequency: $\omega_{B0}^* = 21$ ($\times$), 15 ($\diamond$), 12 ($\bullet$), 9 ($\circ$), and for the single phase flow ($\square$). In all cases $\alpha = 0.01$ and $R^* = 0.01$. (*b*) As (*a*) but for symmetric oscillations of the Bickley jet and for $\omega_{B0}^* = 17.5$ ($\times$), 9.5 ($\diamond$), 7.5 ($\bullet$).

Blumen *et al.* (1975) and Drazin & Davey (1977). According to Blumen, the physical explanation for this stabilizing effect lies in the fact that "a certain amount of basic flow energy must be used to do work against the force due to the elasticity of the medium, before it becomes available to initiate instability". Such an explanation also applies to the present case of a finely dispersed gaseous phase in a liquid medium. In order to further support this physical explanation, Blumen also reported that "investigations of the stability of parallel flow of a density stratified fluid under the action of gravity show a similar stabilizing feature, because some of the available basic flow energy must be used to do work against the buoyancy force" before it can contribute to promote the instability (Drazin 1958).

In addition to the stabilizing effect due to compressibility alone, bubble dynamic damping provides another significant source of energy absorption, which is especially effective at or near resonance where the amplitude of the bubble response is larger. This additional contribution is well exemplified by the different behaviour of the curves shown in figure 7, where the spatial attenuation rate of a bubbly flow with resonant bubble dynamics ($\omega_{B0}^* = 0.25$) is compared with the corresponding simply compressible and incompressible flow solutions for the same value of the void fraction and the other flow parameters. It is evident that bubble dynamic effects are responsible

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

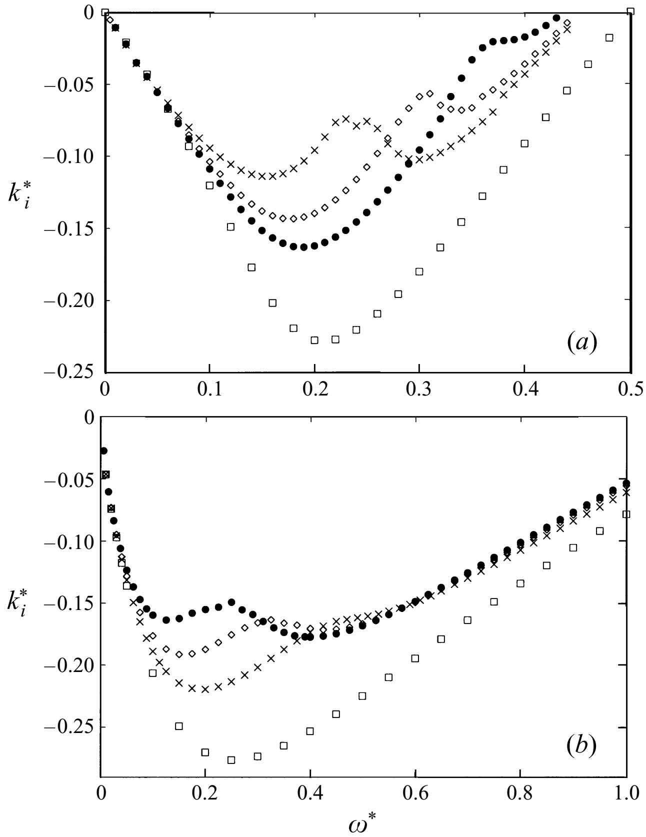

FIGURE 6. (*a*) Data illustrating the effect of bubble resonance ($\omega^* \approx \omega^*_{B0}$) for the hyperbolic tangent shear layer. The attenuation rate, k_i^*,is shown as a function of the perturbation frequency, ω^*, for several values of the bubble resonance frequency: $\omega^*_{B0} = 0.25(\times)$, 0.3 ($\diamond$), and 0.35 ($\bullet$). In all cases $\alpha = 0.003$. The results for incompressible flow ($\alpha = 0$) are also shown for comparison ($\square$). (*b*) As (*a*) but for symmetric oscillations of the Bickley jet and for $\omega^*_{B0} = 0.49(\times)$, 0.38 ($\diamond$), 0.31 ($\bullet$).

for a substantial stabilizing effect in the neighbourhood of resonance conditions and induce a sizeable shift of the most unstable frequency with respect to either one of the barotropic solutions.

7. Conclusions

This study of the stability of bubbly two-dimensional inviscid parallel flows has yielded a number of interesting results. In general, even bubbly mixtures of low void fraction ($\alpha \ll 1$) are considerably more stable than the single-phase fluid. Increasing the void fraction or decreasing the radius and the resonance frequency of the bubbles promote the stability of the flow.

Far from resonance (i.e. for $\omega^*_{B0} > \omega^*$) the observed stabilizing effect is mainly due to the compressibility of the bubbly mixture. This effect is qualitatively consistent with the analogous results of linear temporal stability analyses of compressible flows in an inviscid perfect gas at uniform temperature (Blumen 1970; Blumen *et al.* 1975; Drazin & Davey 1977) and can be explained with the same physical arguments.

At or near resonance conditions ($\omega^*_{B0} \approx \omega^*$), bubble dynamic effects provide a

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

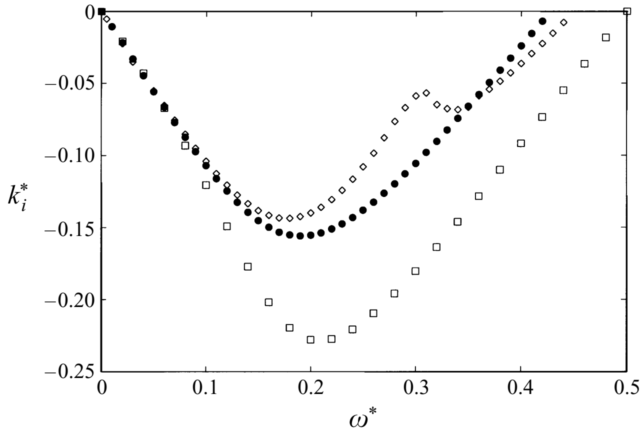

FIGURE 7. Bubble dynamics effects at resonance for the hyperbolic tangent shear layer. The attenuation rate, k_i^*, is shown as a function of the perturbation frequency, ω^*, for a flow with $\omega_{B0}^* = 0.3$ (◇), and for the corresponding compressible flow without bubble dynamics effects (•). In both cases the void fraction is $\alpha = 0.003$. The results for incompressible single-phase flow ($\alpha = 0$) are also shown for comparison (□).

significant additional contribution to the flow stability by effectively subtracting energy from the perturbation field to sustain larger bubble oscillations. Near resonance conditions bubble dynamic effects strongly modify the curves of the spatial attenuation rate as a function of the perturbation frequency, causing them to locally overlap. As a consequence, flows characterized by a lower bubble natural frequency can be less stable over some portion of the perturbation spectrum or have higher values of the most unstable frequency with respect to similar flows with larger bubble resonance frequency. These circumstances have not been observed in flows with small bubble dynamic effects ($\omega_{B0}^* \gg \omega^*$). On the other hand, even in the presence of appreciable bubble dynamic effects, the flows with the highest bubble resonance frequency always are the most unstable. Finally, for gas bubbles in liquids, the bubble natural frequency is likely to be much larger than the most unstable frequency in the majority of practical applications. Hence, in these flows bubble dynamic damping is relatively unimportant and the dominant effect of the bubbles is their contribution to the compressibility of the mixture. We notice, however, that this situation may easily be reversed in bubbly cavitating flows.

The authors gratefully acknowledge the partial support of the Office of Naval Research under grant number N-00014-91-K-1295 and the support of the Italian Ministry of University and Scientific Research.

REFERENCES

D'AGOSTINO, L., D'AURIA, F. & BRENNEN, C. E. 1995 Stability of parallel bubbly and cavitating flows. *Intl Symp. on Cavitation, Deauville, France* (ed. B. Gindroz).

D'AGOSTINO, L. & BRENNEN, C. E. 1983 On the acoustical dynamics of bubble clouds. *ASME Cavitation and Multiphase Flow Forum*, pp. 72–75.

D'AGOSTINO, L. & BRENNEN, C. E. 1988 Acoustical absorption and scattering cross-sections of spherical bubble clouds. *J. Acoust. Soc. Am.* **84**, 2126–2134.

D'AGOSTINO, L. & BRENNEN, C. E. 1989 Linearized dynamics of spherical bubble clouds. *J. Fluid Mech.* **199**, 155–176.

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press

d'Agostino, L., Brennen, C. E. & Acosta, A. J. 1988 Linearized dynamics of two-dimensional bubbly and cavitating flows over slender surfaces. *J. Fluid Mech.* **192**, 485–509.

Arakeri, V. H. & Shanmuganathan, V. 1985 On the evidence for the effect of bubble interference on cavitation noise. *J. Fluid Mech.* **159**, 131–150.

d'Auria, F., d'Agostino, L. & Brennen, C. E. 1994 Linearized dynamics of bubbly and cavitating flows in cylindrical ducts. *ASME Cavitation and Multiphase Flow Forum*, 59–66.

d'Auria, F., d'Agostino, L. & Brennen, C. E. 1995 Inviscid stability of bubbly jets *AIAA Paper* 95-2377.

Betchov, R. & Criminale, W. O. 1966 Spatial instability of the inviscid jet and wake. *Phys. Fluids* **9**, 359–362.

Betchov, R. & Criminale, W. O. 1967 *Stability of Parallel Flows*. Academic Press.

Blumen, W. 1970 Shear layer instability of an inviscid compressible fluid. *J. Fluid Mech.* **40**, 769–781.

Blumen, W., Drazin, P. G. & Billings, D. F. 1975 Shear layer instability of an inviscid compressible fluid. Part 2. *J. Fluid Mech.* **71**, 305–316.

Brennen, C. E. 1995 *Cavitation and Bubble Dynamics*. Oxford University Press.

Ceccio, S. L. & Brennen, C. E. 1990 Observations of the dynamics and acoustics of attached cavities. *ASME Cavitation and Multiphase Flow Forum* (ed. O. Furuya), pp. 79–84.

Drazin, P. G. 1958 The stability of a shear layer in an unbounded heterogeneous inviscid fluid. *J. Fluid Mech.* **4**, 214–224.

Drazin, P. G. & Davey, A. 1977 Shear layer instability of an inviscid compressible fluid. Part 3. *J. Fluid Mech.* **82**, 255–260.

Drazin, P. G. & Howard, L. N. 1966 Hydrodynamic stability of parallel flow of inviscid fluid. *Adv. Appl. Mech.* **9**, 1–89.

Drazin, P. G. & Reid, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

Gates, E. M. 1977 The influence of free stream turbulence, free stream nuclei population and a drag reducing polymer on cavitation inception on two axisymmetric bodies. PhD thesis, California Institute of Technology.

Herbolzeimer, E. 1983 Stability of the flow during sedimentation in inclined channels. *Phys. Fluids* **26**, 2043–2054.

Lees, L. & Reshotko, E. 1962 Stability of the compressible laminar boundary layer. *J. Fluid Mech.* **12**, 555–590.

Mack, L. M. 1987 Review of linear compressible stability theory. In *Stability of Time Depenedent and Spatially Varying Flows* (ed. D. W. Dwoyer & M. Y. Hussaini). Springer.

Michalke, A. 1965 On spatially growing disturbances in an inviscid shear layer. *J. Fluid Mech.* **23**, 521–544.

Plesset, M. S. & Prosperetti, A. 1977 Bubble dynamics and cavitation. *Ann. Rev. Fluid Mech.* **9**, 145–185.

Prosperetti, A. 1982 Bubble dynamics: a review and some recent results. *Appl. Sci. Res.* **3B**, 145–164.

Prosperetti, A. 1984 Bubble phenomena in sound fields: Part One. *Ultrasonics* **3B**, 69–78.

Sato, H. 1960 The stability and transition of a two-dimensional jet. *J. Fluid Mech.* **7**, 53–88.

Sato, H. & Kuriki, K. 1961 Mechanism of transition in the wake of a thin flat plate placed parallel to a uniform flow. *J. Fluid Mech.* **11**, 321–352.

Schlichting, H 1968 *Theory of Boundary Layers*. McGraw Hill.

Shaqfeh, E. S. G. & Acrivos, A. 1986 The effects of inertia on the stability of the convective flow in inclined particle settlers. *Phys. Fluids* **30**, 960–973.

Stoer, J. & Bulirsch, R. 1980 *Introduction to Numerical Analysis*. Springer.

Tatsumi, T. & Kakutani, T. 1958 The stability of a two-dimensional jet. *J. Fluid Mech.* **4**, 261–275.

Wazzan, A. R. 1975 Spatial stability of Tollmien-Schlichting waves. *Prog. Aerospace Sci.* **16**, 99–127.

White, F. M. 1991 *Viscous Fluid Flow*. McGraw Hill.

Wijngaarden, L. van 1972 One-dimensional flow of liquids containing small gas bubbles. *Ann. Rev. Fluid Mech.* **4**, 369–396.

Yang, Y., Chung, J. N., Troutt, T. R. & Crowe C. T. 1990 The influence of particles on the spatial stability of two-phase mixing layers. *Phys. Fluids* **10**, 1839–1845.

https://doi.org/10.1017/S0022112097005211 Published online by Cambridge University Press