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RESEARCH ARTICLE

Proper Lie automorphisms of incidence algebras

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Abstract

Let X be a finite connected poset and K a field. We study the question, when all Lie automorphisms of the incidence algebra $I(X, K)$ are proper. Without any restriction on the length of X, we find only a sufficient condition involving certain equivalence relation on the set of maximal chains of X. For some classes of posets of length one, such as finite connected crownless posets (i.e., without weak crown subposets), crowns, and ordinal sums of two anti-chains, we give a complete answer.

Introduction

A *Lie isomorphism* of associative rings $(R, \cdot)$ and $(S, \cdot)$ is an isomorphism of the corresponding Lie rings $(R, [\ ,\ ])$ and $(S, [\ ,\ ])$, where $[a, b] = a \cdot b - b \cdot a$. If $R = S$, a Lie isomorphism $R \to S$ is called a *Lie automorphism* of R. If, moreover, R and S are algebras, it is natural to require Lie isomorphisms $R \to S$ to be linear. Any bijective map of the form $\phi + \nu$, where ϕ is either an isomorphism $R \to S$ or the negative of an anti-isomorphism $R \to S$, and ν is an additive map on R with values in the center of S whose kernel contains $[R, R]$, is always a Lie isomorphism. Such Lie isomorphisms are called *proper*. In most of the cases studied in the literature, these are the only examples of Lie isomorphisms. Indeed, this is true for Lie automorphisms of full matrix rings $M_n(R)$ over division rings R with $\mathrm{char}(R) \notin \{2, 3\}$ as proved in [14], for Lie isomorphisms of primitive [16], simple [18] and prime rings [17], for Lie automorphisms of upper triangular matrix algebras $T_n(R)$ over commutative rings [7, 9], and for Lie isomorphisms of block-triangular matrix algebras over a UFD [8].

In [13], we described Lie automorphisms of the incidence algebra $I(X, K)$ of a finite connected poset X over a field K. In general, they are not proper as shown in [13, Example 5.20], but, for some classes of posets, every Lie automorphism of $I(X, K)$ is proper. For instance, if X is a chain of cardinality n, then $I(X, K) \cong T_n(K)$, and thus every Lie automorphism of $I(X, K)$ is proper in view of [13, Corollary 5.19] (see also [9, Theorem 6]). So the following question arises.

Question. *What are the necessary and sufficient conditions on a finite connected poset X such that all Lie automorphisms of I(X,K) are proper?*

In this paper, we give a partial answer to this question. Namely, for a general X we find only a sufficient condition (see Corollary 3.12), and for some particular classes of posets of length one X we give a complete answer (see Corollaries 4.7 and 4.12 and Proposition 4.14).

More precisely, our work is organized as follows. Section 1 serves as a background on posets, incidence algebras, and maps on them. In particular, we recall all the necessary definitions from [13] and introduce some new notations. In Section 2, we reduce the question of when all Lie automorphisms of $I(X, K)$ are proper to a purely combinatorial property of X dealing with a certain group $\mathcal{AM}(X)$

of bijections on maximal chains of X (see Theorem 2.10). In our terminology, we prove that every Lie automorphism of $I(X, K)$ is proper if and only if every bijection $\theta \in \mathcal{AM}(X)$ is *proper*. In Section 3, we introduce an equivalence relation $\sim$ on maximal chains of X and show that any $\theta \in \mathcal{AM}(X)$ induces isomorphisms or anti-isomorphisms between certain subsets of X (the so-called *supports* of $\sim$-classes), as proved in Theorem 3.10. Consequently, if all the maximal chains of X are equivalent, then all Lie automorphisms of $I(X, K)$ are proper (see Corollary 3.12). The equivalence $\sim$ is just the equality relation whenever X is of length one, hence this situation is treated separately. This is done in Section 4. We first consider the case when X has no crown subset and give a full description of those X for which any $\theta \in \mathcal{AM}(X)$ is proper (see Corollary 4.7). We then pass to two specific classes of X: n-crowns Cr_n and ordinal sums of two anti-chains $\mathrm{K}_{m,n}$. If $X = \mathrm{Cr}_n$, then we explicitly describe the group $\mathcal{AM}(X)$ (see Proposition 4.10) and its subgroup of proper bijections (see Proposition 4.11). It follows that all $\theta \in \mathcal{AM}(\mathrm{Cr}_n)$ are proper exactly when $n = 2$ (see Corollary 4.12). If $X = \mathrm{K}_{m,n}$, then there are only proper $\theta \in \mathcal{AM}(X)$ as proved in Proposition 4.14.

1. Preliminaries

1.1. Automorphisms and anti-automorphisms

Let A be an algebra. We denote by $\mathrm{Aut}(A)$ the group of (linear) automorphisms of A, by $\mathrm{Aut}^-(A)$ the set of (linear) anti-automorphisms of A and by $\mathrm{Aut}^\pm(A)$ the union $\mathrm{Aut}(A) \cup \mathrm{Aut}^-(A)$. Observe that the union is non-disjoint if and only if A is commutative, in which case $\mathrm{Aut}(A) = \mathrm{Aut}^-(A)$. The set $\mathrm{Aut}^\pm(A)$ is a group under the composition, and moreover, if A is non-commutative and $\mathrm{Aut}^-(A) \neq \emptyset$, then $\mathrm{Aut}(A)$ is a (normal) subgroup of $\mathrm{Aut}^\pm(A)$ of index 2. In particular, $|\mathrm{Aut}(A)| = |\mathrm{Aut}^-(A)|$, whenever A is non-commutative and $\mathrm{Aut}^-(A) \neq \emptyset$. We use the analogous notations $\mathrm{Aut}(X)$, $\mathrm{Aut}^-(X)$ and $\mathrm{Aut}^\pm(X)$ for the group of automorphisms of a poset X, the set of anti-automorphisms of X and the group $\mathrm{Aut}(X) \cup \mathrm{Aut}^-(X)$, respectively. As above, $\mathrm{Aut}(X)$ either coincides with $\mathrm{Aut}^\pm(X)$ or is a subgroup of index 2 in $\mathrm{Aut}^\pm(X)$ (if X is not an anti-chain and $\mathrm{Aut}^-(X) \neq \emptyset$).

1.2. Posets

Let $(X, \leq)$ be a partially ordered set (which we usually shorten to "poset") and $x, y \in X$. The *interval* from x to y is the set $\lfloor x, y \rfloor = \{z \in X : x \leq z \leq y\}$. The poset X is said to be *locally finite* if all the intervals of X are finite. A *chain* in X is a linearly ordered (under the induced order) subset of X. The *length* of a finite chain $C \subseteq X$ is defined to be $|C| - 1$. The *length*[1] of a finite poset X, denoted by $l(X)$, is the maximum length of chains $C \subseteq X$. A *walk* in X is a sequence $x_0, x_1, \ldots, x_m \in X$, such that x_i and x_{i+1} are comparable and $l(\lfloor x_i, x_{i+1} \rfloor) = 1$ (if $x_i \leq x_{i+1}$) or $l(\lfloor x_{i+1}, x_i \rfloor) = 1$ (if $x_{i+1} \leq x_i$) for all $i = 0, \ldots, m-1$. A walk $x_0, x_1, \ldots, x_m$ is *closed* if $x_0 = x_m$. A *path* is a walk satisfying $x_i \neq x_j$ for $i \neq j$. A *cycle* is a closed walk $x_0, x_1, \ldots, x_m = x_0$ in which $m \geq 4$ and $x_i = x_j \Rightarrow \{i, j\} = \{0, m\}$ for $i \neq j$. We say that X is *connected* if for any pair of $x, y \in X$ there is a path $x = x_0, \ldots, x_m = y$. We will denote by $\mathrm{Min}(X)$ (resp. $\mathrm{Max}(X)$) the set of minimal (resp. maximal) elements of X. If X is connected and $|X| > 1$, then $\mathrm{Min}(X) \cap \mathrm{Max}(X) = \emptyset$.

1.3. Incidence algebras

Let X be a locally finite poset and K a field. The *incidence algebra* [20] $I(X, K)$ of X over K is the K-space of functions $f : X \times X \to K$ such that $f(x, y) = 0$ if $x \not\leq y$. This is a unital K-algebra under the convolution product

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y),$$

[1] Often also called the height.

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for any $f, g \in I(X, K)$. Its identity element δ is given by

$$\delta(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Throughout the rest of the paper, X will stand for a connected finite poset. Then $I(X, K)$ admits the standard basis $\{e_{xy} : x \leq y\}$, where

$$e_{xy}(u, v) = \begin{cases} 1, & (u, v) = (x, y), \\ 0, & (u, v) \neq (x, y). \end{cases}$$

We will write $e_x = e_{xx}$. Denote also $B = \{e_{xy} : x < y\}$. It is a well-known fact (see [22, Theorem 4.2.5]) that the Jacobson radical of $I(X, K)$ is

$$J(I(X, K)) = \{f \in I(X, K) : f(x, x) = 0 \text{ for all } x \in X\} = \text{Span}_K B.$$

Diagonal elements of $I(X, K)$ are those $f \in I(X, K)$ satisfying $f(x, y) = 0$ for $x \neq y$. They form a commutative subalgebra $D(X, K)$ of $I(X, K)$ spanned by $\{e_x : x \in X\}$. Clearly, each $f \in I(X, K)$ can be uniquely written as $f = f_D + f_J$ with $f_D \in D(X, K)$ and $f_J \in J(I(X, K))$.

1.4. Decomposition of $\phi \in Aut^{\pm}(I(X, K))$

Now, we recall the descriptions of automorphisms and anti-automorphisms of $I(X, K)$. Firstly, if X and Y are finite posets and $\lambda : X \to Y$ is an isomorphism (resp. anti-isomorphism), then λ *induces* an isomorphism (resp. anti-isomorphism) $\hat{\lambda} : I(X, K) \to I(Y, K)$ defined by $\hat{\lambda}(e_{xy}) = e_{\lambda(x)\lambda(y)}$ (resp. $\hat{\lambda}(e_{xy}) = e_{\lambda(y)\lambda(x)}$), for all $x \leq y$ in X. An element $\sigma \in I(X, K)$ such that $\sigma(x, y) \neq 0$, for all $x \leq y$, and $\sigma(x, y)\sigma(y, z) = \sigma(x, z)$ whenever $x \leq y \leq z$, determines an automorphism M_σ of $I(X, K)$ by $M_\sigma(e_{xy}) = \sigma(x, y)e_{xy}$, for all $x \leq y$. Such automorphisms are called *multiplicative*. Any automorphism (anti-automorphism) of $I(X, K)$ decomposes as

$$\phi = \hat{\lambda} \circ \xi \circ M_\sigma, \tag{1}$$

where $\lambda \in \text{Aut}(X)$ (resp. $\text{Aut}^-(X)$), ξ is an inner automorphism, and M_σ is a multiplicative automorphism of $I(X, K)$. For automorphisms, this was proved in [1, Theorem 5] and for anti-automorphisms in [3, Theorem 5] (for more results on automorphisms and anti-automorphisms of incidence algebras see [3, 4, 5, 6, 12, 15, 21, 23]).

1.5. Lie automorphisms of incidence algebras

In this section, we introduce several new notations and recall some definitions and results from [13].

We denote by $\mathcal{C}(X)$ the set of maximal chains in X. Let $C : u_1 < u_2 < \cdots < u_m$ in $\mathcal{C}(X)$. A bijection $\theta : B \to B$ is *increasing (resp. decreasing) on* C if there exists $D : v_1 < v_2 < \cdots < v_m$ in $\mathcal{C}(X)$ such that $\theta(e_{u_iu_j}) = e_{v_iv_j}$ for all $1 \leq i < j \leq m$ (resp. $\theta(e_{u_iu_j}) = e_{v_{m-j+1}v_{m-i+1}}$ for all $1 \leq i < j \leq m$). In this case, we write $\theta(C) = D$. Moreover, we say that θ *is monotone on maximal chains in* X if, for any $C \in \mathcal{C}(X)$, θ is increasing or decreasing on C. We denote by $\mathcal{M}(X)$ the set of bijections $B \to B$ which are monotone on maximal chains in X. It is easy to see that $\mathcal{M}(X)$ is a subgroup of the symmetric group $S(B)$. Each $\theta \in \mathcal{M}(X)$ induces a bijection on $\mathcal{C}(X)$ which maps C to $\theta(C)$.

Let $\theta : B \to B$ be a bijection and $X^2_< = \{(x, y) \in X^2 : x < y\}$. A map $\sigma : X^2_< \to K^*$ is *compatible* with θ if $\sigma(x, z) = \sigma(x, y)\sigma(y, z)$ whenever $\theta(e_{xz}) = \theta(e_{xy})\theta(e_{yz})$, and $\sigma(x, z) = -\sigma(x, y)\sigma(y, z)$ whenever $\theta(e_{xz}) = \theta(e_{yz})\theta(e_{xy})$.

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Let $\theta : B \to B$ be a bijection and $\Gamma : u_0, u_1, \ldots, u_m = u_0$ a closed walk in X. In [13], we introduced the following 4 functions $X \to \mathbb{N}$:

$$\begin{aligned} s^+_{\theta,\Gamma}(z) &= |\{i : u_i < u_{i+1} \text{ and } \exists w > z \text{ such that } \theta(e_{zw}) = e_{u_iu_{i+1}}\}|, \\ s^-_{\theta,\Gamma}(z) &= |\{i : u_i > u_{i+1} \text{ and } \exists w > z \text{ such that } \theta(e_{zw}) = e_{u_{i+1}u_i}\}|, \\ t^+_{\theta,\Gamma}(z) &= |\{i : u_i < u_{i+1} \text{ and } \exists w < z \text{ such that } \theta(e_{wz}) = e_{u_iu_{i+1}}\}|, \\ t^-_{\theta,\Gamma}(z) &= |\{i : u_i > u_{i+1} \text{ and } \exists w < z \text{ such that } \theta(e_{wz}) = e_{u_{i+1}u_i}\}|. \end{aligned}$$

We call the bijection $\theta : B \to B$ *admissible* if

$$s^+_{\theta,\Gamma}(z) - s^-_{\theta,\Gamma}(z) = t^+_{\theta,\Gamma}(z) - t^-_{\theta,\Gamma}(z), \tag{2}$$

for any closed walk $\Gamma : u_0, u_1, \ldots, u_m = u_0$ in X and for all $z \in X$. In particular, if X is a tree, then any bijection $\theta : B \to B$ is admissible. We denote by $\mathcal{AM}(X)$ the set of those $\theta \in \mathcal{M}(X)$ which are admissible.

Let $X = \{x_1, \ldots, x_n\}$. Given $\theta \in \mathcal{AM}(X)$, a map $\sigma : X^2_< \to K^*$ compatible with θ and a sequence $c = (c_1, \ldots, c_n) \in K^n$ such that $\sum_{i=1}^n c_i \in K^*$, we define in [13, Definition 5.17] the following *elementary Lie automorphism* $\tau = \tau_{\theta,\sigma,c}$ of $I(X, K)$ where, for any $e_{xy} \in B$,

$$\tau(e_{xy}) = \sigma(x, y)\theta(e_{xy}),$$

and $\tau|_{D(X,K)}$ is determined by

$$\tau(e_{x_i})(x_1, x_1) = c_i,$$

$i = 1, \ldots, n$, as in Lemmas 5.8 and 5.16 from [13]. As in [13, Definition 5.15], we say that τ *induces* the pair (θ, σ) and in some situations we write $\theta = \theta_\tau$.

As in [13], we denote by $\mathrm{LAut}(I(X, K))$ the group of Lie automorphisms of $I(X, K)$ and by $\widetilde{\mathrm{LAut}}(I(X, K))$ its subgroup of elementary Lie automorphisms. We will also use the notation $\mathrm{Inn}_1(I(X, K))$ for the subgroup of inner automorphisms consisting of conjugations by $\beta \in I(X, K)$ with $\beta_D = \delta$.

Theorem 1.1. [13, Theorem 4.15] *The group* $\mathrm{LAut}(I(X, K))$ *is isomorphic to the semidirect product* $\mathrm{Inn}_1(I(X, K)) \rtimes \widetilde{\mathrm{LAut}}(I(X, K))$.

2. Proper Lie automorphisms of $I(X, K)$ and proper bijections of B

Let $\varphi \in \mathrm{LAut}(I(X, K))$. Then $\varphi = \psi \circ \tau_{\theta,\sigma,c}$, where $\psi \in \mathrm{Inn}_1(I(X, K))$ and $\tau_{\theta,\sigma,c}$ is an elementary Lie automorphism of $I(X, K)$, by Theorem 1.1. Note that φ is proper if and only if $\tau_{\theta,\sigma,c}$ is proper. Therefore, all Lie automorphisms of $I(X, K)$ are proper if and only if all elementary Lie automorphisms of $I(X, K)$ are proper.

Let $\varphi = \tau_{\theta,\sigma,c}$ be an elementary Lie automorphism of $I(X, K)$. Suppose that φ is proper, $\varphi = \phi + \nu$, where $\phi \in \mathrm{Aut}(I(X, K))$ or $-\phi \in \mathrm{Aut}^-(I(X, K))$ and ν is a linear central-valued map on $I(X, K)$ such that $\nu([I(X, K), I(X, K)]) = \{0\}$. If $x < y$, then $e_{xy} \in J(I(X, K)) = [I(X, K), I(X, K)]$, by [13, Proposition 2.3]. Thus

$$\varphi(e_{xy}) = \phi(e_{xy}), \forall x < y. \tag{3}$$

By [22, Corollary 1.3.15], for each $x \in X$ there is $\alpha_x \in K$ such that

$$\varphi(e_x) = \phi(e_x) + \alpha_x\delta. \tag{4}$$

Suppose firstly that $\phi \in \mathrm{Aut}(I(X, K))$. Then, by (1), $\phi = \hat{\lambda} \circ \xi_f \circ M_\tau$, where $\lambda \in \mathrm{Aut}(X)$, ξ_f is an inner automorphism and M_τ is a multiplicative automorphism of $I(X, K)$. Thus, by (3),

$$\begin{aligned} \theta(e_{xy}) &= \sigma(x, y)^{-1}(\hat{\lambda} \circ \xi_f \circ M_\tau)(e_{xy}) = \sigma(x, y)^{-1}(\hat{\lambda} \circ \xi_f)(\tau(x, y)e_{xy}) \\ &= \sigma(x, y)^{-1}\hat{\lambda}(\tau(x, y)fe_{xy}f^{-1}) = \sigma(x, y)^{-1}\tau(x, y)\hat{\lambda}(f)e_{\lambda(x)\lambda(y)}\hat{\lambda}(f)^{-1}. \end{aligned} \tag{5}$$

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Analogously, if $-\phi \in \mathrm{Aut}^-(I(X,K))$, then, by (1), $\phi = \hat{\lambda} \circ \xi_f \circ M_\tau$, where $\lambda \in \mathrm{Aut}^-(X)$, ξ_f is an inner automorphism and M_τ is a multiplicative automorphism of $I(X,K)$. Thus, by (3),

$$\theta(e_{xy}) = \sigma(x,y)^{-1}(\hat{\lambda} \circ \xi_f \circ M_\tau)(e_{xy}) = \sigma(x,y)^{-1}\tau(x,y)\hat{\lambda}(f)^{-1}e_{\lambda(y)\lambda(x)}\hat{\lambda}(f). \tag{6}$$

Remark 2.1. *Let $x \le y$, $u \le v$ in X, $\alpha \in K^*$ and $h \in I(X,K)$ an invertible element. Note that if $he_{xy}h^{-1} = \alpha e_{uv}$, then $(x,y) = (u,v)$.*

Definition 2.2. *A bijection $\theta : B \to B$ is said to be* proper *if there exists $\lambda \in \mathrm{Aut}^\pm(X)$ such that $\theta(e_{xy}) = \hat{\lambda}(e_{xy})$ for all $e_{xy} \in B$. The proper bijections of B form a group, which we denote by $\mathcal{P}(X)$.*

Proposition 2.3. *If $|X| > 2$, then the group $\mathcal{P}(X)$ is isomorphic to $\mathrm{Aut}^\pm(X)$.*

Proof. The map sending $\lambda \in \mathrm{Aut}(X)$ (resp. $\lambda \in \mathrm{Aut}^-(X)$) to $\theta \in \mathcal{P}(X)$, such that $\theta(e_{xy}) = e_{\lambda(x)\lambda(y)}$ (resp. $\theta(e_{xy}) = e_{\lambda(y)\lambda(x)}$), is an epimorphism from $\mathrm{Aut}^\pm(X)$ to $\mathcal{P}(X)$. We only need to prove that it is injective.

It is injective on $\mathrm{Aut}(X)$. Indeed, take $\lambda, \mu \in \mathrm{Aut}(X)$ such that $(\lambda(x), \lambda(y)) = (\mu(x), \mu(y))$ for all $x < y$ in X. Let x be an arbitrary element of X. Since X is connected and $|X| > 1$, there is $y \in X$ such that either $y < x$ or $y > x$. In both cases, we get $\lambda(x) = \mu(x)$. Thus, $\lambda = \mu$. Similarly, one proves injectivity on $\mathrm{Aut}^-(X)$.

Assume now that there are $\lambda \in \mathrm{Aut}(X)$ and $\mu \in \mathrm{Aut}^-(X)$ such that $(\lambda(x), \lambda(y)) = (\mu(y), \mu(x))$ for all $x < y$ in X. We first show that X must have length at most 1. Indeed, if there were $x < y < z$ in X, then we would have $(\lambda(x), \lambda(y)) = (\mu(y), \mu(x))$ and $(\lambda(y), \lambda(z)) = (\mu(z), \mu(y))$, whence $\lambda(x) = \lambda(z)$, a contradiction. Now, consider a triple $x, y, z \in X$ with $x > z < y$. It follows from $(\lambda(z), \lambda(x)) = (\mu(x), \mu(z))$ and $(\lambda(z), \lambda(y)) = (\mu(y), \mu(z))$ that $\mu(x) = \mu(y)$, a contradiction. Similarly, the existence of a triple $x, y, z \in X$ with $x < z > y$ leads to $\lambda(x) = \lambda(y)$. If there are two incomparable elements $x, y \in X$, then there exists a sequence $x = x_1, \ldots, x_m = y$, where $m \ge 3$ and either $x_1 < x_2 > x_3$ or $x_1 > x_2 < x_3$. In both cases, we come to a contradiction. Thus, X is of length at most 1 and any two elements of X are comparable, which means that X is either a singleton or a chain of length 1. □

Remark 2.4. *If X is a chain of length 1, then $|\mathcal{P}(X)| = 1$, while $|\mathrm{Aut}^\pm(X)| = 2$.*

Proposition 2.5. *We have $\mathcal{P}(X) \subseteq \mathcal{AM}(X)$.*

Proof. Let $\theta \in \mathcal{P}(X)$. Then there exists $\lambda \in \mathrm{Aut}^\pm(X)$ as in Definition 2.2. In both cases, θ is the restriction of $\hat{\lambda}$ to B and, since $\hat{\lambda}$ or $-\hat{\lambda}$ is an elementary Lie automorphism of $I(X,K)$ by [13, Remark 4.8], then $\theta \in \mathcal{AM}(X)$ by [13, Remark 5.10]. □

Lemma 2.6. *Let $\varphi \in \widetilde{\mathrm{LAut}}(I(X,K))$ inducing a pair (θ, σ). Then θ is proper if, and only if, φ is proper.*

Proof. Assume first that $\theta(e_{xy}) = e_{\lambda(x)\lambda(y)}$ for some $\lambda \in \mathrm{Aut}(X)$. Then θ is increasing on any maximal chain in X and thus $\sigma(x,z) = \sigma(x,y)\sigma(y,z)$ for all $x < y < z$. Extending σ to $X^2_\le = \{(x,y) : x \le y\}$ by means of $\sigma(x,x) = 1$ for all $x \in X$, we obtain the multiplicative automorphism $M_\sigma \in \mathrm{Aut}(I(X,K))$. Consider $\psi = \varphi \circ M_\sigma^{-1}$. Notice that $\psi(e_{xy}) = e_{\lambda(x)\lambda(y)}$ for all $x < y$ and $\psi(e_x) = \varphi(e_x)$ for all $x \in X$. It suffices to prove that ψ is proper. Indeed, if $\psi = \phi + \nu$, then $\varphi = \psi \circ M_\sigma = \phi \circ M_\sigma + \nu$ since M_σ is identity on $D(X,R)$.

Clearly, $\psi(e_{xy}) = \hat{\lambda}(e_{xy})$ for all $x < y$, so $\hat{\lambda}$ is a candidate for ϕ. It remains to prove (4) for ψ, that is, to show that $\psi(e_x) = e_{\lambda(x)} + \alpha_x\delta$ for some $\alpha_x \in K$. The latter is equivalent to

$$\psi(e_x)(y,y) = \psi(e_x)(\lambda(x), \lambda(x)) - 1, \tag{7}$$

for all $y \ne \lambda(x)$. Given $x, y \in X$ with $y \ne \lambda(x)$, we choose a path $\lambda(x) = u_0, \ldots, u_m = y$ from $\lambda(x)$ to y. Let $v_i \in X$ such that $\lambda(v_i) = u_i$, $0 \le i \le m$. In particular, $v_0 = x$. Then

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$$\psi(e_x)(y,y) = \psi(e_x)(\lambda(x),\lambda(x)) + \sum_{i=0}^{m-1}(\psi(e_x)(u_{i+1},u_{i+1}) - \psi(e_x)(u_i,u_i)). \tag{8}$$

If $u_0 < u_1$, then $\theta(e_{xv_1}) = e_{u_0u_1}$, so $\psi(e_x)(u_1,u_1) - \psi(e_x)(u_0,u_0) = -1$ by [13, Lemma 5.7]. Similarly, if $u_0 > u_1$, then $\theta(e_{v_1x}) = e_{u_1u_0}$, so $\psi(e_x)(u_0,u_0) - \psi(e_x)(u_1,u_1) = 1$. In any case $\psi(e_x)(u_1,u_1) - \psi(e_x)(u_0,u_0) = -1$. Observe that $v_i \neq x$ for all $i > 0$. Hence $\psi(e_x)(u_{i+1},u_{i+1}) - \psi(e_x)(u_i,u_i) = 0$ for all such i by [13, Lemma 5.7]. It follows that the sum on the right-hand side of (8) has only one non-zero term which equals -1, proving (7).

The case $\theta(e_{xy}) = e_{\lambda(y)\lambda(x)}$, where $\lambda \in \mathrm{Aut}^-(X)$, is similar.

Conversely, suppose that $\varphi = \phi + \nu$, where $\phi \in \mathrm{Aut}(I(X,K))$ (resp. $-\phi \in \mathrm{Aut}^-(I(X,K))$) and ν is a linear central-valued map on $I(X,K)$ annihilating $[I(X,K), I(X,K)]$. It follows from (5) (resp. (6)) and Remark 2.1 that $\theta(e_{xy}) = e_{\lambda(x)\lambda(y)}$ ($\theta(e_{xy}) = e_{\lambda(y)\lambda(x)}$) for all $e_{xy} \in B$, where $\lambda \in \mathrm{Aut}(X)$ ($\lambda \in \mathrm{Aut}^-(X)$). Therefore, $\theta \in \mathcal{P}(X)$. □

Lemma 2.7. *Let $\theta \in \mathcal{M}(X)$. Let $C_1, C_2 \in \mathcal{C}(X)$ such that θ is increasing on C_1 and decreasing on C_2. If there exist $x, y \in C_1 \cap C_2$ and $x < y$, then x is the minimum of C_1 and C_2 and y is the maximum of C_1 and C_2.*

Proof. We first notice that there is $z \in C_1$ such that $z < x$ ($y < z$) if, and only if, there is $z' \in C_2$ such that $z' < x$ ($y < z'$), by the maximality of C_1 and C_2. Suppose that are $z \in C_1$ and $z' \in C_2$ such that $z, z' < x$. Then $\theta(e_{zx})\theta(e_{xy}) = \theta(e_{zy})$ and $\theta(e_{xy})\theta(e_{z'x}) = \theta(e_{z'y})$. Thus, there are $s < u < v < t$ such that $\theta(e_{zx}) = e_{su}$, $\theta(e_{xy}) = e_{uv}$, $\theta(e_{z'x}) = e_{vt}$ and $\theta(e_{zy}) = e_{sv}$, $\theta(e_{z'y}) = e_{ut}$. If θ^{-1} is increasing on a maximal chain containing $s < u < v < t$, then $\theta^{-1}(e_{st}) = \theta^{-1}(e_{su})\theta^{-1}(e_{ut}) = e_{zx}e_{z'y}$ which implies $z' = x$, a contradiction. If θ^{-1} is decreasing on a maximal chain containing $s < u < v < t$, then $\theta^{-1}(e_{st}) = \theta^{-1}(e_{vt})\theta^{-1}(e_{sv}) = e_{z'x}e_{zy}$ which implies $z = x$, a contradiction. Therefore, x is the minimum of C_1 and C_2. Analogously, y is the maximum of C_1 and C_2. □

Lemma 2.8. *For any $\theta \in \mathcal{M}(X)$, there is $\sigma : X^2_< \to K^*$ compatible with θ.*

Proof. Let $\theta \in \mathcal{M}(X)$ and $\mathcal{C}_i$ ($\mathcal{C}_d$) be the set of all maximal chains in X on which θ is increasing (decreasing). Let $(x,y) \in X^2_<$. If $x \in \mathrm{Min}(X)$, we set $\sigma(x,y) = 1$. Otherwise, by Lemma 2.7, both x and y belong only to maximal chains from $\mathcal{C}_i$ or only to maximal chains from $\mathcal{C}_d$. In the former case we set $\sigma(x,y) = 1$ and, in the latter one, we set $\sigma(x,y) = -1$.

Let $x < y < z$ in X. Again, by Lemma 2.7, those three elements can be simultaneously only in maximal chains from $\mathcal{C}_i$ or only in maximal chains from $\mathcal{C}_d$. If they belong to maximal chains from $\mathcal{C}_i$, then $\sigma(x,z) = 1 = \sigma(x,y)\sigma(y,z)$. Otherwise, there are two situations to be considered: $x \in \mathrm{Min}(X)$ or $x \notin \mathrm{Min}(X)$. In the former case, $\sigma(x,y)\sigma(y,z) = 1 \cdot (-1) = -1 = -\sigma(x,z)$. In the latter case, $\sigma(x,y)\sigma(y,z) = (-1)^2 = 1 = -\sigma(x,z)$. Thus, σ is compatible with θ. □

Corollary 2.9. *The image of the group homomorphism $\widetilde{\mathrm{LAut}}(I(X,K)) \to \mathcal{M}(X)$, $\varphi \mapsto \theta_\varphi$, coincides with $\mathcal{AM}(X)$. In particular, $\mathcal{AM}(X)$ is a group.*

Proof. If $\varphi \in \widetilde{\mathrm{LAut}}(I(X,K))$, then $\theta_\varphi \in \mathcal{AM}(X)$ by Lemma 5.4 and Remark 5.10 from [13]. Let now $\theta \in \mathcal{AM}(X)$. By Lemma 2.8, there is $\sigma : X^2_< \to K^*$ compatible with θ. Choose an arbitrary $c = (c_1, \ldots, c_n) \in K^n$ with $\sum_{i=1}^n c_i \in K^*$, where $n = |X|$. Then $\tau = \tau_{\theta,\sigma,c} \in \widetilde{\mathrm{LAut}}(I(X,K))$ such that $\theta_\tau = \theta$. □

Theorem 2.10. *Every Lie automorphism of I(X,K) is proper if and only if $\mathcal{P}(X) = \mathcal{AM}(X)$.*

Proof. Suppose that every Lie automorphism of $I(X,K)$ is proper. Let $\theta \in \mathcal{AM}(X)$. By Lemma 2.8, there is $\sigma : X^2_< \to K^*$ compatible with θ and, by [13, Lemma 5.16], there is $\varphi \in \widetilde{\mathrm{LAut}}(I(X,K))$ inducing (θ,σ). By hypothesis, φ is proper. Therefore, $\theta \in \mathcal{P}(X)$ by Lemma 2.6.

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Conversely, suppose that $\mathcal{P}(X) = \mathcal{AM}(X)$. Let $\varphi \in \widetilde{\mathrm{LAut}}(I(X, K))$ inducing the pair (θ, σ). By hypothesis, θ is proper. Therefore, φ is proper, by Lemma 2.6. Thus, every Lie automorphism of $I(X, K)$ is proper. □

3. Admissible bijections of B and maximal chains in X

Observe that the definitions of the functions $s^{\pm}_{\theta,\Gamma}$ and $t^{\pm}_{\theta,\Gamma}$ make sense for any sequence $\Gamma : u_0, \dots, u_m$ such that either $u_i < u_{i+1}$ or $u_{i+1} < u_i$ for all $0 \le i \le m-1$. We will call such sequences Γ *semiwalks*. If moreover $u_0 = u_m$, then Γ will be called a *closed semiwalk*.

Lemma 3.1. *Let $\theta \in \mathcal{M}(X)$, $z \in X$ and $\Gamma : u_0, u_1, \dots, u_m = u_0$ a closed semiwalk in X. Let $0 \le k < k+l \le m$ such that $u_k < u_{k+1} < \dots < u_{k+l}$ or $u_k > u_{k+1} > \dots > u_{k+l}$ and set $\Gamma' : u_0, \dots, u_k, u_{k+l}, \dots, u_m = u_0$. Then*

$$s^{+}_{\theta,\Gamma}(z) - t^{+}_{\theta,\Gamma}(z) = s^{+}_{\theta,\Gamma'}(z) - t^{+}_{\theta,\Gamma'}(z),\ s^{-}_{\theta,\Gamma}(z) - t^{-}_{\theta,\Gamma}(z) = s^{-}_{\theta,\Gamma'}(z) - t^{-}_{\theta,\Gamma'}(z). \tag{9}$$

Proof. Assume that $u_k < u_{k+1} < \dots < u_{k+l}$. There are two cases.

Case 1. θ^{-1} is increasing on a maximal chain containing $u_k < u_{k+1} < \dots < u_{k+l}$. Then there are $v_k < v_{k+1} < \dots < v_{k+l}$ such that $\theta^{-1}(e_{u_iu_j}) = e_{v_iv_j}$ for all $k \le i < j \le k+l$.

Case 1.1. $z = v_i$ for some $k < i < k+l$. If $\theta(e_{zw}) = e_{u_ju_{j+1}}$ for $w > z$ and $k \le j < k+l$, then $(z, w) = (v_j, v_{j+1})$, which implies that $j = i$ and $w = v_{i+1}$. Similarly $\theta(e_{wz}) = e_{u_ju_{j+1}}$ for $w < z$ and $k \le j < k+l$ yields $w = v_{i-1}$. Since, moreover, $\theta(e_{v_kv_{k+l}}) = e_{u_ku_{k+l}}$ and $z \ne v_k$, there is no $w > z$ such that $\theta(e_{zw}) = e_{u_ku_{k+l}}$. Similarly, there is no $w < z$ such that $\theta(e_{wz}) = e_{u_ku_{k+l}}$. Therefore, $s^{+}_{\theta,\Gamma'}(z) = s^{+}_{\theta,\Gamma}(z) - 1$, $t^{+}_{\theta,\Gamma'}(z) = t^{+}_{\theta,\Gamma}(z) - 1$, $s^{-}_{\theta,\Gamma'}(z) = s^{-}_{\theta,\Gamma}(z)$ and $t^{-}_{\theta,\Gamma'}(z) = t^{-}_{\theta,\Gamma}(z)$.

Case 1.2. $z = v_k$. Again, if $\theta(e_{zw}) = e_{u_ju_{j+1}}$ for $w > z$ and $k \le j < k+l$, then $w = v_{k+1}$. However, there is no $w < z$ such that $\theta(e_{wz}) = e_{u_ju_{j+1}}$ for some $k \le j < k+l$, but there is a unique $w > z$ (namely, $w = v_{k+l}$) such that $\theta(e_{zw}) = e_{u_ku_{k+l}}$. This means that $s^{\pm}_{\theta,\Gamma'}(z) = s^{\pm}_{\theta,\Gamma}(z)$ and $t^{\pm}_{\theta,\Gamma'}(z) = t^{\pm}_{\theta,\Gamma}(z)$.

Case 1.3. $z = v_{k+l}$. This case is similar to Case 1.2. We have $s^{\pm}_{\theta,\Gamma'}(z) = s^{\pm}_{\theta,\Gamma}(z)$ and $t^{\pm}_{\theta,\Gamma'}(z) = t^{\pm}_{\theta,\Gamma}(z)$.

Case 1.4. $z \notin \{v_k, \dots, v_{k+l}\}$. Then there is neither $w < z$ such that $\theta(e_{wz}) = e_{u_ju_{j+1}}$ nor $w > z$ such that $\theta(e_{zw}) = e_{u_ju_{j+1}}$ for some $k \le j < k+l$. Moreover, there is neither $w < z$ such that $\theta(e_{wz}) = e_{u_ku_{k+l}}$ nor $w > z$ such that $\theta(e_{zw}) = e_{u_ku_{k+l}}$. Thus, $s^{\pm}_{\theta,\Gamma'}(z) = s^{\pm}_{\theta,\Gamma}(z)$ and $t^{\pm}_{\theta,\Gamma'}(z) = t^{\pm}_{\theta,\Gamma}(z)$.

Case 2. θ^{-1} is decreasing on a maximal chain containing $u_k < u_{k+1} < \dots < u_{k+l}$. Then everything from Case 1 remains valid with the replacement of the "+"-functions by their " minus;"-analogs and vice versa.

In any case, $s^{+}_{\theta,\Gamma}(z) - t^{+}_{\theta,\Gamma}(z)$ and $s^{-}_{\theta,\Gamma}(z) - t^{-}_{\theta,\Gamma}(z)$ are invariant under the change of Γ for Γ'. When $u_k > u_{k+1} > \dots > u_{k+l}$, the proof is analogous. □

Corollary 3.2. *Let $\theta \in \mathcal{M}(X)$. Then $\theta \in \mathcal{AM}(X)$ if and only if (2) holds for any $z \in X$ and any closed semiwalk $\Gamma : u_0, \dots, u_m = u_0$, $m \ge 2$.*

Proof. The "if" part is trivial. Let us prove the "only if" part. Indeed, the case $m = 2$ is explained in the proof of [13, Lemma 5.13], and if $m \ge 3$, then Γ can be extended to a closed walk Δ by inserting increasing (if $u_i < u_{i+1}$) or decreasing (if $u_i > u_{i+1}$) sequences of elements between u_i and u_{i+1} for all $0 \le i \le m-1$. Since (2) holds for Δ, then by Lemma 3.1 it holds for Γ too. □

Lemma 3.3. *Let $\theta \in \mathcal{AM}(X)$ and $\Gamma : u_0, \dots, u_m = u_0$, $m \ge 2$, a closed semiwalk. Let also $x_i < y_i$, such that $\theta(e_{u_iu_{i+1}}) = e_{x_iy_i}$ for $u_i < u_{i+1}$ and $\theta(e_{u_{i+1}u_i}) = e_{x_iy_i}$ for $u_i > u_{i+1}$.*

i. If $x_i \in \mathrm{Min}(X)$ for some $0 \le i \le m-1$, then there is $j \ne i$ such that $x_i = x_j$.
ii. If $y_i \in \mathrm{Max}(X)$ for some $0 \le i \le m-1$, then there is $j \ne i$ such that $y_i = y_j$.

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Proof. We will prove (i), the proof of (ii) is analogous. Assume that $x_i \neq x_j$ for all $j \neq i$. If $u_i < u_{i+1}$, then $s^+_{\theta^{-1},\Gamma}(x_i) = 1$ and $s^-_{\theta^{-1},\Gamma}(x_i) = 0$, since $\theta^{-1}(e_{x_i y_i}) = e_{u_i u_{i+1}}$ and $\theta^{-1}(e_{x_i w}) \neq e_{u_j u_{j+1}}$ for any $w \neq y_i$ and $j \neq i$ (otherwise x_i would coincide with some x_j for $j \neq i$). Similarly, if $u_i > u_{i+1}$, then $s^+_{\theta^{-1},\Gamma}(x_i) = 0$ and $s^-_{\theta^{-1},\Gamma}(x_i) = 1$. Obviously, $t^{\pm}_{\theta^{-1},\Gamma}(x_i) = 0$, because $x_i \in \mathrm{Min}(X)$. Thus, (2) fails for the triple $(\theta^{-1}, \Gamma, x_i)$, a contradiction. □

Lemma 3.4. *Let $\theta \in \mathcal{M}(X)$. Assume that there exist $C, D \in \mathcal{C}(X)$ such that θ is increasing on C and decreasing on D. If $x \in C \cap D$, then either $x \in \mathrm{Min}(X)$ or $x \in \mathrm{Max}(X)$.*

Proof. Let $C : x_1 < \cdots < x_n$, $D : y_1 < \cdots < y_m$ and $x = x_i = y_j$ for some $1 \le i \le n$ and $1 \le j \le m$. Suppose that $1 < i < n$. Then $1 < j < m$, since otherwise D would not be maximal. There exist maximal chains $C' : u_1 < \cdots < u_n$ and $D' : v_1 < \cdots < v_m$ such that $\theta(e_{x_k x_l}) = e_{u_k u_l}$ for all $1 \le k < l \le n$ and $\theta(e_{y_p y_q}) = e_{v_{m-q+1} v_{m-p+1}}$ for all $1 \le p < q \le m$. In particular, $\theta(e_{x_{i-1} x_i}) = e_{u_{i-1} u_i}$, $\theta(e_{x_i x_{i+1}}) = e_{u_i u_{i+1}}$, $\theta(e_{y_{j-1} y_j}) = e_{v_{m-j+1} v_{m-j+2}}$, $\theta(e_{y_j y_{j+1}}) = e_{v_{m-j} v_{m-j+1}}$. Observe that $x_{i-1} < x_i = y_j < y_{j+1}$. Then either $u_i = v_{m-j}$, or $u_{i-1} = v_{m-j+1}$, depending on whether θ is increasing or decreasing on a maximal chain containing $x_{i-1} < x_i < y_{j+1}$. Similarly, considering $y_{j-1} < y_j = x_i < x_{i+1}$ we obtain $v_{m-j+1} = u_{i+1}$ or $v_{m-j+2} = u_i$. If $u_i = v_{m-j}$, then $v_{m-j+1} = u_{i+1}$, so that $\{u_i, u_{i+1}\} \subseteq C' \cap D'$. However, θ^{-1} is increasing on C' and decreasing on D', so u_i is the common minimum of C' and D' and u_{i+1} is the common maximum of C' and D' by Lemma 2.7. This contradicts the assumption $1 < i < n$. Similarly, $u_{i-1} = v_{m-j+1}$ implies $v_{m-j+2} = u_i$, whence $\{u_{i-1}, u_i\} \subseteq C' \cap D'$ leading to a contradiction.

Thus, $i \in \{1, n\}$. If $i = 1$, then necessarily $j = 1$, as otherwise C would not be maximal. Similarly, if $i = n$, then $j = m$. □

Lemma 3.5. *Let $\theta \in \mathcal{M}(X)$ and $C, D \in \mathcal{C}(X)$, $C : x_1 < \cdots < x_n$, $D : y_1 < \cdots < y_m$. Assume that $x_i = y_j$ for some $1 < i < n$ and $1 < j < m$. If θ is increasing (resp. decreasing) on C, then it is increasing (resp. decreasing) on D. Moreover, if $\theta(C) : u_1 < \cdots < u_n$ and $\theta(D) : v_1 < \cdots < v_m$, then $u_i = v_j$ (resp. $u_{n-i+1} = v_{m-j+1}$).*

Proof. Let θ be increasing on C. Then it is increasing on D by Lemma 3.4. Using the same idea as in the proof of Lemma 3.4, we have $\theta(e_{x_{i-1} x_i}) = e_{u_{i-1} u_i}$, $\theta(e_{x_i x_{i+1}}) = e_{u_i u_{i+1}}$, $\theta(e_{y_{j-1} y_j}) = e_{v_{j-1} v_j}$, $\theta(e_{y_j y_{j+1}}) = e_{v_j v_{j+1}}$. Considering $x_{i-1} < x_i = y_j < y_{j+1}$ we conclude that $u_i = v_j$ or $u_{i-1} = v_{j+1}$. Similarly it follows from $y_{j-1} < y_j = x_i < x_{i+1}$ that $u_i = v_j$ or $v_{j-1} = u_{i+1}$. If $u_i \neq v_j$, then $u_{i-1} = v_{j+1}$ and $v_{j-1} = u_{i+1}$. But this is impossible, since $u_{i-1} < u_{i+1}$ and $v_{j+1} > v_{j-1}$.

The proof for the decreasing case is analogous. □

Definition 3.6. *Let $C, D \in \mathcal{C}(X)$. We say that C and D are* linked *if there exists $x \in C \cap D$ such that $x \notin \mathrm{Min}(X) \sqcup \mathrm{Max}(X)$. Denote by $\sim$ the equivalence relation on $\mathcal{C}(X)$ generated by* $\{(C, D) \in \mathcal{C}(X)^2 : C, D$ are linked$\}$.

Lemma 3.7. *Each $\theta \in \mathcal{M}(X)$ induces a bijection $\widetilde{\theta}$ on $\mathcal{C}(X)/\!\sim$. Moreover, if θ is increasing (resp. decreasing) on $C \in \mathcal{C}(X)$, then it is increasing (resp. decreasing) on any $D \sim C$.*

Proof. Let $C, D \in \mathcal{C}(X)$ be linked. Then $\theta(C)$ and $\theta(D)$ are linked by Lemma 3.5. It follows that $C \sim D$ implies $\theta(C) \sim \theta(D)$, which induces a map $\widetilde{\theta} : \mathcal{C}(X)/\!\sim \to \mathcal{C}(X)/\!\sim$. It is a bijection whose inverse is $\widetilde{\theta^{-1}}$.

Assume that θ is increasing on $C \in \mathcal{C}(X)$. Then by Lemma 3.5 it is increasing on any $D \in \mathcal{C}(X)$ which is linked to C. By the obvious induction, this extends to any $D \sim C$. The decreasing case is similar. □

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Definition 3.8. *Given* $\mathfrak{C} \in \mathcal{C}(X)/\sim$*, we define the* support *of* $\mathfrak{C}$*, denoted* $\mathrm{supp}(\mathfrak{C})$*, as the set* $\{x \in C : C \in \mathfrak{C}\}$.

Remark 3.9. *Let* $\mathfrak{C}, \mathfrak{D} \in \mathcal{C}(X)/\sim$*. If* $\mathfrak{C} \neq \mathfrak{D}$*, then* $\mathrm{supp}(\mathfrak{C}) \cap \mathrm{supp}(\mathfrak{D}) \subseteq \mathrm{Min}(X) \sqcup \mathrm{Max}(X)$.

Indeed, assume that $x \in \mathrm{supp}(\mathfrak{C}) \cap \mathrm{supp}(\mathfrak{D})$, where $x \notin \mathrm{Min}(X)$ and $x \notin \mathrm{Max}(X)$. There are $C \in \mathfrak{C}$ and $D \in \mathfrak{D}$ such that $x \in C \cap D$. But then C and D are linked, so $C \sim D$, whence $\mathfrak{C} = \mathfrak{D}$.

Theorem 3.10. *Let* $\theta \in \mathcal{AM}(X)$ *and* $\mathfrak{C} \in \mathcal{C}(X)/\sim$*. Then there exists an isomorphism or an anti-isomorphism of posets* $\lambda : \mathrm{supp}(\mathfrak{C}) \to \mathrm{supp}(\widetilde{\theta}(\mathfrak{C}))$ *such that for all* $x < y$ *from* $\mathrm{supp}(\mathfrak{C})$ *one has*

$$\theta(e_{xy}) = \hat{\lambda}(e_{xy}). \tag{10}$$

Proof. In view of Lemma 3.7, we may assume that θ is increasing on all $C \in \mathfrak{C}$ or decreasing on all $C \in \mathfrak{C}$. Consider the case of an increasing θ. We are going to construct the corresponding $\lambda : \mathrm{supp}(\mathfrak{C}) \to \mathrm{supp}(\widetilde{\theta}(\mathfrak{C}))$. Let $x \in \mathrm{supp}(\mathfrak{C})$ and $C : x_1 < \cdots < x_n$ a maximal chain from $\mathfrak{C}$ containing x. Denote by $C' : u_1 < \cdots < u_n$ the image of C under θ. If $x = x_i$ for some $1 \le i \le n$, then we put $\lambda(x_i) = u_i$. We still need to show that the definition does not depend on the choice of C.

If $1 < i < n$, then this is true by Lemma 3.5.

If $i = 1$, then $x \in \mathrm{Min}(X)$. If there exists another maximal chain $D : y_1 < \cdots < y_m$ from $\mathfrak{C}$ containing x, then $x = y_1$. We thus need to show that $u_1 = v_1$, where $D' : v_1 < \cdots < v_m$ is the image of D under θ. Since $C \sim D$, there are $C = C_1, \ldots, C_k = D$ such that C_j and C_{j+1} are linked for all $1 \le j \le k-1$. Denote by z_j an element of $C_j \cap C_{j+1}$, $1 \le j \le k-1$, which is neither minimal nor maximal in X. Set also $z_0 = z_k = x$. Observe that $z_j, z_{j+1} \in C_{j+1}$ for all $0 \le j \le k-1$, so that either $z_j \le z_{j+1}$ or $z_j \ge z_{j+1}$. Let $\Gamma : z_0, z_1, \ldots, z_k = z_0$. Clearly, $z_0 \neq z_j$ and $z_j \neq z_k$ for all $1 \le j \le k-1$, as $z_0 = z_k \in \mathrm{Min}(X)$, while $z_j \notin \mathrm{Min}(X)$. Moreover, we will assume that $z_j \neq z_{j+1}$ for all $1 \le j \le k-2$, since otherwise we may just remove the repetitions (and at least 2 elements will remain). Let also $a_j < b_j$ such that $\theta(e_{z_j z_{j+1}}) = e_{a_j b_j}$ if $z_j < z_{j+1}$, and $\theta(e_{z_{j+1} z_j}) = e_{a_j b_j}$ if $z_j > z_{j+1}$, $0 \le j \le k-1$. Observe that $a_0 = u_1$, since $x = z_0 < z_1 \in C$, and $a_{k-1} = v_1$, since $x = z_k < z_{k-1} \in D$. In particular, $a_0, a_{k-1} \in \mathrm{Min}(X)$. Since z_j is not minimal for all $1 \le j \le k-2$, then neither is a_j, so that $a_j \notin \{a_0, a_{k-1}\}$ for such j. But then we must have $a_0 = a_{k-1}$, that is, $u_1 = v_1$, by Lemma 3.3 (i).

The case $i = n$ is similar. The map $\lambda : \mathrm{supp}(\mathfrak{C}) \to \mathrm{supp}(\widetilde{\theta}(\mathfrak{C}))$ is thus constructed.

We now prove that $\lambda(x) < \lambda(y)$ and (10) holds for all $x < y$ from $\mathrm{supp}(\mathfrak{C})$. By construction, this is true for x and y belonging to the same $C \in \mathfrak{C}$. Let now $x < y$ be arbitrary elements of $\mathrm{supp}(\mathfrak{C})$. Choose $C \in \mathfrak{C}$ containing x and $D \in \mathfrak{C}$ containing y. If $x \notin \mathrm{Min}(X)$, then any $C' \in \mathcal{C}(X)$ containing x and y is linked to C, so that $C' \in \mathfrak{C}$. The case when $y \notin \mathrm{Max}(X)$ is similar. Let now $x \in \mathrm{Min}(X)$ and $y \in \mathrm{Max}(X)$. As above, we choose $C = C_1, \ldots, C_k = D$, such that $C_j, C_{j+1} \in \mathfrak{C}$ are linked for all $1 \le j \le k-1$, and $z_j \in C_j \cap C_{j+1}$, $1 \le j \le k-1$, which is neither minimal nor maximal in X. We set $z_0 = x$, $z_k = y$ and $\Gamma : z_0, z_1, \ldots, z_k, z_{k+1} = z_0$. We also assume that $z_j \neq z_{j+1}$ for all $0 \le j \le k$ and denote by $a_j < b_j$ the elements satisfying $\theta(e_{z_j z_{j+1}}) = e_{a_j b_j}$ if $z_j < z_{j+1}$ and $\theta(e_{z_{j+1} z_j}) = e_{a_j b_j}$ if $z_j > z_{j+1}$, $0 \le j \le k$. As above, observe that $a_0, a_k \in \mathrm{Min}(X)$, while $a_1, \ldots, a_{k-1} \notin \mathrm{Min}(X)$. Then $a_0 = a_k$ by Lemma 3.3 (i). Similarly it follows from Lemma 3.3 (ii) that $b_{k-1} = b_k$. But $a_0 = \lambda(x)$ and $b_{k-1} = \lambda(y)$, since $e_{a_0 b_0} = \theta(e_{z_0 z_1}) = e_{\lambda(z_0)\lambda(z_1)} = e_{\lambda(x)\lambda(z_1)}$ and $e_{a_{k-1} b_{k-1}} = \theta(e_{z_{k-1} z_k}) = e_{\lambda(z_{k-1})\lambda(z_k)} = e_{\lambda(z_{k-1})\lambda(y)}$. Hence, $\lambda(x) = a_k < b_k = \lambda(y)$ and $\theta(e_{xy}) = \theta(e_{z_{k+1} z_k}) = e_{a_k b_k} = e_{\lambda(x)\lambda(y)} = \hat{\lambda}(e_{xy})$.

It is clear that λ is a bijection whose inverse is the map $\mu : \mathrm{supp}(\widetilde{\theta}(\mathfrak{C})) \to \mathrm{supp}(\mathfrak{C})$ corresponding to θ^{-1}. Thus, λ is an isomorphism between $\mathrm{supp}(\mathfrak{C})$ and $\mathrm{supp}(\widetilde{\theta}(\mathfrak{C}))$.

The case of a decreasing θ is analogous. □

The following example shows that the admissibility of θ in Theorem 3.10 cannot be dropped.

Example 3.11. *Let* $X = \{1, \ldots, 10, 1', \ldots 9', 7''\}$ *with the following Hasse diagram.*

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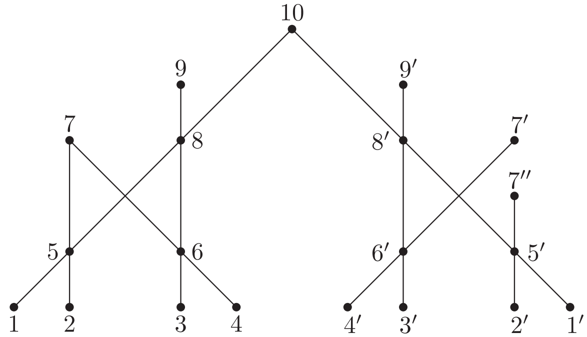


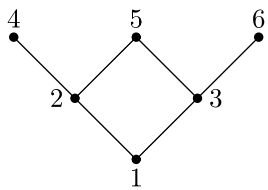
Then $\mathcal{C}(X)/\sim$ consists of 2 classes whose supports are $Y = \{1, \ldots, 10\}$ and $Z = \{1', \ldots, 9', 10, 7''\}$. Observe that there exists $\theta \in \mathcal{M}(X)$ mapping one $\sim$-class to another. It is defined as follows: $\theta(e_{ij}) = e_{i'j'}$ for all $i \le j$ in X with $(i, j) \ne (5, 7)$, $\theta(e_{i'j'}) = e_{ij}$ for all $i' \le j'$ in X with $(i', j') \ne (5', 7'')$, $\theta(e_{57}) = e_{5'7''}$ and $\theta(e_{5'7''}) = e_{57}$ (to make the definition shorter, we set $10' := 10$). However, Y and Z are not isomorphic or anti-isomorphic because $|Y| \ne |Z|$. The reason is that $\theta \notin \mathcal{AM}(X)$. Indeed, for $\Gamma : 5 < 7 > 6 < 8 > 5$ we have $s^{\pm}_{\theta,\Gamma}(7') = 0$, $t^{+}_{\theta,\Gamma}(7') = 0$ and $t^{-}_{\theta,\Gamma}(7') = 1$.

As a consequence of Theorems 2.10 and 3.10, we have the following result which generalizes [13, Corollary 5.19], where X was a chain.

Corollary 3.12. *If* $|\mathcal{C}(X)/\sim| = 1$, *then each* $\varphi \in \mathrm{LAut}(I(X, K))$ *is proper.*

Observe, however, that the condition $|\mathcal{C}(X)/\sim| = 1$ is not necessary for all $\varphi \in \mathrm{LAut}(I(X, K))$ to be proper, as the following example shows.

Example 3.13. *Let* $X = \{1, 2, 3, 4, 5, 6\}$ *with the following Hasse diagram.*



Note that $\mathcal{C}(X)/\sim$ consists of 2 classes whose supports are $Y = \{1, 2, 4, 5\}$ and $Z = \{1, 3, 5, 6\}$. For any $\theta \in \mathcal{AM}(X)$, there are 2 possibilities for the corresponding isomorphisms λ_1 and λ_2 between the supports: either $\lambda_1 : Y \to Y$ and $\lambda_2 : Z \to Z$, or $\lambda_1 : Y \to Z$ and $\lambda_2 : Z \to Y$. In the former case $\lambda_1 = \mathrm{id}_Y$ and $\lambda_2 = \mathrm{id}_Z$, and in the letter case $\lambda_2 = \lambda_1^{-1}$, where λ_1 maps an element $y \in Y$ to the element $z \in Z$ which is symmetric to y with respect to the vertical line passing through the vertices 1 and 5. In both cases, λ_1 and λ_2 are the restrictions of an automorphism of X to Y and Z, respectively.

4. Sets of length one

4.1. Admissible bijections of B and crowns in X

Before proceeding to the case $l(X) = 1$, we will prove a useful fact which holds for X of an arbitrary length.

Definition 4.1. *Let n be an integer greater than 1. By a* weak n-crown *we mean a poset* $P = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ *where*

$$x_i < y_i \text{ for all } 1 \le i \le n,\ x_{i+1} < y_i \text{ for all } 1 \le i \le n-1 \text{ and } x_1 < y_n. \tag{11}$$

An n-crown *is a weak n-crown which has no other pairs of distinct comparable elements except (11). It is thus fully determined by n up to an isomorphism and will be denoted by* Cr_n. *A poset P is called a* weak crown *(resp. crown), if it is a weak n-crown (resp. n-crown) for some* $n \geq 2$. *We say that a poset X has a weak crown (resp. crown) if there is a subset* $Y \subseteq X$ *which is a weak crown (resp. crown) under the induced partial order.*

Posets without crowns are known to satisfy some "good" properties [10, 11, 19].

Lemma 4.2. *Let* $\theta \in \mathcal{M}(X)$. *Then* $\theta \in \mathcal{AM}(X)$ *if and only if (2) holds for any* $z \in X$ *and any weak crown* $\Gamma : u_0, \ldots, u_m = u_0$ *in X.*

Proof. The "only if" part is obvious. For the "if" part take any closed semiwalk $\Gamma : u_0, u_1, \ldots, u_m = u_0$ and $z \in X$. Let $0 \leq k < k + l \leq m$ such that $u_k < u_{k+1} < \cdots < u_{k+l}$. We define $\Gamma' : u_0, \ldots, u_k, u_{k+l}, \ldots, u_m = u_0$. By Corollary 3.2 equality (2) holds for Γ if and only if it holds for Γ'. The same is true for any Γ' obtained from Γ by removing intermediate terms in a decreasing sequence of consecutive vertices. Thus, doing so for all maximal sequences in Γ we finally get Γ' whose vertices form either a sequence $x < y > z$ or a weak crown. However, the case $\Gamma' : x < y > z$ can be ignored, because (2) always holds for such Γ' as shown in the proof of [13, Lemma 5.13]. □

4.2. The crownless case

Let now $l(X) = 1$. Observe that $\mathcal{M}(X) = S(B)$. Moreover, any $C \in \mathcal{C}(X)$ is linked only to itself, so $|\mathcal{C}(X)/\sim| = |\mathcal{C}(X)|$ and Theorem 3.10 becomes useless.

Definition 4.3. *We say that* $\theta \in \mathcal{M}(X)$ is separating *if there exists a pair of non-disjoint* $C, D \in \mathcal{C}(X)$ *such that* $\theta(C)$ *and* $\theta(D)$ *are disjoint.*

Remark 4.4. *Any separating* θ *is not proper.*

Lemma 4.5. *Let* $l(X) = 1$. *If* $|\mathrm{Min}(X)| > 1$ *and* $|\mathrm{Max}(X)| > 1$, *then there are disjoint* $C, D \in \mathcal{C}(X)$.

Proof. Choose arbitrary $x < y$ in X. Obviously, $x \in \mathrm{Min}(X)$ and $y \in \mathrm{Max}(X)$. Let $U = \{u \in \mathrm{Min}(X) \mid u \not\leq y\}$ and $V = \{v \in \mathrm{Max}(X) \mid x \not\leq v\}$. If $U \neq \emptyset$, then take $u \in U$. Clearly, $u \neq x$. Since X is connected, there exists $v > u$, and $v \neq y$ by the definition of U. Then $C : x < y$ and $D : u < v$ are disjoint. The case $V \neq \emptyset$ is similar. Suppose now that $U = V = \emptyset$. This means that $x \leq v$ for any $v \in \mathrm{Max}(X)$ and $y \geq u$ for any $u \in \mathrm{Min}(X)$. Choose $u \in \mathrm{Min}(X) \setminus \{x\}$ and $v \in \mathrm{Max}(X) \setminus \{y\}$. Then $C_1 : x < v$ and $D_1 : u < y$ are disjoint. □

Proposition 4.6. *Let* $l(X) = 1$. *Then* $\mathcal{M}(X) = \mathcal{P}(X)$ *if and only if* $|\mathrm{Min}(X)| = 1$ *or* $|\mathrm{Max}(X)| = 1$.

Proof. If $|\mathrm{Min}(X)| = 1$, say $\mathrm{Min}(X) = \{x\}$, then any $\theta \in \mathcal{M}(X)$ can be identified with a bijection λ of $\mathrm{Max}(X)$ such that $\theta(e_{xy}) = e_{x\lambda(y)}$ for all $y > x$. But λ extends to an automorphism of X by means of $\lambda(x) = x$, so that $\theta(e_{xy}) = e_{\lambda(x)\lambda(y)}$. A symmetric argument works in the case $|\mathrm{Max}(X)| = 1$.

Suppose now that $|\mathrm{Min}(X)| > 1$ and $|\mathrm{Max}(X)| > 1$. By Lemma 4.5 there are disjoint $C : x < y$ and $D : u < v$. Choose a path $x = x_0, x_1, \ldots, x_m = u$. Since $x, u \in \mathrm{Min}(X)$, then $m \geq 2$ and $x_0 < x_1 > x_2$. We define $\theta(e_{x_0x_1}) = e_{xy}$, $\theta(e_{xy}) = e_{x_0x_1}$, $\theta(e_{x_2x_1}) = e_{uv}$, $\theta(e_{uv}) = e_{x_2x_1}$ and $\theta(e_{ab}) = e_{ab}$ for any other $e_{ab} \in B$. Clearly, $\theta \in S(B) = \mathcal{M}(X)$ and it is separating, in particular, not proper. □

If X does not contain a weak crown, then $\mathcal{AM}(X) = \mathcal{M}(X)$ by Lemma 4.2. Hence, we obtain the following.

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Corollary 4.7. *Let $l(X) = 1$ and assume that X does not contain a weak crown. Then $\mathcal{AM}(X) = \mathcal{P}(X)$ if and only if* $|\mathrm{Min}(X)| = 1$ *or* $|\mathrm{Max}(X)| = 1$.

4.3. The crown case

We will now consider two classes of posets of length one which have crowns. We begin with the case of X being a crown and are going to calculate the groups $\mathcal{P}(X)$ and $\mathcal{AM}(X)$ explicitly. Thus, in this subsection $X = \mathrm{Cr}_n = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$.

Definition 4.8. *The chains $x_i < y_i$, $1 \le i \le n$, will be called* odd*, and $x_{i+1} < y_i$, $1 \le i \le n-1$, and $x_1 < y_n$ will be called* even*. Thus, each element of* Cr_n *belongs to exactly one odd chain and to exactly one even chain.*

Lemma 4.9. *Let $\theta \in \mathcal{M}(\mathrm{Cr}_n)$. Then $\theta \in \mathcal{AM}(\mathrm{Cr}_n)$ if and only if for any pair of distinct non-disjoint chains $C, D \in \mathcal{C}(\mathrm{Cr}_n)$, the images $\theta(C)$ and $\theta(D)$ have opposite parities.*

Proof. Observe that (2) is invariant under cyclic shifts of Γ (any such shift does not change the functions $s^{\pm}_{\theta,\Gamma}$ and $t^{\pm}_{\theta,\Gamma}$). Thus, for admissibility it is enough to consider $\Gamma : x_1 < y_1 > x_2 < \cdots < y_n > x_1$, since any other cycle in Cr_n is a cyclic shift of Γ.

The *"only if"* case. Let $\theta \in \mathcal{AM}(\mathrm{Cr}_n)$ and $C, D \in \mathcal{C}(\mathrm{Cr}_n)$, such that $C \cap D = \{z\}$, where $z \in \mathrm{Min}(\mathrm{Cr}_n) \sqcup \mathrm{Max}(\mathrm{Cr}_n)$. Suppose that $z \in \mathrm{Min}(\mathrm{Cr}_n)$. Then there are only two elements $w, w' \in \mathrm{Max}(\mathrm{Cr}_n)$ such that $z < w, w'$. Thus, $t^{\pm}_{\theta,\Gamma}(z) = 0$ and θ is admissible if and only if $s^{+}_{\theta,\Gamma}(z) = s^{-}_{\theta,\Gamma}(z) = 1$, which only occur if $\theta(C)$ and $\theta(D)$ have opposite parities. The case when $z \in \mathrm{Max}(\mathrm{Cr}_n)$ is similar.

The *"if"* case. Let $\theta \in \mathcal{M}(\mathrm{Cr}_n)$ and $z \in \mathrm{Cr}_n$ be arbitrary. Again, we consider the case $z \in \mathrm{Min}(\mathrm{Cr}_n)$, so that $t^{\pm}_{\theta,\Gamma}(z) = 0$. Choose $w, w' \in \mathrm{Max}(\mathrm{Cr}_n)$ with $z < w, w'$ and put $C : z < w$ and $D : z < w'$. Since $\theta(C)$ and $\theta(D)$ have opposite parities, then $s^{\pm}_{\theta,\Gamma}(z) = 1$ and (2) is satisfied. Similarly, one handles the case $z \in \mathrm{Max}(\mathrm{Cr}_n)$. □

Proposition 4.10. *The group $\mathcal{AM}(\mathrm{Cr}_n)$ is isomorphic to $(S_n \times S_n) \rtimes \mathbb{Z}_2$.*

Proof. Denote by $\mathcal{O}$ and $\mathcal{E}$ the subsets of $\mathcal{C}(\mathrm{Cr}_n)$ formed by the odd and even chains, respectively, and let $\mathcal{G} = \{\theta \in \mathcal{M}(\mathrm{Cr}_n) : \theta(\mathcal{O}) = \mathcal{O}$ or $\theta(\mathcal{O}) = \mathcal{E}\}$. We will first prove that $\mathcal{AM}(\mathrm{Cr}_n) = \mathcal{G}$. For any $\theta \in \mathcal{AM}(\mathrm{Cr}_n)$, if $\theta(e_{x_1y_1}) \in \mathcal{O}$, then $\theta(e_{x_2y_1}) \in \mathcal{E}$ by Lemma 4.9. It follows that $\theta(e_{x_2y_2}) \in \mathcal{O}$ by the same reason. Applying this argument consecutively to $e_{x_3y_2}, e_{x_3y_3}, \ldots, e_{x_ky_k}, e_{x_1y_k}$, we obtain $\theta(\mathcal{O}) = \mathcal{O}$. Similarly, if $\theta(e_{x_1y_1}) \in \mathcal{E}$, then $\theta(\mathcal{O}) = \mathcal{E}$. Thus, $\theta \in \mathcal{G}$. On the other hand, let $\theta \in \mathcal{G}$ and $C_1, C_2 \in \mathcal{C}(\mathrm{Cr}_n)$, $C_1 \ne C_2$, such that $C_1 \cap C_2 \ne \emptyset$. Then C_1 and C_2 have opposite parities, say, $C_1 \in \mathcal{O}$ and $C_2 \in \mathcal{E}$. If $\theta(\mathcal{O}) = \mathcal{O}$, then $\theta(\mathcal{E}) = \mathcal{E}$ due to the bijectivity of θ. Analogously, if $\theta(\mathcal{O}) = \mathcal{E}$, then $\theta(\mathcal{E}) = \mathcal{O}$. So, in either case, $\theta(C_1)$ and $\theta(C_2)$ have opposite parities. Therefore, $\theta \in \mathcal{AM}(\mathrm{Cr}_n)$, by Lemma 4.9.

We now prove that $\mathcal{G} \cong (S_n \times S_n) \rtimes \mathbb{Z}_2$. Consider $\mathcal{H} = \{\theta \in \mathcal{M}(\mathrm{Cr}_n) : \theta(\mathcal{O}) = \mathcal{O}\}$. Clearly, $\mathcal{H}$ is a (normal) subgroup of $\mathcal{G}$ of index 2. Since $|\mathcal{O}| = |\mathcal{E}| = n$, we have $\mathcal{H} \cong S_n \times S_n$. Define $\theta \in \mathcal{M}(\mathrm{Cr}_n)$ as follows: $\theta(e_{x_iy_i}) = e_{x_{i+1}y_i}$ and $\theta(e_{x_{i+1}y_i}) = e_{x_iy_i}$ for $1 \le i \le n-1$, $\theta(e_{x_ny_n}) = e_{x_1y_n}$ and $\theta(e_{x_1y_n}) = e_{x_ny_n}$. By definition $\theta \in \mathcal{G} \setminus \mathcal{H}$ and θ has order 2. Therefore, $\mathcal{G} = \mathcal{H} \cdot \langle\theta\rangle \cong (S_n \times S_n) \rtimes \mathbb{Z}_2$. □

Proposition 4.11. *The group $\mathcal{P}(\mathrm{Cr}_n)$ is isomorphic to $\mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$.*

Proof. In view of Proposition 2.3, it suffices to prove that $\mathrm{Aut}^{\pm}(\mathrm{Cr}_n) \cong \mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$. To this end, we will show that $\mathrm{Aut}^{\pm}(\mathrm{Cr}_n) \cong D_{2n}$, where D_{2n} is the group of symmetries of a regular $2n$-gon which is known to be isomorphic to $\mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$. Denote x_i by u_{2i-1} and y_i by u_{2i}, for all $i = 1, \ldots, n$, identifying u_j with the j-th vertex of a regular $2n$-gon, whose vertices are indexed consecutively according to the counterclockwise orientation. For the sake of simplicity, we shall consider the indices modulo $2n$ in the rest of the proof.

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Given $\varphi \in \mathrm{Aut}^{\pm}(\mathrm{Cr}_n)$, set i_φ to be the integer modulo $2n$ such that $\varphi(u_{2n}) = u_{i_\varphi}$. Notice that if i_φ is even then $\varphi \in \mathrm{Aut}(\mathrm{Cr}_n)$, otherwise $\varphi \in \mathrm{Aut}^{-}(\mathrm{Cr}_n)$. In any case, since the only elements of Cr_n comparable with u_{2n}, besides itself, are u_{2n-1} and u_1, then $\varphi(u_1) = u_{i_\varphi \pm 1}$. If $\varphi(u_1) = u_{i_\varphi+1}$, then it can be easily shown inductively that $\varphi(u_j) = u_{i_\varphi+j}$ for any $j = 1, \ldots, 2n$. This corresponds to the counterclockwise rotation by an angle of $i_\varphi\pi/n$ in D_{2n}. If $\varphi(u_1) = u_{i_\varphi-1}$, again by an easy inductive argument, $\varphi(u_j) = u_{i_\varphi-j}$ for all $j = 1, \ldots, 2n$. If i_φ is even, φ corresponds to the reflection across the diagonal containing u_j and u_{j+n}, where $2j = i_\varphi$. Otherwise φ corresponds to the reflection across the line which contains the midpoints of the sides u_ju_{j+1} and $u_{j+n}u_{j+n+1}$, where $i_\varphi = 2j + 1$. Since i_φ can be any of the $2n$ indices of the vertices considered, all the $4n$ elements of D_{2n} ($2n$ rotations and $2n$ reflections) can occur as elements of $\mathrm{Aut}^{\pm}(\mathrm{Cr}_n)$ and we obtain the claimed isomorphism. □

Corollary 4.12. *We have* $\mathcal{P}(\mathrm{Cr}_2) = \mathcal{AM}(\mathrm{Cr}_2)$ *and* $\mathcal{P}(\mathrm{Cr}_n) \neq \mathcal{AM}(\mathrm{Cr}_n)$ *for all* $n > 2$.

Proof. Indeed, $|\mathcal{AM}(\mathrm{Cr}_2)| = |\mathcal{P}(\mathrm{Cr}_2)|$ and $|\mathcal{AM}(\mathrm{Cr}_n)| = 2(n!)^2 > 4n! > 4n = |\mathcal{P}(\mathrm{Cr}_2)|$ for $n > 2$. □

4.4. The case of the ordinal sum of two anti-chains

We will now proceed to the case of sets of length one which have as many crowns as possible.

Definition 4.13. *Given positive integers* m *and* n, *denote by* $\mathrm{K}_{m,n}$ *the poset* $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$, *where* $x_i < y_j$ *for all* $1 \le i \le m$ *and* $1 \le j \le n$, *and there is no other pair of distinct comparable elements.*

Observe that $\mathrm{K}_{m,n}$ is the ordinal sum [24] of two anti-chains of cardinalities m and n. The Hasse diagram of $\mathrm{K}_{m,n}$ is a *complete bipartite graph* [2], so that $\mathrm{Aut}(\mathrm{K}_{m,n}) \cong S_m \times S_n$. It is also clear that $\mathrm{K}_{m,n}$ is anti-isomorphic to $\mathrm{K}_{n,m}$, so we may assume that $m \le n$. The cases $m = 1$ and $m = n = 2$ (a 2-crown) were considered in Proposition 4.6 and Corollary 4.12.

Proposition 4.14. *Let* $2 \le m \le n$. *Then* $\mathcal{P}(\mathrm{K}_{m,n}) = \mathcal{AM}(\mathrm{K}_{m,n})$.

Proof. Let $\theta \in \mathcal{AM}(\mathrm{K}_{m,n})$. Fix $j \in \{1, \ldots, n\}$ and write $\theta(e_{x_iy_j}) = e_{u_iv_i}$, $1 \le i \le m$. Denote by U_j and V_j the sets of all u_i and v_k, respectively. We first prove that for any pair of $u_i \in U_j$ and $v_k \in V_j$ there is l such that $\theta(e_{x_ly_j}) = e_{u_iv_k}$. This is trivial if $u_i = u_k$ or $v_i = v_k$, so let $u_i \neq u_k$ and $v_i \neq v_k$. Consider the cycle $\Gamma : u_i < v{:}i > u_k < v{:}k > u_i$. We have $s^{\pm}_{\theta,\Gamma}(y_j) = 0$ and $t^{+}_{\theta,\Gamma}(y_j) = 2$. Since θ is admissible, we must have $t^{-}_{\theta,\Gamma}(y_j) = 2$. But this means that $\theta(e_{x_ly_j}) = e_{u_iv_k}$ for some $1 \le l \le m$ (and $\theta(e_{x_py_j}) = e_{u_kv_i}$ for some $1 \le p \le m$), as desired. As a consequence, we obtain $|U_j| \cdot |V_j| = m$.

We now prove that $|U_j| = 1$ or $|V_j| = 1$. Assume that $|U_j| \ge 2$ and $|V_j| \ge 2$. Since $|U_j| \cdot |V_j| = m$, we conclude that $|U_j| \le \frac{m}{2}$ and $|V_j| \le \frac{m}{2} \le \frac{n}{2}$. It follows that there exist $z < w$ such that $z \notin U_j$ and $w \notin V_j$. Consider the cycle $\Gamma : u_1 < v_1 > zu_1$. Clearly, $s^{\pm}_{\theta,\Gamma}(y_j) = 0$, $t^{+}_{\theta,\Gamma}(y_j) = 1$ and $t^{-}_{\theta,\Gamma}(y_j) = 0$, a contradiction.

Case 1. $|V_j| = 1$ for all $1 \le j \le n$. Then there exists a bijection λ of $\{y_1, \ldots, y_n\}$ such that $\{\theta(e_{x_1y_j}), \ldots, \theta(e_{x_my_j})\} = \{e_{x_1\lambda(y_j)}, \ldots, e_{x_m\lambda(y_j)}\}$ for all $1 \le j \le n$. We will prove that $\theta(e_{x_iy_j}) = e_{u_i\lambda(y_j)}$ and $\theta(e_{x_iy_k}) = e_{z_i\lambda(y_k)}$ imply $u_i = z_i$ for $j \neq k$. Suppose that $u_i \neq z_i$ and consider the cycle $\Gamma : u_i < \lambda(y_j) > z_i < \lambda(y_k) > u_i$. We have $t^{\pm}_{\theta,\Gamma}(x_i) = 0$, $s^{+}_{\theta,\Gamma}(x_i) = 2$ and $s^{-}_{\theta,\Gamma}(x_i) = 0$, a contradiction. Thus, there exists a bijection μ of $\{x_1, \ldots, x_m\}$ such that $\theta(e_{x_iy_j}) = e_{\mu(x_i)\lambda(y_j)}$ for all $1 \le i \le m$ and $1 \le j \le n$. But this means that θ corresponds to the automorphism of $\mathrm{K}_{m,n}$ acting as λ on $\mathrm{Max}(\mathrm{K}_{m,n})$ and as μ on $\mathrm{Min}(\mathrm{K}_{m,n})$. So, $\theta \in \mathcal{P}(\mathrm{K}_{m,n})$.

Case 2. $|U_j| = 1$ for some $1 \le j \le n$. Then $|V_j| = m$. We will prove that this is possible only if $m = n$. Assume that $m < n$ and let $U_j = \{p_j\}$. Since $\theta(e_{x_iy_j}) = e_{p_jv_i}$ for all $1 \le i \le m$ and $|V_j| < n$, there exist $z < w$ such that $z \neq p_j$ and $w \notin V_j$. Taking the cycle $\Gamma : p_j < v_1 > zp_j$, we obtain $s^{\pm}_{\theta,\Gamma}(y_j) = 0$, $t^{+}_{\theta,\Gamma}(y_j) = 1$ and $t^{-}_{\theta,\Gamma}(y_j) = 0$, a contradiction. Thus, $m = n$. We now prove that $|U_k| = 1$ *for all* $1 \le k \le n$. If $|U_k| \neq 1$, then $k \neq j$ and $|V_k| = 1$, say $V_k = \{q_k\}$. We have $\{\theta(e_{x_1y_j}), \ldots, \theta(e_{x_ny_j})\} = \{e_{p_jy_1}, \ldots, e_{p_jy_n}\}$ and $\{\theta(e_{x_1y_k}), \ldots, \theta(e_{x_ny_k})\} = \{e_{x_1q_k}, \ldots, e_{x_nq_k}\}$. Since $j \neq k$, these sets must be disjoint. But $e_{p_jq_k}$

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belongs to their intersection, a contradiction. Thus, $|U_k| = 1$ for all $1 \leq k \leq n$. Replacing θ by $\theta' \circ \theta$, where $\theta'(e_{xy}) = e_{\mu(y)\mu(x)}$ and μ is the anti-automorphism of X which interchanges x_i and y_i for all $1 \leq i \leq n$, we get the situation of Case 1, so that $\theta' \circ \theta \in \mathcal{P}(X)$. Since $\theta' \in \mathcal{P}(\mathrm{K}_{m,n})$, we conclude that $\theta \in \mathcal{P}(\mathrm{K}_{m,n})$. □

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