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Normalized ground state solutions for critical growth Schrödinger equations with Hardy potential

Song Fan
School of Mathematics and Statistics, Guizhou University, Guiyang 550025, People's Republic of China (Doraemonsong77@gmail.com)

Gui-Dong Li
School of Mathematics and Statistics, Guizhou University, Guiyang 550025, People's Republic of China (bestdong123@163.com)

Chun-Lei Tang
School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China (tangcl@swu.edu.cn) (corresponding author)

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In this article, we study the following Schrödinger equation

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2}u + \lambda u = f(u), & \text{in } \mathbb{R}^N \backslash \{0\}, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, & u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $a > 0$, and $\mu < \frac{(N-2)^2}{4}$. Here $\frac{1}{|x|^2}$ represents the Hardy potential (or 'inverse-square potential'), λ is a Lagrange multiplier, and the nonlinearity function f satisfies the general Sobolev critical growth condition. Our main goal is to demonstrate the existence of normalized ground state solutions for this equation when $0 < \mu < \frac{(N-2)^2}{4}$. We also analyse the behaviour of solutions as $\mu \to 0^+$ and derive the existence of normalized ground state solutions for the limiting case where $\mu = 0$. Finally, we investigate the existence of normalized solutions when $\mu < 0$ and analyse the asymptotic behaviour of solutions as $\mu \to 0^-$.

Keywords: asymptotic behaviour; Hardy potential; Sobolev critical growth; variational methods; normalized solutions

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1. Introduction

In recent decades, significant attention has been directed towards the exploration of standing wave solutions in the context of the time-dependent Schrödinger equation, which is formulated as follows

$$\begin{cases} i\Phi_t + \Delta\Phi + g\left(|\Phi|^2\right)\Phi = 0,\ t \geq 0,\ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\Phi|^2 \mathrm{d}x = a. \end{cases} \tag{1.1}$$

In this context, i represents the imaginary unit, $N \geq 3$, $a > 0$, and $g : [0, \infty) \to \mathbb{R}$ is a nonlinear term. The function $\Phi(t, x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is the wave function. Equation (1.1) arises naturally in the time-dependent Cauchy problem given by

$$\begin{cases} i\Phi_t + \Delta\Phi + g\left(|\Phi|^2\right)\Phi = 0, \\ \Phi(\cdot, 0) = \varphi_0 \in L^2\left(\mathbb{R}^N\right) \setminus \{0\}. \end{cases} \tag{1.2}$$

The L^2-normalization condition in Eq. (1.1) stems from the conservation of the L^2-norm in Eq. (1.2). Indeed multiplying Eq. (1.2) by $\overline{\Phi}$, integrating, and taking the imaginary part leads to $\frac{d}{dt}\int_{\mathbb{R}^N} |\Phi|^2 \mathrm{d}x = 0$ and therefore we can define $\int_{\mathbb{R}^N} |\Phi|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |\varphi_0|^2 \mathrm{d}x = a$, see [12]. The pursuit of solutions with prescribed L^2-norms holds profound significance from both physical and mathematical vantage points. From a physical perspective, the search for solutions characterized by a predetermined L^2-norm is intricately linked with the principle of mass conservation, carrying fundamental physical interpretations across diverse domains. For instance, in the field of nonlinear optics, the L^2-norm corresponds to the power magnitude, while in Bose–Einstein condensates, it encapsulates the particle count and assumes a pivotal role in delineating the system's behaviour (refer to [1, 20, 48]). From a mathematical stance, the examination of solutions with prescribed L^2-norms contributes invaluable insights into the characteristics and dynamics of these solutions, thereby fostering a deeper comprehension of stability and instability phenomena (refer to [8, 15]).

Consider the standing wave solution denoted as $\Phi(t, x) = e^{i\lambda t}u(x)$ in Eq. (1.1), where $u : \mathbb{R}^N \to \mathbb{R}$. Subsequently, we transform Eq. (1.1) into a new form

$$\begin{cases} -\Delta u + \lambda u = f(u), \quad \text{in} \quad \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, \end{cases} \tag{1.3}$$

where $f(u) = g\left(|u|^2\right)u$. Equation (1.3) characterizes the steady-state behaviour of the wave function. In order to analyse Eq. (1.3), we introduce the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x,$$

where $F(u)=\int_0^u f(\tau)d\tau$, and $\mathcal{E}$ belongs to the class C^1 on $H^1\left(\mathbb{R}^N\right)$. A critical point of $\mathcal{E}$ under the mass constraint S_a,

$$S_a=\left\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}|u|^2\mathrm{d}x=a\right\},$$

known as the normalized solution, is a solution of Eq. (1.3).

In the case of Eq. (1.3) with $f(u)=\mu|u|^{q-2}u+|u|^{p-2}u$, $2<q\le p\le 2^*$, the exploration of normalized ground state solutions for Eq. (1.3) was undertaken by Soave in [43, 44]. Building upon the foundational contributions of Soave, subsequent scholarly endeavours have further engaged with Eq. (1.3), as exemplified by works such as [2, 29, 36, 49]. For the general nonlinear terms f, it is noteworthy to mention the investigation carried out by Jeanjean in [27], who assumed that $f:\mathbb{R}\to\mathbb{R}$ satisfies

(H1) $f\in C(\mathbb{R},\mathbb{R})$ and odd.

(H2) There exist $\alpha,\beta\in\mathbb{R}$ satisfying $2+\frac{4}{N}<\alpha\le\beta<2^*=\frac{2N}{N-2}$ such that

$$\alpha F(t)\le f(t)t\le\beta F(t)\ \text{ for any } t\in\mathbb{R}\setminus\{0\}.$$

(H3) The function $\widetilde{F}(t):=f(t)t-2F(t)$ is of class C^1 and satisfies

$$\widetilde{F}'(t)t>\frac{2N+4}{N}\widetilde{F}(t)\ \text{ for any } t\neq 0.$$

and established the existence of normalized ground state solutions to Eq. (1.3) for any $N\ge 1$. Subsequently, for $N\ge 2$, Bartsch and de Valeriola in [4] obtained an infinite number of radial normalized solutions for Eq. (1.3), provided $(H1)$ and $(H2)$. Furthermore, Jeanjean and Lu [31] revisited Eq. (1.3) under the following assumptions:

(H4) $f:\mathbb{R}\to\mathbb{R}$ is continuous.

(H5) $\lim_{s\to 0}\frac{f(s)}{|s|^{1+4/N}}=0$ and $\lim_{s\to\infty}\frac{f(s)}{|s|^{(N+2)/(N-2)}}=0$.

(H6) $\lim_{s\to\infty}\frac{F(s)}{|s|^{2+4/N}}=+\infty$.

(H7) $f(s)s<\frac{2N}{N-2}F(s)$ for all $s\in\mathbb{R}\setminus\{0\}$.

(H8) The function $s\mapsto\frac{\widetilde{F}(s)}{|s|^{2+4/N}}$ is strictly decreasing on $(-\infty,0)$ and strictly increasing on $(0,+\infty)$.

Due to $(H4)$–$(H8)$, which do not require $\widetilde{F}\in C^1$, the authors established the existence of normalized ground state solutions by adapting the argument and employing techniques from Szulkin and Weth [45, 46]. Subsequently, the authors extend the results of Jeanjean [27] regarding the existence of normalized ground state solutions. For readers interested in exploring normalized solutions of Eq. (1.3), we recommend further investigations into works such as [6, 11, 25, 28, 30, 32, 33, 38, 42, 51], along with the references they provide. These works offer deeper insights and additional research pertaining to this subject.

In a parallel vein of research, certain scholars have introduced an external potential V in Eq. (1.3), i.e.

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a. \end{cases} \tag{1.4}$$

For the case where $f(u) = |u|^{p-2}u$ with $2 < p < 2^*$, Pellacci et al. [40] considered the existence of normalized solutions for Eq. (1.4) if V possesses a non-degenerate critical point, who employed the Lyapunov–Schmidt reduction approach to establish the existence of normalized solutions for Eq. (1.4), contingent on the condition that a is sufficiently large and $p < 2 + \frac{4}{N}$, or a is suitably small and $p > 2 + \frac{4}{N}$. Simultaneously, Bartsch et al. [5] employed min-max arguments to establish the existence of normalized solutions for Eq. (1.4) with $2 + \frac{4}{N} < p < 2^*$ and $V(x) \geq 0$ tends to zero at infinity.

Subsequently, the authors in [39] obtained the existence of normalized solutions for Eq. (1.4), when $2 + \frac{4}{N} < p < 2^*$, $V(x) \leq 0$ satisfies $V(x) \leq \limsup_{|x| \to +\infty} V(x) < +\infty$, and

$$\max\left\{ |W|_N, |V|_{\frac{N}{2}} \right\} < M, \quad \text{for some } M \in \mathbb{R}^+, \text{ where } W(x) = V(x)|x|.$$

For the general nonlinearity terms f in Eq. (1.4), Ding and Zhong [18] assumed that f satisfies $(H1)$, $(H2)$, and $(H3')$:

(H3$'$) The functional $\widetilde{F}(s) = f(s)s - 2F(s)$ is of class C^1 and

$$\widetilde{F}'(s)s \geq \alpha \widetilde{F}(s), \text{ for any } s \in \mathbb{R},$$

and V satisfies

(V3) $\lim_{|x| \to +\infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) = 0$ and there exists some $\sigma_1 \in \left[0, \frac{N(\alpha-2)-4}{N(\alpha-2)}\right)$ such that

$$\left| \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x \right| \leq \sigma_1 |\nabla u|_2^2, \text{ for any } u \in H^1\left(\mathbb{R}^N\right).$$

(V4) $\nabla V(x)$ exists a.e. in $\mathbb{R}^N$ and coincides to the weak gradient of V, put $W(x) := \frac{1}{2}\langle \nabla V(x), x \rangle$. There exists some $0 \leq \sigma_2 < \min\left\{ \frac{N(\alpha-2)(1-\sigma_1)}{4} - 1, \frac{N}{\beta} - \frac{N-2}{2} \right\}$ such that

$$\left| \int_{\mathbb{R}^N} W(x)u^2 \mathrm{d}x \right| \leq \sigma_2 |\nabla u|_2^2, \text{ for any } u \in H^1\left(\mathbb{R}^N\right).$$

(V5) $\nabla W(x)$ exists a.e. in $\mathbb{R}^N$ and coincides to the weak gradient of W, put

$$Y(x) := \left(\frac{N}{2}\alpha - N \right) W(x) + \langle \nabla W(x), x \rangle,$$

$\int_{\mathbb{R}^N} Y(x)u^2 \mathrm{d}x$ is well-defined for all $u \in H^1\left(\mathbb{R}^N\right)$ and there exists some $\sigma_3 \in \left[0, \frac{N}{2}\alpha - N - 2\right)$ such that

$$\int_{\mathbb{R}^N} Y_+(x)u^2 \mathrm{d}x \leq \sigma_3 |\nabla u|_2^2, \text{ for any } u \in H^1\left(\mathbb{R}^N\right).$$

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Under $(H1)$, $(H2)$, $(H3')$ and $(V3)$–$(V5)$, the authors proved the existence of normalized solutions for Eq. (1.4) for any given $a > 0$. Li and Zou [35] recently studied the case where $V(x) = -\frac{\mu}{|x|^2}$ and $f(u) = |u|^{2^*-2}u + \nu|u|^{p-2}u$, with $2 < p < 2^*$, in Eq. (1.4), which can be expressed as:

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2}u = \lambda u + |u|^{2^*-2}u + \nu|u|^{p-2}, & N \geq 3, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, & u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.5}$$

and then found several existence results of normalized ground state solutions when $\nu \geq 0$ and non-existence results when $\nu \leq 0$. Furthermore, they also consider the asymptotic behaviour of the normalized solutions as $\mu \to 0$ or $\nu \to 0$. For further findings on Eq. (1.4), please refer to [26, 52] and the corresponding references. We also note that Bieganowski, Mederski, and Schino [12] obtained the existence of normalized solutions for the following singular polyharmonic equation

$$\begin{cases} (-\Delta)^m u + \frac{\mu}{|y|^{2m}}u + \lambda u = g(u), & x = (y, z) \in \mathbb{R}^K \times \mathbb{R}^{N-K}, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0, \end{cases}$$

where g is Sobolev subcritical growth at infinity.

Motivated by the previous studies, we find ourselves inclined to extend our exploration into the realm of normalized solutions for Eq. (1.4) with Hardy potential. Specifically, we investigate the following equation

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} + \lambda u = f(u), & \text{in} \quad \mathbb{R}^N\backslash\{0\}, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, & u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $N \geq 3$, $\lambda \in \mathbb{R}$, $\frac{1}{|x|^2}$ is the Hardy potential, $\mu < \bar{\mu} := \frac{(N-2)^2}{4}$, and f satisfies the following conditions:

(F1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and odd.

(F2) There exist β, η such that $\limsup_{|s|\to 0} \frac{F(s)}{|s|^{2+4/N}} = \beta \in [0, \infty)$ and $\lim_{|s|\to\infty} \frac{f(s)s}{|s|^{2^*}} = 2^*\eta > 0$.

(F3) $\frac{\widetilde{F}(s)}{|s|^{2+\frac{4}{N}}}$ is strictly increasing on $(0, +\infty)$, where $\widetilde{F}(s) = f(s)s - 2F(s)$.

(F4) $f(s)s < 2^*F(s)$ for $s \neq 0$.

(F5) There exist constants $2 + 4/N < p < 2^*$ and $\kappa > 0$ such that

$$F(s) \geq \frac{\kappa}{p}|s|^p.$$

The primary focus of this problem is not only the Sobolev critical growth nonlinear term but also the presence of the so-called 'Hardy potential' (or 'inverse-square potential') in the linear part. The potential with this rate of decay is critical in non-relativistic quantum mechanics, as they represent an inter-mediate threshold between regular potentials (for which there are ordinary stationary states) and singular potentials (for which the energy is not lower-bounded and the particle falls to the centre), for more details see [22]. Besides, it also arises in many other areas such as nuclear physics, molecular physics, and quantum cosmology (see [9, 14, 23, 41]).

The Gagliardo–Nirenberg inequality is crucial to this study. For $2 < p < 2^*$, the inequality is given by:

$$|u|_p \leq C_{N,p}|\nabla u|_2^{\gamma_p}|u|_2^{1-\gamma_p} \text{for } u \in H^1\left(\mathbb{R}^N\right), \tag{1.7}$$

where $C_{N,p} > 0$ represents the optimal constant, and $\gamma_p = N\left(\frac{1}{2} - \frac{1}{p}\right)$. Additionally, $p\gamma_p > 2$ holds if and only if $p > \bar{p} := 2 + \frac{4}{N}$.

We introduce that the corresponding energy functional is of class C^1 in $H^1\left(\mathbb{R}^N\right)$:

$$\mathcal{I}_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \mu\frac{u^2}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

We say that $v \in S_a$ is the normalized ground state solution to Eq. (1.6) if it is a solution of Eq. (1.6) that minimizes the value of $\mathcal{I}_\mu$ among all the normalized solutions of (1.6). Namely, if

$$d\mathcal{I}_\mu|_{S_a}(v) = 0 \quad \text{and} \quad \mathcal{I}_\mu(v) = \inf\left\{\mathcal{I}_\mu(v) : d\mathcal{I}_\mu|_{S_a}(u) = 0, u \in S_a\right\}.$$

Since the functional $\mathcal{I}_\mu$ remains unbounded from below on S_a, we therefore introduce the manifold

$$\mathcal{M}_\mu(a) = \{u \in S_a : P_\mu(u) = 0\},$$

where $P_\mu(u)$ is defined as

$$P_\mu(u) = \int_{\mathbb{R}^N} |\nabla u|^2 - \mu\frac{u^2}{|x|^2}\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} \widetilde{F}(u)\,\mathrm{d}x.$$

It is a widely acknowledged fact that any critical point of $\mathcal{I}_\mu|_{S_a}$ is a member of $\mathcal{M}_\mu(a)$, from an implication of the Pohožaev identity. Furthermore, we delve into the exploration of the minimizing problem

$$m_\mu(a) = \inf_{u\in\mathcal{M}_\mu(a)} \mathcal{I}_\mu(u).$$

We will now delineate the main result of this article.

THEOREM 1.1. *Assume that* $N \geq 3$, $\bar{\mu} > \mu > 0$, $a > 0$, $1 - \frac{\mu}{\bar{\mu}} > 2^* C_{N,\bar{p}}^{\bar{p}}\beta a^{\frac{2}{N}}$, *and* $(F1)$ – $(F5)$ *hold. Then, there exists* $\kappa^* > 0$, *such that for any* $\kappa \geq \kappa^*$ *(*κ *is given in* $(F5)$*), Eq. (1.6) possesses a normalized ground state solution* (u, λ), *where* $u > 0$ *is radial and* $\lambda > 0$.

The solution derived from theorem 1.1 is exponential decay at infinity and potentially blow-up at the origin. This property is stated in the following proposition.

PROPOSITION 1.2. *Let (u, λ) be the solution obtained in theorem 1.1. Then*

(i) $u \in C^2\left(\mathbb{R}^N \backslash\{0\}\right)$.
(ii) *There exist constants $C > 0$ and $R > 0$ such that for $|\alpha| \leq 2$,*

$$|D^\alpha u(x)| \leq C \exp\left(-\sqrt{\frac{1}{2}}|x|\right), \text{for } |x| \geq R.$$

(iii) *There exist constants $C_{r,1} > 0$ and $C_{r,2} > 0$ depend on a sufficiently small $r > 0$ such that*

$$C_{r,2}|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} \leq |u(x)| \leq C_{r,1}|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}, \quad \text{for } x \in B_r \backslash\{0\}.$$

In fact, the limiting equation derived from Eq. (1.6) is as follows

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, & u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.8}$$

and the associated energy functional $\mathcal{I}_\infty : H^1\left(\mathbb{R}^N\right) \to \mathbb{R}$ for Eq. (1.8) is

$$\mathcal{I}_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

Any solution u of Eq. (1.6) belongs to the manifold

$$\mathcal{M}_\infty(a) = \left\{u \in S_a : P_\infty(u) = 0\right\},$$

where $P_\infty(u)$ is defined as

$$P_\infty(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x.$$

Furthermore, we define

$$m_\infty(a) = \inf_{u \in \mathcal{M}_\infty(a)} \mathcal{I}_\mu(u).$$

We then scrutinize the behaviour of solutions as the parameter $\mu \to 0^+$ and derive the existence of solutions for the limiting case, i.e. Eq. (1.8).

THEOREM 1.3. *Assume that $N \geq 3$, $\bar{\mu} > \mu > 0$, $a > 0$, $1 - \frac{\mu}{\bar{\mu}} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, and $(F1)$–$(F5)$ hold. Let $\{(u_{\mu_n}, \lambda_{\mu_n})\}$ in theorem 1.1 with $\mu_n \to 0^+$, then $u_{\mu_n} \to u$ in $H_r^1\left(\mathbb{R}^N\right)$ and $\lambda_{\mu_n} \to \lambda > 0$ as $\mu_n \to 0^+$. Moreover, (u, λ) is a normalized ground state solution of Eq. (1.8).*

Furthermore, we study the existence of solutions for $\mu < 0$.

THEOREM 1.4. *Assume that $N \geq 3$, $0 > \mu$, $a > 0$, $1 > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$ and $(F1)$–$(F5)$ hold. Then $m_\mu(a) = m_\infty(a)$ and $m_\mu(a)$ cannot be achieved. Furthermore, if κ is*

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sufficiently large, Eq. (1.6) admits a mountain pass solution $(u, \lambda) \in H_r^1(\mathbb{R}^N) \times \mathbb{R}^+$ *with* $u > 0$*, whose energy is strictly greater than* $m_\mu(a)$*.*

Remark 1.5. In the case without a mass constraint, when $\mu < 0$, there is no ground state, as demonstrated in [37, theorem 1.1].

It is also of significant interest to investigate the asymptotic behaviour of solutions as $\mu \to 0^-$. Consequently, we present the following theorem.

Theorem 1.6. *Assume that* $N \geq 3$, $0 > \mu$, $a > 0$, $1 > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, *and* $(F1)$–$(F5)$ *hold. Let the positive and radial sequence of solutions* $\{(u_{\mu_n}, \lambda_{\mu_n})\}$ *in theorem 1.4 with* $\mu_n \to 0^-$*, then* $u_{\mu_n} \to u$ *in* $H_r^1(\mathbb{R}^N)$ *and* $\lambda_{\mu_n} \to \lambda > 0$ *as* $\mu_n \to 0^-$*. Moreover,* (u, λ) *is a normalized ground state solution of Eq. (1.8).*

Proposition 1.7. *Let* u *be a solution obtained in either theorem 1.3, theorem 1.4, or theorem 1.6. Then, it can be inferred that* $u \in C^2(\mathbb{R}^N)$*, and there exist* $C > 0$ *and* $R > 0$ *such that for* $|\alpha| \leq 2$*,*

$$|D^\alpha u(x)| \leq C \exp\left(-\sqrt{\frac{1}{2}}|x|\right) \text{ for all } |x| \geq R.$$

Remark 1.8. To illustrate the existence of nonlinear functions that satisfy $(F1)$–$(F5)$, we provide the following example:

$$F(s) = \beta|s|^{2+\frac{4}{N}} + \eta|s|^{2^*} + \frac{\kappa}{p}|s|^p,$$

where $2 + 4/N < p < 2^*$.

The article is structured as follows: In §2, we give a foundation of preliminary concepts and lemmas that will be invoked in subsequent proofs, including the proof of theorem 1.1. The proofs of theorems 1.3, 1.4, and 1.6 are delineated in §3, 4, and 5, respectively.

Notation. Throughout the article, we use the following notations:

- $H^1(\mathbb{R}^N)$ denotes the Sobolev space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \mathrm{d}x\right)^{\frac{1}{2}}.$$

- $H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is the radial function}\}$.
- $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$ denotes the Lebesgue space with the norm

$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad |u|_\infty = \operatorname{ess}\sup_{x \in \mathbb{R}^N} |u(x)|.$$

- $B_r(0) := \{x \in \mathbb{R}^N : |x| < r\}$.
- $S_{r,a} := \{u \in H_r^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a\}$.

- $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right) := \left\{u \in L^{2^*}\left(\mathbb{R}^N\right) : \frac{\partial u}{\partial x_i} \in L^2\left(\mathbb{R}^N\right), i = 1, 2, \ldots, N\right\}$.
- $\mathbb{R}^+ := \{\alpha \in \mathbb{R} : \alpha > 0\}$.
- C denotes a positive constant and is possibly various in different places.

2. Preliminaries

For any $N \geq 3$ and $\mu \in (0, \bar{\mu})$, we define

$$S_\mu := \inf_{u \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\mu}{|x|^2} u^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\right)^{2/2^*}}, \tag{2.1}$$

as in [16, 19]. In particular, when $\mu = 0$, we define

$$S := \inf_{u \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\right)^{2/2^*}}, \tag{2.2}$$

see [47]. Both (2.1) and (2.2) lead to the formulation of an inequality known as the Sobolev inequality. For $2 < p < 2^*$, we recall the Gagliardo–Nirenberg inequality as

$$|u|_p \leq C_{N,p} |\nabla u|_2^{\gamma_p} |u|_2^{1-\gamma_p} \text{ for } u \in H^1\left(\mathbb{R}^N\right),$$

where $C_{N,p} > 0$ represents the optimal constant, $\gamma_p = N\left(\frac{1}{2} - \frac{1}{p}\right)$, and $p\gamma_p > 2$ holds if and only if $p > \bar{p} = 2 + \frac{4}{N}$.

Lemma 2.1. *Assume that $N \geq 3$, and $(F1)$–$(F5)$ hold. Then there exists $c > 0$ such that*

$$f(s)s - \bar{p}F(s) > c|s|^{2^*} \quad \textit{for all } s \neq 0.$$

Proof. By [31, lemma 2.3], we have

$$f(s)s - \bar{p}F(s) > 0 \quad \text{for all } s \neq 0, \tag{2.3}$$

where $\bar{p} = 2 + \frac{4}{N}$. We claim that $\liminf_{s \to 0} \frac{f(s)s - \bar{p}F(s)}{|s|^{2^*}} > 0$. Assume, for the sake of contradiction, that $\liminf_{s \to 0} \frac{f(s)s - \bar{p}F(s)}{|s|^{2^*}} = 0$. Since f is an odd function, we can deduce that

$$\liminf_{s \to 0} \frac{\frac{d}{ds}\left(F(s)/s^{\bar{p}}\right)}{\frac{d}{ds} s^{2^* - p}} = 0. \tag{2.4}$$

From (2.4), it follows that $\liminf_{s \to 0} \frac{F(s)}{s^{2^*}} = 0$, which contradicts $(F5)$. Therefore, we conclude that

$$\liminf_{s \to 0} \frac{f(s)s - \bar{p}F(s)}{s^{2^*}} > 0.$$

This implies there exist $c_1 > 0$ and $\delta > 0$, such that

$$\frac{f(s)s - \bar{p}F(s)}{s^{2^*}} \geq \frac{c_1}{2}, \quad s \in (0, \delta).$$

Furthermore, by $(F1)$, $(F2)$, and (2.3), we have

$$\frac{f(s)s - \bar{p}F(s)}{s^{2^*}} \in C([\delta, \infty), \mathbb{R}^+)$$

with

$$\lim_{s \to \infty} \frac{f(s)s - \bar{p}F(s)}{s^{2^*}} = (2^* - \bar{p})\,\eta > 0.$$

Therefore, there exists $c_2 > 0$ such that $\frac{f(s)s - \bar{p}F(s)}{s^{2^*}} \geq c_2$ on $[\delta, \infty)$. Then there exists $c > 0$ such that

$$f(s)s - \bar{p}F(s) > c|s|^{2^*} \quad \text{for all } s \neq 0.$$

This concludes this proof. □

For convenience, we define

$$t \star u = t^{\frac{N}{2}} u(tx), \quad \text{for any } x \in \mathbb{R}^N, \quad t \in \mathbb{R}^+. \tag{2.5}$$

It is straightforward to verify that $|t \star u|_2 = |u|_2$ for every $t > 0$. Specifically, $u \in S_a$, then $t \star u \in S_a$ for any $t > 0$.

Lemma 2.2. *Assume that* $N \geq 3$, $a > 0$, $\bar{\mu} > \mu$, $\min\left\{1 - \frac{\mu}{\bar{\mu}}, 1\right\} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, *and* $(F1)$–$(F5)$ *hold. Then, for any* $u \in S_a$,

(a) $\mathcal{I}_\mu(t \star u) \to 0^+$ *as* $t \to 0^+$,
(b) $\mathcal{I}_\mu(t \star u) \to -\infty$ *as* $t \to +\infty$.

Proof. We recall the Hardy inequality as presented in [3, theorem 1.72]:

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \mathrm{d}x \leq \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x. \tag{2.6}$$

By $(F2)$, for any $\delta > 0$, there exists $C_{\delta,\eta} > \eta$ such that

$$F(s) \leq (\delta + \beta)\,|s|^{\bar{p}} + C_{\delta,\eta}|s|^{2^*}, \quad \text{for all } s \in \mathbb{R}. \tag{2.7}$$

From (1.7), (2.6), (2.7), and $(F5)$, we derive that

$$\begin{aligned}\mathcal{I}_\mu(t \star u) &\geq \frac{t^2}{2} \min\left\{1, \left(1 - \frac{\mu}{\bar{\mu}}\right)\right\} |\nabla u|_2^2 - (\delta + \beta) t^2 |u|_{\bar{p}}^{\bar{p}} - C_{\delta,\eta} t^{2^*} |u|_{2^*}^{2^*} \\ &\geq \left(\frac{1}{2} \min\left\{1, \left(1 - \frac{\mu}{\bar{\mu}}\right)\right\} - C_{N,\bar{p}}^{\bar{p}} a^{\frac{2}{N}} (\delta + \beta)\right) t^2 |\nabla u|_2^2 - C_{\delta,\eta} t^{2^*} |u|_{2^*}^{2^*},\end{aligned}$$

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and

$$\mathcal{I}_\mu(t\star u)\le \frac{t^2}{2}\max\left\{1,\left(1-\frac{\mu}{\bar\mu}\right)\right\}|\nabla u|_2^2-\frac{\kappa}{p}t^{p\gamma_p}|u|_p^p.$$

The conclusion can be drawn that, given the condition $\bar\mu>\mu$, $p>\bar p$, and a sufficiently small δ,

$$\mathcal{I}_\mu(t\star u)\to 0^+\quad\text{as}\quad t\to 0^+\quad\text{and}\quad \mathcal{I}_\mu(t\star u)\to-\infty\quad\text{as}\quad t\to+\infty.$$

This concludes the proof. □

LEMMA 2.3. *Assume that* $N\ge 3$, $\bar\mu>\mu$, $\min\left\{1-\frac{\mu}{\bar\mu},1\right\}>2^*C_{N,\bar p}^{\bar p}\beta a^{\frac{2}{N}}$, *and* $(F1)$–$(F4)$ *hold. Then, for any* $u\in S_a$, *there exists a unique* $t_u>0$ *such that* $t_u\star u\in\mathcal{M}_\mu(a)$. *Moreover,* $\mathcal{I}_\mu(t_u\star u)>\mathcal{I}_\mu(t\star u)$ *for any* $t>0$ *with* $t\ne t_u$.

Proof. For any $u\in S_a$, we have

$$\mathcal{I}_\mu(t\star u)=\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2}u^2\mathrm{d}x-\int_{\mathbb{R}^N}F(t\star u)\,\mathrm{d}x,$$

and

$$\begin{aligned}P_\mu(t\star u)&=t^2\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2}u^2\mathrm{d}x-\frac{N}{2}\int_{\mathbb{R}^N}\widetilde F(t\star u)\,\mathrm{d}x\\&=t^2\left(\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2}u^2\mathrm{d}x-\frac{N}{2}\int_{\mathbb{R}^N}\frac{\widetilde F\left(t^{\frac{N}{2}}u\right)}{\left|t^{\frac{N}{2}}u\right|^{\bar p}}|u|^{\bar p}\mathrm{d}x\right).\end{aligned}\tag{2.8}$$

It is evident that $\mathcal{I}_\mu(t\star u)$ is of class C^1, and its derivative can be expressed as

$$\frac{d}{dt}\mathcal{I}_\mu(t\star u)=t\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2}u^2\mathrm{d}x-\frac{N}{2}t^{-1}\int_{\mathbb{R}^N}\widetilde F(t\star u)\,\mathrm{d}x=\frac{1}{t}P_\mu(t\star u).$$

With the application of (2.6),

$$t^2\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2}u^2\mathrm{d}x\ge t^2\min\left\{1,1-\frac{\mu}{\bar\mu}\right\}|\nabla u|_2^2.$$

By $(F4),(F5)$, and (2.7), one gets

$$\begin{aligned}\frac{N}{2}\int_{\mathbb{R}^N}\widetilde F(t\star u)\,\mathrm{d}x&<2^*\int_{\mathbb{R}^N}F(t\star u)\,\mathrm{d}x\\&\le 2^*\left((\delta+\beta)t^2\int_{\mathbb{R}^N}|u|^{\bar p}\mathrm{d}x+C_{\delta,\eta}t^{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x\right)\\&=2^*\left(C_{N,\bar p}^{\bar p}a^{\frac{2}{N}}(\delta+\beta)t^2|\nabla u|_2^2+C_2t^{2^*}|\nabla u|_2^{2^*}\right).\end{aligned}$$

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It is apparent that by selecting a sufficiently small δ, we can ensure that:

$$2^{*}C_{N,\bar{p}}^{\bar{p}}a^{\frac{2}{N}}(\delta+\beta)<\min\left\{1-\frac{\mu}{\bar{\mu}},1\right\}.$$

Thus $\frac{d}{dt}\mathcal{I}_{\mu}\left(t\star u\right)(t)>0$ for sufficiently small t. Similar to lemma 2.2, we can conclude that $\frac{d}{dt}\mathcal{I}_{\mu}\left(t\star u\right)(t)\to-\infty$ as $t\to\infty$. Therefore, there exists at least one $t_u\in\mathbb{R}^+$ such that $\frac{d}{dt}\mathcal{I}_{\mu}\left(t\star u\right)(t)=\frac{1}{t_u}P_{\mu}(t_u\star u)=0$, namely, $t_u\star u\in\mathcal{M}_{\mu}(a)$.

Suppose that there exists another t_{u_1} such that $t_{u_1}\star u\in\mathcal{M}_{\mu}(a)$. Combined with (2.8), it yields

$$\int_{\mathbb{R}^N}\frac{\widetilde{F}\left(t_u^{\frac{N}{2}}u\right)}{\left|t_u^{\frac{N}{2}}u\right|^{\bar{p}}}|u|^{\bar{p}}\mathrm{d}x=\int_{\mathbb{R}^N}\frac{\widetilde{F}\left(t_{u_1}^{\frac{N}{2}}u\right)}{\left|t_{u_1}^{\frac{N}{2}}u\right|^{\bar{p}}}|u|^{\bar{p}}\mathrm{d}x,$$

which contradicts $(F3)$. Hence, it is established that $t_u=t_{u_1}$. Furthermore, we can deduce that $\mathcal{I}_{\mu}\left(t_u\star u\right)>\mathcal{I}_{\mu}(t\star u)$ for all $t>0$ with $t\neq t_u$. □

LEMMA 2.4. *Assume that* $N\geq 3$, $\bar{\mu}>\mu$, $a>0$, $\min\left\{1-\frac{\mu}{\bar{\mu}},1\right\}>2^{*}C_{N,\bar{p}}^{\bar{p}}\beta a^{\frac{2}{N}}$, *and* $(F1)$–$(F5)$ *hold. Then,*

(i) *there exists* $\rho>0$*, such that* $\inf\limits_{u\in\mathcal{M}_{\mu}(a)}|\nabla u|_2>\rho$,

(ii) $m_{\mu}(a)=\inf\limits_{u\in\mathcal{M}_{\mu}(a)}\mathcal{I}_{\mu}(u)>0$.

Proof. (i) For any $u\in\mathcal{M}_{\mu}(a)$, combined with (1.7), (2.6), (2.7), and the Sobolev inequality, $P_{\mu}(u)=0$ implies that

$$\begin{aligned}|\nabla u|_2^2&=\int_{\mathbb{R}^N}\frac{\mu}{|x|^2}u^2\mathrm{d}x+\frac{N}{2}\int_{\mathbb{R}^N}\widetilde{F}(u)\mathrm{d}x\\&\leq\max\left\{0,\frac{\mu}{\bar{\mu}}\right\}|\nabla u|_2^2+2^{*}\int_{\mathbb{R}^N}F(u)\mathrm{d}x\\&\leq\max\left\{0,\frac{\mu}{\bar{\mu}}\right\}|\nabla u|_2^2+2^{*}\left(C_{N,\bar{p}}^{\bar{p}}a^{\frac{2}{N}}(\delta+\beta)|\nabla u|_2^2+C_2|\nabla u|_2^{2^{*}}\right).\end{aligned}$$

Let δ enough small such that

$$2^{*}C_{N,\bar{p}}^{\bar{p}}a^{\frac{2}{N}}(\delta+\beta)<\min\left\{1-\frac{\mu}{\bar{\mu}},1\right\}.\tag{2.9}$$

Then there is $\rho>0$ such that $\inf\limits_{u\in\mathcal{M}_{\mu}(a)}|\nabla u|_2>\rho$.

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(ii) For any $u \in \mathcal{M}_\mu(a)$, we can deduce that

$$\begin{aligned}\mathcal{I}_\mu(u) \geq \mathcal{I}_\mu(t \star u) &\geq \frac{t^2}{2} \min\left\{1, 1 - \frac{\mu}{\bar{\mu}}\right\} |\nabla u|_2^2 - \left(t^2(\delta + \beta)|u|_{\bar{p}}^{\bar{p}} + C_{\delta,\eta} t^{2^*} |u|_{2^*}^{2^*}\right) \\ &\geq \frac{t^2}{2} \min\left\{1, 1 - \frac{\mu}{\bar{\mu}}\right\} |\nabla u|_2^2 \\ &\quad - C_2 t^{2^*} |\nabla u|_2^{2^*} - C_{N,\bar{p}}^{\bar{p}} a^{\frac{2}{N}} (\delta + \beta) t^2 |\nabla u|_2^2.\end{aligned}$$

By selecting $t = \frac{\sigma}{|\nabla u|_2}$ with sufficiently small $\sigma > 0$ and taking δ sufficiently small to ensure that (2.9) holds, we can deduce that

$$\mathcal{I}_\mu(u) \geq \frac{\min\left\{1, 1 - \frac{\mu}{\bar{\mu}}\right\} - 2C_{N,\bar{p}}^{\bar{p}} a^{\frac{2}{N}} \beta}{4} \sigma^2 > 0.$$

This concludes the proof. □

Corollary 2.5. *Assume that $N \geq 3$, $\bar{\mu} > \mu$, $a > 0$, $\min\left\{1 - \frac{\mu}{\bar{\mu}}, 1\right\} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, and $(F1)$–$(F5)$ hold. Then, there exists a sufficiently small $\xi > 0$ such that*

$$m_\mu(a) > \sup_{u \in \overline{C(a)}} \mathcal{I}_\mu(u) > 0, \quad \mathcal{I}_\mu(u) > 0, P_\mu(u) > 0 \text{ for any } u \in \overline{C(a)},$$

where $C(a) = \left\{u \in S_a : |\nabla u|_2^2 < \xi\right\}$. Furthermore, $\mathcal{I}_\mu$ has a mountain pass geometry.

Proof. By $(F4)$, (1.7), (2.7), and the Sobolev inequality, we have

$$\begin{aligned}P_\mu(u) &= \int_{\mathbb{R}^N} |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} \widetilde{F}(u) \, \mathrm{d}x \\ &\geq \int_{\mathbb{R}^N} |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \mathrm{d}x - 2^* \int_{\mathbb{R}^N} F(u) \, \mathrm{d}x \\ &\geq \min\left\{1, 1 - \frac{\mu}{\bar{\mu}}\right\} |\nabla u|_2^2 - 2^* \left(C_{N,\bar{p}}^{\bar{p}} a^{\frac{2}{N}} (\delta + \beta) |\nabla u|_2^2 + C_2 |\nabla u|_2^{2^*}\right).\end{aligned}$$

Thus, $P_\mu(u) > 0$ when $u \in \overline{C(a)}$ if $\xi > 0$ is sufficiently small. Similarly, we can obtain that $\mathcal{I}_\mu(u) > 0$ when $u \in \overline{C(a)}$ if $\xi > 0$ is sufficiently small. By $(F5)$, one can see that

$$\mathcal{I}_\mu(u) \leq \left(\frac{1}{2} - \frac{1}{\bar{\mu}} \min\{\mu, 0\}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x,$$

which implies that $m_\mu(a) > \sup_{u \in \overline{C(a)}} \mathcal{I}_\mu(u)$ for $\xi > 0$ small enough. Combined with lemma (2.2), $\mathcal{I}_\mu$ has a mountain pass geometry. □

Set

$$\sigma_\mu(a) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\mu(\gamma(t)),$$

where

$$\Gamma_\mu := \left\{\gamma \in C\left([0,1], S_{r,a}\right) : \gamma(0) \in \overline{C(a)},\ \mathcal{I}_\mu(\gamma(1)) \leq 0\right\}.$$

Following from the strategy in [27], consider the functional $\widetilde{\mathcal{I}}_\mu : \mathbb{R}^+ \times H^1\left(\mathbb{R}^N\right) \to \mathbb{R}$,

$$\widetilde{\mathcal{I}}_\mu(s,u) := \mathcal{I}_\mu(s\star u) = \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\mu}{|x|^2}u^2 \mathrm{d}x - s^{-N}\int_{\mathbb{R}^N} F(s^{\frac{N}{2}}u)\mathrm{d}x. \quad (2.10)$$

Define $\mathcal{I}_\mu^c := \{u \in S_{r,a} : \mathcal{I}_\mu(u) \leq c\}$, and

$$\widetilde{\sigma}_\mu(a) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widetilde{\mathcal{I}}_\mu(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma}_\mu := \left\{\widetilde{\gamma} = (\iota,\zeta) \in C\left([0,1], \mathbb{R}^+ \times S_{r,a}\right) : \widetilde{\gamma}(0) \in \left(1, \overline{C(a)}\right), \widetilde{\gamma}(1) \in \left(1, \mathcal{I}_\mu^0\right)\right\}.$$

For any $u \in S_{r,a}$, since $|\nabla(s\star u)|_2 \to 0$ as $s \to 0$, and $\mathcal{I}_\mu(s\star u) \to -\infty$ as $s \to +\infty$, there exist $0 < s_0 < 1 < s_1$ such that

$$\widetilde{\gamma}_u : t \in [0,1] \mapsto (1, ((1-t)s_0 + ts_1)\star u) \in \mathbb{R}^+ \times S_{r,a}, \quad (2.11)$$

where $\widetilde{\gamma}_u$ is continuous [7, lemma 3.5] and hence forms a path in $\widetilde{\Gamma}_\mu$. Then $\widetilde{\sigma}_\mu(a)$ is well-defined.

LEMMA 2.6. *Assume that* $N \geq 3$, $\bar{\mu} > \mu$, $a > 0$, $\min\left\{1 - \frac{\mu}{\bar{\mu}}, 1\right\} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, *and* $(F1)$*–*$(F5)$ *hold. Then*

$$\widetilde{\sigma}_\mu(a) = m_\mu^r(a) := \inf_{u\in\mathcal{M}_\mu^r(a)} \mathcal{I}_\mu(u),$$

where $\mathcal{M}_\mu^r(a) = \mathcal{M}_\mu(a) \cap H_r^1\left(\mathbb{R}^N\right)$.

Proof. Step 1: $\widetilde{\sigma}_\mu(a) \geq m_\mu^r(a)$. For any $\widetilde{\gamma} = (\iota,\zeta) \in \widetilde{\Gamma}_\mu$, by lemma 2.3, there exists $t_0 \in (0,1)$ such that $\iota(t_0)\star\zeta(t_0) \in \mathcal{M}_\mu^r(a)$. Thus, we have

$$\max_{t\in[0,1]} \widetilde{\mathcal{I}}_\mu(\widetilde{\gamma}(t)) \geq \widetilde{\mathcal{I}}_\mu\left(\widetilde{\gamma}\left(t_0\right)\right) = \mathcal{I}_\mu\left(\iota(t_0)\star\zeta(t_0)\right) \geq \inf_{u\in\mathcal{M}_\mu^r(a)} \mathcal{I}_\mu(u) = m_\mu^r(a).$$

Hence, $\widetilde{\sigma}_\mu(a) \geq m_\mu^r(a)$.

Step 2: $m_\mu^r(a) \geq \widetilde{\sigma}_\mu(a)$. For any $u \in \mathcal{M}_\mu^r(a)$, then $\widetilde{\gamma}_u$ defined in (2.11) is a path in $\widetilde{\Gamma}_\mu$. By lemma 2.3,

$$\mathcal{I}_\mu(u) = \max_{t\in[0,1]} \widetilde{\mathcal{I}}_\mu(\widetilde{\gamma}_u(t)) \geq \widetilde{\sigma}_\mu(a).$$

Thus, $m_\mu^r(a) \geq \widetilde{\sigma}_\mu(a)$. □

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LEMMA 2.7. *Assume that $N \geq 3$, $\bar{\mu} > \mu$, $a > 0$,* $\min\left\{1 - \frac{\mu}{\bar{\mu}}, 1\right\} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$ *and $(F1)$–$(F5)$ hold. Then*

$$\widetilde{\sigma}_\mu(a) = \sigma_\mu(a).$$

Proof. Step 1: $\sigma_\mu(a) \geq \widetilde{\sigma}_\mu(a)$. Let $\gamma \in \Gamma_\mu$, define $\widetilde{\gamma}(t) = (1, \gamma(t)) \in \widetilde{\Gamma}_\mu$. Then $\mathcal{I}_\mu(\gamma(t)) = \widetilde{\mathcal{I}}_\mu(\widetilde{\gamma}(t)) \geq \widetilde{\sigma}_\mu(a)$ for all $t \geq 0$. Hence $\sigma_\mu(a) \geq \widetilde{\sigma}_\mu(a)$.

Step 2: $\widetilde{\sigma}_\mu(a) \geq \sigma_\mu(a)$. For all $\widetilde{\gamma}(t) = (\iota(t), \zeta(t)) \in \widetilde{\Gamma}_\mu$, setting $\gamma(t) = \iota(t) \star \zeta(t)$, then $\gamma \in \Gamma_\mu$ and $\widetilde{\mathcal{I}}_\mu(\widetilde{\gamma}(t)) = \mathcal{I}_\mu(\gamma(t)) \geq \sigma_\mu(a)$. Therefore, $\widetilde{\sigma}_\mu(a) \geq \sigma_\mu(a)$. □

LEMMA 2.8. *Assume that $N \geq 3$, $\bar{\mu} > \mu \geq 0$, $a > 0$, $1 - \frac{\mu}{\bar{\mu}} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$ and $(F1)$–$(F5)$ hold. Then,*

$$m_\mu(a) = m_\mu^r(a).$$

Proof. It suffices to show that $m_\mu(a) \geq m_\mu^r(a)$, since $\mathcal{M}_\mu^r(a) \subset \mathcal{M}_\mu(a)$. For any $u \in \mathcal{M}_\mu(a)$, let $v := |u|^*$ be the Schwarz rearrangement of $|u|$. By the properties of rearrangement, one has

$$\mathcal{I}_\mu(v) \leq \mathcal{I}_\mu(u), \quad P_\mu(v) \leq P_\mu(u) = 0.$$

By lemma 2.3, there exists $t_v > 0$ such that $t_v \star v \in \mathcal{M}_\mu^r(a)$. For any $t > 0$,

$$\begin{aligned}
\mathcal{I}_\mu(t \star v) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{\mu}{|x|^2} v^2 \mathrm{d}x - t^{-N} \int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}} v\right) \mathrm{d}x \\
&\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\mu}{|x|^2} u^2 \mathrm{d}x - t^{-N} \int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}} u\right) \mathrm{d}x \\
&= \mathcal{I}_\mu(t \star u).
\end{aligned}$$

By lemma 2.3, we obtain

$$\mathcal{I}_\mu(u) \geq \mathcal{I}_\mu\left(t_v \star u\right) \geq \mathcal{I}_\mu\left(t_v \star v\right).$$

Thus, $m_\mu(a) = m_\mu^r(a)$. □

LEMMA 2.9. *[7, lemma 3.6] Assume that $N \geq 3$. For any $u \in S_a$ and $t > 0$, the map*

$$T_u S_a \to T_{t*u} S_a, \quad \varphi \mapsto t \star \varphi$$

is a linear isomorphism with inverse

$$T_{t\star u} S_a \to T_u S_a, \quad \psi \mapsto \left(\frac{1}{t}\right) \star \psi.$$

LEMMA 2.10. *Assume that $N \geq 3$, $a > 0$, $\bar{\mu} > \mu$,* $\min\left\{1 - \frac{\mu}{\bar{\mu}}, 1\right\} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, *and $(F1)$–$(F5)$ hold. Then, there exists a Pohožaev–Palais–Smale sequence $\{u_n\} \subset S_{r,a}$ for $\mathcal{I}_\mu$ at the level $\sigma_\mu(a)$,*

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$$\mathcal{I}_\mu(u_n) \to \sigma_\mu(a), \quad \left(\mathcal{I}_\mu|_{S_{r,a}}\right)'(u_n) \to 0, \quad P_\mu(u_n) \to 0, \quad as\ n \to \infty.$$

Proof. By lemmas 2.6 and 2.7, we have

$$\sigma_\mu(a) = m_\mu^r(a) > \sup_{u \in \left(\overline{C(a)} \cup \mathcal{I}_\mu^0\right) \cap S_{r,a}} \mathcal{I}_\mu(u) = \sup_{(s,u) \in \left((1,\overline{C(a)}) \cup \left(1, \mathcal{I}_\mu^0\right)\right) \cap (\mathbb{R} \times S_{r,a})} \widetilde{\mathcal{I}}_\mu(s,u).$$

Using the terminology in [21, Section 5], $\{\widetilde{\gamma}([0,1]) : \widetilde{\gamma} \in \widetilde{\Gamma}_\mu\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_{r,a}$ with extended closed boundary $\left(1, \overline{C(a)}\right) \cup \left(1, \mathcal{I}_\mu^0\right)$. Furthermore, the superlevel set $\{u \in S_{r,a} : \widetilde{\mathcal{I}}_\mu(u) \geq \widetilde{\sigma}_\mu(a)\}$ is a dual set for $\widetilde{\Gamma}_\mu$, meaning that $(F'1)$ and $(F'2)$ in [21, theorem 5.2] are satisfied. Therefore, considering any minimizing sequence $\{\gamma_n = (1, \zeta_n)\} \subset \widetilde{\Gamma}_\mu$ for $\widetilde{\sigma}_\mu(a)$, where $\zeta_n(t) \geq 0$ almost everywhere in $\mathbb{R}^N$ for $t \in [0,1]$, there exists a Palais–Smale sequence $\{(s_n, w_n)\} \subset \mathbb{R}^+ \times S_{r,a}$ for $\widetilde{\mathcal{I}}_\mu\Big|_{\mathbb{R} \times S_{r,a}}$ at level $\widetilde{\sigma}_\mu(a)$, such that as $n \to \infty$,

$$\partial_s \widetilde{\mathcal{I}}_\mu(s_n, w_n) \to 0, \tag{2.12}$$

and

$$\left\| \partial_u \widetilde{\mathcal{I}}_\mu(s_n, w_n) \right\|_{(T_{w_n} S_r)^*} \to 0, \tag{2.13}$$

with the additional property that

$$|s_n - 1| + \|w_n - \zeta_n\| \to 0. \tag{2.14}$$

By (2.10) and (2.12), we have $P_\mu(s_n \star w_n) = o_n(1)$. Also by (2.13) and the boundedness of $\{s_n\}$ due to (2.14), we obtain

$$d\mathcal{I}_\mu(s_n \star w_n)(s_n \star \varphi) = o_n(1)\|\varphi\|, \quad \text{for every } \varphi \in T_{w_n} S_{r,a}. \tag{2.15}$$

Let $u_n := s_n \star w_n$, based on (2.15) and lemma 2.9, $\{u_n\} \subset S_{r,a}$ is a Palais–Smale sequence for $\mathcal{I}_\mu|_{S_{r,a}}$ at the level $\sigma_\mu(a)$. Moreover, $P_\mu(u_n) = o_n(1)$. □

Lemma 2.11. *Assume that $N \geq 3$, $a > 0$, $\bar{\mu} > \mu$, $\min\left\{1 - \frac{\mu}{\bar{\mu}}, 1\right\} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, and $(F1)$–$(F5)$ hold. Then for any $\epsilon > 0$, there exists $\kappa^* > 0$ such that $\sigma_\mu(a) < \epsilon$ as $\kappa > \kappa^*$, where κ appears in $(F5)$.*

Proof. For a fixed $u \in S_{r,a}$, there exist $0 < s_0 < 1 < s_1$ such that

$$\gamma_o(t) = ((1-t)s_0 + ts_1) \star u \in \Gamma_\mu.$$

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By $(F5)$ and lemmas 2.6–2.8, we observe that

$$\begin{aligned}\sigma_\mu(a) &\leq \max_{t\in[0,1]} \mathcal{I}_\mu(\gamma_o(t))\\ &\leq \max_{t\geq 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2}u^2\mathrm{d}x-\frac{\kappa}{p}t^{p\gamma_p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\right\}\\ &\leq \max_{t\geq 0}\left\{\frac{1}{2}\max\left\{1,1-\frac{\mu}{\bar{\mu}}\right\}t^2\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x-\frac{\kappa}{p}t^{p\gamma_p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\right\}\\ &\leq C\left(\frac{1}{\kappa}\right)^{\frac{2}{p\gamma_p-2}},\end{aligned}$$

which deduces that $\sigma_\mu(a)<\epsilon$ for any $\kappa>\kappa^*$ by noting that $p>\bar{p}=2+\frac{4}{N}$. □

Proof of theorem 1.1. Consider the sequence $\{u_n\}$ arising from lemma 2.10. As the functional $\mathcal{I}_\mu$ exhibits even symmetry with respect to u, we can assume u_n is nonnegative.

We claim that $\{u_n\}$ is bounded in $H_r^1\left(\mathbb{R}^N\right)$. From $(F2)$ and lemma 2.1, $\mathcal{I}_\mu\left(u_n\right)=\sigma_\mu(a)+o_n(1)$ combined with $P_\mu\left(u_n\right)=o_n(1)$ implies that there is a small enough $c>0$ such that

$$\begin{aligned}\sigma_\mu(a)+o_n(1)=\mathcal{I}_\mu\left(u_n\right)-\frac{1}{2}P_\mu(u_n)&=\int_{\mathbb{R}^N}\frac{N}{4}f\left(u_n\right)u_n-\frac{N+2}{2}F\left(u_n\right)\mathrm{d}x\\ &\geq c\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x.\end{aligned}$$

Also from $P\left(u_n\right)=o_n(1)$, we have

$$\begin{aligned}\left(1-\frac{\mu}{\bar{\mu}}\right)|\nabla u_n|_2^2\leq\int_{\mathbb{R}^N}|\nabla u_n|^2-\mu\frac{u_n^2}{|x|^2}\mathrm{d}x&=\frac{N}{2}\int_{\mathbb{R}^N}\widetilde{F}\left(u_n\right)\mathrm{d}x\\ &\leq 2^*C_{N,\bar{p}}^{\bar{p}}a^{\frac{2}{N}}(\delta+\beta)\left|\nabla u_n\right|_2^2+C_2\left|u_n\right|_{2^*}^{2^*},\end{aligned}$$

which shows that $\{|\nabla u_n|_2\}$ is bounded, and $\{u_n\}\subset S_{r,a}$, so that $\{u_n\}$ is bounded in $H_r^1\left(\mathbb{R}^N\right)$.

Therefore, there exists $u\in H_r^1\left(\mathbb{R}^N\right)$ such that, up to a subsequence, $u_n\rightharpoonup u$ in $H_r^1\left(\mathbb{R}^N\right)$, $u_n\to u$ in $L^p\left(\mathbb{R}^N\right)$ for $p\in(2,2^*)$, and $u_n(x)\to u(x)$ almost everywhere in $\mathbb{R}^N$. We claim that $u\neq 0$, by contradiction, that $u=0$. By utilizing the Strauss inequality [50, lemma 4.5] for the sequence $\{u_n\}$ in $H_r^1\left(\mathbb{R}^N\right)$, it follows that

$$|u_n(x)|\leq C_N|u_n|_2^{\frac{1}{2}}|\nabla u_n|_2^{\frac{1}{2}}|x|^{\frac{1-N}{2}}\text{a.e. on }\mathbb{R}^N.$$

Consequently, it can be deduced that $u_n(x)\to 0$ as $|x|\to\infty$.

By using $(F2)$, we establish

$$\lim_{s\to\infty}\frac{\frac{N}{2}\widetilde{F}(s)-2^*\eta|s|^{2^*}}{|s|^{2^*}+|s|^2}=0,\quad\lim_{s\to 0}\frac{\frac{N}{2}\widetilde{F}(s)-2^*\eta|s|^{2^*}}{|s|^2+|s|^{2^*}}=0.$$

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By the boundedness of $\{|u_n|_{2^*}\}$ and the fact that $\{u_n\} \in S_{r,a}$, we have

$$\int_{\mathbb{R}^N} |u_n|^2 + |u_n|^{2^*} \mathrm{d}x \le M, \quad \text{for some positive } M.$$

Consequently, by Lions Lemma [10, theorem A.I.], we can say that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{N}{2}\widetilde{F}(u_n) - 2^*\eta|u_n|^{2^*}\mathrm{d}x = 0. \tag{2.16}$$

Then by $P(u_n) \to 0$, we can deduces that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2}\mathrm{d}x + o_n(1) &= \frac{N}{2}\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x \\ &= 2^*\eta\int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x + \int_{\mathbb{R}^N} \frac{N}{2}\widetilde{F}(u_n) - 2^*\eta|u_n|^{2^*}\mathrm{d}x \\ &\le 2^*\eta S_\mu^{\frac{N}{2-N}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2}\mathrm{d}x\right)^{\frac{N}{N-2}}.\end{aligned} \tag{2.17}$$

By using lemma 2.4, we can assume that, up to a subsequence,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2}\mathrm{d}x \to l^* > 0.$$

By (2.17), we find that $l^* \ge (2^*\eta)^{\frac{2-N}{2}} S_\mu^{\frac{N}{2}}$. Similarly as (2.16), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n - 2^*F(u_n)\mathrm{d}x = 0.$$

This allows us to derive

$$\begin{aligned} m_\mu(a) + o_n(1) &= \mathcal{I}_\mu(u_n) - \frac{1}{2^*}P_\mu(u_n) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2}\mathrm{d}x + \frac{N-2}{4}\int_{\mathbb{R}^N} f(u_n)u_n - 2^*F(u_n)\mathrm{d}x \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2}\mathrm{d}x + o_n(1) \\ &\ge \frac{1}{N}(2^*\eta)^{\frac{2-N}{2}} S_\mu^{\frac{N}{2}} + o_n(1), \end{aligned}$$

which contradicts lemma 2.11. Hence, $u \ne 0$. By the weak lower semi-continuity, we deduce

$$\int_{\mathbb{R}^N} |u|^2\mathrm{d}x = a_0 \in (0, a].$$

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Since $\{u_n\}$ is a Palais–Smale sequence of $\mathcal{I}_\mu|_{S_{r,a}}$, there exists $\{\lambda_n\}$ such that for any $\varphi \in H^1\left(\mathbb{R}^N\right)$

$$\int_{\mathbb{R}^N}\left(\nabla u_n \nabla \varphi-\mu \frac{u_n \varphi}{|x|^2}+\lambda_n u_n \varphi-f\left(u_n\right) \varphi\right) \mathrm{d} x=o_n(1)\|\varphi\|. \tag{2.18}$$

Setting $\varphi=u_n$ and by the boundedness of $\{u_n\}$ in $H^1\left(\mathbb{R}^N\right)$, we have

$$-\lambda_n a=\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2-\frac{\mu}{|x|^2} u_n^2 \mathrm{d} x-\int_{\mathbb{R}^N} f\left(u_n\right) u_n \mathrm{d} x+o_n(1).$$

Moreover, we can infer that the boundedness of λ_n by the boundedness of $\{u_n\}$. Therefore, up to a subsequence, $\lambda_n \rightarrow \lambda \in \mathbb{R}$. By (2.18),

$$\int_{\mathbb{R}^N}(\nabla u \nabla \varphi-\mu \frac{u \varphi}{|x|^2}+\lambda u \varphi-f(u) \varphi) \mathrm{d} x=0 \tag{2.19}$$

implies that (u, λ) satisfies

$$-\Delta u-\frac{\mu}{|x|^2} u+\lambda u=f(u). \tag{2.20}$$

Thus, one has

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2-\frac{\mu}{|x|^2} u^2+\lambda|u|^2-f(u) u\right) \mathrm{d} x=0, \tag{2.21}$$

and

$$\frac{N-2}{2 N} \int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2} u^2 \mathrm{d} x+\frac{\lambda}{2} \int_{\mathbb{R}^N}|u|^2 \mathrm{d} x-\int_{\mathbb{R}^N} F(u) \mathrm{d} x=0. \tag{2.22}$$

Combined with (2.21) and (2.22), we can infer that

$$\int_{\mathbb{R}^N}|\nabla u|^2-\frac{\mu}{|x|^2} u^2 \mathrm{d} x-\frac{N}{2} \int_{\mathbb{R}^N}(f(u) u-2 F(u)) \mathrm{d} x=0.$$

i.e. $P_\mu(u)=0$.

Defining $v_n:=u_n-u \rightharpoonup 0$ in $H^1\left(\mathbb{R}^N\right)$, we can utilize the Brézis–Lieb lemma [13] to state that

$$\mathcal{I}_\mu(u_n)=\mathcal{I}_\mu(u)+\mathcal{I}_\mu(v_n)+o_n(1), \quad P_\mu(u_n)=P_\mu(u)+P_\mu(v_n)+o_n(1).$$

We claim that $v_n \rightarrow 0$ in $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$. Let us proceed by assuming, for the sake of contradiction, that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N}\left|\nabla v_n\right|^2-\mu \frac{v_n^2}{|x|^2} \mathrm{d} x>0.$$

Since $P_\mu(u)=0$, we have $P_\mu(v_n)=o_n(1)$. This implies

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla v_n|^2-\mu\frac{v_n^2}{|x|^2}\mathrm{d}x+o_n(1)&=\frac{N}{2}\int_{\mathbb{R}^N}\widetilde{F}(v_n)\mathrm{d}x\\&=2^*\eta\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{N}{2}\widetilde{F}(v_n)-2^*\eta|v_n|^{2^*}\mathrm{d}x\\&\le 2^*\eta S_\mu^{\frac{N}{2-N}}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2-\mu\frac{v_n^2}{|x|^2}\mathrm{d}x\right)^{\frac{N}{N-2}}+o_n(1).\end{aligned}$$

Similarly, we can deduce

$$\lim_{n\to\infty}\mathcal{I}_\mu(v_n)\ge\frac{1}{N}\left(2^*\eta\right)^{\frac{2-N}{2}}S_\mu^{\frac{N}{2}}.$$

Furthermore,

$$\mathcal{I}_\mu(u)=\mathcal{I}_\mu(u)-\frac{1}{2}P_\mu(u)=\int_{\mathbb{R}^N}\frac{N}{4}\widetilde{F}(u)-F(u)\mathrm{d}x>0.$$

As a consequence, we arrive at

$$\sigma_\mu(a)=\mathcal{I}_\mu\left(v_n\right)+\mathcal{I}_\mu\left(u\right)+o_n(1)\ge\frac{1}{N}\left(2^*\eta\right)^{\frac{2-N}{2}}S_\mu^{\frac{N}{2}},$$

which contradicts lemma 2.11, so that $u_n\to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x=\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x,\tag{2.23}$$

which leads to

$$\int_{\mathbb{R}^N}F\left(u_n\right)\mathrm{d}x\to\int_{\mathbb{R}^N}F(u)\mathrm{d}x,\quad\int_{\mathbb{R}^N}f\left(u_n\right)u_n\mathrm{d}x\to\int_{\mathbb{R}^N}f(u)u\mathrm{d}x.\tag{2.24}$$

Furthermore, by (2.20),

$$-\Delta u-\mu\frac{u}{|x|^2}+\lambda u=f(u).$$

Since (2.21), (2.22), and $(F4)$, one obtains

$$\begin{aligned}\lambda&=\frac{1}{a}\left(\int_{\mathbb{R}^N}f(u)u\mathrm{d}x-\int_{\mathbb{R}^N}|\nabla u|^2-\mu\frac{u^2}{|x|^2}\mathrm{d}x\right)\\&=\frac{1}{a}\left\{\int_{\mathbb{R}^N}\frac{2-N}{2}f(u)u+NF(u)\mathrm{d}x\right\}>0.\end{aligned}$$

Thus, $\lambda>0$ and $u\in S_a$ by (2.18) and (2.19). According to lemmas 2.6 and 2.8, $(u,\lambda)\in H_r^1\left(\mathbb{R}^N\right)\times\mathbb{R}^+$ is the normalized ground state solution of (1.6). We can further establish that $u>0$ through the strong maximum principle. This concludes the proof. □

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Here, we provide the proof of proposition 1.2, which has been previously established in [34]. However, for completeness, we will only prove (i) and (ii). It is worth noting that the proof of (iii) has already been established in prior works [17, 24, 34].

Proof of proposition 1.2. Since $\frac{1}{|x|^2} \in C^2\left(\mathbb{R}^N \backslash B_{r_0}(0)\right)$ for any small $r_0 > 0$. Then by using a standard elliptic regularity argument, we establish that $u \in C^2\left(\mathbb{R}^N \backslash \{0\}\right)$. We now turn our attention to proving the exponential decay of the solution. Since $u \in C^2\left(\mathbb{R}^N \backslash \{0\}\right)$ and $u \in S_{r,a}$, then $u(x) \to 0$ as $|x| \to +\infty$.

Consequently, there exists $R > 0$ such that

$$-\Delta u(x) = \frac{\mu}{|x|^2} u(x) + f(u(x)) - \lambda u(x) \leq -\frac{\lambda}{2} u(x) \quad \text{for all } |x| \geq R. \tag{2.25}$$

Define $\phi(x) = M \exp\left(-\sqrt{\frac{\lambda}{2}}|x|\right)$, where M is chosen to satisfy

$$M \exp\left(-\sqrt{\frac{\lambda}{2}} R\right) \geq u(x) \text{ for all } |x| = R.$$

By direct calculation, it follows

$$\Delta \phi = \left(\frac{\lambda}{2} - \frac{N-1}{r}\sqrt{\frac{\lambda}{2}}\right)\phi, \text{ for all } \ x \neq 0, \text{ where } r = |x|.$$

This leads to the immediate conclusion

$$\Delta \phi \leq \frac{\lambda}{2}\phi \quad \text{for all } x \neq 0. \tag{2.26}$$

Combining (2.25) with (2.26), it becomes evident that the function $\varphi = \phi - u$ fulfils

$$\begin{cases} -\Delta \varphi + \frac{\lambda}{2}\varphi \geq 0 & \text{in } \ |x| \geq R, \\ \varphi(x) \geq 0 & \text{in } \ |x| = R, \\ \lim\limits_{|x| \to \infty} \varphi(x) = 0. \end{cases}$$

In accordance with the maximum principle, it follows that $\varphi(x) \geq 0$ holds true for all $|x| \geq R$. Consequently,

$$u(x) \leq M \exp\left(-\sqrt{\frac{\lambda}{2}}|x|\right), \ |x| \geq R. \tag{2.27}$$

Further, based on $(F1)$–$(F2)$ in conjunction with the exponential decay of u, it is evident that for sufficiently large $|x|$,

$$m_1 u \leq |f(u) - \lambda u| \leq m_2 u,$$

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where $m_2 \geq m_1 > 0$. As u satisfies Eq. (1.6),

$$-u_{rr} - \frac{N-1}{r}u_r - \frac{\mu}{r^2}u = f(u) - \lambda u, \quad r \in (r_0, +\infty),\ r_0 > 0, \tag{2.28}$$

with $u_r = \frac{\partial u}{\partial r}, u_{rr} = \frac{\partial^2 u}{\partial r^2}, r = |x|$.

It is a known fact that the equation

$$-\left(r^{N-1}u_r\right)_r = \mu r^{N-3}u + r^{N-1}f(u) - \lambda r^{N-1}u, \quad r \in (r_0, +\infty),\ r_0 > 0, \tag{2.29}$$

can be integrated over the interval (r, R), using (2.27), and then letting $r, R \to +\infty$. This integration demonstrates that $r^{N-1}u_r$ possesses a limit as $r \to \infty$, which, according to (2.27), must be zero. Furthermore, integrating (2.29) over $(r, +\infty)$ implies exponential decay of u_r (also referenced in [10]). Finally, the exponential decay of u_{rr}, and consequently of $|D^\alpha u(x)|$ for $|\alpha| \leq 2$, directly follows from (2.28). This concludes the proof. □

3. Proof of theorem 1.3

In this section, we delve into the asymptotic behaviour of the solution to Eq. (1.6) as $\mu \to 0^+$.

Lemma 3.1. *Assume that $N \geq 3$, $a>0$, $1 - \frac{\mu}{\bar{\mu}} > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$, and (F1)–(F5) hold. Then, for any sequence $\mu_n \in (0, \bar{\mu})$ with $\mu_n \to 0^+$ as $n \to \infty$, we have $\lim\limits_{n\to\infty} m_{\mu_n}(a) = m_\infty(a)$.*

Proof. For any $0 < \mu_n < \bar{\mu}$,

$$\mathcal{I}_\infty(u) = \mathcal{I}_{\mu_n}(u) + \mu_n \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\mathrm{d}x,$$

and consequently, by lemma 2.3

$$m_\infty(a) = \inf_{u \in S_a} \max_{t>0} \mathcal{I}_\infty(t \star u) \geq \inf_{u \in S_a} \max_{t>0} \mathcal{I}_{\mu_n}(t \star u) = m_{\mu_n}(a).$$

We now proceed to assert that

$$m_\infty(a) \leq \lim_{n\to\infty} m_{\mu_n}(a).$$

For each $n \geq 1$, let $u_n \in \mathcal{M}_{\mu_n}(a)$ be such that

$$\mathcal{I}_{\mu_n}(u_n) = m_{\mu_n}(a) < m_\infty(a) + \frac{1}{n}.$$

Consequently, $|\nabla u_n|_2^2 \leq C$ for all $n \geq 1$, ensuring that u_n is bounded in $H^1\left(\mathbb{R}^N\right)$. Let t_n be determined according to lemma 2.3 such that $t_n \star u_n \in \mathcal{M}_\infty(a)$. Moreover,

$$P_{\mu_n}(u_n) = \int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu_n \frac{u_n^2}{|x|^2}\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x = 0,$$

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and

$$P_{\infty}\left(t_n \star u_n\right)=t_n^2 \int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 \mathrm{~d} x-\frac{N}{2} t_n^{-N} \int_{\mathbb{R}^N} \widetilde{F}\left(t_n^{\frac{N}{2}} u_n\right) \mathrm{d} x=0 .$$

As $n \rightarrow \infty$, we establish

$$\int_{\mathbb{R}^N} \frac{\widetilde{F}\left(t_n^{\frac{N}{2}} u_n\right)}{\left|t_n^{\frac{N}{2}} u_n\right|^{2+\frac{4}{N}}}\left|u_n\right|^{2+\frac{4}{N}} \mathrm{~d} x=\int_{\mathbb{R}^N} \frac{\widetilde{F}\left(u_n\right)}{\left|u_n\right|^{2+\frac{4}{N}}}\left|u_n\right|^{2+\frac{4}{N}} \mathrm{~d} x+o_n(1) .$$

Furthermore, based on $(F3)$, it follows that $t_n \rightarrow 1$ as $n \rightarrow \infty$. By [7, lemma 3.5], we have $\left\|t_n \star u_n-u_n\right\| \rightarrow 0$, and consequently,

$$\mathcal{I}_{\infty}\left(t_n \star u_n\right)-\mathcal{I}_{\infty}\left(u_n\right) \rightarrow 0, \quad \text { as } n \rightarrow \infty .$$

This entails

$$\begin{aligned}
m_{\infty}(a) \leq \mathcal{I}_{\infty}\left(t_n \star u_n\right) & =\mathcal{I}_{\infty}\left(u_n\right)+o_n(1) \\
& =\mathcal{I}_{\mu_n}\left(u_n\right)+\mu_n \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^2} \mathrm{d} x+o_n(1) \leq m_{\mu_n}(a)+o_n(1) .
\end{aligned}$$

Hence, we conclude that $m_{\infty}(a) \leq \lim\limits_{n \rightarrow \infty} m_{\mu_n}(a)$. This concludes the proof. □

Proof of theorem 1.3. Assume that $\left(u_n, \lambda_n\right)$ is obtained in theorem 1.1 with $m_{\mu_n}(a)$, where $\bar{\mu}>\mu_n$ and $\mu_n \rightarrow 0^{+}$. In other words, $\left(u_n, \lambda_n\right)$ satisfies

$$-\Delta u_n-\frac{\mu_n}{|x|^2} u_n+\lambda_n u_n=f\left(u_n\right). \tag{3.1}$$

Consequently, $P_{\mu_n}\left(u_n\right)=0$.

Similarly to the proof of theorem 1.1, $\left\{u_n\right\}$ is bounded in $H_r^1\left(\mathbb{R}^N\right)$, so that there exists a nonnegative function $u \in H_r^1\left(\mathbb{R}^N\right)$ such that $u_n \rightharpoonup u$ in $H_r^1\left(\mathbb{R}^N\right)$, $u_n \rightarrow u$ in $L^p\left(\mathbb{R}^N\right)$ for $p \in\left(2,2^*\right)$, and $u_n(x) \rightarrow u(x)$ almost everywhere in $\mathbb{R}^N$. Consequently, by taking $n \rightarrow \infty$ in (3.1), we have

$$-\Delta u+\lambda u=f(u), \tag{3.2}$$

which shows that $P_{\infty}(u)=0$.

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Claim 1: $u \neq 0$. If not, $u=0$, and $P_{\mu_n}(u_n) = 0$ combined with (2.16) implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x &= \frac{N}{2}\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x + \mu_n \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^2}\mathrm{d}x \\
&= 2^*\eta \int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x + \int_{\mathbb{R}^N} \frac{N}{2}\widetilde{F}(u_n) - 2^*\eta|u_n|^{2^*}\mathrm{d}x + \mu_n \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^2}\mathrm{d}x \\
&= 2^*\eta \int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x + o_n(1) \\
&\leq 2^*\eta S^{\frac{N}{2-N}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x\right)^{\frac{N}{N-2}} + o_n(1).
\end{aligned} \tag{3.3}$$

Combined with lemma 2.4, we deduce that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x \geq (2^*\eta)^{\frac{2-N}{2}} S^{\frac{N}{2}} + o_n(1).$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\mu_n \to 0^+$, we have

$$\begin{aligned}
m_{\mu_n}(a) &= \mathcal{I}_{\mu_n}(u_n) - \frac{1}{2^*}P_{\mu_n}(u_n) \\
&= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \mu_n \frac{u_n^2}{|x|^2}\mathrm{d}x + \frac{N-2}{4}\int_{\mathbb{R}^N} f(u_n)u_n - 2^*F(u_n)\mathrm{d}x \\
&\geq \frac{1}{N}(2^*\eta)^{\frac{2-N}{2}} S^{\frac{N}{2}} + o_n(1).
\end{aligned} \tag{3.4}$$

Thus,

$$\lim_{n\to\infty} m_{\mu_n}(a) = m_\infty(a) \geq \frac{1}{N}(2^*\eta)^{\frac{2-N}{2}} S^{\frac{N}{2}},$$

which contradicts lemma 2.11, and then $u \neq 0$. Based on Eq. (3.2), we have

$$\lambda|u|_2^2 = \int_{\mathbb{R}^N} f(u)u\mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x > 0,$$

which holds due to $(F4)$.

Claim 2: $u_n \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Let us proceed by contradiction and assume that

$$\nu := \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x > 0,$$

where $v_n = u_n - u$. Since $P_{\mu_n}(u_n) = 0$, we can infer that $P_{\mu_n}(v_n) = 0$. Similarly, one can see that

$$\mathcal{I}_{\mu_n}(v_n) \geq \frac{1}{N}(2^*\eta)^{\frac{2-N}{2}} S^{\frac{N}{2}} + o_n(1),$$

and by lemma 2.1,

$$\mathcal{I}_\infty(u) = \mathcal{I}_\infty(u) - \frac{1}{2}P_\infty(u) = \int_{\mathbb{R}^N} \frac{N}{4}\widetilde{F}(u) - F(u)\mathrm{d}x > 0.$$

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Consequently, we arrive at

$$m_{\mu_n}(a) = \mathcal{I}_{\mu_n}(u_n) = \mathcal{I}_{\mu_n}(v_n) + \mathcal{I}_\infty(u) + o_n(1) > \frac{1}{N}\left(2^*\eta\right)^{\frac{2-N}{2}} S^{\frac{N}{2}} + o_n(1).$$

This leads to

$$\lim_{n\to\infty} m_{\mu_n}(a) = m_\infty(a) \geq \frac{1}{N}\left(2^*\eta\right)^{\frac{2-N}{2}} S^{\frac{N}{2}},$$

which is a contradiction. Thus, we conclude that $u_n \to u$ in $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$.

Furthermore, we can establish that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x = \int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^N} F\left(u_n\right)\mathrm{d}x \to \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

Consequently,

$$\lim_{n\to\infty}\lambda_n\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x = \lambda\int_{\mathbb{R}^N}u^2\mathrm{d}x,$$

and given that $\lambda > 0$, so that $u_n \to u$ in $H_r^1(\mathbb{R}^N)$. Thus, by lemma 3.1, $(u,\lambda) \in H_r^1(\mathbb{R}^N)\times\mathbb{R}^+$ is a normalized ground state of Eq. (1.8). Moreover, $u > 0$ by the strong maximum principle. □

4. Proof of theorem 1.4

In this section, we focus on the existence of normalized solutions for Eq. (1.6) when $\mu < 0$.

LEMMA 4.1. *Assume that $N \geq 3$, $0 > \mu$, and $(F1)$–$(F5)$ hold. Then, $m_\mu(a) = m_\infty(a)$. Additionally, $m_\mu(a)$ cannot be attained.*

Proof. When $\mu < 0$, it becomes evident that $m_\infty(a) \leq m_\mu(a)$. According to theorem 1.3, Eq. (1.8) possesses a ground state solution $v \in \mathcal{M}_\infty(a)$, achieving $m_\infty(a)$, i.e. $\mathcal{I}_\infty(v) = m_\infty(a)$ and $P_\infty(v) = 0$. Moreover, due to the exponential decay of v, we have

$$v(x) \leq M\exp\left(-\sqrt{\frac{\lambda}{2}}|x|\right), |x| > R, \text{ for some } R > 0.$$

Consequently, we can introduce $v_n(x) = v(x - y_n e_1)$, where $e_1 = (1, 0, \ldots, 0)$, $y_n \in \mathbb{R}^+$ and $y_n \to +\infty$ as $n \to \infty$. Furthermore, given any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that,

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$$\frac{1}{|x|^2} \leq \epsilon, \text{ for all } |x| \geq R_\epsilon.$$

Since $y_n \to +\infty$ as $n \to \infty$, there exists $R_\epsilon > 0$, such that as $n \to \infty$,

$$v_n(x) \leq M \exp\left(-\sqrt{\frac{\lambda}{2}}\,|x-y_n|\right) \leq \frac{C}{|x-y_n|},\ x \in B_{R_\epsilon}(0).$$

This results in

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{v_n^2}{|x|^2}\mathrm{d}x &= \int_{B_{R_\epsilon}(0)} \frac{v_n^2}{|x|^2}\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_{R_\epsilon}(0)} \frac{v_n^2}{|x|^2}\mathrm{d}x \\ &\leq \frac{C}{|y_n - R_\epsilon|^2} \int_{B_{R_\epsilon}(0)} \frac{1}{|x|^2}\mathrm{d}x + \epsilon \int_{\mathbb{R}^N\setminus B_{R_\epsilon}(0)} v_n^2 \mathrm{d}x \\ &\leq \frac{C_1}{|y_n - R_\epsilon|^2} + \epsilon a \to 0,\ \text{as } n \to \infty.\end{aligned}$$

Since $v_n \in \mathcal{M}_\infty(a)$,

$$P_\infty(v_n) = \int_{\mathbb{R}^N} |\nabla v_n|^2 \,\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} [f(v_n)v_n - 2F(v_n)]\,\mathrm{d}x = 0,$$

as deduced from lemma 2.3, there exists a unique $t_n > 0$ satisfying

$$P(t_n \star v_n) = t_n^2 \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \mu \frac{v_n^2}{|x|^2}\right)\mathrm{d}x - \frac{N}{2} t_n^{-N} \int_{\mathbb{R}^N} \widetilde{F}\left(t_n^{\frac{N}{2}} v_n\right)\mathrm{d}x = 0.$$

Therefore, as $y_n \to \infty$, we establish

$$\int_{\mathbb{R}^N} \frac{\widetilde{F}\left(t_n^{\frac{N}{2}} v_n\right)}{\left|t_n^{\frac{N}{2}} v_n\right|^{2+\frac{4}{N}}} |v_n|^{2+\frac{4}{N}}\,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{\widetilde{F}(v_n)}{|v_n|^{2+\frac{4}{N}}} |v_n|^{2+\frac{4}{N}}\,\mathrm{d}x + o_n(1).$$

Furthermore, based on $(F3)$, it follows that $t_n \to 1$ as $n \to \infty$. We now proceed to demonstrate that $m_\mu(a) \leq m_\infty(a)$. As $\{t_n \star v_n\} \subset \mathcal{M}_\mu(a)$ and $t_n \to 1$ as $n \to \infty$, it follows that

$$\begin{aligned}m_\mu(a) &\leq \mathcal{I}_\mu(t_n \star v_n) = \mathcal{I}_\mu(t_n \star v_n) - \frac{1}{2^*}P_\mu(t_n \star v_n) \\ &= \frac{t_n^2}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2 - \frac{\mu}{|x|^2}v_n^2 \mathrm{d}x + \int_{\mathbb{R}^N} \frac{N-2}{4} f(t_n \star v_n)\, t_n \star v_n \\ &\quad - \frac{N}{2}F(t_n \star v_n)\,\mathrm{d}x \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \frac{N-2}{4} f(v_n)\,v_n - \frac{N}{2}F(v_n)\,\mathrm{d}x + o_n(1) \\ &= \mathcal{I}_\infty(v_n) - \frac{1}{2^*}P_\infty(v_n) + o_n(1) \\ &= \mathcal{I}_\infty(v_n) + o_n(1) = \mathcal{I}_\infty(v) + o_n(1) = m_\infty(a) + o_n(1).\end{aligned}$$

This concludes the establishment that $m_\mu(a) = m_\infty(a)$.

Now, we proceed to prove that $m_\mu(a)$ cannot be achievement. By proof by contradiction, we assume that $u_a \in \mathcal{M}_\mu(a)$ attains $m_\mu(a)$. By lemma 2.3, there exists a unique $t_{u_a} > 0$ such that $t_{u_a} \star u_a \in \mathcal{M}_\infty(a)$. It can be seen that

$$
\begin{aligned}
m_\infty(a) \le \mathcal{I}_\infty(t_{u_a} \star u_a) &= \mathcal{I}_\mu(t_{u_a} \star u_a) + \frac{t_{u_a}^2}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u_a^2 \mathrm{d}x \\
&< \mathcal{I}_\mu(t_{u_a} \star u_a) \le \mathcal{I}_\mu(u_a) = m_\mu(a) = m_\infty(a),
\end{aligned}
$$

which creates a contradiction. □

Proof of theorem 1.4. The first part of theorem 1.4 has already been established in lemma 4.1. Next, we will prove that Eq. (1.6) has normalized solutions.

By lemma 2.10, there exists a Pohožaev–Palais–Smale sequence $\{u_n\} \subset S_{r,a}$ for $\mathcal{I}_\mu$ at level of $\sigma_\mu(a)$. That is,

$$
\mathcal{I}_\mu(u_n) \to \sigma_\mu(a), \quad \left(\mathcal{I}_\mu|_{S_{r,a}}\right)'(u_n) \to 0, \quad P_\mu(u_n) \to 0, \quad \text{as } n \to \infty.
$$

Similarly as the proof of theorem 1.1, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$, and there exists $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, such that $u_n \to u$ in $H^1(\mathbb{R}^N)$, and there exists a $\lambda > 0$, such that $(u, \lambda) \in H_r^1(\mathbb{R}^N) \times \mathbb{R}^+$ is a normalized solution of Eq. (1.6). Furthermore, $u > 0$ by the strong maximum principle. By lemma 4.1, we can deduce that $\sigma_\mu(a) > m_\mu(a)$.
□

5. Proof of theorem 1.6

LEMMA 5.1. *Assume that $N \ge 3$, $a > 0$, and $1 > 2^* C_{N,\bar{p}}^{\bar{p}} \beta a^{\frac{2}{N}}$. Then, for any sequence $\mu_n \le 0$ with $\mu_n \to 0^-$ as $n \to \infty$, we have $\lim_{n\to\infty} \sigma_{\mu_n}(a) = m_\infty(a)$.*

Proof. For the sake of clarity in our presentation, let's define:

$$
m_{r,\infty}(a) := \inf_{v \in \mathcal{M}_\infty^r(a)} \mathcal{I}_\infty(v),
$$

where

$$
\mathcal{M}_\infty^r(a) := \{v \in S_{r,a} : P_\infty(v) = 0\}.
$$

It's clear that $m_{r,\infty}(a) = m_\infty(a)$ by lemma 2.8. Hence, we just need to prove $\lim_{n\to\infty} \sigma_{\mu_n}(a) = m_{r,\infty}(a)$. For $\mu \le 0$, it's evident that $\sigma_\mu(a) \ge m_\infty(a)$ and $\sigma_\mu(a)$ is non-increasing with respect to μ. Therefore, we just need to prove that $m_\infty(a)$ is the greatest lower bound of $\{\sigma_{\mu_n}(a)\}$.

Assume that $\omega \in \mathcal{M}_\infty^r(a)$ is the function that achieves $m_{r,\infty}(a)$, implying $\mathcal{I}_\infty(\omega) = m_{r,\infty}(a)$. By lemma 2.3, we can find $0 < s_0 < 1 < s_1$ such that

$$
\gamma_o(t) = ((1-t)s_0 + ts_1) \star w \in \Gamma_\mu, \quad \gamma_o(t) \cap \mathcal{M}_\infty^r(a) \ne \emptyset.
$$

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Furthermore, for any given $\epsilon > 0$, there exists a positive integer N_ϵ such that, for every $n > N_\epsilon$,

$$-\frac{1}{2}s_1^2\int_{\mathbb{R}^N}\frac{\mu_n}{|x|^2}\omega^2\mathrm{d}x \le \epsilon.$$

We can deduce that

$$\begin{aligned}\sigma_{\mu_n}(a) &\le \max_{t\in[0,1]}\mathcal{I}_{\mu_n}(\gamma_o(t)) = \max_{t\in[0,1]}\mathcal{I}_\infty(\gamma_o(t)) - \frac{s_1^2}{2}\int_{\mathbb{R}^N}\frac{\mu_n}{|x|^2}\omega^2\mathrm{d}x\\ &= \mathcal{I}_\infty(\omega) + \epsilon = m_{r,\infty}(a) + \epsilon.\end{aligned}$$

This clearly indicates that $m_{r,\infty}(a)$ is the infimum of $\{\sigma_{\mu_n}(a)\}$, and by lemma 2.8, $\lim_{n\to\infty}\sigma_{\mu_n}(a) = m_{r,\infty}(a) = m_\infty(a)$. □

Proof of theorem 1.6. Let (u_n, λ_n) be the solution in theorem 1.4, which satisfies

$$-\Delta u_n - \frac{\mu_n}{|x|^2}u_n + \lambda_n u_n = f(u_n), \tag{5.1}$$

where $\mu_n \to 0^-$ as $n \to \infty$, and then $P_{\mu_n}(u_n) = 0$. As in the proof of theorem 1.4, we can establish that $u_n \to u$ in $H^1(\mathbb{R}^N)$ and $\lambda_n \to \lambda > 0$, and $(u,\lambda) \in H_r^1(\mathbb{R}^N)\times\mathbb{R}^+$ is the normalized ground state solution of Eq. (1.8). Additionally, by the strong maximum principle, $u > 0$. □

Proof of proposition 1.7. The proof of proposition 1.7 can be derived from the proof of proposition 1.2 by applying the same method. To avoid unnecessary repetition, we do not provide the proof here. □

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