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Titchmarsh's Method for the Approximate Functional Equations for $\zeta'(s)^2$, $\zeta(s)\zeta''(s)$, and $\zeta'(s)\zeta''(s)$

Jun Furuya, T. Makoto Minamide, and Yoshio Tanigawa

Abstract. Let $\zeta(s)$ be the Riemann zeta function. In 1929, Hardy and Littlewood proved the approximate functional equation for $\zeta^2(s)$ with error term $O(x^{1/2-\sigma}((x+y)/|t|)^{1/4}\log|t|)$, where $-1/2<\sigma<3/2$, $x,y\ge 1$, $xy=(|t|/2\pi)^2$. Later, in 1938, Titchmarsh improved the error term by removing the factor $((x+y)/|t|)^{1/4}$. In 1999, Hall showed the approximate functional equations for $\zeta'(s)^2, \zeta(s)\zeta''(s)$, and $\zeta'(s)\zeta''(s)$ (in the range $0<\sigma<1$) whose error terms contain the factor $((x+y)/|t|)^{1/4}$. In this paper we remove this factor from these three error terms by using the method of Titchmarsh.

1 Introduction

Hardy and Littlewood left us much important work on the Riemann zeta function $\zeta(s)$. The following approximate functional equation for $\zeta(s)^2$ is one of them [12, p. 90, Theorem 2].

Theorem A *Let A be a positive number, and $s=\sigma+it$. If $-1/2\le\sigma\le 3/2$, $x>A$, $y>A$, and $xy=(t/2\pi)^2$, then*

$$\zeta(s)^2=\sum_{n\le x}\frac{d(n)}{n^s}+\chi(s)^2\sum_{n\le y}\frac{d(n)}{n^{1-s}}+O\Big(x^{1/2-\sigma}\Big(\frac{x+y}{|t|}\Big)^{1/4}\log|t|\Big) \tag{1.1}$$

with

$$\chi(s)=2(2\pi)^{s-1}\sin\Big(\frac{s\pi}{2}\Big)\Gamma(1-s) \tag{1.2}$$

and $d(n)=\sum_{d|n}1$.

In the case $s=1/2+it$, $t>2$, $x=y=t/2\pi$, formula (1.1) has the form

$$\zeta\Big(\frac12+it\Big)^2=\sum_{n\le\frac{t}{2\pi}}\frac{d(n)}{n^{\frac12+it}}+i\Big(\frac{t}{2\pi e}\Big)^{-2it}\sum_{n\le\frac{t}{2\pi}}\frac{d(n)}{n^{\frac12-it}}+O(\log t) \tag{1.3}$$

(for $i(t/2\pi e)^{-2it}$, see (2.19) below).

Ingham [14] applied formula (1.3) to obtain an asymptotic formula of the fourth power moment of $\zeta(s)$ on the critical line, though Hardy and Littlewood [11] had

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already proved its upper bound by using their approximate functional equation of $\zeta(s)$. It seems that (1.3) was already known before the publication of [12] (see [14, Introduction]). As is well known, Ingham's result is fundamental in studies on zero density estimates for $\zeta(s)$ or the divisor problem. Heath-Brown improved Ingham's result [13].

In 1938, Titchmarsh succeeded in improving the error term of (1.1).

Theorem B ([22, p. 109]) *Let σ and t be the same notation of Theorem A. If $0 \le \sigma \le 1$, $x > A$, $y > A$, and $xy = (|t|/2\pi)^2$, then*

$$\zeta(s)^2 = \sum_{n \le x} \frac{d(n)}{n^s} + \chi(s)^2 \sum_{n \le y} \frac{d(n)}{n^{1-s}} + O(x^{1/2-\sigma} \log |t|). \tag{1.4}$$

Titchmarsh [22] introduced an exact formula for $\zeta(s)^2$ involving the integral

$$\int_{4\pi\sqrt{nx}}^{\infty} \frac{K_1(\nu) + \frac{\pi}{2} Y_1(\nu)}{\nu^{2s}}\, d\nu, \tag{1.5}$$

where $K_1(\nu)$ and $Y_1(\nu)$ are the Bessel functions. Analyzing the integral and applying [12, Lemma β], he could remove the factor $((x+y)/|t|)^{1/4}$ from (1.1). Another proof was given by Ivić [15] using the Voronoï summation formula (see also [15, Chapter 4 Note] for much valuable information on the approximate functional equation for the Riemann zeta function).

As another aspect of the study on $\zeta(s)$, there is much research on the derivative of $\zeta(s)$, for example, Ingham [14], Berndt [5], Conrey [6], Gonek [9], Hall [10], Levinson and Montgomery [16], Speiser [18], and Spira [19–21], More recently the related subjects were also studied in Akatsuka [1], Aoki and Minamide [3], Banerjee and Minamide [4], Furuya, Minamide, and Tanigawa [7, 8] and Minamide [17].

Hall studied the distribution of real positive zeros of Hardy's function

$$Z(t) := e^{i\theta(t)} \zeta\big(\tfrac{1}{2} + it\big) = \Big\{ \pi^{-it} \frac{\Gamma\big(\frac{1}{4} + i\frac{t}{2}\big)}{\Gamma\big(\frac{1}{4} - i\frac{t}{2}\big)} \Big\}^{1/2} \zeta\big(\tfrac{1}{2} + it\big).$$

In fact, he showed [10, Theorem 1] that

$$\limsup_{n \to \infty} \frac{t_{n+1} - t_n}{(2\pi/\log T)} \ge \Big(\frac{105}{4}\Big)^{1/4},$$

where t_n $(n = 1, 2, 3, \dots)$ denote distinct positive zeros of $Z(t)$ that are arranged in non-decreasing order. For this result he compared the mean values $\int_0^T Z(t)^4\, dt$ and $\int_0^T Z'(t)^4\, dt$, in particular, he used the following asymptotic formula [10, Theorem 5]:

$$\int_0^T Z'(t)^4\, dt = \frac{1}{1120\pi^2} T \log^8 T + O(T \log^7 T).$$

To show this, he applied Ingham's method to the approximate functional equation of $Z'(s)^2$, which is derived by those of $\zeta(s)\zeta'(s)$ and $\zeta'(s)^2$. The approximate functional equation of $\zeta'(s)^2$ is formulated as follows.

Theorem C ([10, p. 298, Theorem 8]) *Let $0<\sigma<1$, $t\geq 2\pi$, $4\pi^2xy=t^2$, $x\geq 1$, $y\geq 1$. Then*

$$\begin{aligned}(1.6)\quad \zeta'(s)^2 &= \sum_{n\leq x}\frac{D_{(1)}(n)}{n^s}\\ &\quad+\chi(s)^2\sum_{n\leq y}\frac{1}{n^{1-s}}\Big(D_{(1)}(n)-d(n)\log n\log\frac{t}{2\pi}+d(n)\log^2\frac{t}{2\pi}\Big)\\ &\quad+O\Big(x^{1/2-\sigma}\Big(\frac{x+y}{t}\Big)^{1/4}\log^3 t\Big),\end{aligned}$$

where $D_{(1)}(n)$ is the function defined by $D_{(1)}(n)=\sum_{d|n}(\log d)(\log\frac{n}{d})$.

The approximate functional equation for $Z'(s)^2$ becomes the following.

Theorem D ([10, p. 307, Lemma 7 (186)]) *Let $t\geq 2\pi$, $x\geq 1$, $y\geq 1$, $4\pi^2xy=t^2$. Then*

(1.7)

$$\begin{aligned}Z'(t)^2 &= -e^{2i\theta(t)}\sum_{n\leq x}\frac{1}{n^{1/2+it}}\Big\{D_{(1)}(n)-\frac{1}{2}d(n)\log n\log\frac{t}{2\pi}+\frac{1}{4}d(n)\log^2\frac{t}{2\pi}\Big\}\\ &\quad-e^{-2i\theta(t)}\sum_{n\leq y}\frac{1}{n^{1/2-it}}\Big\{D_{(1)}(n)-\frac{1}{2}d(n)\log n\log\frac{t}{2\pi}+\frac{1}{4}d(n)\log^2\frac{t}{2\pi}\Big\}\\ &\quad+O\Big(\Big(\frac{x+y}{t}\Big)^{1/4}\log^3 t\Big).\end{aligned}$$

Hall employed the method of Hardy and Littlewood for the proof of (1.6), hence the factor $((x+y)/|t|)^{1/4}$ appears in the error term. Though he used the formula in the case $x=y=t/2\pi$ in the application of the mean value of $Z'(t)^4$, he remarked that it would be desirable to remove the factor $((x+y)/|t|)^{1/4}$ in the above error term by applying the method of Titchmarsh [22]. However, he mentioned that "the matter is not straightforward because Titchmarsh's method leads us to certain functions, arising as integrals, for which asymptotic formulae are required and which in our case are not simple combinations of Bessel functions". In this paper we tackle this difficulty. Combining the methods of [9, 17, 22], we will prove the following theorem.

Theorem 1.1 *Let $0<\sigma<1$, $|t|\geq 2\pi$, $xy=(t/2\pi)^2$, $x\geq 1$, $y\geq 1$. Then, uniformly in σ, we have*

$$\begin{aligned}(1.8)\quad \zeta'(s)^2 &= \sum_{n\leq x}\frac{D_{(1)}(n)}{n^s}\\ &\quad+\chi(s)^2\sum_{n\leq y}\frac{1}{n^{1-s}}\Big(D_{(1)}(n)-d(n)\log n\log\frac{|t|}{2\pi}+d(n)\log^2\frac{|t|}{2\pi}\Big)\\ &\quad+O(x^{1/2-\sigma}\log^3|t|).\end{aligned}$$

By (1.8) it is possible to remove the factor $((x+y)/t)^{1/4}$ from the error term in (1.7).

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For $\zeta(s)\zeta'(s)$ Hall mentioned in the proof of Lemma 3 [10, (63)] that

(1.9)
$$2\zeta(s)\zeta'(s) = -\sum_{n\le x}\frac{d(n)\log n}{n^s} + \chi(s)^2\sum_{n\le y}\frac{d(n)\log(4\pi^2 n/t^2)}{n^{1-s}} + O(x^{1/2-\sigma}\log^2|t|).$$

This formula can also be proved by the method of Titchmarsh and it will be used in the proof of (1.8).

Furthermore, Hall proved the approximate functional equations of $\zeta(s)\zeta''(s)$ and $\zeta'(s)\zeta''(s)$ and of $Z(t)Z''(t)$ and $Z'(t)Z''(t)$, whose error terms contain the factor $((x+y)/|t|)^{1/4}$ as in (1.6). Applying the method of Titchmarsh we can remove this factor. In fact, the approximate functional equation of $\zeta(s)\zeta''(s)$ and $\zeta'(s)\zeta''(s)$ have the following forms.

Theorem 1.2 *Let $0<\sigma<1$, $|t|\ge 2\pi$, $xy=(t/2\pi)^2$, $x\ge1$, $y\ge1$. Then we have*

$$\begin{aligned}\zeta(s)\zeta''(s) &= \sum_{n\le x}\frac{d_{(0,2)}(n)}{n^s}\\ &\quad + \chi(s)^2\sum_{n\le y}\frac{1}{n^{1-s}}\Big(d_{(0,2)}(n) - d(n)\log n\cdot\log\frac{|t|}{2\pi} + d(n)\log^2\frac{|t|}{2\pi}\Big)\\ &\quad + O(x^{1/2-\sigma}\log^3|t|),\end{aligned}\tag{1.10}$$

and

$$\begin{aligned}\zeta'(s)\zeta''(s) = \sum_{n\le x}\frac{d_{(1,2)}(n)}{n^s} + \chi(s)^2\sum_{n\le y}\frac{1}{n^{1-s}}\bigg(&-d_{(1,2)}(n) + d_{(0,2)}(n)\log\frac{|t|}{2\pi}\\ &- d(n)\log^2 n\log\frac{|t|}{2\pi} + \frac{3}{2}d(n)\log n\log^2\frac{|t|}{2\pi}\\ &- d(n)\log^3\frac{|t|}{2\pi}\bigg) + O(x^{1/2-\sigma}\log^4|t|),\end{aligned}\tag{1.11}$$

where

$$d_{(0,2)}(n) = \sum_{d|n}\log^2 d,\tag{1.12}$$

$$d_{(1,2)}(n) = \sum_{d|n}\log^2 d\log\frac{n}{d}.\tag{1.13}$$

In this paper, we focus upon the proof of Theorem 1.1. We will omit the details of the proof of Theorem 1.2, since it can be proved by similar arguments. However we will show the exact formulas for these zeta functions in Theorem 3.2.

For simplicity, we use the following notation in Section 2. Let $F(w)$ be a function defined in an appropriate region and let c be a real number. We define

$$\int_{(c)}F(w)\,dw := \int_{c-i\infty}^{c+i\infty}F(w)\,dw, \qquad ⨏_{(c)}F(w)\,dw := \Big(\int_{c-i\infty}^{c-2i} + \int_{c+2i}^{c+i\infty}\Big)F(w)\,dw,$$
$$⨏_{-\infty}^{\infty}F(c+iy)i\,dy := \Big(\int_{-\infty}^{-2} + \int_{2}^{\infty}\Big)F(c+iy)i\,dy.$$

2 Preliminaries

In this section, following [22], we derive certain expressions of $\zeta'(s)^2$, $\zeta(s)\zeta''(s)$, and $\zeta'(s)\zeta''(s)$ in order to derive their exact formulas.

The arithmetical functions $d_{(0,2)}(n)$ and $d_{(1,2)}(n)$ defined by (1.12) and (1.13) are the coefficients of $\zeta(s)\zeta''(s)$ and $\zeta'(s)\zeta''(s)$, respectively. More generally, we can define $\zeta^{(k)}(s)\zeta^{(l)}(s) = \sum_{n=1}^{\infty} \frac{d_{(k,l)}(n)}{n^s}$, where $\zeta^{(k)}(s)$ denotes the k-th derivatives of $\zeta(s)$ and $\operatorname{Re} s > 1$. Clearly we have $D_{(1)}(n) = d_{(1,1)}(n)$. Since $\zeta'(s)^2$ is the main object of this paper, we will use the notation $D_{(1)}(n)$ for the coefficients of $\zeta'(s)^2$.

The following properties are easily proved.

Lemma 2.1 *Let $d_{(k,l)}(n)$ (k and l are non-negative integers) be defined by the above. Then we have*

$$d_{(0,1)}(n) = -\frac{1}{2}d(n)\log n, \tag{2.1}$$

$$D_{(1)}(n) = -d_{(0,1)}(n)\log n - d_{(0,2)}(n) = \frac{1}{2}d(n)\log^2 n - d_{(0,2)}(n), \tag{2.2}$$

$$d_{(1,2)}(n) = \frac{1}{2}d_{(0,2)}(n)\log n - \frac{1}{4}d(n)\log^3 n = -\frac{1}{2}D_{(1)}(n)\log n. \tag{2.3}$$

We need the asymptotic formulas of the sum of these functions.

Lemma 2.2 ([7], [17, p. 346]) *We have*

$$\sum_{n\le x} d_{(0,1)}(n) = -\frac{1}{2}x\log^2 x - (\gamma-1)x\log x + (\gamma-1)x + O(x^{1/3}), \tag{2.4}$$

$$\sum_{n\le x} D_{(1)}(n) = \frac{x\log^3 x}{3!} - \frac{x\log^2 x}{2!} + \frac{(1-2\gamma_1)x\log x}{1!} + (2\gamma_1 - 4\gamma_2 - 1)x + O(x^{1/3}), \tag{2.5}$$

$$\sum_{n\le x} d_{(0,2)}(n) = \frac{1}{3}x\log^3 x + (\gamma-1)x\log^2 x - 2(\gamma-\gamma_1-1)x\log x + 2(\gamma-\gamma_1+2\gamma_2-1)x + O(x^{1/3}), \tag{2.6}$$

$$\sum_{n\le x} d_{(1,2)}(n) = -\frac{1}{12}x\log^4 x + \frac{1}{3}x\log^3 x + (\gamma_1-1)x\log^2 x - 2(\gamma_1-\gamma_2-1)x\log x + 2(\gamma_1-\gamma_2-1)x + O(x^{1/3}), \tag{2.7}$$

where γ is the Euler constant and γ_j are the coefficients of the Laurent expansion of $\zeta(s)$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \gamma_2(s-1)^2 + \cdots. \tag{2.8}$$

We prepare the following functions.

Definition 2.3 Let $0 < \varepsilon < 1/4$, $\alpha = 1 + \varepsilon$, $\mathrm{Re}\, s = \sigma > -\varepsilon$, $x > 2$, and n is a positive integer. We define

$$\mathcal{L}_1(s;n) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\chi(1-w)^2}{n^w}\frac{sx^{1-w-s}}{(1-w)(s-1+w)}\,dw,$$
$$\mathcal{L}_2(s;n) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\chi(1-w)\chi'(1-w)}{n^w}\frac{sx^{1-w-s}}{(1-w)(s-1+w)}\,dw,$$
$$\mathcal{L}_3(s;n) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\chi'(1-w)^2}{n^w}\frac{sx^{1-w-s}}{(1-w)(s-1+w)}\,dw,$$
$$\mathcal{L}_4(s;n) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\chi(1-w)\chi''(1-w)}{n^w}\frac{sx^{1-w-s}}{(1-w)(s-1+w)}\,dw,$$
$$\mathcal{L}_5(s;n) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\chi'(1-w)\chi''(1-w)}{n^w}\frac{sx^{1-w-s}}{(1-w)(s-1+w)}\,dw.$$

Remark 2.4 In fact, by the Stirling formula and the formula (2.20) below, we observe that the integrals $\mathcal{L}_j(s;n)$ $(j = 1,\dots,5)$ are absolutely convergent for $0 < \alpha < 1$ and $\alpha \neq 1-\sigma$, and we can see that these integrals are convergent in the range $0 < \alpha < 3/2$, $\alpha \neq 1$ and $\alpha \neq 1-\sigma$. In this paper, we will use these integrals over the line (α) with $\alpha = 1+\varepsilon$. The assumption $\sigma > -\varepsilon$ is not needed for the existence of the above integrals, however, it is necessary for the following lemma.

Henceforth, we will discuss several representations of $\zeta'(s)^2$, $\zeta(s)\zeta''(s)$, $\zeta'(s)\zeta''(s)$.

Lemma 2.5 *For $s = \sigma + it \neq 1$, $\sigma > -\varepsilon$ $(0 < \varepsilon < 1/4)$, $|t| \geq 2$, and $x > 2$, we have*

$$\zeta'(s)^2 = \sum_{n\leq x}\frac{D_{(1)}(n)}{n^s} + \sum_{n=1}^{\infty} D_{(1)}(n)\mathcal{L}_1(s;n) - 2\sum_{n=1}^{\infty} d_{(0,1)}(n)\mathcal{L}_2(s;n) + \sum_{n=1}^{\infty} d(n)\mathcal{L}_3(s;n) + O(|t|^{-1}x^{1-\sigma}\log^3 x) + O(x^{1/3-\sigma}), \tag{2.9}$$

$$\zeta(s)\zeta''(s) = \sum_{n\leq x}\frac{d_{(0,2)}(n)}{n^s} + \sum_{n=1}^{\infty} d_{(0,2)}(n)\mathcal{L}_1(s;n) - 2\sum_{n=1}^{\infty} d_{(0,1)}(n)\mathcal{L}_2(s;n) + \sum_{n=1}^{\infty} d(n)\mathcal{L}_4(s;n) + O(|t|^{-1}x^{1-\sigma}\log^3 x) + O(x^{1/3-\sigma}), \tag{2.10}$$

$$\begin{aligned}\zeta'(s)\zeta''(s) = {} & \sum_{n\leq x}\frac{d_{(1,2)}(n)}{n^s} - \sum_{n=1}^{\infty} d_{(1,2)}(n)\mathcal{L}_1(s;n) \\ & + \sum_{n=1}^{\infty}(d_{(0,2)}(n) + 2D_{(1)}(n))\mathcal{L}_2(s;n) \\ & - \sum_{n=1}^{\infty} d_{(0,1)}(n)(2\mathcal{L}_3(s;n) + \mathcal{L}_4(s;n)) + \sum_{n=1}^{\infty} d(n)\mathcal{L}_5(s;n) \\ & + O(|t|^{-1}x^{1-\sigma}\log^4 x) + O(x^{1/3-\sigma}),\end{aligned} \tag{2.11}$$

where these implied constants are independent of σ.

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Proof First we assume that $\sigma > 1$ and $1 < c < \sigma$. Let

$$I^{(k,l)} = \frac{1}{2\pi i}\int_{(c)} \zeta^{(k)}(w)\zeta^{(l)}(w)\frac{sx^{w-s}}{w(s-w)}\,dw,$$

where k and l are non-negative integers. It is easy to see that

$$I^{(k,l)} = \zeta^{(k)}(s)\zeta^{(l)}(s) - \sum_{n\le x}\frac{d_{(k,l)}(n)}{n^s} + x^{-s}\sum_{n\le x} d_{(k,l)}(n).$$

Let ε be a constant satisfying $0 < \varepsilon < 1/4$. Shifting the line of integration from (c) to $(-\varepsilon)$, we find that

$$I^{(k,l)} = \frac{1}{2\pi i}\int_{(-\varepsilon)} \zeta^{(k)}(w)\zeta^{(l)}(w)\frac{sx^{w-s}}{w(s-w)}\,dw + \zeta^{(k)}(0)\zeta^{(l)}(0)x^{-s} + \operatorname*{Res}_{w=1}\zeta^{(k)}(w)\zeta^{(l)}(w)\frac{sx^{w-s}}{w(s-w)}.$$

Let us consider the case $k = l = 1$. By computing the residues explicitly, we have

$$\begin{aligned}\zeta'(s)^2 = {}&\sum_{n\le x}\frac{D_{(1)}(n)}{n^s} + \frac{1}{2\pi i}\int_{(-\varepsilon)}\zeta'(w)^2\frac{sx^{w-s}}{w(s-w)}\,dw \\ &- x^{-s}\sum_{n\le x}D_{(1)}(n) + \zeta'(0)^2x^{-s} \\ &+ \Big(\frac{1}{s-1}+1\Big)\frac{x^{1-s}\log^3 x}{3!} + \Big(\frac{1}{(s-1)^2}-1\Big)\frac{x^{1-s}\log^2 x}{2!} \\ &+ \Big(\frac{1}{(s-1)^3} - \frac{2\gamma_1}{s-1} - 2\gamma_1 + 1\Big)\frac{x^{1-s}\log x}{1!} \\ &+ \Big(\frac{1}{(s-1)^4} - \frac{2\gamma_1}{(s-1)^2} - \frac{4\gamma_2}{s-1} + 2\gamma_1 - 4\gamma_2 - 1\Big)x^{1-s},\end{aligned} \tag{2.12}$$

where γ and γ_j are defined by (2.8). By analytic continuation, (2.12) is valid for $\sigma > -\varepsilon$, except for $s = 1$. Substituting (2.5) in the right-hand side of (2.12) we find that

$$\begin{aligned}\zeta'(s)^2 = {}&\sum_{n\le x}\frac{D_{(1)}(n)}{n^s} + \frac{1}{2\pi i}\int_{(-\varepsilon)}\zeta'(w)^2\frac{sx^{w-s}}{w(s-w)}\,dw \\ &+ O(|t|^{-1}x^{1-\sigma}\log^3 x) + O(x^{1/3-\sigma}).\end{aligned} \tag{2.13}$$

It is well known that the Riemann zeta function satisfies the functional equation:

$$\zeta(w) = \chi(w)\zeta(1-w), \tag{2.14}$$

where $\chi(s)$ is the function defined by (1.2). Differentiating both sides of (2.14) and squaring, we have

$$\zeta'(w)^2 = \chi(w)^2\zeta'(1-w)^2 - 2\chi(w)\chi'(w)\zeta(1-w)\zeta'(1-w) + \chi'(w)^2\zeta(1-w)^2. \tag{2.15}$$

Substituting (2.15) in the integrand of (2.13), replacing w by $1-w$, and expanding zeta functions that appear we obtain the formula (2.9).

The other cases are similar. In fact, we have

$$\begin{aligned}\zeta(s)\zeta''(s) &= \sum_{n\le x}\frac{d_{(0,2)}(n)}{n^s} + \frac{1}{2\pi i}\int_{(-\varepsilon)}\zeta(w)\zeta''(w)\frac{sx^{w-s}}{w(s-w)}\,dw \\ &\quad - x^{-s}\sum_{n\le x} d_{(0,2)}(n) + \zeta(0)\zeta''(0)x^{-s} \\ &\quad + \frac{1}{3}\Big(\frac{1}{s-1}+1\Big)x^{1-s}\log^3 x + \Big(\frac{1}{(s-1)^2}+\frac{\gamma}{s-1}-1+\gamma\Big)x^{1-s}\log^2 x \\ &\quad + 2\Big(\frac{1}{(s-1)^3}+\frac{\gamma}{(s-1)^2}+\frac{\gamma_1}{(s-1)}+1-\gamma+\gamma_1\Big)x^{1-s}\log x \\ &\quad + 2\Big(\frac{2\gamma_2}{(s-1)^4}+\frac{\gamma}{(s-1)^3}+\frac{\gamma_1}{(s-1)^2}+\frac{\gamma_2}{s-1}-1+\gamma-\gamma_1+\gamma_2\Big)x^{1-s} \\ &= \sum_{n\le x}\frac{d_{(0,2)}(n)}{n^s} + \frac{1}{2\pi i}\int_{(-\varepsilon)}\zeta(w)\zeta''(w)\frac{sx^{w-s}}{w(s-w)}\,dw \\ &\quad + O(|t|^{-1}x^{1-\sigma}\log^3 x) + O(x^{1/3-\sigma}),\end{aligned}$$

and

$$\begin{aligned}\zeta'(s)\zeta''(s) &= \sum_{n\le x}\frac{d_{(1,2)}(n)}{n^s} + \frac{1}{2\pi i}\int_{(-\varepsilon)}\zeta'(w)\zeta''(w)\frac{sx^{w-s}}{w(s-w)}\,dw \\ &\quad - x^{-s}\sum_{n\le x} d_{(1,2)}(n) + \zeta'(0)\zeta''(0)x^{-s} \\ &\quad - \frac{1}{12}\Big(\frac{1}{s-1}+1\Big)x^{1-s}\log^4 x - \frac{1}{3}\Big(\frac{1}{(s-1)^2}-1\Big)x^{1-s}\log^3 x \\ &\quad - \Big(\frac{1}{(s-1)^3}-\frac{\gamma_1}{s-1}-\gamma_1+1\Big)x^{1-s}\log^2 x \\ &\quad - 2\Big(\frac{1}{(s-1)^4}-\frac{\gamma_1}{(s-1)^2}-\frac{\gamma_2}{s-1}-1+\gamma_1-\gamma_2\Big)x^{1-s}\log x \\ &\quad - 2\Big(\frac{1}{(s-1)^5}-\frac{\gamma_1}{(s-1)^3}-\frac{\gamma_2}{(s-1)^2}+1-\gamma_1+\gamma_2\Big)x^{1-s} \\ &= \sum_{n\le x}\frac{d_{(1,2)}(n)}{n^s} + \frac{1}{2\pi i}\int_{(-\varepsilon)}\zeta'(w)\zeta''(w)\frac{sx^{w-s}}{w(s-w)}\,dw \\ &\quad + O(|t|^{-1}x^{1-\sigma}\log^4 x) + O(x^{1/3-\sigma}),\end{aligned}$$

where in the second and fourth equalities we have used (2.6) and (2.7), respectively. Substituting the functional equations

$$\begin{aligned}\zeta(w)\zeta''(w) &= \chi(w)^2\zeta(1-w)\zeta''(1-w) - 2\chi(w)\chi'(w)\zeta(1-w)\zeta'(1-w) \\ &\quad + \chi(w)\chi''(w)\zeta(1-w)^2,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\zeta'(w)\zeta''(w) &= -\chi(w)^2\zeta'(1-w)\zeta''(1-w) \\ &\quad + \chi(w)\chi'(w)\Big(\zeta(1-w)\zeta''(1-w) + 2\zeta'(1-w)^2\Big) \\ &\quad - \Big(2\chi'(w)^2 + \chi(w)\chi''(w)\Big)\zeta(1-w)\zeta'(1-w) \\ &\quad + \chi'(w)\chi''(w)\zeta(1-w)^2\end{aligned} \tag{2.17}$$

in the above integrands and expanding the resulting Dirichlet series, we have (2.10) and (2.11). ∎

We recall the asymptotic formulas for $\chi(s)$, $\chi(s)^2$, and $\chi^{(k)}(s)$ to modify $\mathcal{L}_j(s;n)$ to a more suitable form.

Lemma 2.6 (Titchmarsh [23, p. 78], Gonek [9, p. 9]) *For $\sigma \in [a,b]$ (a bounded interval), $|t| \geq 2$, and $|(\sigma-1)/t| < 1$, we have*

$$\chi(\sigma+it) = \Big(\frac{|t|}{2\pi}\Big)^{\frac{1}{2}-\sigma-it} e^{i(t+\operatorname{sgn}(t)\frac{\pi}{4})}\Big(1+O\Big(\frac{1}{|t|}\Big)\Big), \tag{2.18}$$

$$\chi(\sigma+it)^2 = \Big(\frac{|t|}{2\pi}\Big)^{1-2\sigma} e^{iE(t)}\Big(1+O\Big(\frac{1}{|t|}\Big)\Big), \tag{2.19}$$

$$\chi^{(k)}(\sigma+it) = \chi(\sigma+it)\Big(-\log\frac{|t|}{2\pi}\Big)^k + O\Big(|t|^{-\frac{1}{2}-\sigma}\Big(\log|t|\Big)^{k-1}\Big), \tag{2.20}$$

where

$$E(t) = -2t\log\frac{|t|}{2\pi} + 2t + \operatorname{sgn}(t)\frac{\pi}{2}, \tag{2.21}$$

and $\operatorname{sgn}(t) = t/|t|$ *is the signature of t. The implied constants are independent of σ.*

Moreover, we need the following lemma, which can be easily shown.

Lemma 2.7 *Let k be a non-negative integer, $s = \sigma + it$, $\alpha = 1+\varepsilon$, $0 < \varepsilon < 1/4$, and $\sigma \in (-\varepsilon, b]$ (b is a constant) and $|t| \geq 2$. We have*

$$⨍_{-\infty}^{\infty} |\beta|^{2\varepsilon}\log^k|\beta|\Big|\frac{s}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}\Big|\,d\beta \ll |t|^{2\varepsilon}\log^{k+1}|t|, \tag{2.22}$$

where the implied constant in the symbol $\ll$ depends on k, σ, and ε. Here the meaning of $⨍_{-\infty}^{\infty} F(w)\,dw$ is given at the end of Section 1.

In order to express $\mathcal{L}_j(s;n)$ in a more appropriate form, it is convenient to introduce the following functions.

Definition 2.8 Let $s = \sigma + it$, $\alpha = 1+\varepsilon$, $0 < \varepsilon < 1/4$, and $\sigma \in (-\varepsilon, b]$ (b is a constant) and $|t| > 2$. We define

$$L_j(s;n) = \frac{1}{2\pi i}⨍_{-\infty}^{\infty}\frac{1}{n^{\alpha+i\beta}}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha} e^{iE(-\beta)}\frac{sx^{1-\alpha-i\beta-s}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)^j}\,i\,d\beta,$$

where $E(t)$ is the function defined by (2.21).

The relations among $\mathcal{L}_j(s;n)$ and $L_j(s;n)$ are given in the next lemma.

Lemma 2.9 *Let $\alpha = 1+\varepsilon$, $0 < \varepsilon < 1/4$, and $s = \sigma + it$ $(|t| \geq 2)$. Assume that $\sigma \in (-\varepsilon, b]$ (b is a constant). Then we have*

(2.23)
$$\mathcal{L}_1(s;n) = L_1(s;n) + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|}{n^\alpha}\Big),$$
(2.24)
$$\mathcal{L}_2(s;n) = -\frac{1}{2}\big(\log nx\, L_1(s;n) + L_2(s;n)\big) + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log^2|t|}{n^\alpha}\Big),$$
(2.25)
$$\begin{aligned}\mathcal{L}_3(s;n) &= \frac{1}{4}(\log nx)^2 L_1(s;n) + \frac{1}{2}\log nx\, L_2(s;n) + \frac{1}{2}L_3(s;n)\\ &\quad + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|(\log^2|t| + \log nx)}{n^\alpha}\Big),\end{aligned}$$
(2.26)
$$\begin{aligned}\mathcal{L}_4(s;n) &= \frac{1}{4}(\log nx)^2 L_1(s;n) + \frac{1}{2}\log nx\, L_2(s;n) + \frac{1}{2}L_3(s;n)\\ &\quad + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|(\log^2|t| + \log nx)}{n^\alpha}\Big),\end{aligned}$$
(2.27)
$$\begin{aligned}\mathcal{L}_5(s;n) &= -\frac{1}{8}(\log nx)^3 L_1(s;n) - \frac{3}{8}(\log nx)^2 L_2(s;n) - \frac{3}{4}\log nx L_3(s;n)\\ &\quad - \frac{3}{4}L_4(s;n) + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|\big(\log^3|t| + \log|t|\cdot\log nx + \log^2 nx\big)}{n^\alpha}\Big).\end{aligned}$$

Proof To deduce the formula (2.23), we use (2.19) and (2.22). Then we have
$$\begin{aligned}\mathcal{L}_1(s;n) &= \frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{1}{n^{\alpha+i\beta}}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}e^{iE(-\beta)}\frac{sx^{1-\alpha-i\beta-s}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\,d\beta\\ &\quad + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|}{n^\alpha}\Big).\end{aligned}$$

For (2.24), by (2.18), (2.19), and (2.20), we first observe that
$$\begin{aligned}\chi(s)\chi'(s) &= \chi(s)^2\Big(-\log\frac{|t|}{2\pi}\Big) + O\Big(|\chi(s)||t|^{-\frac{1}{2}-\sigma}\Big)\\ &= \Big(\frac{|t|}{2\pi}\Big)^{1-2\sigma}e^{iE(t)}\Big(1 + O\Big(\frac{1}{|t|}\Big)\Big)\Big(-\log\frac{|t|}{2\pi}\Big) + O\Big(|t|^{-2\sigma}\Big)\\ &= \Big(\frac{|t|}{2\pi}\Big)^{1-2\sigma}e^{iE(t)} + O\Big(|t|^{-2\sigma}\log|t|\Big).\end{aligned}$$
Then replacing s by $1-w = -\alpha - i\beta = -\varepsilon - i\beta$ and applying Lemma 2.7, we get
$$\begin{aligned}\mathcal{L}_2(s;n) &= \frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{1}{n^{\alpha+i\beta}}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\Big(-\log\frac{|\beta|}{2\pi}\Big)e^{iE(-\beta)}\\ &\quad\times\frac{sx^{1-\alpha-i\beta-s}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\,d\beta + O\Big(\frac{x^{1-\sigma-\alpha}|t|^{2\varepsilon}\log^2|t|}{n^\alpha}\Big).\end{aligned}$$

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Now noting $\frac{d}{d\beta}(-e^{iE(-\beta)}/2i) = (-\log(|\beta|/2\pi))e^{iE(-\beta)}$ and integrating by parts we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{1}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\Big(-\frac{e^{iE(-\beta)}}{2i}\Big)'\frac{sx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\,d\beta\\
&=\Big[\frac{1}{2\pi i}\frac{1}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\frac{e^{iE(-\beta)}}{-2i}\frac{sx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\Big]_{-\infty}^{-2}\\
&+\Big[\frac{1}{2\pi i}\frac{1}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\frac{e^{iE(-\beta)}}{-2i}\frac{sx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\Big]_{2}^{\infty}\\
&-\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{-1+2\alpha}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-2+2\alpha}\Big(\frac{\operatorname{sgn}(\beta)}{2\pi}\Big)\frac{e^{iE(-\beta)}}{-2i}\frac{sx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\,d\beta\\
&-\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{1}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\frac{e^{iE(-\beta)}}{-2i}\frac{(-i\log nx)sx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)}i\,d\beta\\
&-\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{1}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\frac{e^{iE(-\beta)}}{-2i}\frac{isx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)^2(s-1+\alpha+i\beta)}i\,d\beta\\
&-\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\frac{1}{n^\alpha}\Big(\frac{|\beta|}{2\pi}\Big)^{-1+2\alpha}\frac{e^{iE(-\beta)}}{-2i}\frac{-isx^{1-\alpha-s}(nx)^{-i\beta}}{(1-\alpha-i\beta)(s-1+\alpha+i\beta)^2}i\,d\beta.
\end{aligned}$$

We note that, by Lemma 2.7, the first, second, third, and fifth terms on the right-hand side of the above formula are bounded by $O(\frac{x^{1-\sigma-\alpha}|t|^{2\varepsilon}\log|t|}{n^\alpha})$. We observe that the fourth and sixth terms are $\log nx\, L_1(s;n)$ and $L_2(s;n)$, respectively. Hence we have (2.24).

As for others, we have (2.25) and (2.26) by integrating by parts twice, and (2.27) by integrating by parts three times. Note that $\mathcal{L}_3(s;n)$ and $\mathcal{L}_4(s;n)$ have the same form since their integrands have the same asymptotic behaviours. ∎

Remark 2.10 The O-terms in (2.24)–(2.27) are obtained by Lemma 2.7. It should be noted that the implied constant in (2.22) depends on k, σ, ε and involves the factor $1/(\sigma+\varepsilon)^j$, hence the implied constants in the O-terms in (2.24)–(2.27) also have the same factors. Trivially, if we consider only $0 \le \sigma \le 1$, they do not depend on σ.

It must be remarked that the sixth term after integration by parts contributes to the main term in $\mathcal{L}_2(s;n)$ and similarly in $\mathcal{L}_j(s;n)$ $(j = 3,4,5)$.

Combining Lemma 2.5 and Lemma 2.9 we get the following formulas which will be used in the next section.

Proposition 2.11 *Assume that $s = \sigma + it \neq 1$, $-\varepsilon < \sigma \le b$ (b is a constant), $0 < \varepsilon < 1/4$, and $t \ll x \ll t^2$. We have*

(2.28)

$$\begin{aligned}
\zeta'(s)^2 &= \sum_{n\le x}\frac{D_{(1)}(n)}{n^s} + \sum_{n=1}^{\infty} D_{(1)}(n)L_1(s;n)\\
&\quad + \sum_{n=1}^{\infty} d_{(0,1)}(n)(\log nx\cdot L_1(s;n) + L_2(s;n))
\end{aligned}$$

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$$+\sum_{n=1}^{\infty} d(n)\Big(\frac{1}{4}\log^2 nx\, L_1(s;n)+\frac{1}{2}\log nx\, L_2(s;n)+\frac{1}{2}L_3(s;n)\Big)$$
$$+O(x^{1/2-\sigma}\log^3|t|),$$

(2.29)

$$\zeta(s)\zeta''(s)=\sum_{n\le x}\frac{d_{(0,2)}(n)}{n^s}+\sum_{n=1}^{\infty} d_{(0,2)}(n)L_1(s;n)$$
$$+\sum_{n=1}^{\infty} d_{(0,1)}(n)(\log nx\, L_1(s;n)+L_2(s;n))$$
$$+\sum_{n=1}^{\infty} d(n)\Big(\frac{1}{4}(\log nx)^2 L_1(s;n)+\frac{1}{2}\log nx\, L_2(s;n)+\frac{1}{2}L_3(s;n)\Big)$$
$$+O(x^{1/2-\sigma}\log^3|t|),$$

(2.30)

$$\zeta'(s)\zeta''(s)=\sum_{n\le x}\frac{d_{(1,2)}(n)}{n^s}-\sum_{n=1}^{\infty} d_{(1,2)}(n)L_1(s;n)$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}(2d_{(0,1)}(n)\log n+d_{(0,2)}(n))(\log nx\, L_1(s;n)+L_2(s;n))$$
$$-3\sum_{n=1}^{\infty} d_{(0,1)}(n)\Big(\frac{1}{4}(\log nx)^2 L_1(s;n)+\frac{1}{2}\log nx L_2(s;n)+\frac{1}{2}L_3(s;n)\Big)$$
$$+\sum_{n=1}^{\infty} d(n)\Big(-\frac{1}{8}(\log nx)^3 L_1(s;n)-\frac{3}{8}(\log nx)^2 L_2(s;n)$$
$$-\frac{3}{4}\log nx\, L_3(s;n)-\frac{3}{4}L_4(s;n)\Big)$$
$$+O(x^{1/2-\sigma}\log^4|t|).$$

Remark 2.12 The implied constants in (2.28)–(2.30) contain the factor $1/(\sigma+\varepsilon)^j$ by the error in Lemma 2.9 and also contain the sums like $\sum_{n=1}^{\infty} D_{(1)}(n)/n^{1+\varepsilon}$. If we take σ in [0,1], then those constants are independent of σ.

3 The Exact Formulas

Here we will deduce the exact formulas for $\zeta'(s)^2$, $\zeta(s)\zeta''(s)$, and $\zeta'(s)\zeta''(s)$, which are important as in [22]. To state them we introduce the functions $U_j(s;n)$ for $j=0,1,2,3$.

Definition 3.1 Let $x>0$ be a real number, n a positive integer, and $s=\sigma+it$ a complex variable as above. We define

$$U_j(s;n)=-\frac{2^{4s}\pi^{2s-2}}{n^{1-s}}\int_{4\pi\sqrt{nx}}^{\infty}(\log v)^j\frac{K_1(v)+\frac{\pi}{2}Y_1(v)}{v^{2s}}\,dv \quad (j=0,1,2,3), \tag{3.1}$$

where $K_1(v)$ and $Y_1(v)$ are the Bessel functions.

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Theorem 3.2 *Assume that $s = \sigma + it \neq 1$, $-\varepsilon < \sigma \leq b$ (where b is a constant), $0 < \varepsilon < 1/4$, and $t \ll x \ll t^2$. Then we have*

$$(3.2)\quad \zeta'(s)^2 = \sum_{n\leq x}\frac{D_{(1)}(n)}{n^s} + s\sum_{n=1}^{\infty}D_{(1)}(n)U_0(s;n) + \frac{s}{4}\sum_{n=1}^{\infty}d(n)(\log^2(2^4\pi^2) + 2\log(2^4\pi^2)\log n)U_0(s;n) - s\sum_{n=1}^{\infty}d(n)(\log(2^4\pi^2) + \log n)U_1(s;n) + s\sum_{n=1}^{\infty}d(n)U_2(s;n) + O(x^{1/2-\sigma}\log^3|t|),$$

$$(3.3)\quad \zeta(s)\zeta''(s) = \sum_{n\leq x}\frac{d_{(0,2)}(n)}{n^s} + s\sum_{n=1}^{\infty}d_{(0,2)}(n)U_0(s;n) + \frac{s}{4}\sum_{n=1}^{\infty}d(n)(\log^2(2^4\pi^2) + 2\log(2^4\pi^2)\log n)U_0(s;n) - s\sum_{n=1}^{\infty}d(n)(\log(2^4\pi^2) + \log n)U_1(s;n) + s\sum_{n=1}^{\infty}d(n)U_2(s;n) + O(x^{1/2-\sigma}\log^3|t|),$$

$$(3.4)\quad \zeta'(s)\zeta''(s) = \sum_{n\leq x}\frac{d_{(1,2)}(n)}{n^s} - \sum_{n=1}^{\infty}d_{(1,2)}(n)sU_0(s;n) + \sum_{n=1}^{\infty}d(n)\Big(\frac{1}{8}(\log 2^4\pi^2)^3 + \frac{3}{8}(\log 2^4\pi^2)^2\log n + \frac{1}{2}(\log 2^4\pi^2)\log^2 n\Big)sU_0(s;n) - \sum_{n=1}^{\infty}d(n)\Big(\frac{3}{4}(\log 2^4\pi^2)^2 + \frac{3}{2}\log 2^4\pi^2\log n + \log^2 n\Big)sU_1(s;n) + \frac{3}{2}\sum_{n=1}^{\infty}d(n)(\log 2^4\pi^2 + \log n)sU_2(s;n) - \sum_{n=1}^{\infty}d(n)sU_3(s;n) + \sum_{n=1}^{\infty}d_{(0,2)}(n)\Big(-\frac{1}{2}\big(\log 2^4\pi^2\big)sU_0(s;n) + sU_1(s;n)\Big) + O(x^{1/2-\sigma}\log^4|t|).$$

To derive approximate functional equations for $\zeta'(s)^2$, $\zeta(s)\zeta''(s)$, and $\zeta'(s)\zeta''(s)$ from Theorem 3.2, we need Titchmarsh's fundamental lemma on (1.5).

Lemma 3.3 (Titchmarsh [22, p. 111]) *For $0 < \alpha < 3/2$ and $\sigma > -1/4$ we have*

$$(3.5)\quad \frac{1}{2\pi i}\int_{(\alpha)}\frac{\chi(1-w)^2}{n^w}\frac{sx^{1-w-s}}{(1-w)(s-1+w)}\,dw =$$

$$-\frac{s2^{4s}\pi^{2s-2}}{n^{1-s}}\int_{4\pi\sqrt{nx}}^{\infty}\frac{K_1(v)+\frac{\pi}{2}Y_1(v)}{v^{2s}}\,dv.$$

Formula (3.5) is equivalent to the formula of Titchmarsh [22, p. 111, l.2]. He proved his formula for $\sigma > 1$ first and then for $\sigma > -\frac{1}{4}$ by analytic continuation. The condition for α is needed for the convergence of the integral.

We remark that in our notation formula (3.5) is expressed as

$$\mathcal{L}_1(s;n) = sU_0(s;n). \tag{3.6}$$

Furthermore, we can get the relations among $L_j(s;n)$ and $U_j(s;n)$ from Lemma 3.3.

Lemma 3.4 *Let $-\varepsilon < \sigma \le b$ (b is a constant), $|t| \ge 2$, $0 < \varepsilon < 1/4$, and $\alpha = 1+\varepsilon$. Then we have*

$$L_1(s;n) = sU_0(s;n) + O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|}{n^{\alpha}}\Big), \tag{3.7}$$

$$\begin{aligned}\log xL_1(s;n) + L_2(s;n)\\ &= -s(\log 2^4\pi^2 n)U_0(s;n) + 2sU_1(s;n)\\ &\quad + O\Big(\frac{x^{1-\alpha-\sigma}(\log x)|t|^{2\varepsilon}\log|t|}{n^{\alpha}}\Big),\end{aligned} \tag{3.8}$$

$$\begin{aligned}(\log x)^2L_1(s;n) + 2\log xL_2(s;n) + 2L_3(s;n)\\ &= s(\log 2^4\pi^2 n)^2U_0(s;n)\\ &\quad - 4s(\log 2^4\pi^2 n)U_1(s;n) + 4sU_2(s;n)\\ &\quad + O\Big(\frac{x^{1-\alpha-\sigma}(\log x)^2|t|^{2\varepsilon}\log|t|}{n^{\alpha}}\Big),\end{aligned} \tag{3.9}$$

$$\begin{aligned}-(\log x)^3L_1(s;n) - 3(\log x)^2L_2(s;n) - 6\log xL_3(s;n) - 6L_4(s;n)\\ &= (\log 2^4\pi^2 n)^3sU_0(s;n) - 6(\log 2^4\pi^2 n)^2sU_1(s;n)\\ &\quad + 12(\log 2^4\pi^2 n)sU_2(s;n)\\ &\quad - 8sU_3(s;n) + O\Big(\frac{x^{1-\alpha-\sigma}(\log x)^3|t|^{2\varepsilon}\log|t|}{n^{\alpha}}\Big).\end{aligned} \tag{3.10}$$

Proof The first assertion (3.7) is obtained by (3.6) and (2.23) immediately. For other assertions, divide (3.6) by s and differentiate with respect to s repeatedly. Thus we get

$$\frac{1}{2\pi i}\int_{(\alpha)}\frac{\chi(1-w)^2}{n^w}\frac{x^{1-w}}{1-w}\Big(\frac{-(\log x)x^{-s}}{s-1+w} - \frac{x^{-s}}{(s-1+w)^2}\Big)\,dw = \frac{d}{ds}U_0(s,n),$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\alpha)}\frac{\chi(1-w)^2}{n^w}\frac{x^{1-w}}{1-w}\Big(\frac{(\log x)^2x^{-s}}{s-1+w} + \frac{2(\log x)x^{-s}}{(s-1+w)^2} + \frac{2x^{-s}}{(s-1+w)^3}\Big)\,dw\\ = \frac{d^2}{ds^2}U_0(s;n),\end{aligned}$$

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$$\frac{1}{2\pi i}\int_{(\alpha)}\frac{\chi(1-w)^2}{n^w}\frac{x^{1-w}}{1-w}\Big(\frac{-(\log x)^3x^{-s}}{s-1+w}-\frac{3(\log x)^2x^{-s}}{(s-1+w)^2}-\frac{6(\log x)x^{-s}}{(s-1+w)^3}-\frac{6x^{-s}}{(s-1+w)^4}\Big)\,dw=\frac{d^3}{ds^3}U_0(s;n),$$

By multiplying $\pm s$ on both sides and using (2.19) and Lemma 2.7, we can see that the left-hand sides of the above formulas can be written as a linear combination of $L_j(s;n)$ as in the left-hand sides of (3.8), (3.9), and (3.10), plus the error terms there. On the other hand, the right-hand sides above can be calculated by

$$\frac{d^j}{ds^j}U_0(s;n)=-\frac{\pi^{-2}}{n}\int_{4\pi\sqrt{nx}}^{\infty}\Big(\frac{2^4\pi^2n}{v^2}\Big)^s(\log 2^4\pi^2n-2\log v)^j\Big(K_1(v)+\frac{\pi}{2}Y_1(v)\Big)\,dv.$$

Now by a simple calculation we get the assertions (3.8), (3.9), and (3.10). ∎

By Lemma 2.7 we observe that implied constants are independent of σ in $[0,1]$. Finally, we will prove the exact formulas for $\zeta'(s)^2$ and others.

Proof of Theorem 3.2 First we consider the case $\zeta'(s)^2$. In (2.28), we substitute the identity $d_{(0,1)}(n)=-\frac{1}{2}d(n)\log n$ and get

$$\begin{aligned}\zeta'(s)^2&=\sum_{n\le x}\frac{D_{(1)}(n)}{n^s}+\sum_{n=1}^{\infty}D_{(1)}(n)L_1(s;n)\\&\quad+\sum_{n=1}^{\infty}d(n)\Big\{-\frac{1}{2}\log n(\log nxL_1(s;n)+L_2(s;n))+\frac{1}{4}(\log nx)^2L_1(s;n)\\&\quad+\frac{1}{2}\log nx\,L_2(s;n)+\frac{1}{2}L_3(s;n)\Big\}\\&\quad+O(x^{1/2-\sigma}\log^3|t|)\\&=\sum_{n\le x}\frac{D_{(1)}(n)}{n^s}+\sum_{n=1}^{\infty}D_{(1)}(n)L_1(s;n)-\frac{1}{4}\sum_{n=1}^{\infty}d(n)\log^2 n\,L_1(s;n)\\&\quad+\frac{1}{4}\sum_{n=1}^{\infty}d(n)(\log^2 xL_1(s;n)+2\log xL_2(s;n)+2L_3(s;n))\\&\quad+O(x^{1/2-\sigma}\log^3|t|).\end{aligned}$$

Now we apply (3.7) and (3.9) in the right-hand side of the above formula. Then the coefficient of $d(n)$ from the third and the fourth terms in the right-hand side above becomes

$$\begin{aligned}&-\frac{1}{4}\log^2 n\Big(sU_0(s;n)+O\Big(\frac{x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|}{n^\alpha}\Big)\Big)\\&+\frac{1}{4}\Big(s(\log^2(2^4\pi^2)+\log^2 n+2\log 2^4\pi^2\cdot\log n)U_0(s;n)\\&\qquad-4s(\log 2^4\pi^2+\log n)U_1(s;n)+4sU_2(s;n)+O\Big(\frac{x^{1-\alpha-\sigma}(\log x)^2|t|^{2\varepsilon}\log|t|}{n^\alpha}\Big)\Big)\end{aligned}$$

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$$= \frac{s}{4}(\log^2(2^4\pi^2) + 2\log 2^4\pi^2 \cdot \log n)U_0(s;n) - s(\log(2^4\pi^2) + \log n)U_1(s;n) + sU_2(s;n) + O\Big(\frac{(\log n)^2 x^{1-\alpha-\sigma}|t|^{2\varepsilon}\log|t|}{n^\alpha}\Big) + O\Big(\frac{x^{1-\alpha-\sigma}(\log x)^2|t|^{2\varepsilon}\log|t|}{n^\alpha}\Big).$$

This proves (3.2).

Similarly we have

$$\zeta(s)\zeta''(s) = \sum_{n\le x}\frac{d_{(0,2)}(n)}{n^s} + \sum_{n=1}^{\infty} d_{(0,2)}(n)L_1(s;n) - \sum_{n=1}^{\infty}\frac{d(n)\log^2 n}{4}L_1(s;n) + \sum_{n=1}^{\infty}\frac{d(n)}{4}(\log^2 xL_1(s;n) + 2\log xL_2(s;n) + 2L_3(s;n)) + O(x^{1/2-\sigma}\log^3|t|)$$

and

$$\zeta'(s)\zeta''(s) = \sum_{n\le x}\frac{d_{(1,2)}(n)}{n^s} - \frac{1}{8}\sum_{n=1}^{\infty} d(n)\log^2 n(\log xL_1(s;n) + L_2(s;n)) - \frac{1}{8}\sum_{n=1}^{\infty} d(n)\Big(\log^3 xL_1(s;n) + 3\log^2 xL_2(s;n) + 6\log xL_3(s;n) + 6L_4(s;n)\Big) + \frac{1}{2}\sum_{n=1}^{\infty} d_{(0,2)}(n)(\log xL_1(s;n) + L_2(s;n)) + O(x^{1/2-\sigma}\log^4|t|).$$

Applying (3.7) and (3.9) to the case $\zeta(s)\zeta''(s)$, as well as (3.8) and (3.10) to the case $\zeta'(s)\zeta''(s)$, we get (3.3) and (3.4). These implied constants in the arguments are independent of σ in $[0,1]$. ■

Remark 3.5 By a similar method we can obtain the exact formula for $\zeta(s)\zeta'(s)$:

$$\zeta(s)\zeta'(s) = \sum_{n\le x}\frac{d_{(0,1)}(n)}{n^s} + \frac{s}{2}\sum_{n=1}^{\infty} d(n)(\log(2^4\pi^2) + \log n)U_0(s;n) - s\sum_{n=1}^{\infty} d(n)U_1(s;n) + O(x^{1/2-\sigma}\log^2|t|).$$

4 Lemma β of Hardy and Littlewood

We need to know the behaviour of the integrals

$$\int_{4\pi\sqrt{nx}}^{\infty}(\log v)^j\frac{\cos(v-\pi/4)}{v^{2s+1/2}}\,dv$$

for $j = 1,2,3$. Following [22] we investigate these functions by an extension of Lemma β of Hardy and Littlewood. As in [12, (2.22)], let

$$I_j = I_j(\xi,s) = \int_\xi^\infty(\log v)^j v^{-s}\cos\Big(v-\frac{\pi}{4}\Big)\,dv \tag{4.1}$$

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for $j = 0, 1, 2, 3$. Since s should be replaced by $2s + 1/2$ in (4.1), we need to consider $I_j(\xi, s)$ in the range $1/2 < \sigma = \operatorname{Re} s < 5/2$. It is carried out by integration by parts as in [22]. We get the following lemma.

Lemma 4.1 *Suppose that A denotes a positive constant, which may be different at each occurrence.* [1] *Let $1/2 < \sigma < 5/2$ and $t > A$, $\xi > A$. Then we have*

(i) $I_j \ll \xi^{-\sigma} \log^j \xi$ *for $t < At < \xi$,*

(ii)
$$I_j = \frac{e^{\pi i/4}}{2} \frac{\xi^{1-s} e^{i\xi} \log^j \xi}{\xi - t} + O\Big(\frac{t\xi^{1-\sigma} \log^j \xi}{(\xi - t)^3}\Big) + O(\xi^{-\sigma} \log^j \xi)$$

for $t + A < \xi < At$,

(iii)
$$I_j = \Gamma(1-s) \cos \frac{\pi}{2}\Big(s - \frac{1}{2}\Big) \log^j t + O(\delta_j t^{-\frac{1}{2}-\sigma} \log^{j-1} t) + O\Big(\frac{\xi^{1-\sigma} \log^j \xi}{t}\Big)$$

for $\xi < At < t$,

(iv)
$$I_j = \Gamma(1-s) \cos \frac{\pi}{2}\Big(s - \frac{1}{2}\Big) \log^j t + O(\delta_j t^{-\frac{1}{2}-\sigma} \log^{j-1} t) + \frac{e^{\pi i/4}}{2(1-s)(2-s)} \frac{\xi^{3-s} e^{i\xi} \log^j \xi}{t - \xi} + O\Big(\frac{\xi^{2-\sigma} \log^j \xi}{(t-\xi)^3}\Big) + O\Big(\frac{\xi^{1-\sigma} \log^j \xi}{t}\Big)$$

for $At < \xi < t - A$,

(v) $I_j \ll t^{1/2} \xi^{-\sigma} \log^j(\max\{\xi, t\})$ *for all cases,*

where

$$\delta_j = \begin{cases} 0 & \text{for } j = 0, \\ 1 & \text{for } j = 1, 2, 3. \end{cases}$$

These O-constants are independent of σ ($1/2 < \sigma < 5/2$).

Proof (i), (ii), and (v) are proved by the similar method from [12].

We consider the other cases. First we suppose that $0 < \sigma < 1$ and we have

$$I_j = \Big\{ \int_0^\infty - \int_0^\xi \Big\} (\log v)^j v^{-s} \cos\Big(v - \frac{\pi}{4}\Big)\, dv =: J_{1j} + J_{2j},$$

[1] We follow [12, Lemma β] for the use of a constant A in the conditions on ξ and t. As is noted in [12], "the inequalities on ξ and t sometimes restrict the possible values of A."

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say. For J_{2j}, by repeating integration by parts and using the method of [12, Lemma β], we have

$$J_{2j} = \begin{cases} O\Big(\dfrac{\xi^{1-\sigma}\log^j \xi}{t}\Big) & \text{for } \xi < At < t \\ \dfrac{e^{\pi i/4}}{2(1-s)(2-s)}\dfrac{\xi^{3-s}e^{i\xi}\log^j \xi}{t-\xi} & \\ \qquad +O\Big(\dfrac{\xi^{2-\sigma}\log^j \xi}{(t-\xi)^3}\Big)+O\Big(\dfrac{\xi^{1-\sigma}\log^j \xi}{t}\Big) & \text{for } At < \xi < t-A. \end{cases} \tag{4.2}$$

Note that formula (4.2) is valid when $\frac{1}{2} \le \sigma \le \frac{5}{2}$.

To evaluate the integral J_{1j} ($j = 0, 1, 2, 3$), we recall the well-known formula [2, p. 50]:

$$\int_0^\infty \frac{\cos(v-\frac{\pi}{4})}{v^s}\,dv = \Gamma(1-s)\cos\frac{\pi(s-\frac{1}{2})}{2}, \tag{4.3}$$

which is valid for $0 < \operatorname{Re} s < 1$ and is uniformly convergent in a small neighbourhood of s in this range. Differentiating both sides of (4.3) with respect to s repeatedly, we have

$$J_{11} = \Gamma(1-s)\left\{\psi(1-s)\cos\frac{\pi(s-\frac{1}{2})}{2}+\frac{\pi}{2}\sin\frac{\pi(s-\frac{1}{2})}{2}\right\}, \tag{4.4}$$

$$\begin{aligned} J_{12} = \Gamma(1-s)\bigg\{&\left(\psi'(1-s)+\psi^2(1-s)\right)\cos\frac{\pi(s-\frac{1}{2})}{2} \\ &+\pi\psi(1-s)\sin\frac{\pi(s-\frac{1}{2})}{2}-\frac{1}{4}\pi^2\cos\frac{\pi(s-\frac{1}{2})}{2}\bigg\}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} J_{13} = \Gamma(1-s)\bigg\{&\Big(\psi^3(1-s)+3\psi(1-s)\psi'(1-s)+\psi''(1-s) \\ &-\frac{3}{4}\pi^2\psi(1-s)\Big)\cos\frac{\pi(s-\frac{1}{2})}{2} \\ &+\Big(\frac{3\pi}{2}\psi^2(1-s)+\frac{3\pi}{2}\psi'(1-s)-\frac{\pi^3}{8}\Big)\sin\frac{\pi(s-\frac{1}{2})}{2}\bigg\}, \end{aligned} \tag{4.6}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function. By (4.3), (4.4), (4.5), and (4.6), $I_j = J_{1j} + J_{2j}$ is valid for $1/2 \le \sigma \le 5/2$.

It remains to evaluate the right-hand sides of (4.4), (4.5), and (4.6). From [2, Corollary 1.4.5] we have

$$\psi(1-s) = \log t - \frac{\pi i}{2} + O\left(\frac{1}{t}\right) \tag{4.7}$$

for $t \gg 1$ and hence

$$\psi(1-s)\cos\frac{\pi(s-\frac{1}{2})}{2}+\frac{\pi}{2}\sin\frac{\pi(s-\frac{1}{2})}{2} = \cos\frac{\pi(s-\frac{1}{2})}{2}\log t + O\Big(\frac{e^{\pi t/2}}{t}\Big).$$

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Therefore, by the Stirling formula of Γ function $\Gamma(\sigma+it)=O(|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|})$, we have

$$J_{11}=\int_0^\infty \log v\frac{\cos(v-\frac{\pi}{4})}{v^s}dv=\Gamma(1-s)\cos\frac{\pi(s-\frac{1}{2})}{2}\log t+O\left(t^{-\frac{1}{2}-\sigma}\right). \tag{4.8}$$

For J_{12}, squaring both sides of (4.7), substituting it into (4.5), and using the well-known estimate $\psi'(1-s)=O(1/t)$ [2, (1.2.14)], we find that

(4.9)

$$\begin{aligned}J_{12}=\Gamma(1-s)\Big\{&\Big(\log^2 t-\pi i\log t-\frac{\pi^2}{4}\Big)\cos\frac{\pi(s-\frac{1}{2})}{2}+\pi\Big(\log t-\frac{\pi i}{2}\Big)\sin\frac{\pi(s-\frac{1}{2})}{2}\\&-\frac{\pi^2}{4}\cos\frac{\pi(s-\frac{1}{2})}{2}+O\Big(\frac{\log t}{t}e^{\frac{\pi}{2}t}\Big)\Big\}\\=\Gamma(1-s)\Big\{&\log^2 t\,\cos\frac{\pi(s-\frac{1}{2})}{2}-\Big(\pi i\log t+\frac{\pi}{2}\Big)e^{\frac{\pi i(s-\frac{1}{2})}{2}}+O\Big(\frac{\log t}{t}e^{\frac{\pi}{2}t}\Big)\Big\}\\=\Gamma(1-s)&\cos\frac{\pi(s-\frac{1}{2})}{2}\log^2 t+O(t^{-1/2-\sigma}\log t).\end{aligned}$$

By noting $\psi''(1-s)=O(1/t^2)$, similarly we have that

$$J_{13}=\Gamma(1-s)\cos\frac{\pi(s-\frac{1}{2})}{2}\log^3 t+O(t^{-1/2-\sigma}\log^2 t). \tag{4.10}$$

Finally, from (4.2), (4.8), (4.9), and (4.10) we get the assertions (iii) and (iv). ■

5 The Approximate Functional Equation

In order to prove the approximate functional equations for $\zeta'(s)^2, \zeta(s)\zeta''(s)$, and $\zeta'(s)\zeta''(s)$, we approximate $U_j(s;n)$ in Theorem 3.2 by $I_j(4\pi\sqrt{nx}, 2s+1/2)$, which we discussed in the previous section. It is well known that

$$K_1(v)+\frac{\pi}{2}Y_1(v)=\Big(\frac{\pi}{2v}\Big)^{1/2}\Big(-\cos\Big(v-\frac{\pi}{4}\Big)+\frac{3}{8v}\sin\Big(v-\frac{\pi}{4}\Big)+O\Big(\frac{1}{|v|^2}\Big)\Big)$$

[22, p. 111], hence by (3.1) and (4.1) we find that

(5.1)

$$\begin{aligned}U_j(s,n)=\frac{2^{4s-1/2}\pi^{2s-3/2}}{n^{1-s}}\Big\{&I_j\Big(4\pi\sqrt{nx},2s+\frac{1}{2}\Big)-\frac{3}{8}\int_{4\pi\sqrt{nx}}^\infty\frac{(\log v)^j\sin(v-\pi/4)}{v^{2s+3/2}}dv\\&+O((nx)^{-\sigma-3/4}\log^j(nx))\Big\},\end{aligned}$$

where O-constant is independent of σ if $\sigma>-3/2+\varepsilon_1$ (for any $\varepsilon_1>0$). Let us consider the case $\zeta'(s)^2$. There is no loss of generality in assuming that $t>2\pi$.

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First we treat the case $1 \le y \le x$. Since $xy = (t/2\pi)^2$, this implies $y \le t/2\pi \le x \le (t/2\pi)^2$. In view of (3.2) of Theorem 3.2, we need to evaluate the series

$$
\begin{aligned}
(5.2)\quad & s\sum_{n=1}^{\infty} D_{(1)}(n)U_0(s;n) + \frac{s}{4}\sum_{n=1}^{\infty} d(n)(\log^2(2^4\pi^2) + 2\log(2^4\pi^2)\log n)U_0(s;n) \\
& - s\sum_{n=1}^{\infty} d(n)(\log(2^4\pi^2) + \log n)U_1(s;n) + s\sum_{n=1}^{\infty} d(n)U_2(s;n) \\
& =: S_1 + S_2 - S_3 + S_4.
\end{aligned}
$$

Evaluation of S_4 First, we consider the sum $S_4 = s\sum_{n=1}^{\infty} d(n)U_2(s;n)$ closely. By (5.1), we have

$$S_4 = S_4^{(1)} + S_4^{(2)} + S_4^{(3)},$$

where

$$
\begin{aligned}
S_4^{(1)} &= 2^{4s-\frac{1}{2}}\pi^{2s-\frac{3}{2}} s\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-s}} I_2(4\pi\sqrt{nx}, 2s+1/2), \\
S_4^{(2)} &= -\frac{3}{8}2^{4s-\frac{1}{2}}\pi^{2s-\frac{3}{2}} s\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-s}}\int_{4\pi\sqrt{nx}}^{\infty}(\log v)^2\frac{\sin(v-\frac{\pi}{4})}{v^{2s+\frac{3}{2}}}\,dv, \\
S_4^{(3)} &= O\Big(t\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-\sigma}}(nx)^{-\sigma-3/4}\log^2(nx)\Big).
\end{aligned}
$$

If $\sigma > 0$ is bounded, the O-constant is independent of σ. It is easy to see that

$$S_4^{(3)} \ll x^{1/4-\sigma}\log^2 x.$$

To evaluate $S_4^{(1)}$, we follow the method of [22], where Lemma 4.1 is essentially used. We divide the infinite sum in $S_4^{(1)}$ into five parts as

$$
\begin{aligned}
\sum_{n=1}^{\infty} &= \sum_{n<\frac{y}{8}} + \sum_{\frac{y}{8}\le n<y-\sqrt{y}} + \sum_{y-\sqrt{y}\le n\le y+\sqrt{y}} + \sum_{y+\sqrt{y}<n\le 4y} + \sum_{n>4y} \\
&=: \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5.
\end{aligned}
$$

For Σ_3, we apply Lemma 4.1 (v) and get

$$
\begin{aligned}
\Sigma_3 &\ll \sum_{y-\sqrt{y}\le n\le y+\sqrt{y}}\frac{d(n)}{n^{1-\sigma}} t^{1/2}(nx)^{-\sigma-1/4}\log^2((nx)^{1/2}+t) \\
&\ll t^{1/2}x^{-\sigma-1/4}(\log t)^2 y^{-5/4}\sum_{y-\sqrt{y}\le n\le y+\sqrt{y}} d(n) \\
&\ll t^{1/2}x^{-\sigma-1/4}y^{-3/4}\log^3 t \\
&\ll t^{-1}x^{1/2-\sigma}\log^3 t.
\end{aligned}
$$

For Σ_5, we note that $4\pi\sqrt{nx} \geq 2\cdot 2t$. Hence we can apply Lemma 4.1 (i) and get

$$\begin{aligned}\Sigma_5 &\ll \sum_{n>4y} \frac{d(n)}{n^{1-\sigma}}(nx)^{-\sigma-1/4}\log^2(nx)\\ &\ll x^{-\sigma-1/4}\sum_{n>4y}\frac{d(n)}{n^{5/4}}(\log^2 n + 2\log n\log x + \log^2 x)\\ &\ll x^{-\sigma-1/4}y^{-1/4}(\log^3 y + \log^2 x\log y)\\ &\ll t^{-1}x^{1/2-\sigma}\log^3 t.\end{aligned}$$

For Σ_4, we note that $2t+1 < 4\pi\sqrt{nx} < 2\cdot 2t$. Using Lemma 4.1 (ii) we have

$$\begin{aligned}\Sigma_4 &= \sum_{y+\sqrt{y}<n\leq 4y}\frac{d(n)}{n^{1-s}}\bigg\{\frac{e^{\pi i/4}}{2}\frac{(4\pi\sqrt{nx})^{1/2-2s}e^{4\pi i\sqrt{nx}}(\log 4\pi\sqrt{nx})^2}{4\pi\sqrt{nx}-2t}+\\ &\quad + O\Big(\frac{t(nx)^{1/4-\sigma}(\log 4\pi\sqrt{nx})^2}{(4\pi\sqrt{nx}-2t)^3}\Big) + O\big((nx)^{-\sigma-1/4}(\log 4\pi\sqrt{nx})^2\big)\bigg\}\\ &=: \Sigma_{41}+\Sigma_{42}+\Sigma_{43}.\end{aligned}$$

Note that these O-constants are independent of σ if σ is bounded. The term Σ_{42} is bounded by

$$\begin{aligned}\Sigma_{42} &\ll tx^{1/4-\sigma}y^{-3/4}\log^2 t\sum_{y+\sqrt{y}<n\leq 4y}\frac{d(n)}{(2\pi\sqrt{nx}-t)^3}\\ &\ll tx^{-\sigma-5/4}y^{3/4}\log^2 t\sum_{y+\sqrt{y}<n\leq 4y}\frac{d(n)}{(n-y)^3}\\ &\ll tx^{-\sigma-5/4}y^{-1/4}\log^3 t\\ &\ll x^{1/2-\sigma}t^{-1}\log^3 t,\end{aligned}$$

where we have used the estimate $\sum_{y+\sqrt{y}<n\leq 4y}\frac{d(n)}{(n-y)^3} \ll y^{-1}\log y$ from [22, p. 112]. It is easy to see that Σ_{43} is also bounded by $O(x^{1/2-\sigma}t^{-1}\log^3 t)$.

In the case $\zeta(s)^2$, Titchmarsh took a genius approach to the corresponding sum to Σ_{41} by approximating it with an exponential sum. However, we take a slightly different approach from that of Titchmarsh. We will evaluate

$$\Sigma_{41} \ll x^{-\sigma-1/4}\bigg|\sum_{y+\sqrt{y}<n\leq 4y}\frac{d(n)e^{4\pi i\sqrt{nx}}(\log 4\pi\sqrt{x}+\frac{1}{2}\log n)^2}{n^{3/4}(\sqrt{n}-\sqrt{y})}\bigg|. \tag{5.3}$$

Write $y_1 = y+\sqrt{y}$ for simplicity; let

$$\begin{aligned}W &= \sum_{y_1<n\leq 4y}\frac{d(n)e^{4\pi i\sqrt{nx}}(\log n)^2}{n^{3/4}(\sqrt{n}-\sqrt{y})},\\ G(w) &= \sum_{y_1<n\leq y_1+w} d(n)e^{4\pi i\sqrt{nx}} = \sum_{y_1<\mu\nu\leq y_1+w} e^{4\pi i\sqrt{\mu\nu x}},\end{aligned}$$

for $0 \le w \le 4y - y_1$. By partial summation we find that

$$W = G(4y - y_1)F(4y - y_1) - \int_0^{4y-y_1} G(w)F'(w)\,dw, \tag{5.4}$$

where we put

$$F(w) = \frac{\log^2(y_1 + w)}{(y_1 + w)^{3/4}(\sqrt{y_1 + w} - \sqrt{y})}.$$

Now we must evaluate the sum $G(w)$. By the symmetry for μ and ν, we see that

$$G(w) = 2 \sum_{\mu \le 2\sqrt{y}} \sum_{\substack{\frac{y_1}{\mu} < \nu \le \frac{y_1 + w}{\mu} \\ \mu < \nu}} e^{2\pi i f_\mu(\nu)} + O\Big(\frac{w}{\sqrt{y}} + 1\Big)$$

with $f_\mu(\nu) = 2\sqrt{\mu\nu x}$. To estimate the above exponential sum, we recall van der Corput's theorem [23, Theorem 5.9].

Lemma 5.1 *If $f(x)$ is real and twice differentiable, and*

$$0 < \lambda_2 \le f''(x) \le h\lambda_2 \quad (\text{or } \lambda_2 \le -f''(x) \le h\lambda_2)$$

throughout the interval $[a, b]$, and $b \ge a + 1$, then

$$\sum_{a < n \le b} e^{2\pi i f(n)} = O(h(b - a)\lambda_2^{1/2}) + O(\lambda_2^{-1/2}).$$

Since $|f_\mu''(\nu)| \asymp \mu^2 x^{1/2} y^{-3/2}$, we find by this lemma that

$$\sum_{\substack{\frac{y_1}{\mu} < \nu \le \frac{y_1 + w}{\mu} \\ \mu < \nu}} e^{2\pi i f_\mu(\nu)} \ll w x^{1/4} y^{-3/4} + \mu^{-1} x^{-1/4} y^{3/4} + 1. \tag{5.5}$$

Thus we get

$$\begin{aligned} G(w) &\ll w x^{1/4} y^{-1/4} + x^{-1/4} y^{3/4} \log y + y^{1/2} \\ &\ll w x^{1/4} y^{-1/4} + y^{1/2} \log y. \end{aligned} \tag{5.6}$$

Using (5.6), the first term on the right-hand side of (5.4) is bounded by $x^{1/4} y^{-1/2} \log^2 t$. On the other hand, the second term on the right-hand side of (5.4) is bounded by

$$\begin{aligned} x^{1/4} y^{-1/4} &\int_0^{4y-y_1} w|F'(w)|\,dw + y^{1/2} \log y \int_0^{4y-y_1} |F'(w)|\,dw \\ &= x^{1/4} y^{-1/4} \Big| \int_0^{4y-y_1} wF'(w)\,dw \Big| + y^{1/2} \log y \Big| \int_0^{4y-y_1} F'(w)\,dw \Big| \\ &\ll x^{1/4} y^{-1/2} \log^3 y + y^{-1/4} \log^3 y \\ &\ll x^{1/4} y^{-1/2} \log^3 t. \end{aligned}$$

In the first equality, we used that $F(w)$ is monotonically decreasing. We have also used the assumption $y \le x$ in the last inequality. Therefore we get

$$W \ll x^{1/4} y^{-1/2} \log^3 t.$$

The other terms in (5.3) that involve $\log^2 x$ and $\log x \log n$ are treated similarly and we find that they are all bounded by $x^{1/4}y^{-1/2}\log^3 t$. Hence we finally get

$$\Sigma_{41} \ll x^{-\sigma-1/4}\cdot x^{1/4}y^{-1/2}\log^3 t \ll x^{1/2-\sigma}(xy)^{-1/2}\log^3 t \ll x^{1/2-\sigma}t^{-1}\log^3 t.$$

Combining these estimates we have $\Sigma_4 \ll x^{1/2-\sigma}t^{-1}\log^3 t$.

Next we consider Σ_1 and Σ_2. It follows by Lemma 4.1 (iii)and (iv) that

$$\begin{aligned}\Sigma_1 &= \sum_{n<\frac{y}{8}} \frac{d(n)}{n^{1-s}}\Big\{\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\log^2(2t) + O(t^{-1-2\sigma}\log t)\\ &\qquad + O\Big(\frac{(nx)^{-\sigma+1/4}(\log nx)^2}{t}\Big)\Big\}\\ &=: \Sigma_{11}+\Sigma_{12}+\Sigma_{13}\end{aligned}$$

and

$$\begin{aligned}\Sigma_2 &= \sum_{\frac{y}{8}\le n<y-\sqrt{y}} \frac{d(n)}{n^{1-s}}\Big\{\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\,\log^2(2t) + O(t^{-1-2\sigma}\log t)\\ &\qquad + \frac{e^{\pi i/4}}{2(\frac{1}{2}-2s)(\frac{3}{2}-2s)}\frac{(4\pi\sqrt{nx})^{5/2-2s}e^{4\pi i\sqrt{nx}}(\log 4\pi\sqrt{nx})^2}{2t-4\pi\sqrt{nx}}\\ &\qquad + O\Big(\frac{(nx)^{-\sigma+3/4}(\log nx)^2}{x^{3/2}(\sqrt{y}-\sqrt{n})^3}\Big) + O\Big(\frac{(nx)^{-\sigma+1/4}(\log nx)^2}{t}\Big)\Big\}\\ &=: \Sigma_{21}+\Sigma_{22}+\Sigma_{23}+\Sigma_{24}+\Sigma_{25}.\end{aligned}$$

By the similar method for Σ_{42}, we can see easily that

$$\Sigma_{12}, \Sigma_{22} \ll t^{-1}\Big(\frac{y}{t^2}\Big)^{\sigma}\log^2 t \ll t^{-1}x^{-\sigma}\log^2 t$$

and

$$\Sigma_{13}, \Sigma_{24}, \Sigma_{25} \ll x^{1/2-\sigma}t^{-1}\log^3 t$$

On the other hand,

$$\Sigma_{23} \ll \frac{x^{-\sigma+3/4}}{t^2}\Big|\sum_{\frac{y}{8}\le n<y-\sqrt{y}}\frac{d(n)n^{1/4}e^{4\pi i\sqrt{nx}}(\log 4\pi\sqrt{nx})^2}{\sqrt{y}-\sqrt{n}}\Big|.$$

We will estimate

$$\widetilde{W} := \sum_{\frac{y}{8}\le n<y-\sqrt{y}}\frac{d(n)n^{1/4}e^{4\pi i\sqrt{nx}}\log^2 n}{\sqrt{y}-\sqrt{n}}.$$

For this purpose, instead of $G(w)$ in Σ_{41}, we define, by writing $y_2 = y-\sqrt{y}$,

$$\widetilde{G}(w) = \sum_{y_2-w\le n<y_2} d(n)e^{4\pi i\sqrt{nx}} \quad\text{and}\quad \widetilde{F}(w) = \frac{(y_2-w)^{1/4}(\log(y_2-w))^2}{\sqrt{y}-\sqrt{y_2-w}}$$

for $0<w\le\frac{7}{8}y-\sqrt{y}$. By partial summation we have

$$\widetilde{W} = \widetilde{G}\Big(\frac{7}{8}y-\sqrt{y}\Big)\widetilde{F}\Big(\frac{7}{8}y-\sqrt{y}\Big) - \int_0^{\frac{7}{8}y-\sqrt{y}}\widetilde{G}(t)\widetilde{F}'(t)\,dt.$$

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As in $G(w)$, we have

$$\widetilde{G}(w) \ll wx^{1/4}y^{-1/4} + x^{-1/4}y^{3/4}\log y + y^{1/2} \ll wx^{1/4}y^{-1/4} + y^{1/2}\log y.$$

Similarly to the case W, by noting that $\widetilde{F}(t)$ is monotonically decreasing, we obtain $\widetilde{W} \ll x^{1/4}y^{1/2}\log^3 y$. The other terms involving $\log^2 x$ and $\log n \log x$ are the same and we get

$$\Sigma_{23} \ll t^{-2}x^{-\sigma+3/4}\cdot x^{1/4}y^{1/2}\log^3 y \ll t^{-1}x^{1/2-\sigma}\frac{(xy)^{1/2}}{t}\log^3 y \ll x^{1/2-\sigma}t^{-1}\log^3 t.$$

In Σ_{21}, we will extend the range of n to y with the error

$$\Big|\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\log^2(2t)\Big|\sum_{y-\sqrt{y}<n\le y}\frac{d(n)}{n^{1-\sigma}} \ll t^{-2\sigma}\log^2 t\cdot y^{\sigma-1/2}\log y$$
$$\ll t^{-1}x^{1/2-\sigma}\log^3 t.$$

Hence we get

$$S_4^{(1)} = 2^{4s-1/2}\pi^{2s-3/2}s\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\sum_{n\le y}\frac{d(n)(\log 2t)^2}{n^{1-s}} + O(x^{1/2-\sigma}\log^3 t),$$

where the O-constant is independent of σ if σ is bounded.

As for $S_4^{(2)}$, it is treated more easily since there exists the extra factor $\frac{3}{8}v^{-1}$ in the asymptotic expansion of $K_1(v)+\frac{\pi}{2}Y_1(v)$. In fact, first by integration by parts and then by Lemma 4.1 (v) we have

$$\int_\xi^\infty(\log v)^2\frac{\sin(v-\frac{\pi}{4})}{v^{2s+3/2}}\,dv \ll t^{-1/2}\xi^{-(2\sigma+\frac{1}{2})}\log^2(t+\xi).$$

Hence

$$S_4^{(2)} \ll t^{1/2}\sum_{n=1}^{\infty}d(n)n^{\sigma-1}(nx)^{-(\sigma+\frac{1}{4})}(\log^2 t+\log t\log n+\log^2 n) \ll x^{1/4-\sigma}\log^2 t.$$

Now combining the estimates of $S_4^{(1)}$, $S_4^{(2)}$, and $S_4^{(3)}$, we finally get

$$S_4 = 2^{4s-1/2}\pi^{2s-3/2}s\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\sum_{n\le y}\frac{d(n)(\log 2t)^2}{n^{1-s}} + O(x^{1/2-\sigma}\log^3 t). \tag{5.7}$$

Evaluations of S_2 and S_3 The terms S_2 and S_3 in (5.2) are treated similarly to S_4, that is,

(5.8)

$$S_2 = 2^{4s-1/2}\pi^{2s-3/2}s\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\sum_{n\le y}\frac{d(n)\big(\frac{1}{4}\log(2^4\pi^2)^2+\frac{1}{2}\log(2^4\pi^2)\log n\big)}{n^{1-s}}$$
$$+ O(x^{1/2-\sigma}\log^3 t),$$

(5.9)

$$S_3 = 2^{4s-1/2}\pi^{2s-3/2}s\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\sum_{n\le y}\frac{d(n)(\log(2^4\pi^2)+\log n)\log 2t}{n^{1-s}}$$
$$+ O(x^{1/2-\sigma}\log^3 t).$$

Evaluation of S_1 For S_1, we follow the same lines as in S_4. Substituting (5.1) we have $S_1 = S_1^{(1)} + S_1^{(2)} + S_1^{(3)}$, where

$$S_1^{(1)} = 2^{4s-1/2}\pi^{2s-3/2}s\sum_{n=1}^{\infty}\frac{D_{(1)}(n)}{n^{1-s}}I_0\Big(4\pi\sqrt{nx}, 2s+\frac{1}{2}\Big),$$
$$S_1^{(2)} = -\frac{3}{8}2^{4s-1/2}\pi^{2s-3/2}s\sum_{n=1}^{\infty}\frac{D_{(1)}(n)}{n^{1-s}}\int_{4\pi\sqrt{nx}}^{\infty}\frac{\sin\big(v-\frac{\pi}{4}\big)}{v^{2s+3/2}}\,dv,$$
$$S_1^{(3)} = O\Big(t\sum_{n=1}^{\infty}\frac{D_{(1)}(n)}{n^{1-\sigma}}(nx)^{-\sigma-3/4}\Big).$$

Divide the sum in $S_1^{(1)}$ into five parts as in $S_4^{(1)}$. Almost all parts are a repetition of the arguments of $S_4^{(1)}$ but we use Lemma 2.2 (2.5) this time. However, the exponential sums corresponding to Σ_{41} and Σ_{23} in the case $S_4^{(1)}$ are different. In fact, we need to evaluate the sums

$$V_1 := x^{-\sigma-1/4}\Big|\sum_{y+\sqrt{y}<n\le 4y}\frac{D_{(1)}(n)e^{4\pi i\sqrt{nx}}}{n^{3/4}(\sqrt{n}-\sqrt{y})}\Big|,$$
$$\widetilde{V}_1 := \frac{x^{-\sigma+3/4}}{t^2}\Big|\sum_{\frac{y}{8}<n\le y-\sqrt{y}}\frac{D_{(1)}(n)n^{1/4}e^{4\pi i\sqrt{nx}}}{\sqrt{y}-\sqrt{n}}\Big|$$

in this case. For V_1, let

$$G_1(w) = \sum_{y_1<n\le y_1+w}D_{(1)}(n)e^{4\pi i\sqrt{nx}},\ F_1(v) = \frac{1}{v^{3/4}(\sqrt{v}-\sqrt{y})},$$

where $y_1 = y+\sqrt{y}$. As in the case Σ_{41}, we have, by partial summation,

$$\sum_{y_1<n\le 4y}\frac{D_{(1)}(n)e^{4\pi i\sqrt{nx}}}{n^{3/4}(\sqrt{n}-\sqrt{y})} = G_1(4y-y_1)F_1(4y) - \int_0^{4y-y_1}G_1(v)F_1'(y_1+v)\,dv. \tag{5.10}$$

If we can show that

$$G_1(w) \ll (wx^{1/4}y^{-1/4} + y^{1/2}\log y)\log^2 y, \tag{5.11}$$

which is similar to (5.6), we get the desired result $V_1 \ll t^{-1}x^{1/2-\sigma}\log^3 t$ from (5.10) by noticing that $F_1(w)$ is monotonically decreasing. To show (5.11) we rewrite $G_1(w)$ as

$$\begin{aligned}G_1(w) &= \sum_{y_1<\mu\nu\le y_1+w}\log\mu\log\nu\ e^{4\pi i\sqrt{\mu\nu x}}\\ &= 2\sum_{\mu}\log\mu\sum_{\substack{\frac{y_1}{\mu}<\nu\le\frac{y_1+w}{\mu}\\ \mu<\nu}}\log\nu\ e^{4\pi i\sqrt{\mu\nu x}} + O\Big(\Big(\frac{w}{\sqrt{y}}+1\Big)\log^2 y\Big).\end{aligned}$$

By (5.5) and partial summation, we have

$$\sum_{\frac{y_1}{\mu}<\nu\le\frac{y_1+w}{\mu}}\log\nu\ e^{4\pi i\sqrt{\mu\nu x}} \ll \log y(wx^{1/4}y^{-3/4} + \mu^{-1}x^{-1/4}y^{3/4}+1)$$

and hence (5.11). We have similarly that $\widetilde{V}_1 \ll t^{-1}x^{1/2-\sigma}\log^3 t$. Thus we get

$$S_1^{(1)} = 2^{4s-1/2}\pi^{2s-3/2}s\,\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\sum_{n\le y}\frac{D_{(1)}(n)}{n^{1-s}} + O(x^{1/2-\sigma}\log^3 t).$$

It is easy to see that $S_1^{(2)}, S_1^{(3)} \ll x^{1/2-\sigma}$.
Collecting these estimate we have

$$S_1 = 2^{4s-1/2}\pi^{2s-3/2}s\,\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\sum_{n\le y}\frac{D_{(1)}(n)}{n^{1-s}} + O(x^{1/2-\sigma}\log^3 t). \tag{5.12}$$

By (5.7), (5.8), (5.9), and (5.12), equation (5.2) becomes

$$2^{4s-1/2}\pi^{2s-3/2}s\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s\Big\{\sum_{n\le y}\frac{D_{(1)}(n)}{n^{1-s}} + \sum_{n\le y}\frac{d(n)}{n^{1-s}}H(n)\Big\} + O(x^{1/2-\sigma}\log^3 t), \tag{5.13}$$

where $H(n)$ is given by

(5.14)

$$\begin{aligned} H(n) &= \frac{1}{4}(\log 2^4\pi^2)^2 + \frac{1}{2}\log 2^4\pi^2\cdot\log n - (\log 2^4\pi^2 + \log n)\log 2t + (\log 2t)^2 \\ &= (\log\frac{t}{2\pi})^2 - \log\frac{t}{2\pi}\cdot\log n. \end{aligned}$$

We remark further that

$$2^{4s-1/2}\pi^{2s-3/2}s\,\Gamma\Big(\frac{1}{2}-2s\Big)\cos\pi s = \chi^2(s) + O(t^{-2\sigma}) \tag{5.15}$$

(see [22, p. 114]) and

$$t^{-2\sigma}\sum_{n\le y}\frac{d(n)}{n^{1-\sigma}} \ll t^{-2\sigma}y^{\sigma}\log y \ll x^{-\sigma}\log y \tag{5.16}$$

for $0<\sigma<1$.

By (3.2), (5.13), (5.14), (5.15), and (5.16) we finally get (1.8) in the case $x \ge y$.

We will prove (1.8) in the case $x<y$. Applying the approximate functional equation for $x<y$ and replacing s by $1-s$, we know that

$$\zeta'(1-s)^2 = \sum_{n\le y}\frac{D_{(1)}(n)}{n^{1-s}} + \chi(1-s)^2\sum_{n\le x}\frac{1}{n^s}\Bigg(D_{(1)}(n) - d(n)\log n\log\frac{t}{2\pi} + d(n)\log^2\frac{t}{2\pi}\Bigg) + O(y^{\sigma-1/2}\log^3 t). \tag{5.17}$$

We use the functional equation (2.15), namely

$$\zeta'(s)^2 = \chi^2(s)\zeta'(1-s)^2 - 2\chi(s)\chi'(s)\zeta(1-s)\zeta'(1-s) + \chi'(s)^2\zeta(1-s)^2. \tag{5.18}$$

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By (2.20) and (1.4) we have

$$(5.19)\quad \chi'(s)^2\zeta^2(1-s) = \Big(\chi^2(s)\log^2\frac{t}{2\pi} + O(t^{-2\sigma}\log t)\Big) \times \Big(\sum_{n\le y}\frac{d(n)}{n^{1-s}} + \chi^2(1-s)\sum_{n\le x}\frac{d(n)}{n^s} + O(y^{\sigma-1/2}\log t)\Big) = \chi^2(s)\log^2\frac{t}{2\pi}\sum_{n\le y}\frac{d(n)}{n^{1-s}} + \log^2\frac{t}{2\pi}\sum_{n\le x}\frac{d(n)}{n^s} + O(x^{1/2-\sigma}\log^3 t).$$

By (2.20) and (1.9) we have similarly that

$$(5.20)\qquad 2\chi(s)\chi'(s)\zeta(1-s)\zeta'(1-s) = \chi^2(s)\log\frac{t}{2\pi}\sum_{n\le y}\frac{d(n)\log n}{n^{1-s}} - \log\frac{t}{2\pi}\sum_{n\le x}\frac{d(n)\log\frac{4\pi^2 n}{t^2}}{n^s} + O(x^{1/2-\sigma}\log^3 t).$$

Substituting (5.17), (5.19), and (5.20) in (5.18) we find that the same formula (1.8) holds for $x \le y$.

6 Remarks on the Proof of Theorem 1.2

The proofs of (1.10) and (1.11) are similar to that of (1.8). So we only give some simple remarks on them.

Comparing (3.2) and (3.3), we observe that the difference between them is the coefficients $D_{(1)}(n)$ and $d_{(0,2)}(n)$ in the first two series on their right-hand side. As in the case of $G_1(w)$ in the previous section, we obtain

$$\sum_{y_1<n\le y_1+w} d_{(0,2)}(n)e^{4\pi i\sqrt{nx}} \ll (wx^{1/4}y^{-1/4} + y^{1/2}\log y)\log^2 y.$$

Furthermore, since the sums of $D_{(1)}(n)$ and $d_{(0,2)}(n)$ have the same asymptotic order (see Lemma 2.2 (2.5) and (2.6)), we can see that (1.10) holds true in the case $x \ge y$. In the case $x \le y$, we can prove that the same formula (1.10) holds by applying (1.10) for $x \le y$ with s replaced by $1-s$, and utilizing the functional equation (2.16) and the approximate functional equations for $\zeta(s)^2$ and $\zeta(s)\zeta'(s)$ ((1.4) and (1.9), respectively).

Finally we consider (1.11) of Theorem 1.2. Let $y \le x$. If we apply the similar argument to the right-hand side of (3.4), we find that the terms from the second sum to the sixth sum there reduce (up to the error $O(x^{1/2-\sigma}\log^4 t)$) to the sum over $1 \le n \le y$ with $sU_j(s,n)$ replaced by $2^{4s-1/2}\pi^{2s-3/2}s\Gamma(1/2-2s)\cos\pi s(\log 2t)^j n^{s-1}$. Hence we have

$$\zeta'(s)\zeta''(s) = \sum_{n\le x}\frac{d_{(1,2)}(n)}{n^s} + 2^{4s-1/2}\pi^{2s-3/2}s\Gamma(1/2-2s)\cos\pi s \times \sum_{n\le y}\frac{-d_{(1,2)}(n) + d_{(0,2)}(n)\log\frac{t}{2\pi} + d(n)K(n)}{n^{1-s}} + O(x^{1/2-\sigma}\log^4 t),$$

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where $K(n)$ is given by

$$K(n) = \Big(\frac{1}{8}(\log 2^4\pi^2)^3 + \frac{3}{8}(\log 2^4\pi^2)^2\log n + \frac{1}{2}\log 2^4\pi^2\cdot\log^2 n\Big) - \Big(\frac{3}{4}(\log 2^4\pi^2)^2 + \frac{3}{2}\log 2^4\pi^2\cdot\log n + \log^2 n\Big)\log 2t + \frac{3}{2}(\log 2^4\pi^2 + \log n)(\log 2t)^2 - (\log 2t)^3.$$

It is easy to see that

$$K(n) = -\log\frac{t}{2\pi}\cdot\log^2 n + \frac{3}{2}\Big(\log\frac{t}{2\pi}\Big)^2\log n - \Big(\log\frac{t}{2\pi}\Big)^3;$$

hence by noting (5.15), we get (1.11) in the case $y \le x$.

In the case $x \le y$, similarly to the other two cases, we first apply (1.11) for the case $x \le y$ with s replaced by $1-s$, utilize the functional equation (2.17) and the approximate functional equations for $\zeta(s)^2$, $\zeta(s)\zeta'(s)$, $\zeta'(s)^2$, and $\zeta(s)\zeta''(s)$ ((1.4), (1.9), (1.8), and (1.10), respectively), and we can see that the same formula also holds for the case $x \le y$. Note that we need all the other approximate functional equations and (2.3).

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Department of Integrated Human Sciences (Mathematics), Hamamatsu University School of Medicine, Handayama 1-20-1, Shizuoka 431-3192, Japan
e-mail: jfuruya@hama-med.ac.jp

Graduate School of Sciences and Technology for Innovation, Yamaguchi University, Yoshida 1677-1, Yamaguchi 753-8512, Japan
e-mail: minamide@yamaguchi-u.ac.jp

Graduate School of Mathematics, Nagoya University, Furo-cho, Nagoya 464-8602, Japan
e-mail: tanigawa@math.nagoya-u.ac.jp