

The second moment of symmetric square L -functions over Gaussian integers

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We prove a new upper bound on the second moment of Maass form symmetric square L -functions defined over Gaussian integers. Combining this estimate with the recent result of Balog–Biro–Cherubini–Laaksonen, we improve the error term in the prime geodesic theorem for the Picard manifold.

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1. Introduction

Consider the three-dimensional hyperbolic space

$$\mathbf{H}^3 = \{(z, r); \quad z = x + iy \in \mathbf{C}; \quad r > 0\}$$

and the Picard group defined over Gaussian integers

$$\Gamma = \mathbf{PSL}(2, \mathbf{Z}[i]).$$

The prime geodesic theorem for the Picard manifold $\Gamma \backslash \mathbf{H}^3$ provides an asymptotic formula for the function $\pi_\Gamma(X)$, which counts the number of primitive hyperbolic or loxodromic elements in Γ with norm less than or equal to X . In 1983 Sarnak [15] proved that

$$\pi_\Gamma(X) = \text{Li}(X^2) + O(X^{5/3+\epsilon}). \quad (1.1)$$

Since then the function $\pi_\Gamma(X)$ was intensively studied, see [1, 3, 9, 11]. The currently best known result due to Balog–Biro–Cherubini–Laaksonen [4] states that the error term in (1.1) can be replaced by

$$O(X^{3/2+4\theta/7+\epsilon}), \quad (1.2)$$

where $\theta = 1/6$ (see [12]) is the best known subconvexity exponent for quadratic Dirichlet L -functions defined over Gaussian integers. In the current paper, we improve this result further.

THEOREM 1.1. For any $\epsilon > 0$

$$\pi_{\Gamma}(X) = \text{Li}(X^2) + O(X^{3/2+((32\theta^2+28\theta-1)/(46+40\theta))+\epsilon}). \tag{1.3}$$

REMARK 1.2. Substituting $\theta = 1/6$, we obtain that the error term in (1.3) is

$$O(X^{3/2+41/474+\epsilon}) = O(X^{1.586\dots}),$$

while (1.2) is equal to

$$O(X^{3/2+2/21+\epsilon}) = O(X^{1.595\dots}).$$

Theorem 1.1 is a direct consequence of the following estimate on the second moment of Maass form symmetric square L -functions defined over Gaussian integers.

THEOREM 1.3. For $s = 1/2 + it$, $|t| \ll T^\epsilon$ we have

$$\sum_{T < r_j < 2T} \alpha_j |L(\text{sym}^2 u_j, s)|^2 \ll T^{3+4\theta+\epsilon}, \tag{1.4}$$

where α_j is the standard normalizing coefficient, see (2.6).

Theorem 1.3 improves [1, theorem 3.3], where it was shown that the left-hand side of (1.4) can be bounded by $T^{4+\epsilon}$. The main new ingredient that leads to the improvement is a more careful treatment of sums of Kloosterman sums.

The strategy for proving [1, theorem 3.3] consists of using an approximate functional equation for both L -functions followed by the application of the Kuznetsov trace formula. This results in expressions containing sums of Kloosterman sums multiplied by some complicated weight function. Then [1, theorem 3.3] is proved by estimating the Kloosterman sums using Weil’s bound and analysing the weight function thoroughly.

The proof of theorem 1.3 is quite different. As the first step we apply an approximate functional equation only for one L -function, which reduces our problem to the investigation of the first twisted moment of symmetric square L -functions. For this moment we prove an explicit formula which is similar to the one derived in [3]. The advantage of such hybrid approach is that it allows us to evaluate sums of Kloosterman sums by replacing them with sums of Zagier L -series weighted by a double integral of the Gauss hypergeometric function. The Zagier L -series can be estimated using the subconvexity result of Nelson [12]. Consequently, the main difficulty of our approach is the analysis of the sums of the weight function given by the following expression

$$\sum_{|l| \ll T} \sum_{|n| \ll |l|} |n^2 - 4l^2|^{2\theta} \times \int_0^1 \int_T^{2T} F(1 - s - ir, 1 - s + ir, 1; -x_{\pm}(n/l, y)) \frac{r^2 dr dy}{(1 - y^2)^{3/2-s}}, \tag{1.5}$$

where for $\vartheta = \arg(z)$ we have

$$x_{\pm}(z, y) = \frac{f_{\pm}(z, y)}{1 - y^2}, \quad f_{\pm}(z, y) = y^2 \pm |z|y \cos \vartheta + |z/2|^2.$$

When $s = 1/2$, the asymptotic formula for the hypergeometric function in (1.5) as $r \rightarrow \infty$ was proved by Jones [8] and Farid Khwaja–Olde Daalhuis [5]. However, for the application to the prime geodesic theorem, it is required to consider $s = 1/2 + it$. For this reason, we prove a uniform version of [5, theorem 3.1]. Applying the resulting asymptotic formula for the hypergeometric function in (1.5) and evaluating the integral over r , we show that the contribution of the summands with $|x_{\pm}(n/l, y)| \gg T^{\epsilon-2}$ in (1.5) is negligible. The final result comes from the opposite case: $|x_{\pm}(n/l, y)| \ll T^{\epsilon-2}$. In this case the hypergeometric function is approximately 1, and therefore, there is no cancellations in the r -integral. Luckily, the situation when $|x_{\pm}(n/l, y)| \ll T^{\epsilon-2}$ is sufficiently rare so that we can detect all such cases by dividing the sums over n and l into many different ranges. Analysing (1.5) carefully in these ranges, we complete the proof of theorem 1.3.

The paper is organized as follows. All required preliminary results and notation are collected in §2. The explicit formula for the first twisted moment of symmetric square L -functions over Gaussian integers is given in §3. Section 4 is devoted to the generalization of the results of Farid Khwaja and Olde Daalhuis concerning a uniform asymptotic formula for the Gauss hypergeometric function. Finally, theorem 1.3 is proved in §5 and theorem 1.1 in §6.

2. Notation and preliminary results

Let $\mathbf{k} = \mathbf{Q}(i)$ be the Gaussian number field. All sums in this paper are over Gaussian integers unless otherwise indicated. For $\Re(s) > 1$, the Dedekind zeta function is defined as

$$\zeta_{\mathbf{k}}(s) = 4^{-1} \sum_{n \neq 0} |n|^{-2s}.$$

Let $\sigma_{\alpha}(n) = 4^{-1} \sum_{d|n} |d|^{2\alpha}$. For $\Re(s) > 1$ and $r \in \mathbf{R}$ we have

$$\frac{1}{4} \sum_{n \neq 0} \frac{\sigma_{ir}(n^2)}{|n|^{2s+2ir}} = \frac{\zeta_{\mathbf{k}}(s)\zeta_{\mathbf{k}}(s + ir)\zeta_{\mathbf{k}}(s - ir)}{\zeta_{\mathbf{k}}(2s)}. \tag{2.1}$$

Let $[n, x] = \Re(n\bar{x})$ and $e[x] = \exp(2\pi i\Re(x))$. For $m, n, c \in \mathbf{Z}[i]$ with $c \neq 0$ the Kloosterman sum is defined by

$$S(m, n; c) = \sum_{\substack{a \pmod{c} \\ (a, c) = 1}} e \left[m \frac{a}{c} + n \frac{a^*}{c} \right], \quad aa^* \equiv 1 \pmod{c}.$$

For $m \in \mathbf{Z}$, $\xi \in \mathbf{C}$ and $\Re(s) > 1$ let

$$\zeta_{\mathbf{k}}(s; m, \xi) = \sum_{n+\xi \neq 0} \left(\frac{n + \xi}{|n + \xi|} \right)^m \frac{1}{|n + \xi|^{2s}}.$$

LEMMA 2.1. *If $m \neq 0$ then $\zeta_{\mathbf{k}}(s; m, \xi)$ is entire function of variable s . Otherwise, it is regular in s except for a simple pole at $s = 1$ with residue π . For $\text{Re}(s) < 0$ we have*

$$\zeta_{\mathbf{k}}(s; m, \xi) = (-i)^{|m|} \pi^{2s-1} \frac{\Gamma(1-s+|m|/2)}{\Gamma(s+|m|/2)} \sum_{n \neq 0} \left(\frac{n}{|n|}\right)^{-m} \frac{e[n\xi]}{|n|^{2(1-s)}}. \tag{2.2}$$

Proof. Equation (2.2) follows from [10, lemma 2] by making the change of variable $n \rightarrow \bar{n}$ on the right-hand side of [10, (4.1)] and using the fact that $e[\bar{n}\bar{\xi}] = e[n\xi]$. \square

Let $\Gamma(z)$ be the Gamma function. By Stirling’s formula we have

$$\begin{aligned} \Gamma(\sigma + it) &= \sqrt{2\pi}|t|^{\sigma-1/2} \exp(-\pi|t|/2) \exp\left(i\left(t \log|t| - t + \frac{\pi t(\sigma - 1/2)}{2|t|}\right)\right) \\ &\times (1 + O(|t|^{-1})) \end{aligned} \tag{2.3}$$

for $|t| \rightarrow \infty$ and any fixed σ . Evaluating sufficiently many terms in the asymptotic expansion, it is possible to replace $O(|t|^{-1})$ in (2.3) by an arbitrarily accurate approximation.

Let $\{\kappa_j = 1 + r_j^2, j = 1, 2, \dots\}$ be the non-trivial discrete spectrum of the hyperbolic Laplacian on $L^2(\Gamma \backslash \mathbf{H}^3)$, and $\{u_j\}$ be the orthonormal basis of the space of *Maass cusp* forms consisting of common eigenfunctions of all Hecke operators and the hyperbolic Laplacian. Each function u_j has the following Fourier expansion

$$u_j(z) = y \sum_{n \neq 0} \rho_j(n) K_{ir_j}(2\pi|n|y) e[nx],$$

where $K_\nu(z)$ is the K-Bessel function of order ν .

The corresponding Rankin–Selberg L -function is defined as

$$L(u_j \otimes u_j, s) = \sum_{n \neq 0} \frac{|\rho_j(n)|^2}{|n|^{2s}}, \quad \Re(s) > 1.$$

Using the relation to the Hecke eigenvalues $\rho_j(n) = \rho_j(1)\lambda_j(n)$, we obtain

$$L(u_j \otimes u_j, s) = |\rho_j(1)|^2 \frac{\zeta_{\mathbf{k}}(s)}{\zeta_{\mathbf{k}}(2s)} L(\text{sym}^2 u_j, s),$$

where

$$L(\text{sym}^2 u_j, s) = \zeta_{\mathbf{k}}(2s) \sum_{n \neq 0} \frac{\lambda_j(n^2)}{|n|^{2s}}. \tag{2.4}$$

The symmetric square L -function is an entire function (see [16]) which satisfies the functional equation

$$L(\text{sym}^2 u_j, s) \gamma(s, r_j) = L(\text{sym}^2 u_j, 1-s) \gamma(1-s, r_j),$$

where

$$\gamma(s, r_j) = \pi^{-3s} \Gamma(s) \Gamma(s + ir_j) \Gamma(s - ir_j). \tag{2.5}$$

The standard normalizing coefficient is defined as follows

$$\alpha_j = \frac{r_j |\rho_j(1)|^2}{\sinh(\pi r_j)}. \tag{2.6}$$

LEMMA 2.2. *The following approximate functional equation holds*

$$L(\text{sym}^2 u_j, s) = \sum_{l \neq 0} \frac{\lambda_j(l^2)}{|l|^{2s}} V(|l|, r_j, s) + \sum_{l \neq 0} \frac{\lambda_j(l^2)}{|l|^{2-2s}} V(|l|, r_j, 1-s), \tag{2.7}$$

where for any $y > 0$ and $a > 0$

$$V(y, r_j, s) = \frac{1}{2\pi i} \int_{(a)} \frac{\gamma(s+z, r_j)}{\gamma(s, r_j)} \zeta_{\mathbf{k}}(2s+2z) \mathfrak{F}(z) y^{-2z} \frac{dz}{z}, \tag{2.8}$$

$$\mathfrak{F}(z) = \exp(z^2) P_n(z^2)$$

with P_n being a polynomial of degree n such that $P_n(0) = 1$.

Proof. The proof is similar to [13, lemma 7.2.1]. □

Using (2.5) and (2.3) we prove the following estimates (see [13, lemma 7.2.2]).

LEMMA 2.3. *Let $r_j \sim T$ and $s = 1/2 + it$ with $|t| \ll T^\epsilon$. For any positive numbers y and A we have*

$$V(y, r_j, s) \ll \left(\frac{r_j^2 |t|}{y^2} \right)^A. \tag{2.9}$$

For any positive integer N and $1 \leq y \ll r_j^{1+\epsilon}$

$$V(y, r_j, s) = \frac{1}{2\pi i} \int_{(a)} \left(\frac{r_j}{\pi^{3/2} y} \right)^{2z} \frac{\Gamma(s+z)}{\Gamma(s)} \zeta_{\mathbf{k}}(2s+2z) \mathfrak{F}(z) \\ \times \left(1 + \sum_{k=1}^{N-1} \frac{p_{2k}(v+t)}{r_j^k} \right) \frac{dz}{z} + O(r_j^{-N+\epsilon}), \tag{2.10}$$

where $v = \Im(z)$ and $p_n(v)$ is a polynomial of degree n .

Let

$$\mathcal{L}_{\mathbf{k}}(s; n) = \frac{\zeta_{\mathbf{k}}(2s)}{\zeta_{\mathbf{k}}(s)} \sum_{q \neq 0} \frac{\rho_q(n)}{|q|^{2s}}, \quad \Re(s) > 1, \tag{2.11}$$

$$\rho_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}.$$

We have (see [17] and [3, (3.20)])

$$\mathcal{L}_{\mathbf{k}}(s; 0) = 4\zeta_{\mathbf{k}}(2s-1). \tag{2.12}$$

Furthermore, the following subconvexity bound holds (see [3, Eq. (3.12)])

$$\mathcal{L}_{\mathbf{k}}(1/2 + it; n) \ll (1 + |t|)^A |n|^{2\theta + \epsilon}, \tag{2.13}$$

where θ can be taken as $1/6$ according to the result of Nelson [12]. As an analogue of [2, lemma 4.1], we derive the following equation relating sums of Kloosterman sums and the L-series (2.11)

$$\sum_{q \neq 0} \frac{1}{|q|^{2+2s}} \sum_{c \pmod{q}} S(l^2, c^2; q) e \left[n \frac{c}{q} \right] = \frac{\mathcal{L}_{\mathbf{k}}(s; n^2 - 4l^2)}{\zeta_{\mathbf{k}}(2s)}. \tag{2.14}$$

3. Explicit formula for the first moment

In this section, we evaluate the first twisted moment

$$\mathcal{M}_1(l, s; h) := \sum_j h(r_j) \alpha_j \lambda_j(l^2) L(\text{sym}^2 u_j, s), \tag{3.1}$$

where $h(t)$ is an even function, holomorphic in any fixed horizontal strip and satisfying the conditions

$$h(\pm(n - 1/2)i) = 0, \quad h(\pm ni) = 0 \quad \text{for } n = 1, 2, \dots, N, \tag{3.2}$$

$$h(r) \ll \exp(-c|r|^2) \tag{3.3}$$

for some fixed N and $c > 0$.

THEOREM 3.1. *For $1/2 \leq \Re(s) < 1$ and any even function $h(t)$ that is holomorphic in any fixed horizontal strip and satisfies the conditions (3.2) and (3.3), we have*

$$\mathcal{M}_1(l, s; h) = MT(l, h; s) + CT(l, h; s) + ET(l, h; s) + \Sigma(l, h; s),$$

where

$$MT(l, h; s) = \frac{4\zeta_{\mathbf{k}}(2s)}{\pi^2 |l|^{2s}} \int_{-\infty}^{\infty} r^2 h(r) \tanh(\pi r) dr, \tag{3.4}$$

$$CT(l, h; s) = -8\pi \zeta_{\mathbf{k}}(s) \int_{-\infty}^{\infty} h(r) \frac{\sigma_{ir}(l^2)}{|l|^{2ir}} \frac{\zeta_{\mathbf{k}}(s + ir) \zeta_{\mathbf{k}}(s - ir)}{\zeta_{\mathbf{k}}(1 + ir) \zeta_{\mathbf{k}}(1 - ir)} dr, \tag{3.5}$$

$$ET(l, h; s) = -16\pi^2 h(i(s - 1)) \frac{\zeta_{\mathbf{k}}(2s - 1)}{\zeta_{\mathbf{k}}(2 - s)} \left(\frac{\sigma_{1-s}(l^2)}{|l|^{2-2s}} + \frac{\sigma_{s-1}(l^2)}{|l|^{2s-2}} \right), \tag{3.6}$$

$$\Sigma(l, h; s) = \Sigma_0(l, h; s) + \Sigma_2(l, h; s) + \Sigma_{gen}(l, h; s), \tag{3.7}$$

$$\Sigma_0(l, h; s) = \frac{8(2\pi)^{2s-1}}{\pi^2 |l|^{2-2s}} \mathcal{L}_{\mathbf{k}}(s; -4l^2) \int_0^{\pi/2} I(0, \tau, s) d\tau,$$

$$\Sigma_2(l, h; s) = \frac{32(2\pi)^{2s-1} \zeta_{\mathbf{k}}(2s - 1)}{\pi^2 |l|^{2-2s}} \int_0^{\pi/2} \sum_{\pm} I(\pm 2, \tau, s) d\tau,$$

$$\Sigma_{gen}(l, h; s) = \frac{8(2\pi)^{2s-1}}{\pi^2 |l|^{2-2s}} \sum_{n \neq 0, \pm 2l} \mathcal{L}_{\mathbf{k}}(s; n^2 - 4l^2) \int_0^{\pi/2} I\left(\frac{n}{l}, \tau, s\right) d\tau, \tag{3.8}$$

$$I(z, \tau, s) = \frac{1}{4} \int_{-\infty}^{\infty} r^2 \check{h}(r) \cosh(\pi r) \sum_{\pm} \frac{\Gamma(1-s-ir)\Gamma(1-s+ir)}{4(\cos \tau)^{2-2s}} \times F(1-s+ir, 1-s-ir, 1; -x_{\pm}(z, \tau)) dr, \tag{3.9}$$

where for $\vartheta = \arg(z)$ we define

$$x_{\pm}(z, \tau) = \frac{|z|^2 + 4 \sin^2 \tau \pm 4|z| \sin \tau \cos \vartheta}{(2 \cos \tau)^2}.$$

Proof. This result is a generalization of [3, theorem 4.13] and can be proved in the same way. We will indicate the required changes in the proof of [3, theorem 4.13].

We start by assuming that $\Re(s) > 3/2$. Substituting (2.4) into (3.1), applying the Kuznetsov formula (see [3, theorem 3.2]) and using (2.1), we obtain an analogue of [3, lemma 4.2], namely

$$\mathcal{M}_1(l, s; h) = MT(l, h; s) + CT(l, h; s) + \Sigma(s)$$

with

$$\Sigma(s) = \zeta_{\mathbf{k}}(2s) \sum_{n \neq 0} \frac{1}{|n|^{2s}} \sum_{q \neq 0} \frac{S(l^2, n^2; q)}{|q|^2} \check{h}\left(\frac{2\pi l n}{q}\right), \tag{3.10}$$

where $\check{h}(z)$ is defined in [3, (3.15)]. Note that the main difference between (3.10) and [3, (4.4)] is in the arguments of Kloosterman sums. Furthermore, (3.10) contains the additional multiple $\zeta_{\mathbf{k}}(2s)$.

The next step is to adapt the proof of [3, lemma 4.6] to our case. The main changes occur while proving an analogue of [3, (4.44)]. Using (2.2) and (2.14) we obtain

$$\begin{aligned} & \sum_{q \neq 0} \frac{1}{|q|^{2+2s}} \sum_{c \pmod{q}} S(l^2, c^2; q) \zeta_{\mathbf{k}}(s + w/2; 2m, c/q) \\ &= (-i)^{|2m|} \pi^{2s+w-1} \frac{\Gamma(1-s-w/2+|m|)}{\Gamma(s+w/2+|m|)} \\ & \quad \times \sum_{n \neq 0} \left(\frac{n}{|n|}\right)^{-2m} |n|^{2s+w-2} \sum_{q \neq 0} \frac{1}{|q|^{2+2s}} \\ & \quad \times \sum_{c \pmod{q}} S(l^2, c^2; q) e[nc/q] \\ &= (-1)^{|m|} \pi^{2s+w-1} \frac{\Gamma(1-s-w/2+|m|)}{\Gamma(s+w/2+|m|)} \\ & \quad \times \sum_{n \neq 0} \left(\frac{n}{|n|}\right)^{-2m} |n|^{2s+w-2} \frac{\mathcal{L}_{\mathbf{k}}(s; n^2 - 4l^2)}{\zeta_{\mathbf{k}}(2s)}. \end{aligned} \tag{3.11}$$

Applying (3.11) in place of [3, (4.44)], we finally derive the following analogue of [3, (4.40)]

$$\begin{aligned} \Sigma(s) &= \frac{8(2\pi)^{2s-1}}{\pi^2 |l|^{2-2s}} \int_0^{\pi/2} \mathcal{L}_{\mathbf{k}}(s; -4l^2) I(0, \tau, s) d\tau \\ &\quad + \frac{8(2\pi)^{2s-1}}{\pi^2 |l|^{2-2s}} \int_0^{\pi/2} \sum_{n \neq 0} \mathcal{L}_{\mathbf{k}}(s; n^2 - 4l^2) I(n/l, \tau, s) d\tau, \end{aligned}$$

where $I(0, \tau, s)$ and $I(z, \tau, s)$ are defined by [3, (4.39)] and [3, (4.38)]. The representation (3.9) is proved in [3, lemma 4.10]. Using (2.12) we show that for $\Re(s) > 1/2$

$$\Sigma(s) = \Sigma_0(l, h; s) + \Sigma_2(l, h; s) + \Sigma_{gen}(l, h; s).$$

The final step is to continue analytically the term $CT(l, h; s)$ to the region $\Re(s) < 1$. Doing so, we obtain the additional summand $ET(l, h; s)$ defined by (3.6) which comes from the poles of $\zeta_{\mathbf{k}}(s + ir)\zeta_{\mathbf{k}}(s - ir)$. Thus the theorem is proved for $1/2 < \Re(s) < 1$. In order to extend our result to the critical line $\Re(s) = 1/2$ and specially to the point $s = 1/2$, we proceed in the same way as in [3, theorem 4.13, remark 4.14]. □

4. Special functions

According to (3.9), in order to estimate $I(z, \tau, s)$ it is required to investigate the asymptotic behaviour of the Gauss hypergeometric function

$$F(1/2 - it + ir, 1/2 - it - ir, 1; -x) \tag{4.1}$$

for $r \sim T$, $|t| \ll T^\epsilon$ as $T \rightarrow +\infty$ and uniformly for $0 < x < \infty$. When $t = 0$, the asymptotic formula for (4.1) was proved by Jones [8] using the Liouville–Green method. Furthermore, this result was reproved by Farid Khwaja and Olde Daalhuis [5, theorem 3.1] by means of the saddle point method.

For our application, it is required to have an asymptotic formula for (4.1) for any small t . This can be achieved by generalizing [5, theorem 3.1]. Accordingly, we introduce two new variables $\lambda = ir$ and $\alpha = t/r$ and consider

$$F(1/2 + \lambda(1 - \alpha), 1/2 - \lambda(1 + \alpha), 1; -x)$$

for $\lambda \rightarrow \infty$ in $|\arg(\lambda)| \leq \pi/2$ and $|\alpha| \ll |\lambda|^{-1+\epsilon}$. Consequently, we derive the following result.

THEOREM 4.1. *For $0 < x < \infty$ and $|\alpha| \ll |\lambda|^{-1+\epsilon}$ we have*

$$\begin{aligned} &F\left(\frac{1}{2} + \lambda(1 - \alpha), \frac{1}{2} - \lambda(1 + \alpha), 1; -x\right) \\ &= -e^{\lambda\eta} \left(I_0(\lambda\xi) \sum_{j=0}^{n-1} \frac{a_j}{\lambda^j} + \frac{2}{\xi} I_1(\lambda\xi) \sum_{j=1}^{n-1} \frac{b_j}{\lambda^j} + O(\Phi_n(\lambda, \xi)) \right) \end{aligned} \tag{4.2}$$

as $\lambda \rightarrow \infty$ in $|\arg(\lambda)| \leq \pi/2$, where $a_j = a_j(x, \alpha, \xi)$ and $b_j = b_j(x, \alpha, \xi)$, $I_0(x)$ and $I_1(x)$ are *I*-Bessel functions, and

$$\eta = \alpha \log(1 + x), \tag{4.3}$$

$$\begin{aligned} \xi = \alpha \log(1 + x) - (1 + \alpha) \log\left(\frac{\alpha x + \sqrt{x^2 + x(1 - \alpha^2)}}{x + \sqrt{x^2 + x(1 - \alpha^2)}}\right) \\ + (1 - \alpha) \log\left(\frac{1 - \alpha + x + \sqrt{x^2 + x(1 - \alpha^2)}}{1 - \alpha}\right), \end{aligned} \tag{4.4}$$

$$a_0 = -\frac{\xi^{1/2}}{2^{1/2}(x^2 + x(1 - \alpha^2))^{1/4}}, \tag{4.5}$$

$$\Phi_n(\lambda, \xi) = \frac{1}{|\lambda|^n} \left(|I_0(\lambda\xi)| + \frac{|I_1(\lambda\xi)|}{|\xi|} \right).$$

Proof. Using [5, (3.14)] with $a = 1/2 - \lambda\alpha$, $z = 1 + 2x$ we obtain

$$F\left(\frac{1}{2} + \lambda(1 - \alpha), \frac{1}{2} - \lambda(1 + \alpha), 1; -x\right) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} g(t)e^{\lambda f(t)} dt,$$

where

$$\begin{aligned} f(t) &= (1 + \alpha) \log\left(\frac{(1 + x) - t}{1 - t}\right) - (1 - \alpha) \log(t), \\ g(t) &= \frac{1}{t^{1/2}(1 - t)} \left(\frac{1 - t}{1 + x - t}\right)^{1/2}. \end{aligned}$$

As in [5] the branch-cuts are $(-\infty, 0)$ and $(1, 1 + x)$. Since

$$f'(t) = (1 + \alpha) \left(\frac{1}{t - 1 - x} - \frac{1}{t - 1}\right) - \frac{1 - \alpha}{t},$$

the saddle points (the solutions of $f'(t) = 0$) are

$$\mathbf{sp}_{\pm} = \frac{x + 1 - \alpha \pm \sqrt{x^2 + x(1 - \alpha^2)}}{1 - \alpha}.$$

Similarly to [5, (3.18)], we make the change of variable

$$f(t) = \tau + \frac{\xi^2}{4\tau} + \eta. \tag{4.6}$$

If $x = \alpha = 0$ this transformation reduces to $-\log t = \tau$, and therefore, the point $t = 1$ corresponds to $\tau = 0$. Thus under the transformation (4.6) we have that as τ grows from $-\infty$ to 0, the variable t decays from $+\infty$ to $1 + x$, and $\tau = -\xi/2$ corresponds to $t = \mathbf{sp}_+$. As τ grows from 0 to $+\infty$, the variable t decays from

1 to 0, and $\tau = \xi/2$ corresponds to $t = \mathbf{sp}_-$. An important consequence of these observations is that $dt/d\tau < 0$. Solving the following system

$$\begin{cases} f(\mathbf{sp}_+) = -\xi + \eta \\ f(\mathbf{sp}_-) = \xi + \eta \end{cases},$$

we obtain the equality (4.4) for ξ and (4.3) for η . Finally, we prove an analogue of [5, (3.20), (3.25)], namely

$$F\left(\frac{1}{2} + \lambda(1 - \alpha), \frac{1}{2} - \lambda(1 + \alpha), 1; -x\right) = -\frac{e^{\lambda\eta}}{2\pi i} \int_{\mathfrak{C}} G_0(\tau) e^{\lambda(\tau + \xi^2/4\tau)} \frac{d\tau}{\tau},$$

where (see [5, (3.21)])

$$G_0(\tau) = g(t) \frac{dt}{d\tau} \tau \tag{4.7}$$

and \mathfrak{C} is the steepest descent contour (see [5, figure 3]).¹ Following [5] we obtain (4.2). It is left to evaluate a_0, b_0 which are defined by (see [5, (3.31)])

$$a_0 = \frac{1}{2} (G_0(\xi/2) + G_0(-\xi/2)), \quad b_0 = \frac{\xi}{4} (G_0(\xi/2) - G_0(-\xi/2)). \tag{4.8}$$

As a consequence of (4.7) we find that

$$G_0(\pm\xi/2) = \pm \frac{\xi}{2} g(\mathbf{sp}_{\mp}) \frac{dt}{d\tau} \Big|_{\pm\xi/2}. \tag{4.9}$$

Next, we evaluate $g(\mathbf{sp}_{\mp})$. Since $0 < \mathbf{sp}_- < 1$ we have

$$g(\mathbf{sp}_-) = \frac{1}{\sqrt{\mathbf{sp}_-(1 - \mathbf{sp}_-)(1 + x - \mathbf{sp}_-)}} = \left(\frac{1 - \alpha}{1 + \alpha}\right)^{1/2} \frac{1}{x^{1/2} \cdot \mathbf{sp}_-}. \tag{4.10}$$

Since $\mathbf{sp}_+ > 1 + x$ the following holds

$$g(\mathbf{sp}_+) = \frac{-1}{\sqrt{\mathbf{sp}_+(\mathbf{sp}_+ - 1)(\mathbf{sp}_+ - 1 - x)}} = -\left(\frac{1 - \alpha}{1 + \alpha}\right)^{1/2} \frac{1}{x^{1/2} \cdot \mathbf{sp}_+}. \tag{4.11}$$

Differentiating (4.6) we obtain $f'(t)(dt/d\tau) = 1 - \xi^2/4\tau^2$. Taking one more derivative we infer

$$f''(t) \left(\frac{dt}{d\tau}\right)^2 + f'(t) \frac{d^2t}{d\tau^2} = \frac{\xi^2}{2\tau^3} \implies \left(\frac{dt}{d\tau}\right)^2 \Big|_{\pm\xi/2} = \frac{\pm 4}{\xi f''(\mathbf{sp}_{\mp})}.$$

Evaluating the second derivative at the saddle points

$$f''(\mathbf{sp}_{\pm}) = \mp 2 \frac{1 - \alpha}{1 + \alpha} \frac{\sqrt{x^2 + x(1 - \alpha^2)}}{x \cdot \mathbf{sp}_{\pm}^2}$$

¹Note that there is a typo in [5, (3.20),(3.25)]: the minus sign is missed. This is because the contour \mathfrak{C} is taken in the opposite direction to the contour obtained after the change of variable (4.6).

and using the fact that $dt/d\tau < 0$, we have

$$\left. \frac{dt}{d\tau} \right|_{\pm\xi/2} = - \left(\frac{1+\alpha}{1-\alpha} \right)^{1/2} \frac{(2x)^{1/2} \cdot \mathbf{sp}_{\mp}}{\xi^{1/2}(x^2+x(1-\alpha^2))^{1/4}}. \tag{4.12}$$

Substituting (4.10)–(4.12) into (4.9), we conclude that

$$G_0(\pm\xi/2) = - \frac{(\xi/2)^{1/2}}{(x^2+x(1-\alpha^2))^{1/4}}. \tag{4.13}$$

Substituting (4.13) into (4.8), we prove (4.5) and show that $b_0 = 0$. □

We will apply theorem 4.1 with $\lambda = ir$. In this case I -Bessel functions can be written in terms of J -Bessel functions (see [14, 10.27.6])

$$I_0(ir\xi) = J_0(r\xi), \quad I_1(ir\xi) = iJ_1(r\xi). \tag{4.14}$$

Furthermore, it follows from (4.4) that as $x \rightarrow 0$ and $x \rightarrow \infty$ we respectively have

$$\xi = 2\sqrt{x(1-\alpha^2)} + O(x^{3/2}), \quad \xi = \log(x) + O(1). \tag{4.15}$$

Also it will be required to study ξ as a function of α . In this case, as $\alpha \rightarrow 0$ we have

$$\xi(\alpha) = \xi(0) + O(\alpha^2), \quad \xi(0) = \log(1+2x+2\sqrt{x^2+x}). \tag{4.16}$$

5. Proof of theorem 1.3

Following the paper of Ivic and Jutila [7], let us define

$$\omega_T(r) = \frac{1}{G\pi^{1/2}} \int_T^{2T} \exp\left(-\frac{(r-K)^2}{G^2}\right) dK. \tag{5.1}$$

For an arbitrary $A > 1$ and some $c > 0$ we have (see [7])

$$\begin{aligned} \omega_T(r) &= 1 + O(r^{-A}) \text{ if } T + cG\sqrt{\log T} < r < 2T - cG\sqrt{\log T}, \\ \omega_T(r) &= O((|r|+T)^{-A}) \text{ if } r < T - cG\sqrt{\log T} \text{ or } r > 2T + cG\sqrt{\log T}, \end{aligned}$$

and otherwise

$$\omega_T(r) = 1 + O(G^3(G + \min(|r-T|, |r-2T|))^{-3}).$$

To prove theorem 1.3 we consider

$$\mathcal{M}_2(T) := \sum_{r_j} \alpha_j \omega_T(r_j) L(\text{sym}^2 u_j, 1/2 + it) L(\text{sym}^2 u_j, 1/2 - it),$$

where $\omega_T(r)$ is defined by (5.1) with $G = T^{1-\epsilon}$ and $|t| \ll T^\epsilon$. Applying the approximate functional equation (2.7) for $L(\text{sym}^2 u_j, 1/2 - it)$ and using (2.9), we

infer

$$\mathcal{M}_2(T) \ll \sum_{|l| \ll T^{1+\epsilon}} \frac{1}{|l|} \left| \sum_{r_j} \alpha_j \omega_T(r_j) V(|l|, r_j, 1/2 \pm it) \lambda_j(l^2) L(\text{sym}^2 u_j, 1/2 + it) \right|.$$

Then it follows from (5.1) that

$$\mathcal{M}_2(T) \ll \sum_{|l| \ll T^{1+\epsilon}} \frac{1}{|l|} \left| \frac{1}{G\pi^{1/2}} \int_T^{2T} \mathcal{M}_1(l, s, h(\cdot) V(|l|, \cdot, 1/2 \pm it)) \, dK \right|, \tag{5.2}$$

where $s = 1/2 + it$ and as in (3.1) we have

$$\mathcal{M}_1(l, s; h(\cdot) V(|l|, \cdot, 1/2 \pm it)) := \sum_j h(r_j) V(|l|, r_j, 1/2 \pm it) \alpha_j \lambda_j(l^2) L(\text{sym}^2 u_j, s) \tag{5.3}$$

with

$$h(r) = q_N(r) \exp\left(-\frac{(r - K)^2}{G^2}\right) + q_N(r) \exp\left(-\frac{(r + K)^2}{G^2}\right), \tag{5.4}$$

$$q_N(r) = \frac{(r^2 + 1/4) \cdots (r^2 + (N - 1/2)^2)}{(r^2 + 100N^2)^N} \frac{(r^2 + 1) \cdots (r^2 + N^2)}{(r^2 + 100N^2)^N}.$$

To evaluate (5.3) we apply theorem 3.1 and then substitute the result in (5.2). This way the contribution of (3.4)–(3.6) is bounded by $O(T^{3+\epsilon})$. Thus to prove theorem 1.3 it is required to show that

$$\mathcal{M}_2(T) \ll \sum_{|l| \ll T^{1+\epsilon}} \frac{1}{|l|} \left| \frac{1}{G\pi^{1/2}} \int_T^{2T} \Sigma(l, s, h(\cdot) V(|l|, \cdot, 1/2 \pm it)) \, dK \right| \ll T^{3+4\theta+\epsilon},$$

where $\Sigma(l, h; s)$ is defined by (3.7).

The most difficult part is to estimate the term which involves the $\Sigma_{gen}(l, h; s)$ summand of $\Sigma(l, h; s)$. The contribution of the term with $\Sigma_0(l, h; s)$ can be estimated similarly (see also [3, (6.42), (6.43)]). The summand $\Sigma_2(l, h; s)$ is a part of the main term (see [3, p. 24]), and therefore, is of size $O(T^{3+\epsilon})$. Furthermore, it is sufficient to consider only one ('+' or '-') case for $V(|l|, \cdot, 1/2 \pm it)$ since both cases can be treated in the same way. Consequently, using (2.13) and (3.8) and making the change of variable $y = \sin \tau$, we prove that

$$\mathcal{M}_2(T) \ll T^{3+\epsilon} + \sum_{|l| \ll T^{1+\epsilon}} \sum_{\pm} \frac{S_{\pm}(l)}{|l|^2}, \tag{5.5}$$

where

$$S_{\pm}(l) = \sum_{n \neq 0, 2l, -2l} |n^2 - 4l^2|^{2\theta} \left| \int_0^1 \frac{1}{G\pi^{1/2}} \int_T^{2T} I_{\pm}\left(\frac{n}{l}, y, s\right) \, dK \frac{dy}{(1 - y^2)^{1/2}} \right| \tag{5.6}$$

and (see (3.9))

$$I_{\pm}(z, y, s) = \int_{-\infty}^{\infty} r^2 h(r) V(|l|, r, s) \cosh(\pi r) \frac{\Gamma(1-s-ir)\Gamma(1-s+ir)}{(1-y^2)^{1-s}} \times F(1-s+ir, 1-s-ir, 1; -x_{\pm}(z, y)) dr. \tag{5.7}$$

Here for $\vartheta = \arg(z)$ we have

$$x_{\pm}(z, y) = \frac{f_{\pm}(z, y)}{1-y^2}, \quad f_{\pm}(z, y) = y^2 \pm |z|y \cos \vartheta + |z/2|^2. \tag{5.8}$$

According to [3, lemma 4.11]

$$I_{\pm}(z, y, s) = \frac{2}{(x_{\pm}(z, y)(1-y^2))^{1-s}} \int_{-\infty}^{\infty} r^2 h(r) \cosh(\pi r) x_{\pm}^{-ir}(z, y) \times \frac{\Gamma(1-s+ir)\Gamma(-2ir)}{\Gamma(s-ir)} \times F\left(1-s+ir, 1-s+ir, 1+2ir; \frac{-1}{x_{\pm}(z, y)}\right) dr. \tag{5.9}$$

First, we show that the contribution of large $|n|$ in (5.5) is negligible. To this end, we prove an analogue of [3, lemma 4.12]. The only difference in our case is the presence of the function $V(|l|, r, s)$ in (5.7). Moving the line of integration in (2.8) to $\Re(z) = M$ we obtain

$$V(|l|, r - iM, s) \ll \left(\frac{r}{|l|}\right)^{2M}.$$

Using this estimate and following the proof of [3, lemma 4.12] we infer

$$\int_0^1 \frac{1}{G\pi^{1/2}} \int_T^{2T} I_{\pm}(z, \tau, s) dK \frac{dy}{(1-y^2)^{1/2}} \ll T^3 \left(\frac{T}{|lz|}\right)^{2N+1}.$$

Consequently, the contribution of n such that $|n| \gg T^\epsilon |l|$ to (5.6) is negligible.

Next, we consider the contribution of small n . Let

$$J_{\pm}\left(\frac{n}{l}, y, T\right) := \frac{1}{G\pi^{1/2}} \int_T^{2T} I_{\pm}\left(\frac{n}{l}, y, s\right) dK.$$

Then we have

$$S_{\pm}(l) \ll \sum_{\substack{n \neq 0, 2l, -2l \\ |n| \ll T^\epsilon |l|}} |n^2 - 4l^2|^{2\theta} \left| \int_0^1 J_{\pm}\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \right| + T^{-A}. \tag{5.10}$$

LEMMA 5.1. *Suppose that for $1 - T^{-B} < y < 1$ the following inequalities hold: $f_{\pm}(z, y) > T^{-a}$ and $B - 5 > a > 0$. Then*

$$\int_{1-T^{-B}}^1 J_{\pm}\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \ll T^{(5+a-B)/2}. \tag{5.11}$$

Proof. It follows from (5.8) and the statement of the lemma that $x_{\pm}(z, y)^{-1} \ll T^{a-B} < T^{-5}$. Thus the hypergeometric function in (5.9) can be estimated by a constant. Consequently,

$$J_{\pm} \left(\frac{n}{l}, y, T \right) \ll \frac{T^{5/2}}{f_{\pm}(z, y)^{1/2}} \ll T^{(5-a)/2}.$$

Then the lemma follows by substituting this estimate in (5.11). □

LEMMA 5.2. For $0 \leq y < 1$, $|x_{\pm}(z, y)| \gg T^{\epsilon-2}$ and an arbitrary fixed $A > 0$ we have

$$J_{\pm} \left(\frac{n}{l}, y, T \right) \ll \frac{T^{-A}}{(1-y^2)^{1/2}}. \tag{5.12}$$

For $0 \leq y < 1$, $|x_{\pm}(z, y)| \ll T^{\epsilon-2}$ we have

$$J_{\pm} \left(\frac{n}{l}, y, T \right) \ll \frac{T^{3+\epsilon}}{(1-y^2)^{1/2}}. \tag{5.13}$$

Proof. To prove the required estimates we substitute (4.2) to (5.7). All terms coming from (4.2) can be estimated in the same way and the error term is negligible. Using (4.14) we obtain

$$J_{\pm} \left(\frac{n}{l}, y, T \right) = \frac{e^{it \log(1+x_{\pm}(z,y))}}{(1-y^2)^{1-s}} \frac{1}{G\pi^{1/2}} \int_T^{2T} \int_{-\infty}^{\infty} r^2 h(r) V(|l|, r, s) \times \cosh(\pi r) \Gamma(1-s-ir) \Gamma(1-s+ir) J_0(r\xi) dr dK + \dots \tag{5.14}$$

Suppose that $|x_{\pm}(z, y)| \ll T^{\epsilon-2}$. Since $r \sim T$ it follows from (4.15) that $r\xi \ll T^{\epsilon}$. Estimating everything on the right-hand side of (5.14) trivially we prove (5.13).

Suppose that $|x_{\pm}(z, y)| \gg T^{\epsilon-2}$. In this case we have $r\xi \gg T^{\epsilon}$. Thus we can apply the asymptotic formula [6, 8.451.1] for $J_0(r\xi)$ (again it is enough to consider only the main term). Consequently, it is required to evaluate

$$\frac{1}{\xi^{1/2}} \int_{-\infty}^{\infty} r^{3/2} h(r) V(|l|, r, s) \cosh(\pi r) \Gamma(1-s-ir) \Gamma(1-s+ir) e^{ir\xi} dr.$$

Using (5.4) and the Stirling formula (2.3) for the Gamma functions, we infer (see [3, (6.12)])

$$J_{\pm} \left(\frac{n}{l}, y, T \right) \ll \frac{(1-y^2)^{-1/2}}{G\pi^{1/2}} \int_T^{2T} K^{2it} \int_{-\infty}^{\infty} \frac{r^{3/2}}{\xi^{1/2}} V(|l|, r, s) \times \exp \left(-\frac{(r-K)^2}{G^2} + ir\xi \right) dr dK.$$

In the integral over r we first make the change of variable $r = K + Gv$. Note that ξ depends on α (and therefore depends on r). To overcome this difficulty, we expand

it in Taylor series (4.16) at $\alpha = 0$ with sufficiently many terms. As a result, we obtain

$$\begin{aligned}
 J_{\pm} \left(\frac{n}{l}, y, T \right) &\ll \frac{\xi(0)^{-1/2}}{(1-y^2)^{1/2}} \int_T^{2T} K^{3/2+2it} e^{iK\xi(0)} \\
 &\times \int_{-\infty}^{\infty} V(|l|, K + Gv, s) \exp(-v^2 + iGv\xi(0)) \, dv dK, \quad (5.15)
 \end{aligned}$$

where

$$\xi(0) = \log(1 + 2x_{\pm}(z, y) + 2\sqrt{x_{\pm}(z, y)^2 + x_{\pm}(z, y)}).$$

In order to evaluate the integral over v , it is convenient to use the representation (2.10) for $V(|l|, K + Gv, s)$. Before doing this, at the cost of a negligible error term we truncate the integral over v at $v = \pm(\log T)^2$. Now we apply (2.10) (with a being chosen as a small fixed $\epsilon_1 > 0$). It is sufficient to consider only the main term from (2.10) since all other terms can be treated in the same way and are smaller in size. The error term coming from the remainder in (2.10) is negligible. Therefore, we have

$$\begin{aligned}
 &\int_{-\infty}^{\infty} V(|l|, K + Gv, s) \exp(-v^2 + iGv\xi(0)) \, dv \\
 &= \int_{-(\log T)^2}^{(\log T)^2} \exp(-v^2 + iGv\xi(0)) \frac{1}{2\pi i} \int_{(\epsilon_1)} \left(\frac{K + Gv}{\pi^{3/2}|l|} \right)^{2u} \\
 &\quad \times \frac{\Gamma(s + u)}{\Gamma(s)} \zeta_{\mathbf{k}}(2s + 2u) \mathfrak{F}(u) \frac{du}{u} \, dv + \dots
 \end{aligned}$$

Since the function $\mathfrak{F}(u)$ decays exponentially we can truncate the integral over u at $|\Im u| = \log T$ with a negligibly small error term. Consequently, we have $|Gvu/K| \ll T^{-\epsilon}$, and therefore, it is possible to replace $(K + Gv)^u$ by K^u since

$$(K + Gv)^u = K^u \left(1 + \frac{Gvu}{K} + \dots \right).$$

As a result,

$$\begin{aligned}
 &\int_{-\infty}^{\infty} V(|l|, K + Gv, s) \exp(-v^2 + iGv\xi(0)) \, dv \\
 &= \frac{1}{2\pi i} \int_{(\epsilon_1)} \left(\frac{K}{\pi^{3/2}|l|} \right)^{2u} \frac{\Gamma(s + u)}{\Gamma(s)} \\
 &\quad \times \zeta_{\mathbf{k}}(2s + 2u) \mathfrak{F}(u) \int_{-(\log T)^2}^{(\log T)^2} \exp(-v^2 + iGv\xi(0)) \, dv \frac{du}{u} + \dots
 \end{aligned}$$

Now we can extend the integral over v to the whole real line at a cost of a negligible error term. Evaluating the resulting integral, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} V(|l|, K + Gv, s) \exp(-v^2 + iGv\xi(0)) \, dv \\ &= \pi^{1/2} \exp(-G^2\xi(0)^2) \frac{1}{2\pi i} \int_{(\epsilon_1)} \left(\frac{K}{\pi^{3/2}|l|}\right)^{2u} \frac{\Gamma(s+u)}{\Gamma(s)} \\ & \quad \times \zeta_{\mathbf{k}}(2s+2u) \mathfrak{F}(u) \frac{du}{u} + \dots \end{aligned} \tag{5.16}$$

Substituting (5.16) into (5.15) and estimating the integrals over u and K trivially, we prove that

$$J_{\pm}\left(\frac{n}{l}, y, T\right) \ll \frac{\exp(-G^2\xi^2(0)) + T^{-A}}{(1-y^2)^{1/2}\xi(0)^{1/2}} T^{5/2+\epsilon}.$$

Finally, since $G = T^{1-\epsilon_2}$ and $|x_{\pm}(z, y)| \gg T^{\epsilon-2}$ we have $G\xi(0) \gg T^{\epsilon_3}$. This completes the proof of (5.12). □

The next step is to apply lemmas 5.1 and 5.2 for estimating (5.6). To prove theorem 1.3 it is enough to show (see (5.5)) that

$$\begin{aligned} \sum_{|l| \ll T^{1+\epsilon}} \sum_{\pm} \frac{S_{\pm}(l)}{|l|^2} &\ll \sum_{|l| \ll T^{1+\epsilon}} \sum_{\pm} \frac{1}{|l|^2} \sum_{\substack{n \neq 0, \pm 2l, -2l, \\ |n| \ll T^{\epsilon}|l|}} |n^2 - 4l^2|^{2\theta} \\ &\quad \times \left| \int_0^1 J_{\pm}\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \right| \ll T^{3+4\theta+\epsilon}, \end{aligned}$$

where for $S_{\pm}(l)$ we used the estimate (5.10). We consider further only $S_{-}(l)$ since the sum $S_{+}(l)$ can be estimated similarly. Moreover, without loss of generality we may assume that l belongs to the first quadrant (we denote this as $l \in I$). To prove theorem 1.3 it is enough to show that

$$\begin{aligned} \sum_{\substack{|l| \ll T^{1+\epsilon} \\ l \in I}} \frac{S_{-}(l)}{|l|^2} &\ll T^{3+4\theta+\epsilon}, \\ S_{-}(l) &= \sum_{n \in N(l, T)} |n^2 - 4l^2|^{2\theta} \left| \int_0^1 J_{-}\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \right|, \end{aligned}$$

where $N(l, T) = \{n : n \neq 0, \pm 2l, |n| \ll T^{\epsilon}|l|\}$.

First, we decompose the sum $S_-(l)$ into two parts depending on whether $\cos \vartheta = \cos(\arg(n/l)) \leq 0$ or $\cos \vartheta = \cos(\arg(n/l)) > 0$, namely

$$S_-(l) = S_-^1(l) + S_-^2(l),$$

$$S_-^1(l) = \sum_{\substack{n \in N(l,T) \\ \cos \vartheta \leq 0}} |n^2 - 4l^2|^{2\theta} \left| \int_0^1 J_- \left(\frac{n}{l}, y, T \right) \frac{dy}{(1 - y^2)^{1/2}} \right|,$$

$$S_-^2(l) = \sum_{\substack{n \in N(l,T) \\ \cos \vartheta > 0}} |n^2 - 4l^2|^{2\theta} \left| \int_0^1 J_- \left(\frac{n}{l}, y, T \right) \frac{dy}{(1 - y^2)^{1/2}} \right|.$$

LEMMA 5.3. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_-^1(l)}{|l|^2} \ll T^{2+4\theta+\epsilon}. \tag{5.17}$$

Proof. Since $\cos \vartheta = \cos(\arg(n/l)) \leq 0$ we have (see (5.8))

$$x_-(n/l, y) \geq \frac{y^2 + |n/(2l)|^2}{1 - y^2}. \tag{5.18}$$

Let B be a fixed large number. Since $|n/(2l)| \gg T^{-1-\epsilon}$ lemma 5.1 implies that

$$\int_{1-T^{-B}}^1 J_{\pm} \left(\frac{n}{l}, y, T \right) \frac{dy}{(1 - y^2)^{1/2}} \ll T^{-A},$$

where A can be made arbitrary large by taking sufficiently large B .

Let us now consider the case $y < 1 - T^{-B}$. Due to (5.12) the contribution of n such that $x_-(n/l, y) \gg T^{-2+\epsilon}$ is negligible. When $x_-(n/l, y) \ll T^{-2+\epsilon}$ we apply (5.13). It follows from (5.18) that in this case $y \ll T^{-1+\epsilon}$ and $|n| \ll |l|T^{-1+\epsilon} \ll T^\epsilon$. Therefore, the contribution of such n can be estimated by

$$S_-^1(l) \ll \sum_{|n| \ll T^\epsilon} |n^2 - 4l^2|^{2\theta} \int_0^{T^{-1+\epsilon}} \frac{T^{3+\epsilon}}{1 - y} dy \ll |l|^{4\theta} T^{2+\epsilon}. \tag{5.19}$$

Summing (5.19) over $|l| \ll T^{1+\epsilon}$ we obtain (5.17). □

Next, we consider the case $\cos \vartheta = \cos(\arg(n/l)) > 0$. Let

$$2l = a + ib, \quad n = c + id, \tag{5.20}$$

where a, b, c, d are integers (and $a, b \geq 0$). Since $\vartheta = \arg(n/2l)$ we have

$$y_n := \Re \left(\frac{n}{2l} \right) = \left| \frac{n}{2l} \right| \cos \vartheta = \frac{ac + bd}{a^2 + b^2}, \tag{5.21}$$

$$s_n := \Im \left(\frac{n}{2l} \right) = \left| \frac{n}{2l} \right| \sin \vartheta = \frac{ad - bc}{a^2 + b^2}, \tag{5.22}$$

$$y^2 - 2y \left| \frac{n}{2l} \right| \cos \vartheta + \left| \frac{n}{2l} \right|^2 = (y - y_n)^2 + s_n^2. \tag{5.23}$$

Since $|2l|^2 = a^2 + b^2 \ll T^{2+\epsilon}$ either $y_n = 1, s_n = 1$ or

$$|y_n - 1|, |s_n - 1| \gg \frac{1}{T^{2+\epsilon}}. \tag{5.24}$$

Using (5.21)–(5.23) it is possible to rewrite (5.8) as

$$f_-(n/l, y) = (y - y_n)^2 + s_n^2, \quad x_-(n/l, y) = \frac{(y - y_n)^2 + s_n^2}{1 - y^2}. \tag{5.25}$$

It follows from lemma 5.2 that the main contribution to $S_-^2(l)$ comes from n such that $x_-(n/l, y) \ll T^{-2+\epsilon}$. This may happen (see (5.25)) only if $(y - y_n)^2 + s_n^2 \ll T^{-2+\epsilon}$. Furthermore, the contribution of y close to 1 should be treated separately by lemma 5.1. We split the sum over n in $S_-^2(l)$ into several parts depending on values of y_n and s_n as follows

$$S_-^2(l) = \sum_{j=1}^9 S_j(l), \quad S_j(l) = \sum_{\substack{n \in N_j(l, T) \\ \cos \vartheta > 0}} |n^2 - 4l^2|^{2\theta} \left| \int_0^1 J_- \left(\frac{n}{l}, y, T \right) \frac{dy}{(1 - y^2)^{1/2}} \right|, \tag{5.26}$$

where

$$\begin{aligned} N_1(l, T) &= \left\{ n \in N(l, T) \mid |s_n| \gg T^{-1+\epsilon} \right\}, \\ N_2(l, T) &= \left\{ n \in N(l, T) \mid y_n = 1, |s_n| \ll T^{-1+\epsilon} \right\}, \\ N_3(l, T) &= \left\{ n \in N(l, T) \mid y_n \leq 1, s_n = 0 \right\}, \\ N_4(l, T) &= \left\{ n \in N(l, T) \mid y_n < 1 - \epsilon_1, 0 < |s_n| \ll T^{-1+\epsilon} \right\}, \\ N_5(l, T) &= \left\{ n \in N(l, T) \mid 1 - T^{-2+\epsilon} < y_n < 1, 0 < |s_n| \ll T^{-1+\epsilon} \right\}, \\ N_6(l, T) &= \left\{ n \in N(l, T) \mid 1 - \epsilon_1 < y_n < 1 - T^{-2+\epsilon}, 0 < |s_n| \ll T^{-1+\epsilon} \right\}, \\ N_7(l, T) &= \left\{ n \in N(l, T) \mid y_n > 1 + T^{-2+\epsilon}, 0 < |s_n| \ll T^{-1+\epsilon} \right\}, \\ N_8(l, T) &= \left\{ n \in N(l, T) \mid 1 < y_n < 1 + T^{-2+\epsilon}, 0 < |s_n| \ll T^{-1+\epsilon} \right\}, \\ N_9(l, T) &= \left\{ n \in N(l, T) \mid 1 < y_n, s_n = 0 \right\}. \end{aligned}$$

Furthermore, we will separately estimate the integrals over $y < 1 - T^{-B}$ and over $1 - T^{-B} < y < 1$.

LEMMA 5.4. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_1(l)}{|l|^2} \ll T^{-A}.$$

Proof. Since $s_n \gg T^{-1+\epsilon}$ we have $f_-(n/l, y) \gg T^{-2+\epsilon}$. Consequently, by lemma 5.1

$$\int_{1-T^{-B}}^1 J_-\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \ll T^{-A}.$$

Next, we consider the part of $S_1(l)$ with the integral over $y < 1 - T^{-B}$. In this case $x_-(n/l, y) \gg T^{-2+\epsilon}$, and therefore, the contribution of this part to $S_1(l)$ is $O(T^{-A})$ by (5.12). □

LEMMA 5.5. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_2(l)}{|l|^2} \ll T^{-A}.$$

Proof. In this case $y_n = 1$ and $0 < |s_n| \ll T^{-1+\epsilon}$. Using (5.21), (5.22), we obtain

$$|n| \leq |2l| \sqrt{1 + O(T^{-2+\epsilon})}. \tag{5.27}$$

It follows from (5.21) that $y_n = 1$ is equivalent to $ac + bd = a^2 + b^2$. The set of solutions of this linear equation is as follows

$$c_j = a + \frac{bj}{(a, b)}, \quad d_j = b - \frac{aj}{(a, b)}, \quad j \in \mathbf{Z}.$$

Let $n_j = c_j + id_j$. Then $s_{n_j} = -j/(a, b)$. Thus (5.25) can be rewritten as

$$x_-(n_j/l, y) \gg 1 - y + \frac{s_{n_j}^2}{1 - y} \gg |s_{n_j}| = \frac{|j|}{(a, b)} \gg \frac{1}{T^{1+\epsilon}}.$$

Furthermore, $|n_j| = |2l| \sqrt{1 + j^2/(a, b)^2}$. Consequently, from (5.27) we conclude that $|j| \ll T^\epsilon$. Therefore, the contribution to $S_2(l)$ of the integral over $y < 1 - T^{-B}$ for such n_j is negligible.

To deal with the integral over $1 - T^{-B} < y < 1$ we note that

$$f_-(n_j/l, y) = (1 - y)^2 + s_{n_j}^2 \gg \frac{j^2}{(a, b)^2} \gg \frac{1}{T^{2+\epsilon}}.$$

Applying lemma 5.1 we again obtain $O(T^{-A})$. □

LEMMA 5.6. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_3(l)}{|l|^2} \ll T^{2+4\theta+\epsilon}.$$

Proof. According to (5.22) the condition $s_n = 0$ is equivalent to $ad - bc = 0$. The set of solutions of this linear equation is given by

$$c_j = \frac{aj}{(a, b)}, \quad d_j = \frac{bj}{(a, b)}, \quad j \in \mathbf{Z}, \quad n_j = c_j + id_j.$$

In this case, (5.25) can be rewritten as

$$x_-(n_j/l, y) = \frac{(y - y_{n_j})^2}{1 - y^2}, \quad f_-(n_j/l, y) = (y - y_{n_j})^2. \tag{5.28}$$

By (5.21) we have

$$y_{n_j} = \frac{j}{(a, b)}, \quad 0 \leq j < (a, b) \Rightarrow 1 - y_{n_j} \gg \frac{1}{T^{1+\epsilon}}. \tag{5.29}$$

Applying lemma 5.1 and using (5.29) to estimate $f_-(n_j/l, y)$ from below, we obtain

$$\int_{1-T^{-B}}^1 J_-\left(\frac{n_j}{l}, y, T\right) \frac{dy}{(1 - y^2)^{1/2}} \ll T^{-A}$$

if B is sufficiently large. To estimate the remaining integral over $0 < y < 1 - T^{-B}$ we decompose it into two parts. The first part is over

$$Y_1 = \left\{ y \mid 0 < y < 1 - \frac{1}{TB}, \frac{(y - y_{n_j})^2}{1 - y} \gg T^{-2+\epsilon} \right\}$$

and the second one is over

$$Y_2 = \left\{ y \mid 0 < y < 1 - \frac{1}{TB}, \frac{(y - y_{n_j})^2}{1 - y} \ll T^{-2+\epsilon} \right\}.$$

The integral over Y_1 is negligible due to (5.28) and (5.12). Using (5.13), the integral over Y_2 can be bounded by

$$\int_{Y_2} \frac{T^3 dy}{1 - y} \ll \int_{|t| \ll T^{-1+\epsilon} \sqrt{1 - y_{n_j}}} \frac{T^3 dt}{1 - y_{n_j} - t} \ll \frac{T^{2+\epsilon}}{\sqrt{1 - y_{n_j}}}, \tag{5.30}$$

where the estimate (5.29) was used. It remains to sum (5.30) over n_j and over $|l| \ll T^{1+\epsilon}$ (see (5.26)). As a result,

$$\begin{aligned} \sum_{|l| \ll T^{1+\epsilon}} \frac{S_3(l)}{|l|^2} &\ll \sum_{|l| \ll T^{1+\epsilon}} \sum_{j=1}^{(a,b)-1} \frac{|n_j^2 - 4l^2|^{2\theta}}{|l|^2} \frac{T^{2+\epsilon}}{\sqrt{1 - y_{n_j}}} \\ &\ll T^{2+\epsilon} \sum_{|l| \ll T^{1+\epsilon}} |l|^{4\theta-2} \sum_{j=1}^{(a,b)-1} \left(1 - \frac{j}{(a, b)}\right)^{2\theta-1/2} \\ &\ll T^{2+\epsilon} \sum_{a^2+b^2 \ll T^{2+\epsilon}} (a, b)(a^2 + b^2)^{2\theta-1} \ll T^{2+4\theta+\epsilon}. \end{aligned}$$

□

LEMMA 5.7. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_4(l)}{|l|^2} \ll T^{3+4\theta+\epsilon}.$$

Proof. Since $y_n < 1 - \epsilon_1$ for some $\epsilon_1 > 0$ it follows from (5.25) that for $1 - T^{-B} < y < 1$ we have $f_-(n/l, y) \gg \epsilon_1^2$. Therefore, by lemma 5.1

$$\int_{1-T^{-B}}^1 J_-\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \ll T^{-A}.$$

To estimate the remaining integral over $0 < y < 1 - T^{-B}$ we decompose it into two parts. The first part is over y such that $x_-(n/l, y) \gg T^{-2+\epsilon}$ and the second one is over

$$x_-(n/l, y) = \frac{(y - y_n)^2 + s_n^2}{1 - y^2} \ll T^{-2+\epsilon}.$$

The first part is negligible due to (5.12). Using (5.13) to estimate the second part and enlarging it to the domain $|y - y_n| \ll T^{-1+\epsilon}$ we obtain

$$\int_{|y-y_n| \ll T^{-1+\epsilon}} \frac{T^3 dy}{1-y} \ll T^{2+\epsilon}. \tag{5.31}$$

Next, we sum (5.31) over n and over $|l| \ll T^{1+\epsilon}$ (see (5.26)). Since $y_n < 1 - \epsilon_1$ and $0 < |s_n| \ll T^{-1+\epsilon}$ it follows from (5.21) and (5.22) that $|n| < |2l|(1 - \epsilon)$. Therefore,

$$\begin{aligned} \sum_{|l| \ll T^{1+\epsilon}} \frac{S_4(l)}{|l|^2} &\ll \sum_{|l| \ll T^{1+\epsilon}} \sum_{\substack{|n| < |2l|(1-\epsilon) \\ 0 < |s_n| \ll T^{-1+\epsilon}}} \frac{|n^2 - 4l^2|^{2\theta}}{|l|^2} T^{2+\epsilon} \\ &\ll T^{2+\epsilon} \sum_{|l| \ll T^{1+\epsilon}} |l|^{4\theta-2} \sum_{(c,d) \in \Omega_{a,b}} 1, \end{aligned} \tag{5.32}$$

where (see (5.20) and (5.22))

$$\Omega_{a,b} = \left\{ (c, d) \left| c^2 + d^2 < (a^2 + b^2)(1 - \epsilon), |ad - bc| < (a^2 + b^2)T^{-1+\epsilon} \right. \right\}.$$

The sum over $(c, d) \in \Omega_{a,b}$ can be estimated by the double integral over $\Omega_{a,b}$. To evaluate this integral we make the change of variables

$$c = x \cos \vartheta_l - y \sin \vartheta_l, \quad d = x \sin \vartheta_l + y \cos \vartheta_l, \quad \vartheta_l = \arg(2l),$$

and obtain

$$\sum_{(c,d) \in \Omega_{a,b}} 1 \ll \iint_{\Omega_{|2l|}} 1 dx dy \ll |2l|^2 T^{-1+\epsilon}, \tag{5.33}$$

where

$$\Omega_{|2l|} = \left\{ (x, y) \mid x^2 + y^2 < (1 - \epsilon)|2l|^2, |y| < |2l|T^{-1+\epsilon} \right\}.$$

Substituting (5.33) to (5.32), we infer

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_4(l)}{|l|^2} \ll T^{1+\epsilon} \sum_{|l| \ll T^{1+\epsilon}} |l|^{4\theta} \ll T^{3+4\theta+\epsilon}.$$

□

LEMMA 5.8. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_5(l)}{|l|^2} \ll T^{-A}.$$

Proof. Since $s_n \gg T^{-2-\epsilon}$ using (5.25) we have $f_-(n/l, y) \gg T^{-4-\epsilon}$. Therefore, by lemma 5.1

$$\int_{1-T^{-B}}^1 J_-\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \ll T^{-A}.$$

Next, we are going to show that s_n cannot be very small if y_n is close to 1, namely

$$1 - T^{-2+\epsilon} < y_n < 1 \Rightarrow |s_n| \gg T^{-1-\epsilon}. \tag{5.34}$$

Using (5.21) and the fact that $0 < 1 - y_n < T^{-2+\epsilon}$, we obtain

$$0 < \frac{a(a-c) + b(b-d)}{a^2 + b^2} < T^{-2+\epsilon}.$$

This implies that $a^2 + b^2 > T^{2-\epsilon}$. Furthermore, $a^2 + b^2 = |2l|^2 \ll T^{2+\epsilon}$. Let $c_1 = a - c$, $d_1 = b - d$. Then it follows from the condition $1 - T^{-2+\epsilon} < y_n < 1$ that

$$T^{2-\epsilon} < a^2 + b^2 \ll T^{2+\epsilon}, \quad 0 < ac_1 + bd_1 \ll T^\epsilon. \tag{5.35}$$

According to (5.22)

$$s_n = \frac{bc_1 - ad_1}{a^2 + b^2}.$$

Note that since we assumed that l belongs to the first quadrant, we have $a, b \geq 0$.

Let $c_1 d_1 \geq 0$. Thus $c_1, d_1 \geq 0$ since $ac_1 + bd_1 > 0$. Suppose that both a and b are larger than T^ϵ . Since $ac_1 + bd_1 \ll T^\epsilon$ this implies that $c_1 = d_1 = 0$, which is impossible since $n \neq 2l$. Therefore, without loss of generality, we assume that $a \gg T^\epsilon$ and $b \ll T^\epsilon$. It follows from the first double inequality in (5.35) that $T^{1-\epsilon} \ll a \ll T^{1+\epsilon}$. Hence c_1 can only be 0 (since $ac_1 + bd_1 \ll T^\epsilon$). Therefore,

$$|s_n| = \frac{ad_1}{a^2 + b^2} \gg \frac{1}{a} \gg T^{-1-\epsilon}.$$

Let $c_1 d_1 \leq 0$. Suppose that $c_1 \geq 0$ and $d_1 \leq 0$ (the other case can be treated similarly). It follows from the second double inequality in (5.35) that $c_1 > b(-d_1)/a$.

Therefore,

$$s_n > \frac{-d_1}{a} \gg T^{-1-\epsilon}.$$

Since $1 - T^{-2+\epsilon} < y_n < 1$ and $|s_n| \gg T^{-1-\epsilon}$ we conclude that for all $0 < y < 1$ we have

$$x_-(n/l, y) = \frac{(y - y_n)^2 + s_n^2}{1 - y^2} \gg \frac{T^\epsilon}{T^2}.$$

Therefore, we can apply (5.13) showing that the contribution of this case to $S_5(l)$ is negligibly small. □

LEMMA 5.9. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_6(l)}{|l|^2} \ll T^{3+4\theta+\epsilon}.$$

Proof. In this setting $1 - \epsilon_1 < y_n < 1 - T^{-2+\epsilon}$ and $0 < |s_n| \ll T^{-1+\epsilon}$. Here in contrast to the case when $y_n < 1 - \epsilon_1$ it is required to be more accurate in estimating the integral in (5.31). In particular, $(1 - y) \sim (1 - y_n)$ can be very small (like $T^{-2-\epsilon}$). Nevertheless, due to (5.24) and since $s_n \gg T^{-2-\epsilon}$ we have $f_-(n/l, y) \gg T^{-4-\epsilon}$ for $1 - T^{-B} < y < 1$ (see (5.25)). Therefore, by lemma 5.1

$$\int_{1-T^{-B}}^1 J_-\left(\frac{n}{l}, y, T\right) \frac{dy}{(1 - y^2)^{1/2}} \ll T^{-A}.$$

To estimate the remaining integral over $0 < y < 1 - T^{-B}$ we decompose it into two parts. The first part is over

$$Y_1 = \left\{ y \mid 0 < y < 1 - \frac{1}{T^B}, \frac{(y - y_n)^2 + s_n^2}{1 - y} \gg T^{-2+\epsilon} \right\}$$

and the second one is over

$$Y_2 = \left\{ y \mid 0 < y < 1 - \frac{1}{T^B}, \frac{(y - y_n)^2 + s_n^2}{1 - y} \ll T^{-2+\epsilon} \right\}.$$

The integral over Y_1 is negligible due to (5.28) and (5.12). To estimate the integral over Y_2 we first simplify the set Y_2 . Since $1 - \epsilon_1 < y_n < 1 - T^{-2+\epsilon}$ we show that

$$Y_2 \subset \left\{ y \mid 0 < y < 1 - \frac{1}{T^B}, |y - y_n| \ll T^{-1+\epsilon} \sqrt{1 - y_n} \right\}.$$

Applying (5.13), we prove that the integral over Y_2 can be bounded by

$$\int_{Y_2} \frac{T^3 dy}{1 - y} \ll \int_{|t| \ll T^{-1+\epsilon} \sqrt{1 - y_n}} \frac{T^3 dt}{1 - y_n - t} \ll \frac{T^{2+\epsilon}}{\sqrt{1 - y_n}}. \tag{5.36}$$

It remains to sum (5.36) over n and over $|l| \ll T^{1+\epsilon}$ (see (5.26)). Note that

$$|n|^2 = |2l|^2(y_n^2 + s_n^2) < |2l|^2(1 + O(T^{-2+\epsilon})).$$

Therefore, we obtain

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_6(l)}{|l|^2} \ll \sum_{|l| \ll T^{1+\epsilon}} \sum_{\substack{|n| < |2l|(1+\epsilon), |s_n| \ll T^{-1+\epsilon} \\ T^{-2+\epsilon} < 1-y_n < \epsilon_1}} \frac{|n^2 - 4l^2|^{2\theta}}{|l|^2} \frac{T^{2+\epsilon}}{\sqrt{1-y_n}}. \tag{5.37}$$

As in the previous case, let $c_1 = a - c$, $d_1 = b - d$. Then

$$1 - y_n = \frac{ac_1 + bd_1}{a^2 + b^2}, \quad s_n = \frac{bc_1 - ad_1}{a^2 + b^2}, \quad |2l - n|^2 = c_1^2 + d_1^2.$$

Moreover,

$$|n|^2 = |2l|^2 + c_1^2 + d_1^2 - 2(ac_1 + bd_1).$$

The condition on y_n in (5.37) can be rewritten (and slightly simplified) as

$$0 < ac_1 + bd_1 \ll \epsilon_1 |2l|^2.$$

The condition on $|n|$ in (5.37) can be rewritten (and slightly simplified) as

$$c_1^2 + d_1^2 \ll \epsilon |2l|^2.$$

So (5.37) can be rewritten as

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_6(l)}{|l|^2} \ll T^{2+\epsilon} \sum_{|l| \ll T^{1+\epsilon}} |l|^{-1+2\theta} \sum_{(c_1, d_1) \in \Omega(a, b)} \frac{(c_1^2 + d_1^2)^\theta}{(ac_1 + bd_1)^{1/2}}, \tag{5.38}$$

where

$$\Omega(a, b) = \left\{ (c_1, d_1) \mid \begin{aligned} &0 < ac_1 + bd_1 \ll \epsilon_1 |2l|^2, \\ &c_1^2 + d_1^2 \ll \epsilon |2l|^2, |bc_1 - ad_1| \ll |2l|^2 T^{-1+\epsilon} \end{aligned} \right\}.$$

The sum over $(c_1, d_1) \in \Omega(a, b)$ can be estimated by the double integral over $\Omega(a, b)$. To evaluate this integral we make the change of variables

$$c_1 = x \cos \vartheta_l - y \sin \vartheta_l, \quad d_1 = x \sin \vartheta_l + y \cos \vartheta_l, \quad \vartheta_l = \arg(2l).$$

Using the fact that $a = |2l| \cos \vartheta_l$ and $b = |2l| \sin \vartheta_l$, we infer

$$\sum_{(c_1, d_1) \in \Omega_{a, b}} \frac{(c_1^2 + d_1^2)^{2\theta}}{(ac_1 + bd_1)^{1/2}} \ll \iint_{\Omega(|2l|)} \frac{(x^2 + y^2)^\theta}{(x|2l|)^{1/2}} dx dy,$$

where

$$\Omega(|2l|) = \left\{ (x, y) \mid x^2 + y^2 < \epsilon|2l|^2, |y| \ll |2l|T^{-1+\epsilon}, x > 0 \right\}.$$

Estimating the integrand $x^2 + y^2$ by $|2l|^2$, we obtain

$$\sum_{(c_1, d_1) \in \Omega_{a,b}} \frac{(c_1^2 + d_1^2)^{2\theta}}{(ac_1 + bd_1)^{1/2}} \ll |2l|^{2\theta-1/2} \int_{|y| \ll |2l|T^{-1+\epsilon}} \int_0^{|2l|} \frac{dx dy}{x^{1/2}} \ll |2l|^{1+2\theta} T^{-1+\epsilon}. \tag{5.39}$$

Substituting (5.39) to (5.38), we conclude that

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_6(l)}{|l|^2} \ll T^{1+\epsilon} \sum_{|l| \ll T^{1+\epsilon}} |l|^{4\theta} \ll T^{3+4\theta+\epsilon}.$$

□

LEMMA 5.10. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_7(l)}{|l|^2} \ll T^{-A}.$$

Proof. Since $y_n > 1 + T^{-2+\epsilon}$ and $0 < |s_n| \ll T^{-1+\epsilon}$ we have

$$\min_{0 < y < 1} f_-(n/l, y) = (y_n - 1)^2 + s_n^2 \gg T^{-4-\epsilon},$$

see (5.25) and (5.24). Thus for $1 - T^{-B} < y < 1$ we can apply lemma 5.1 showing that

$$\int_{1-T^{-B}}^1 J_-\left(\frac{n}{l}, y, T\right) \frac{dy}{(1-y^2)^{1/2}} \ll T^{-A}.$$

To estimate the remaining integral over $0 < y < 1 - T^{-B}$ we are going to prove that $x_-(n/l, y) \gg T^{-2+\epsilon}$ (according to (5.12) this would imply that the integral is negligible). For $1 - T^{-2+\epsilon/2} < y < 1$ the following estimate holds

$$\frac{(y - y_n)^2}{1 - y^2} > \frac{T^{-4+2\epsilon}}{T^{-2+\epsilon/2}} = T^{-2+3\epsilon/2}. \tag{5.40}$$

For $0 < y < 1 - T^{-2+\epsilon/2}$ we have

$$\frac{(y - y_n)^2}{1 - y^2} > \frac{(1 + T^{-2+\epsilon} - y)^2}{1 - y} > 1 - y > T^{-2+\epsilon/2}. \tag{5.41}$$

Now the required estimate $x_-(n/l, y) \gg T^{-2+\epsilon}$ follows from (5.25), (5.40) and (5.41). □

LEMMA 5.11. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_8(l)}{|l|^2} \ll T^{-A}.$$

Proof. As in lemma 5.8 (see the proof of (5.34)) we show that

$$1 < y_n < 1 + T^{-2+\epsilon} \Rightarrow |s_n| \gg T^{-1-\epsilon}.$$

Similarly to the previous case, the contribution of the integral over $1 - T^{-B} < y < 1$ is negligible. Furthermore, by (5.25) we have

$$x_-(n/l, y) = \frac{(y_n - y)^2 + s_n^2}{1 - y^2} > \frac{(1 - y)^2 + s_n^2}{1 - y} = 1 - y + \frac{s_n^2}{1 - y} \gg s_n \gg T^{-1-\epsilon}.$$

Thus by (5.12) the contribution of the integral over $0 < y < 1 - T^{-B}$ is also negligible. □

LEMMA 5.12. *The following estimate holds*

$$\sum_{|l| \ll T^{1+\epsilon}} \frac{S_9(l)}{|l|^2} \ll T^{-A}.$$

Proof. As in lemma 5.6 we obtain that

$$y_n - 1 \gg \frac{1}{T^{1+\epsilon}}, \tag{5.42}$$

and therefore, we conclude that the contribution of $1 - T^{-B} < y < 1$ is negligibly small. Next, using (5.25) and (5.42) we obtain

$$x_-(n/l, y) = \frac{(y_n - y)^2 + s_n^2}{1 - y^2} > \frac{((1 - y) + (y_n - 1))^2}{1 - y} > 1 - y + (y_n - 1) \gg T^{-1-\epsilon}.$$

Finally, by (5.12) the contribution of the integral over $0 < y < 1 - T^{-B}$ is negligibly small. □

6. Proof of theorem 1.1

Assume that for some $\alpha > 0$

$$\sum_{T < r_j < 2T} \alpha_j |L(\text{sym}^2 u_j, s)|^2 \ll T^{3+2\alpha}.$$

Then following the arguments on [1, p. 5363] we obtain

$$\sum_{0 < r_j \leq T} X^{ir_j} \ll T^{(7+2\alpha)/4+\epsilon} X^{1/4+\epsilon} + T^2.$$

Using this estimate and following the proof of [4, corollary 1.4] we conclude that the error term for $\pi_\Gamma(X)$ is bounded by

$$O\left(\frac{X^{2+\alpha/2+\epsilon}}{Y^{3/4+\alpha/2}} + X^{(4\theta+6)/5+\epsilon} Y^{2/5} + \frac{X^2}{Y} + X^{1+\epsilon}\right).$$

Choosing

$$Y = X^{(10\alpha+16(1-\theta))/(23+10\alpha)},$$

we have

$$O(X^{3/2+(\alpha(2+16\theta)+24\theta-1)/(46+20\alpha)+\epsilon}).$$

Theorem 1.3 allows us to take $\alpha = 2\theta$ which yields theorem 1.1.

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