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A NOTE ON A GENERALIZED EHRENFEST URN MODEL: ANOTHER LOOK AT THE MEAN TRANSITION TIMES

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Abstract

This note is motivated by Blom's work in 1989. We consider a generalized Ehrenfest urn model in which a randomly-chosen ball has a positive probability of moving from one urn to the other urn. We use recursion relations between the mean transition times to derive formulas in terms of finite sums, which are shown to be equivalent to the definite integrals obtained by Blom.

Keywords: Ehrenfest urn model; transition time

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1. Introduction

The Ehrenfest urn model has two urns, urn I and urn II with M balls distributed in the two urns. At time $t = 1, 2, 3 \ldots,$ one of the M balls is selected at random and placed in the other urn. The number of balls in urn I is recorded as a *state* at each time t. Blom [1] represented the mean transition time from a state to another state under the Ehrenfest urn model as a definite integral and justified the result by claiming the definite integral satisfies the recursion relations between the mean transition times. In fact, those recursion relations can directly lead to a finite sum formula for the mean transition time between two consecutive states. In addition to deriving this finite sum formula and showing its equivalence to the definite integral, we generalize the result by allowing the chosen ball to move from one urn to the other with a probability p, $0 < p \leq 1$.

2. Finite sum formula for the mean transition time

Let e_k denote the mean transition time, the expected number of steps needed to move from state k to state $k + 1$ $(0 \leq k \leq M - 1)$ for the first time. It is easy to see that under the generalized Ehrenfest urn model $e_0 = 1/p$, and for $k = 1, 2, \ldots, M - 1$, e_k can be expressed recursively as

$$e_k = \frac{M + pke_{k-1}}{p(M - k)}, \qquad k = 1, 2, \ldots, M - 1. \tag{1}$$

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2.1. Finite sum formula for e_k

We use the recursion relation in (1) to write the first few terms of e_k explicitly:

$$
\begin{aligned}
e_0 &= \frac{1}{p},\\
e_1 &= \frac{M+1}{p(M-1)},\\
e_2 &= \frac{M+2pe_1}{p(M-2)} = \frac{M(M-1)+2(M+1)}{p(M-1)(M-2)} = \frac{M(M-1)+2M+2}{p(M-1)(M-2)},\\
e_3 &= \frac{M+3pe_2}{p(M-3)} = \frac{M(M-1)(M-2)+3M(M-1)+3\times 2M+3\times 2)}{p(M-1)(M-2)(M-3)},\\
e_4 &= \frac{M+4pe_3}{p(M-4)}\\
&= \frac{M(M-1)(M-2)(M-3)+4M(M-1)(M-2)+4\times 3M(M-1)+4\times 3\times 2M+4\times 3\times 2}{p(M-1)(M-2)(M-3)(M-4)}.
\end{aligned}
$$

Note that for each $k \geq 1$, the numerator of e_k is a sum of $k+1$ terms with each term a polynomial in M. After examining those terms and taking the denominator into consideration, we express e_k for $k = 1, 2, \ldots, M-1$ as

$$
e_k = \frac{1}{p}\sum_{j=0}^{k} \frac{k!}{(k-j)!}\,\frac{M!/(M-k+j)!}{(M-1)!/(M-k-1)!} = \frac{1}{p\binom{M-1}{k}}\sum_{j=0}^{k}\binom{M}{j}. \tag{2}
$$

Note that this finite sum formula holds for $e_0 = 1/p$ as well.

2.2. Equivalence to Blom's definite integral

Now we show that the finite sum formula in (2) with $p = 1$ is equivalent to the definite integral

$$
e_k = M\int_0^1 x^{M-k-1}(2-x)^k \mathrm{d}x
$$

obtained by Blom [1]. We can express this definite integral as

$$
\begin{aligned}
e_k &= M\int_0^1 x^{M-k-1}[1+(1-x)]^k \mathrm{d}x\\
&= M\int_0^1 x^{M-k-1}\sum_{j=0}^{k}\binom{k}{j}(1-x)^j \mathrm{d}x\\
&= M\sum_{j=0}^{k}\binom{k}{j}\int_0^1 x^{M-k-1}(1-x)^j \mathrm{d}x\\
&= M\sum_{j=0}^{k}\binom{k}{j}\frac{(M-k-1)!\,j!}{(M-k+j)!} \qquad (3)\\
&= \sum_{j=0}^{k}\frac{k!}{(k-j)!}\,\frac{M!/(M-k+j)!}{(M-1)!/(M-k-1)!}. \qquad (4)
\end{aligned}
$$

We use the beta function to obtain the equality in (3). It can be seen clearly that the finite sum in (4) is equal to that in (2) with $p = 1$, and this establishes the equivalence.

3. Concluding remarks

The finite sum formula in (2) was also derived by Palacios [2] with $p = 1$ via electric networks. Palacios remarks that it seems harder to derive the finite sum formula as the solution to the recursion relation in (1). In this note we respond to his remark. Moreover, in this note we have shown that we can simply multiply the finite sum by $1/p$ to obtain the formula for the mean transition time between two consecutive states under the generalized Ehrenfest urn model. This result can be seen through Palacios' electric network approach by increasing the effective resistance between each pair of vertices in the electric network by a constant factor of $1/p$.

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