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ON GENERALIZED GAUDUCHON METRICS

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Abstract We study a class of Hermitian metrics on complex manifolds, recently introduced by Fu, Wang and Wu, which are a generalization of Gauduchon metrics. This class includes the class of Hermitian metrics for which the associated fundamental 2-form is $\partial\bar{\partial}$-closed. Examples are given on nilmanifolds, on products of Sasakian manifolds, on S^1-bundles and via the twist construction introduced by Swann.

Keywords: Hermitian structure; Gauduchon metric; Kähler manifolds

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1. Introduction

Let (M, J, g) be a Hermitian manifold of real dimension $2n$. If the fundamental 2-form $\Omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is d-closed, then the metric g is Kähler. In the literature, weaker conditions on Ω have been studied, and they involve the closure with respect to the $\partial\bar{\partial}$-operator of the (k,k)-form $\Omega^k = \Omega \wedge \cdots \wedge \Omega$.

If $\partial\bar{\partial}\Omega = 0$, then the Hermitian structure (J, g) is said to be *strong Kähler with torsion* and g is called *SKT* (or *pluriclosed*; see, for example, [**12**]). In this case the Hermitian structure is characterized by the condition that the torsion 3-form $c = Jd\Omega$ of the Bismut connection is d-closed. SKT metrics have recently been studied by many authors and they also have applications in type II string theory and in two-dimensional supersymmetric σ-models [**12**, **18**, **27**]. Moreover, they also have links with generalized Kähler structures (see, for instance, [**3**, **12**, **16**, **17**]). New simply connected SKT examples have been constructed by Swann [**28**] via the twist construction, by reproducing the six-dimensional examples found previously in [**14**]. Recently, Streets and Tian introduced a Hermitian Ricci flow under which the SKT condition is preserved [**26**].

If $\partial\bar{\partial}\Omega^{n-2} = 0$, then the Hermitian metric g on M is said to be *astheno-Kähler*. Jost and Yau used this condition in [**19**] to study Hermitian harmonic maps and to extend Siu's Rigidity Theorem to non-Kähler manifolds.

On a complex surface any Hermitian metric is automatically astheno-Kähler, and in complex dimension $n = 3$ the notion of astheno-Kähler metric coincides with that of SKT. Astheno–Kähler structures on products of Sasakian manifolds and then, in particular, on Calabi–Eckmann manifolds have been constructed in [**21**]. For $n > 3$ other examples of astheno-Kähler manifolds have been found in [**9**] via the twist construction [**28**], blow-ups and resolutions of orbifolds.

In [**8**] it was shown that the blow-up of an SKT manifold M at a point or along a compact complex submanifold Y is still SKT, as in the Kähler case (see, for example, [**4**]). Moreover, by [**9**] the same property holds for the blow-ups of complex manifolds endowed with a J-Hermitian metric such that $\partial\bar{\partial}\Omega = 0$ and $\partial\bar{\partial}\Omega^2 = 0$.

If Ω^{n-1} is $\partial\bar{\partial}$-closed or equivalently if the Lee form is co-closed, then the Hermitian metric g is called *standard* or a *Gauduchon metric* [**13**]. Recently, in [**11**] Fu *et al.* introduced a generalization of the Gauduchon metrics on complex manifolds. Let (M, J) be a complex manifold M of complex dimension n and let k be an integer such that $1 \leqslant k \leqslant n-1$; a J-Hermitian metric g on M is called *kth Gauduchon* if

$$\partial\bar{\partial}\Omega^k \wedge \Omega^{n-k-1} = 0. \tag{1.1}$$

Then, by definition, the notion of the $(n-1)$th Gauduchon metric coincides with that of the usual Gauduchon metrics. In [**11**], there is associated to any Hermitian structure Ω on a complex manifold (M, J) a unique constant $\gamma_k(\Omega)$, which is invariant by biholomorphisms and which depends smoothly on Ω. It is also proved that $\gamma_k(\Omega) = 0$ if and only if there exists a kth Gauduchon metric in the conformal class of Ω. Moreover, the first examples of first Gauduchon metrics were constructed in [**11**] on the three-dimensional complex manifolds constructed by Calabi and on $S^5 \times S^1$.

In this paper we will study first Gauduchon metrics, i.e. Hermitian metrics for which

$$\partial\bar{\partial}\Omega \wedge \Omega^{n-2} = 0.$$

This class of metrics includes the SKT metrics as particular case, so it is natural to study which of the properties that hold for the SKT metrics are still valid in the case of first Gauduchon metrics.

Astheno–Kähler and SKT metrics on compact complex manifolds cannot be balanced for $n > 2$ unless they are Kähler (see [**1**, **22**]), where by 'balanced' one means that the Lee form vanishes. If the Lee form is exact, then the Hermitian structure is called conformally balanced. By [**9**, **18**, **24**] a conformally balanced SKT (or astheno-Kähler) structure on a compact manifold of complex dimension n whose Bismut connection has (restricted) holonomy contained in $\mathrm{SU}(n)$ is necessarily Kähler. In § 2 we prove similar results for first Gauduchon metrics.

In complex dimension 3, invariant SKT structures on *nilmanifolds*, i.e. on compact quotients of nilpotent Lie groups by uniform discrete subgroups, were studied in [**10**, **30**], showing that the existence of such structures depends only on the left-invariant complex structure on the Lie group and that the Lie group is 2-step. Nilmanifolds of any even dimension endowed with a left-invariant complex structure J and an SKT J-Hermitian metric were studied in [**5**] showing that they are all 2-step and they are indeed the total space of holomorphic principal torus bundles over complex tori.

In § 3 we prove that on a six-dimensional nilmanifold endowed with an invariant complex structure J, an invariant J-Hermitian metric is first Gauduchon if and only if it is SKT. However, by using the results in [**11**], we show that there are complex six-dimensional nilmanifolds admitting non-invariant first Gauduchon metrics. A family J_t, $t \in (0, 1]$, of complex structures admitting compatible first Gauduchon metrics for any t and such that J_t admits SKT metrics only for $t = 1$ is also given on a six-dimensional nilmanifold. Moreover, we show that this family of complex structures on the previous nilmanifold does not have any compatible (invariant or not) balanced metric.

The situation is different in higher dimensions, since in § 4 we give examples in dimension 8 of complex nilmanifolds admitting invariant first Gauduchon metrics and no SKT metrics. These examples are products of (quasi-)Sasakian manifolds. We construct first Gauduchon metrics on S^1-bundles over quasi-Sasakian manifolds and on the product of two Sasakian manifolds endowed with the complex structure introduced in [**21**]. In § 5 we study the existence of first Gauduchon metrics on the blow-up of a complex manifold at a point or along a compact submanifold. In the last section, by using the twist construction [**28**], we find examples of simply connected six-dimensional compact complex manifolds endowed with first Gauduchon metrics.

2. First Gauduchon metrics and relation with the balanced condition

In this section we show that first Gauduchon metrics, like the SKT ones, are complementary to the balanced condition. Let us start by reviewing the definition and the properties of the kth Gauduchon metrics contained in [**11**].

We recall the following definition.

Definition 2.1 (Fu *et al.* [11]). Let (M, J) be a complex manifold of complex dimension n and let k be an integer such that $1 \leqslant k \leqslant n-1$. A J-Hermitian metric g on (M, J) is called *kth Gauduchon* if its fundamental 2-form Ω satisfies the condition (1.1).

For $k = n - 1$ one gets the classical standard metric. Moreover, according to the previous definition, an SKT metric is a first Gauduchon metric and an astheno-Kähler metric is a $(n-2)$th Gauduchon metric.

Extending the result proved by Gauduchon in [**13**] for standard metrics, it is shown in [**11**] that if (M, J, g, Ω) is an n-dimensional compact Hermitian manifold, then, for any integer $1 \leqslant k \leqslant n-1$, there exists a unique constant $\gamma_k(\Omega)$ and a (unique up to a constant) function $v \in \mathcal{C}^\infty(M)$ such that

$$\tfrac{1}{2}\mathrm{i}\partial\bar{\partial}(\mathrm{e}^v\Omega^k) \wedge \Omega^{n-k-1} = \gamma_k(\Omega)\mathrm{e}^v\Omega^n.$$

If (M, J, g, Ω) is Kähler, then $\gamma_k(\Omega) = 0$ and v is a constant function for any $1 \leqslant k \leqslant n-1$.

The constant $\gamma_k(\Omega)$ is invariant under biholomorphisms and by [**11**, Proposition 11] the sign of $\gamma_k(\Omega)$ is invariant in the conformal class of Ω.

To compute the sign of the constant $\gamma_k(\Omega)$ one can use the following.

Proposition 2.2 (Fu *et al.* [11]). *For a Hermitian structure (J, g, Ω) on an n-dimensional complex manifold (M, J), the number $\gamma_k(\Omega) > 0$ (respectively, $=0$, <0) if and only if there exists a metric $\tilde{g}$ in the conformal class of g such that*

$$\tfrac{1}{2}\mathrm{i}\partial\bar{\partial}\tilde{\Omega}^k \wedge \tilde{\Omega}^{n-k-1} > 0 \quad (\textit{respectively}, \ = 0, < 0),$$

where $\tilde{\Omega}$ is the fundamental 2-form associated to $(J, \tilde{g})$.

For $n = 3$, by [**11**, Theorem 6] one has the following.

Proposition 2.3 (Fu *et al.* [11]). *On any compact three-dimensional complex manifold there exists a Hermitian metric g such that its fundamental 2-form Ω has $\gamma_1(\Omega) > 0$.*

So, for $n = 3$, if one finds a Hermitian metric $\tilde{g}$ such that its fundamental 2-form $\tilde{\Omega}$ has $\gamma_1(\tilde{\Omega}) < 0$, then, by using [**11**, Corollary 10], there exists a first Gauduchon metric.

We recall that the Lee form of a Hermitian manifold (M, J, g) of complex dimension n is the 1-form

$$\theta = J * d * \Omega = Jd^*\Omega,$$

where d^* is the formal adjoint of d with respect to g. The formula $*\Omega = \Omega^{n-1}/(n-1)!$ implies that $d(\Omega^{n-1}) = \theta \wedge \Omega^{n-1}$. The Hermitian structure (J, g) is called *balanced* if θ vanishes or, equivalently, if $d(\Omega^{n-1}) = 0$, and such structures belong to the class W_3 in the well-known Gray–Hervella classification [**15**].

In real dimension 4, the SKT condition is equivalent to $d^*\theta = 0$, but in higher dimensions both the SKT condition and the astheno-Kähler condition on a compact Hermitian manifold are complementary to the balanced condition [**1**,**22**]. In the case of first Gauduchon metrics, we can prove the following.

Proposition 2.4. *Let (M, J) be a compact complex manifold of complex dimension $n \geqslant 3$. If g is a J-Hermitian metric that is first Gauduchon and balanced, then g is Kähler.*

Proof. For a compact Hermitian manifold (M, J, g), one has the two natural linear operators acting on differential forms:

$$L\varphi = \Omega \wedge \varphi,$$

and the adjoint operator L^* of L with respect to the global scalar product defined by

$$\langle \varphi, \psi \rangle = p! \int_M (\varphi, \psi) \,\mathrm{vol}_g,$$

where Ω is the fundamental 2-form of (J, g), (φ, ψ) is the pointwise g-scalar product and vol_g is the volume form. By [**22**], one has

$$L^{*3}(2\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega) = 96(n-2)[2d^*\theta + 2\|\theta\|^2 - \|T\|^2], \tag{2.1}$$

where θ is the Lee form, $d^*\theta$ its co-differential, $\|\theta\|$ its g-norm and T is the torsion of the Chern connection ∇^{C}.

By using the formula

$$L^{*r}L^s\varphi = L^sL^{*r}\varphi + \sum_{i=1}^{r} 4^i(i!)^2 \binom{s}{i}\binom{r}{i}\binom{n-p-s+r}{i} L^{s-i}L^{*r-i}\varphi,$$

which holds for any p-form φ and any positive integer $r \leqslant s$, one gets

$$\begin{aligned} L^{*n}(2\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega^{n-2}) &= L^{*n}(L^{n-3}(2\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega)) \\ &= 4^n\frac{n!}{3!}(n-3)!L^{*3}(2\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega). \end{aligned} \tag{2.2}$$

Therefore, if (J, g) is balanced, then $\theta = 0$ and, consequently, the first Gauduchon condition implies that $T = 0$, i.e. g is Kähler. □

A Hermitian structure is called *conformally balanced* if the Lee form θ is d-exact. By [**9**, **18**, **24**], a conformally balanced SKT (or astheno-Kähler) structure on a compact manifold of complex dimension n whose Bismut connection has (restricted) holonomy contained in $\mathrm{SU}(n)$ is necessarily Kähler. We can now prove a similar result for the first Gauduchon metrics.

Theorem 2.5. *A conformally balanced first Gauduchon structure (J, g) on a compact manifold of complex dimension $n \geqslant 3$ whose Bismut connection has (restricted) holonomy contained in $\mathrm{SU}(n)$ is necessarily Kähler and therefore it is a Calabi–Yau structure.*

Proof. Since the Hermitian structure is first Gauduchon, by (2.1) and (2.2) we have

$$2d^*\theta + 2\|\theta\|^2 - \|T\|^2 = 0.$$

Therefore,

$$d^*\theta = \tfrac{1}{2}\|T\|^2 - \|\theta\|^2. \tag{2.3}$$

By [**1**, (2.11)], the trace $2u$ of the Ricci form of the Chern connection is related to the trace b of the Ricci form of the Bismut connection by the equation

$$2u = b + 2d^*\theta + 2\|\theta\|^2. \tag{2.4}$$

Since the condition that the Bismut connection has (restricted) holonomy contained in $\mathrm{SU}(n)$ implies that the Ricci form of the Bismut connection vanishes, by using (2.3) and (2.4), we obtain

$$2u = \|T\|^2.$$

Now, if $u > 0$ then it follows from [**18**, Theorem 4.1 and (4.4)] that all plurigenera of (M, J) vanish. But, since (J, g) is conformally balanced, there exists a nowhere-vanishing holomorphic $(n, 0)$-form by [**24**, **27**]. Therefore, u must vanish and g is Kähler. □

3. Generalized Gauduchon metrics on 6-nilmanifolds

In this section we study first Gauduchon metrics on complex nilmanifolds $(M = \Gamma\backslash G, J)$ of real dimension 6 endowed with an invariant complex structure J, that is, G is a simply connected nilpotent Lie group and Γ is a lattice in G of maximal rank, and J arises from a left-invariant complex structure on the Lie group G.

In [**25**] Salamon proved that, up to isomorphism, there are exactly 18 nilpotent Lie algebras of real dimension 6 admitting complex structures. In [**30**] it was shown that in fact there are two special and disjoint types of complex equations for the six-dimensional nilpotent Lie algebras, depending on the 'nilpotency' of the complex structure. We recall that a complex structure J on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called nilpotent if there is a basis of $(1,0)$-forms $\{\omega_j\}_{j=1}^n$ satisfying $d\omega^1 = 0$ and

$$d\omega^j \in \Lambda^2\langle\omega^1, \dots, \omega^{j-1}, \bar\omega^1, \dots, \bar\omega^{j-1}\rangle, \quad j = 2, \dots, n.$$

In dimension 6, one has the following.

Proposition 3.1 (Ugarte [30]; Ugarte and Villacampa [31]). *Let J be an invariant complex structure on a nilmanifold M of real dimension 6.*

(i) *If J is nilpotent, then there is a global basis $\{\omega^j\}_{j=1}^3$ of invariant $(1,0)$-forms on M satisfying*

$$\left.\begin{aligned} d\omega^1 &= 0,\\ d\omega^2 &= \epsilon\omega^{1\bar1},\\ d\omega^3 &= \rho\omega^{12} + (1-\epsilon)A\omega^{1\bar1} + B\omega^{1\bar2} + C\omega^{2\bar1} + (1-\epsilon)D\omega^{2\bar2}, \end{aligned}\right\} \tag{3.1}$$

where $A, B, C, D \in \mathbb{C}$, and $\epsilon, \rho \in \{0, 1\}$.

(ii) *If J is non-nilpotent, then there is a global basis $\{\omega^j\}_{j=1}^3$ of invariant $(1,0)$-forms on M satisfying*

$$\left.\begin{aligned} d\omega^1 &= 0,\\ d\omega^2 &= \omega^{13} + \omega^{1\bar3},\\ d\omega^3 &= \mathrm{i}\epsilon\omega^{1\bar1} \pm \mathrm{i}(\omega^{1\bar2} - \omega^{2\bar1}), \end{aligned}\right\} \tag{3.2}$$

with $\epsilon = 0, 1$.

The fundamental 2-form Ω of any invariant J-Hermitian metric g on M is then given by

$$\Omega = \sum_{j,k=1}^{3} x_{j\bar k}\omega^{j\bar k}, \tag{3.3}$$

where $x_{j\bar k} \in \mathbb{C}$ and $\bar x_{k\bar j} = -x_{j\bar k}$. Note that the positive definiteness of the metric g implies in particular that

$$-\mathrm{i}x_{j\bar j} \in \mathbb{R}^+, \quad \mathrm{i}\det(x_{j\bar k}) > 0.$$

The following result is proved by a direct calculation, so we omit the proof.

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Lemma 3.2. *Let (J, g) be a Hermitian structure on a six-dimensional nilpotent Lie algebra $\mathfrak{g}$, and let Ω be its fundamental form.*

(i) *If J is nilpotent, then, in terms of the basis $\{\omega^j\}_{j=1}^3$ satisfying (3.1), the $(2,2)$-form $\partial\bar\partial\Omega$ is given by*

$$\partial\bar\partial\Omega = x_{3\bar 3}(\rho + |B|^2 + |C|^2 - 2(1-\epsilon)\operatorname{Re}(A\bar D))\omega^{1\bar 1 2\bar 2}.$$

This implies that

$$\tfrac{1}{2}\mathrm{i}\partial\bar\partial\Omega \wedge \Omega = \tfrac{1}{2}\mathrm{i}\frac{x_{3\bar 3}^2}{\det(x_{j\bar k})}(\rho + |B|^2 + |C|^2 - 2(1-\epsilon)\operatorname{Re}(A\bar D))\Omega^3,$$

and therefore the sign of $\gamma_1(\Omega)$ depends only on the complex structure.

(ii) *If J is non-nilpotent, then in terms of the basis $\{\omega^j\}_{j=1}^3$ satisfying (3.2), the form $\partial\bar\partial\Omega$ is given by*

$$\partial\bar\partial\Omega = 2x_{3\bar 3}\omega^{1\bar 1 2\bar 2} + 2x_{2\bar 2}\omega^{1\bar 1 3\bar 3}.$$

This implies that

$$\tfrac{1}{2}\mathrm{i}\partial\bar\partial\Omega \wedge \Omega = \mathrm{i}\frac{x_{2\bar 2}^2 + x_{3\bar 3}^2}{\det(x_{j\bar k})}\Omega^3,$$

and therefore $\gamma_1(\Omega) > 0$ for any Ω.

As an immediate consequence we get the following.

Proposition 3.3. *Let (M, J) be a complex nilmanifold of real dimension 6 endowed with an invariant complex structure J. An invariant J-Hermitian metric on M is first Gauduchon if and only if it is SKT.*

The classification of nilmanifolds of real dimension 6 admitting an invariant SKT metric is given in [**10**]. It is clear from Lemma 3.2 that J has to be nilpotent and ϵ must be zero in order to have an invariant Hermitian structure Ω such that $\gamma_1(\Omega) < 0$. By [**30**], it follows that if $\epsilon = 0$ and J is not bi-invariant, then there is a basis $\{\omega^j\}_{j=1}^3$ such that

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \rho\omega^{12} + \omega^{1\bar 1} + B\omega^{1\bar 2} + (x + \mathrm{i}y)\omega^{2\bar 2}, \tag{3.4}$$

where $B, x + \mathrm{i}y \in \mathbb{C}$, and $\rho = 0, 1$. The underlying real Lie algebra $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:

$$\begin{aligned}
&\mathfrak{h}_2 = (0,0,0,0,12,34), && \mathfrak{h}_3 = (0,0,0,0,0,12+34),\\
&\mathfrak{h}_4 = (0,0,0,0,12,14+23), && \mathfrak{h}_5 = (0,0,0,0,13+42,14+23),\\
&\mathfrak{h}_6 = (0,0,0,0,12,13), && \mathfrak{h}_8 = (0,0,0,0,0,12),
\end{aligned}$$

where, for instance, by $(0,0,0,0,0,12)$ we denote the Lie algebra with structure equations $de^j = 0$, $j = 1, \dots, 5$, $de^6 = e^1 \wedge e^2$. The classes of isomorphisms can be distinguished by the following conditions.

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(a) If $|B| = \rho$, then the Lie algebra $\mathfrak{g}$ is isomorphic to

(1) $\mathfrak{h}_2$, for $y \neq 0$,

(2) $\mathfrak{h}_3$, for $\rho = y = 0$ and $x \neq 0$,

(3) $\mathfrak{h}_4$, for $\rho = 1$, $y = 0$ and $x \neq 0$,

(4) $\mathfrak{h}_6$, for $\rho = 1$ and $x = y = 0$,

(5) $\mathfrak{h}_8$, for $\rho = x = y = 0$.

(b) If $|B| \neq \rho$, then the Lie algebra $\mathfrak{g}$ is isomorphic to

(1) $\mathfrak{h}_2$, for $4y^2 > (\rho - |B|^2)(4x + \rho - |B|^2)$,

(2) $\mathfrak{h}_4$, for $4y^2 = (\rho - |B|^2)(4x + \rho - |B|^2)$,

(3) $\mathfrak{h}_5$, for $4y^2 < (\rho - |B|^2)(4x + \rho - |B|^2)$.

Remark 3.4. Note that $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa manifold. The bi-invariant complex structure J_0 on $\mathfrak{h}_5$ corresponds to $\rho = 1$ and $\epsilon = A = B = C = D = 0$ in (3.1). It is proved in [**11**] that the natural balanced metric on the Iwasawa manifold has $\gamma_1 > 0$, and by Lemma 3.2 (i) we see that the same holds for any other invariant J_0-Hermitian structure Ω on the Iwasawa manifold.

Proposition 3.5. *Let (M, J) be a complex nilmanifold of real dimension 6 endowed with an invariant complex structure J. Then, there is an invariant J-Hermitian metric Ω such that $\gamma_1(\Omega) < 0$ if and only if M corresponds to $\mathfrak{h}_2$, $\mathfrak{h}_3$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$.*

Proof. If J is bi-invariant, then $\gamma_1(\Omega)$ cannot be negative, for any invariant Ω. By the previous discussion J must be nilpotent and there exists a basis of invariant $(1,0)$-forms satisfying the reduced equations (3.4). From Lemma 3.2 (i) it follows that there is an invariant J-Hermitian structure Ω such that $\gamma_1(\Omega) < 0$ if and only if

$$2x > \rho + |B|^2. \tag{3.5}$$

Since this implies $x > 0$, the Lie algebras $\mathfrak{h}_6$ and $\mathfrak{h}_8$ are excluded because they correspond to cases (a4) and (a5) above. Given ρ and B, we can choose $x, y \in \mathbb{R}$ satisfying (3.5) and one of the conditions (b1), (b2) or (b3). This shows the existence of a Ω with $\gamma_1(\Omega) < 0$ on the Lie algebras $\mathfrak{h}_2, \mathfrak{h}_4$ and $\mathfrak{h}_5$. Finally, the result for $\mathfrak{h}_3$ follows from (a2). □

Given a complex nilmanifold (M, J) with invariant J, the existence of an SKT or a balanced Hermitian metric implies the existence of an invariant one obtained by the symmetrization process [**7**]. From Propositions 2.3 and 3.5 we conclude the following.

Theorem 3.6. *Let M be a nilmanifold of real dimension 6 with underlying Lie algebra isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_3$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$. Then, there is an invariant complex structure J on M admitting a (non-invariant) first Gauduchon metric and having no J-Hermitian SKT metrics.*

The following example gives a deformation of an SKT structure in a family of complex structures that do not admit any compatible SKT metric, but have compatible first Gauduchon metrics.

Example 3.7. Let M be a nilmanifold with underlying Lie algebra $\mathfrak{h}_4$, and let $\{e^1, \dots, e^6\}$ be a basis of invariant 1-forms on M such that

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12}, \quad de^6 = e^{14} + e^{23}.$$

We consider the family of complex structures J_t, $t \neq 0$, defined by

$$\omega^1 = e^1 + \mathrm{i}e^4, \qquad \omega^2 = e^2 + \mathrm{i}t(e^3 - e^4), \qquad \omega^3 = 2(e^5 - \mathrm{i}e^6).$$

Since the complex equations for J_t are

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + \frac{1}{t}\omega^{2\bar{2}}, \tag{3.6}$$

by (a3) above the complex structure J_t is defined on M for any $t \neq 0$. Moreover, by Lemma 3.2 (i) the complex nilmanifold (M, J_t) has a compatible SKT metric if and only if $t = 1$, and admits an invariant Hermitian metric Ω with $\gamma_1(\Omega) < 0$ if and only if $t < 1$.

Therefore, J_t, $t \in (0, 1]$, is a deformation of the complex structure J_1 such that the complex manifold (M, J_t) has a first Gauduchon metric for any t but admits a compatible SKT metric only for $t = 1$.

On the other hand, because of the form of the complex equations (3.6), the fundamental form of any invariant J_t-Hermitian metric is equivalent to that given by (3.3) with $x_{1\bar{3}} = x_{2\bar{3}} = 0$. Then, the balanced condition $d\Omega^2 = 0$ reduces to

$$x_{1\bar{1}} + t x_{2\bar{2}} = t x_{1\bar{2}}.$$

Since $x_{1\bar{1}} = \mathrm{i}\lambda$ and $x_{2\bar{2}} = \mathrm{i}\mu$, for some $\lambda, \mu > 0$, we have that $x_{1\bar{2}} = \mathrm{i}(\mu + \lambda/t)$. This implies that

$$\det \begin{pmatrix} x_{1\bar{1}} & x_{1\bar{2}} \\ -\bar{x}_{2\bar{1}} & x_{2\bar{2}} \end{pmatrix} = \left(\mu + \frac{\lambda}{t}\right)^2 - \lambda\mu = \mu^2 + \frac{2-t}{t}\lambda\mu + \frac{\lambda^2}{t^2}$$

is positive for any $t \in (0, 1]$, which is a contradiction to the positive definiteness of the metric. By [**7**] we conclude that for any $t \in (0, 1]$ the complex manifold (M, J_t) does not admit any balanced (invariant or not) metric.

4. Products of Sasakian manifolds and circle bundles

Here we construct first Gauduchon metrics on products of Sasakian manifolds and on certain circle bundles over quasi-Sasakian manifolds.

We recall that an almost contact metric manifold $(N^{2n-1}, \varphi, \xi, \eta, g)$ is called *quasi-Sasakian* if it is normal and its fundamental form $\Phi(\cdot, \cdot) = g(\varphi\cdot, \cdot)$ is closed. If, in particular, $d\eta = \Phi$, then the almost contact metric structure is said to be *Sasakian*.

Let $(M_1, \varphi_1, \xi_1, \eta_1, g_1)$ and $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$ be two Sasakian manifolds of dimensions $2n_1 + 1$ and $2n_2 + 1$, respectively. On the product manifold $M = M_1 \times M_2$ we consider the family of complex structures J given by Tsukada [**29**] and used in [**21**] to construct astheno-Kähler structures:

$$J(X,Y) = \left(\varphi_1(X) - \frac{a}{b}\eta_1(X)\xi_1 - \frac{a^2+b^2}{b}\eta_2(Y)\xi_1, \varphi_2(Y) + \frac{1}{b}\eta_1(X)\xi_2 + \frac{a}{b}\eta_2(Y)\xi_2\right),$$

for $X \in \mathfrak{X}(M_1)$ and $Y \in \mathfrak{X}(M_2)$, where $a, b \in \mathbb{R}$ and $b \neq 0$.

For any $t \in \mathbb{R}^*$ such that $t/b > 0$, we consider on M the J-Hermitian metric

$$\Omega = \Phi_1 + \Phi_2 + t\eta_1 \wedge \eta_2, \tag{4.1}$$

where Φ_j denotes the fundamental 2-form on $(M_j, \varphi_j, \xi_j, \eta_j, g_j)$ for each $j = 1, 2$.

Theorem 4.1. *Let us suppose that $2n = 2(n_1 + n_2 + 1) > 6$. The Hermitian structure Ω given by (4.1) on M is first Gauduchon if and only if it is astheno-Kähler if and only if $n_1(n_1 - 1) + 2an_1n_2 + (a^2 + b^2)n_2(n_2 - 1) = 0$.*

Proof. Since $d\Omega = t(\Phi_1 \wedge \eta_2 - \eta_1 \wedge \Phi_2)$, we have that

$$d^c\Omega = JdJ\Omega = Jd\Omega = tJ(\Phi_1 \wedge \eta_2 - \eta_1 \wedge \Phi_2).$$

The definition of the complex structure implies that

$$J\eta_1 = \frac{a}{b}\eta_1 + \frac{a^2+b^2}{b}\eta_2 \quad \text{and} \quad J\eta_2 = -\frac{1}{b}\eta_1 - \frac{a}{b}\eta_2,$$

which imply

$$\begin{aligned} d^c\Omega &= t(\Phi_1 \wedge J\eta_2 - J\eta_1 \wedge \Phi_2) \\ &= -\frac{t}{b}[\Phi_1 \wedge \eta_1 + a\Phi_1 \wedge \eta_2 + a\eta_1 \wedge \Phi_2 + (a^2+b^2)\eta_2 \wedge \Phi_2]. \end{aligned}$$

Thus,

$$dd^c\Omega = \frac{t}{b}[\Phi_1 \wedge \Phi_1 + 2a\Phi_1 \wedge \Phi_2 + (a^2+b^2)\Phi_2 \wedge \Phi_2].$$

On the other hand,

$$\begin{aligned} \Omega^{n-2} &= \sum_{r=0}^{n-2} \binom{n-2}{r} (\Phi_1 + \Phi_2)^{n-2-r} \wedge (t\eta_1 \wedge \eta_2)^r \\ &= (\Phi_1 + \Phi_2)^{n-3} \wedge [\Phi_1 + \Phi_2 + t(n-2)\eta_1 \wedge \eta_2] \\ &= \sum_{s=0}^{n-3} \binom{n-3}{s} \Phi_1^{n-3-s} \wedge \Phi_2^s \wedge [\Phi_1 + \Phi_2 + t(n-2)\eta_1 \wedge \eta_2], \end{aligned}$$

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because $(\eta_1 \wedge \eta_2)^r$ is non-zero only for $r = 0, 1$. Therefore,

$$dd^c\Omega \wedge \Omega^{n-2} = \frac{t}{b}\sum_{s=0}^{n-3}\binom{n-3}{s}[\Phi_1^2 + 2a\Phi_1 \wedge \Phi_2 + (a^2+b^2)\Phi_2^2] \wedge \Phi_1^{n-3-s} \wedge \Phi_2^s \wedge [\Phi_1 + \Phi_2 + t(n-2)\eta_1 \wedge \eta_2]$$
$$= \frac{t}{b}\sum_{s=0}^{n-1} C(n,s)\Phi_1^{n-1-s} \wedge \Phi_2^s \wedge [\Phi_1 + \Phi_2 + t(n-2)\eta_1 \wedge \eta_2],$$

where $C(n,0) = 1$, $C(n,1) = n-3+2a$, $C(n,n-2) = 2a+(a^2+b^2)(n-3)$, $C(n,n-1) = a^2+b^2$ and

$$C(n,s) = \binom{n-3}{s} + 2a\binom{n-3}{s-1} + (a^2+b^2)\binom{n-3}{s-2},$$

for $2 \leqslant s \leqslant n-3$.

Now, by the same argument as in [**21**] we get that

$$dd^c\Omega \wedge \Omega^{n-2} = \frac{t}{b}C(n,n_2)\Phi_1^{n_1} \wedge \Phi_2^{n_2} \wedge [\Phi_1 + \Phi_2 + t(n-2)\eta_1 \wedge \eta_2]$$
$$= \frac{t^2}{b}(n-2)C(n,n_2)\Phi_1^{n_1} \wedge \Phi_2^{n_2} \wedge \eta_1 \wedge \eta_2.$$

It is clear that Ω is first Gauduchon if and only if $C(n,n_2) = 0$, which is equivalent to $n_1(n_1-1) + 2an_1n_2 + (a^2+b^2)n_2(n_2-1) = 0$. The latter condition is also equivalent to Ω being astheno-Kähler by [**21**, Theorem 4.1]. □

Remark 4.2. By a calculation similar to that in the proof of the previous theorem, for $n > 3$ we get that

$$\Omega^n = nt\left[\binom{n-3}{n_2} + 2\binom{n-3}{n_2-1} + \binom{n-3}{n_2-2}\right]\Phi_1^{n_1} \wedge \Phi_2^{n_2} \wedge \eta_1 \wedge \eta_2,$$

and using $dd^c\Omega = -2\mathrm{i}\partial\bar\partial\Omega$ we conclude that if the metric Ω is not first Gauduchon, then

$$\tfrac{1}{2}\mathrm{i}\partial\bar\partial\Omega \wedge \Omega^{n-2} = \frac{(n-2)t}{nb}\left[\frac{n_1(n_1-1) + 2n_1n_2a + n_2(n_2-1)(a^2+b^2)}{n_1(n_1-1) + 2n_1n_2 + n_2(n_2-1)}\right]\Omega^n.$$

Proposition 4.3. *Let us suppose that $n_1 = n_2 = 3$, that is, the real dimension of M is equal to $2n = 6$. Then, the Hermitian metric Ω given by (4.1) is first Gauduchon if and only if it is SKT if and only if $a = 0$. Moreover, $\gamma_1(\Omega) < 0$ if and only if $a < 0$, which implies the existence of a first Gauduchon metric on (M, J_a) for any $a < 0$ whenever the manifold is compact.*

Proof. Since M_j is three dimensional,

$$dd^c\Omega = \frac{2at}{b}\Phi_1 \wedge \Phi_2 \quad \text{and} \quad dd^c\Omega \wedge \Omega = \frac{2at^2}{b}\Phi_1 \wedge \Phi_2 \wedge \eta_1 \wedge \eta_2.$$

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Since $\Omega^3 = 6t\Phi_1 \wedge \Phi_2 \wedge \eta_1 \wedge \eta_2$ and $dd^c\Omega = -2\mathrm{i}\partial\bar{\partial}\Omega$, we conclude that

$$\tfrac{1}{2}\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega = \frac{at}{3b}\Omega^3.$$

The last assertion follows from Proposition 2.3 and [**11**, Corollary 10]. □

Next we construct first Gauduchon metrics on circle bundles over certain almost contact metric manifolds. Let $(N, \varphi, \xi, \eta, g)$ be a $(2n-1)$-dimensional almost contact metric manifold and let F be a closed 2-form on N which represents an integral cohomology class on N. From the well-known result of Kobayashi [**20**], we can consider the circle bundle $S^1 \hookrightarrow P \to N$, with connection 1-form θ on P whose curvature form is $d\theta = \pi^*(F)$, where $\pi : P \to N$ is the projection.

By using a normal almost contact structure (φ, ξ, η) on N and a connection 1-form θ on P such that

$$F(\varphi X, Y) + F(X, \varphi Y) = 0, \quad F(\xi, X) = 0, \quad \text{for all } X, Y \in \chi(N),$$

one can define a complex structure J on P as follows (see [**23**]). For any right-invariant vector field X on P, JX is given by

$$\theta(JX) = -\pi^*(\eta(\pi_* X)), \qquad \pi_*(JX) = \varphi(\pi_* X) + \tilde{\theta}(X)\xi, \tag{4.2}$$

where $\tilde{\theta}(X)$ is the unique function on N such that $\pi^*\tilde{\theta}(X) = \theta(X)$.

The above definition can be extended to arbitrary vector fields X on P, since X can be written in the form $X = \sum_j f_j X_j$, with f_j smooth functions on P and X_j right-invariant vector fields. Then $JX = \sum_j f_j JX_j$.

Moreover, a Riemannian metric h on P compatible with J (see [**23**]) is given by

$$h(X, Y) = \pi^* g(\pi_* X, \pi_* Y) + \theta(X)\theta(Y), \tag{4.3}$$

for any right-invariant vector fields X, Y.

It was shown in [**6**] that if $(N, \varphi, \xi, \eta, g)$ is quasi-Sasakian, h is SKT if and only if

$$d\eta \wedge d\eta + F \wedge F = 0.$$

For first Gauduchon metrics we prove the following.

Theorem 4.4. *Let $(N, \varphi, \xi, \eta, g)$ be a $(2n-1)$-dimensional quasi-Sasakian manifold, $n > 2$, and let F be a closed 2-form on N that represents an integral cohomology class. Consider the circle bundle $S^1 \hookrightarrow P \xrightarrow{\pi} N$ with connection 1-form θ whose curvature form is $d\theta = \pi^*(F)$. If $d\theta$ is J-invariant, then the almost Hermitian structure (J, h) on P, defined by (4.2) and (4.3), is first Gauduchon if and only if*

$$(d\eta \wedge d\eta + F \wedge F) \wedge \Phi^{n-3} = 0, \tag{4.4}$$

where Φ denotes the fundamental form of the quasi-Sasakian structure (φ, ξ, η, g).

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Proof. Since $\Omega = \pi^*\Phi + \pi^*\eta \wedge \theta$ is the fundamental 2-form associated to (J, h) and the structure is quasi-Sasakian, we have

$$Jd\Omega = J(\pi^*(d\eta)) \wedge J\theta - J(\pi^*(\eta)) \wedge J(d\theta).$$

Since $J\theta = \pi^*(\eta)$ and the almost contact structure is normal, we have

$$J(\pi^*(d\eta)) = \pi^*(d\eta), \quad d\eta(\xi, X) = 0, \quad \text{for all } X \in \chi(N),$$

and therefore

$$\begin{aligned} dJd\Omega &= d(\pi^*(d\eta) \wedge \pi^*(\eta) + \theta \wedge J(\pi^*(F))) \\ &= \pi^*(d\eta) \wedge \pi^*(d\eta) + d\theta \wedge J(\pi^*(F)) - \theta \wedge d[J(\pi^*(F))] \\ &= \pi^*(d\eta) \wedge \pi^*(d\eta) + \pi^*(F) \wedge \pi^*(F), \end{aligned}$$

where in the last equality we used the fact that $d\theta = \pi^*(F)$ is invariant by the complex structure J. Therefore,

$$\begin{aligned} dJd\Omega \wedge \Omega^{n-2} &= \pi^*(d\eta \wedge d\eta + F \wedge F) \wedge (\pi^*\Phi + \pi^*\eta \wedge \theta)^{n-2} \\ &= \pi^*(d\eta \wedge d\eta + F \wedge F) \wedge [\pi^*\Phi^{n-2} + (n-2)\pi^*(\Phi^{n-3} \wedge \eta) \wedge \theta] \\ &= (n-2)\pi^*[(d\eta \wedge d\eta + F \wedge F) \wedge \Phi^{n-3} \wedge \eta] \wedge \theta \end{aligned}$$

and $dJd\Omega \wedge \Omega^{n-2}$ vanishes if and only if (4.4) is satisfied. □

Example 4.5. Consider the five-dimensional solvable Lie group S with structure equations

$$\begin{aligned} de^i &= 0, \quad i = 1, 4, \\ de^2 &= e^{13}, \\ de^3 &= -e^{12}, \\ de^5 &= e^{14} + e^{23}, \end{aligned}$$

endowed with the left-invariant Sasakian structure (φ, ξ, η, g) given by

$$\varphi(e_1) = e_4, \qquad \varphi(e_2) = -e_3, \qquad \eta = e^5, \qquad g = \sum_{i=1}^{5} (e^i)^2.$$

By [**2**, Corollary 4.2] the solvable Lie group S admits a compact quotient by a uniform discrete subgroup Γ. If we consider the S^1-bundle (P, J) over the compact quotient $\Gamma \setminus S$ with connection 1-form θ such that

$$d\theta = \pi^*(2e^{14} - 2e^{23}),$$

we have that $\Omega = \phi^*\Phi + \pi^*\eta \wedge \theta$ is a J-Hermitian structure on the S^1-bundle over $\Gamma \setminus S$ such that $\gamma_1(\Omega) < 0$. Therefore, by Proposition 2.3 the complex manifold (P, J) admits a first Gauduchon metric.

Remark 4.6. As a consequence of Theorem 4.4 we have that on a trivial S^1-bundle over a Sasakian manifold the metric h compatible with the complex structure J given by (4.2) cannot be first Gauduchon.

We show next that in real dimension 8 there are non-SKT complex nilmanifolds having invariant first Gauduchon metrics. The complex equations

$$d\omega^1 = d\omega^2 = d\omega^3 = 0, \qquad d\omega^4 = A\omega^{1\bar{1}} - \omega^{2\bar{2}} - \omega^{3\bar{3}} \tag{4.5}$$

define a complex eight-dimensional nilmanifold (M_A, J_A) for any $A = p + \mathrm{i}q \in \mathbb{C}$. It is easy to see that M_A is a quotient of $H_3 \times H_5$ if $q \neq 0$, a quotient of $H_7 \times \mathbb{R}$ if $q = 0$ and a quotient of $H_5 \times \mathbb{R}^3$ if $p = q = 0$. Here we denote by H_{2n+1} the generalized real Heisenberg group of dimension $2n + 1$.

Lemma 4.7. *Let M be a nilmanifold of real dimension $2n$ endowed with an invariant complex structure J, and let Ω be any invariant J-Hermitian structure. For $k = 1, \dots, [\frac{1}{2}n] - 1$, the structure Ω is kth Gauduchon if and only if it is $(n-k-1)$th Gauduchon.*

Proof. For any $k = 1, \dots, [\frac{1}{2}n] - 1$, we use the formula

$$\int_M \partial\bar{\partial}\Omega^k \wedge \Omega^{n-k-1} = \int_M \partial\bar{\partial}\Omega^{n-k-1} \wedge \Omega^k, \tag{4.6}$$

which holds for a general compact complex manifold. Indeed,

$$\partial\bar{\partial}\Omega^k \wedge \Omega^{n-k-1} = d(\bar{\partial}\Omega^k \wedge \Omega^{n-k-1}) + \bar{\partial}\Omega^k \wedge \partial\Omega^{n-k-1}$$

and by Stokes's Theorem

$$\int_M \partial\bar{\partial}\Omega^k \wedge \Omega^{n-k-1} = \int_M \bar{\partial}\Omega^k \wedge \partial\Omega^{n-k-1}. \tag{4.7}$$

On the other hand, in a similar way we get

$$\int_M \partial\bar{\partial}\Omega^{n-k-1} \wedge \Omega^k = \int_M \bar{\partial}\Omega^k \wedge \partial\Omega^{n-k-1}. \tag{4.8}$$

Simplifying (4.7) and (4.8), we obtain (4.6).

Since Ω is invariant on the complex nilmanifold (M, J), there exist $\lambda, \mu \in \mathbb{R}$ such that $\frac{1}{2}\mathrm{i}\partial\bar{\partial}\Omega^k \wedge \Omega^{n-k-1} = \lambda\Omega^n$ and $\frac{1}{2}\mathrm{i}\partial\bar{\partial}\Omega^{n-k-1} \wedge \Omega^k = \mu\Omega^n$. Therefore,

$$\lambda \int_M \Omega^n = \tfrac{1}{2}\mathrm{i} \int_M \partial\bar{\partial}\Omega^k \wedge \Omega^{n-k-1} = \tfrac{1}{2}\mathrm{i} \int_M \partial\bar{\partial}\Omega^{n-k-1} \wedge \Omega^k = \mu \int_M \Omega^n,$$

which implies that $\lambda = 0$ if and only if $\mu = 0$, i.e. Ω is kth Gauduchon if and only if it is $(n-k-1)$th Gauduchon. □

Proposition 4.8. *Let (M_A, J_A) be the complex nilmanifold given by (4.5) and let*

$$\Omega = \sum_{j,k=1}^{4} x_{j\bar{k}} \omega^{j\bar{k}}$$

(where $x_{j\bar{k}} \in \mathbb{C}$ and $\bar{x}_{k\bar{j}} = -x_{j\bar{k}}$) be the fundamental 2-form of any invariant J_A-Hermitian metric g on M_A. Then, Ω is never SKT, and Ω is first Gauduchon if and only if it is second Gauduchon if and only if it is astheno-Kähler.

Moreover,

(i) *Ω is first Gauduchon if and only if*

$$p \det \begin{pmatrix} x_{2\bar{2}} & x_{2\bar{4}} \\ -\bar{x}_{2\bar{4}} & x_{4\bar{4}} \end{pmatrix} + p \det \begin{pmatrix} x_{3\bar{3}} & x_{3\bar{4}} \\ -\bar{x}_{3\bar{4}} & x_{4\bar{4}} \end{pmatrix} = \det \begin{pmatrix} x_{1\bar{1}} & x_{1\bar{4}} \\ -\bar{x}_{1\bar{4}} & x_{4\bar{4}} \end{pmatrix},$$

(ii) *Ω is balanced if and only if $q = 0$ and*

$$p \det \begin{pmatrix} x_{2\bar{2}} & x_{2\bar{3}} & x_{2\bar{4}} \\ -\bar{x}_{2\bar{3}} & x_{3\bar{3}} & x_{3\bar{4}} \\ -\bar{x}_{2\bar{4}} & -\bar{x}_{3\bar{4}} & x_{4\bar{4}} \end{pmatrix} = \det \begin{pmatrix} x_{1\bar{1}} & x_{1\bar{2}} & x_{1\bar{4}} \\ -\bar{x}_{1\bar{2}} & x_{2\bar{2}} & x_{2\bar{4}} \\ -\bar{x}_{1\bar{4}} & -\bar{x}_{2\bar{4}} & x_{4\bar{4}} \end{pmatrix} + \det \begin{pmatrix} x_{1\bar{1}} & x_{1\bar{3}} & x_{1\bar{4}} \\ -\bar{x}_{1\bar{3}} & x_{3\bar{3}} & x_{3\bar{4}} \\ -\bar{x}_{1\bar{4}} & -\bar{x}_{3\bar{4}} & x_{4\bar{4}} \end{pmatrix}.$$

In particular, if $p < 0$, then the complex manifold (M_A, J_A) does not admit either invariant balanced metrics or first Gauduchon metrics.

Proof. From the complex equations (4.5) if follows that Ω satisfies

$$\partial\bar{\partial}\Omega = x_{4\bar{4}}(A + \bar{A})\omega^{1\bar{1}2\bar{2}} + x_{4\bar{4}}(A + \bar{A})\omega^{1\bar{1}3\bar{3}} - 2x_{4\bar{4}}\omega^{2\bar{2}3\bar{3}},$$

which never vanishes because $x_{4\bar{4}} \neq 0$. Therefore, any invariant J_A-Hermitian metric is not SKT.

Moreover, a direct calculation shows that

$$\partial\bar{\partial}\Omega \wedge \Omega^2 = 2x_{4\bar{4}}[(A + \bar{A})(x_{2\bar{2}}x_{4\bar{4}} + x_{3\bar{3}}x_{4\bar{4}} + |x_{2\bar{4}}|^2 + |x_{3\bar{4}}|^2) - 2(x_{1\bar{1}}x_{4\bar{4}} + |x_{1\bar{4}}|^2)]\omega^{1\bar{1}2\bar{2}3\bar{3}4\bar{4}}$$

and therefore the metric is first Gauduchon if and only if

$$(A + \bar{A})(x_{2\bar{2}}x_{4\bar{4}} + x_{3\bar{3}}x_{4\bar{4}} + |x_{2\bar{4}}|^2 + |x_{3\bar{4}}|^2) - 2(x_{1\bar{1}}x_{4\bar{4}} + |x_{1\bar{4}}|^2) = 0. \tag{4.9}$$

It is easy to check that the condition for astheno-Kähler is satisfied if and only if (4.9) holds. This condition is precisely that given in (i), so the proof of (i) follows from Lemma 4.7.

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The balanced condition for Ω, i.e. $d\Omega^3 = 0$, can be seen to be equivalent to the system given by the equation

$$A \det \begin{pmatrix} x_{2\bar{2}} & x_{2\bar{3}} & x_{2\bar{4}} \\ -\bar{x}_{2\bar{3}} & x_{3\bar{3}} & x_{3\bar{4}} \\ -\bar{x}_{2\bar{4}} & -\bar{x}_{3\bar{4}} & x_{4\bar{4}} \end{pmatrix} = \det \begin{pmatrix} x_{1\bar{1}} & x_{1\bar{2}} & x_{1\bar{4}} \\ -\bar{x}_{1\bar{2}} & x_{2\bar{2}} & x_{2\bar{4}} \\ -\bar{x}_{1\bar{4}} & -\bar{x}_{2\bar{4}} & x_{4\bar{4}} \end{pmatrix} + \det \begin{pmatrix} x_{1\bar{1}} & x_{1\bar{3}} & x_{1\bar{4}} \\ -\bar{x}_{1\bar{3}} & x_{3\bar{3}} & x_{3\bar{4}} \\ -\bar{x}_{1\bar{4}} & -\bar{x}_{3\bar{4}} & x_{4\bar{4}} \end{pmatrix}$$

and its conjugate. The positive definiteness of the metric g implies in particular that

$$-\mathrm{i}x_{j\bar{j}}, \qquad -\det \begin{pmatrix} x_{j\bar{j}} & x_{j\bar{k}} \\ -\bar{x}_{j\bar{k}} & x_{k\bar{k}} \end{pmatrix}, \qquad \mathrm{i}\det \begin{pmatrix} x_{j\bar{j}} & x_{j\bar{k}} & x_{j\bar{l}} \\ -\bar{x}_{j\bar{k}} & x_{k\bar{k}} & x_{k\bar{l}} \\ -\bar{x}_{j\bar{l}} & -\bar{x}_{k\bar{l}} & x_{l\bar{l}} \end{pmatrix}$$

are strictly positive real numbers for $1 \leqslant j < k < l \leqslant 4$. Therefore, the balanced condition for Ω is satisfied if and only if A is real and (ii) holds.

Finally, the last assertion in the proposition follows easily from the positive definiteness of the metric g. □

Remark 4.9. The family (4.5) belongs to the more general family of astheno-Kähler manifolds given in [**9**].

As a consequence of the above proposition and [**7**], we get that the complex manifold (M_A, J_A) does not admit any SKT Hermitian structure, and if $p < 0$ or $q \neq 0$, then (M_A, J_A) does not possess any balanced Hermitian metric.

Corollary 4.10. *Let M be a compact quotient of $H_7 \times \mathbb{R}$ by a lattice of maximal rank. Then, there exists a complex structure on M that does not admit any SKT Hermitian structure, but has balanced Hermitian metrics and kth Gauduchon metrics for $k = 1$ and 2.*

Note that a result similar to the above corollary holds for $H_5 \times \mathbb{R}^3$, which corresponds to the case $A = 0$.

Corollary 4.11. *Let M be a compact quotient of $H_3 \times H_5$ by a lattice of maximal rank. Then, there exists a complex structure on M which does not admit either SKT or balanced Hermitian structure, but has invariant kth Gauduchon metrics for $k = 1$ and 2.*

Remark 4.12. Note that for $p = q = 0$ one has on M_A the complex structure coming from the product $H_5 \times \mathbb{R}^3$ of a Sasakian manifold with a cosymplectic one. If $p < 0$ (respectively, $p > 0$) and $q = 0$, then the complex structure comes from a product of a Sasakian (respectively, quasi-Sasakian) manifold with $\mathbb{R}$. Finally, for $q \neq 0$, the complex structure comes from a product $H_5 \times H_3$ of two quasi-Sasakian manifolds.

5. Blow-ups

In [**8**] it was shown that the SKT condition is preserved by the blow-up of the complex manifold at a point or along a compact submanifold. We can prove the following.

Proposition 5.1. *Let (M, J) be a complex manifold of complex dimension $n > 2$ and endowed with a J-Hermitian metric g such that its fundamental 2-form Ω satisfies the condition $\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega^{n-2} > 0$ (or < 0). Then, the blow-up $\tilde{M}_p$ at a point $p \in M$ and the blow-up $\tilde{M}_Y$ along a compact submanifold Y admit a Hermitian metric such that its fundamental 2-form $\tilde{\Omega}$ satisfies the condition $\mathrm{i}\partial\bar{\partial}\tilde{\Omega} \wedge \tilde{\Omega}^{n-2} > 0$ (or < 0).*

Proof. Let us start the proof in the case of the blow-up of M at a point p. We now briefly recall the construction of the blow-up. Let $z = (z_1, \dots, z_n)$ be holomorphic coordinates in an open set U centred around the point $p \in M$. The blow-up $\tilde{M}_p$ of M is the complex manifold obtained by adjoining to $M \setminus \{p\}$ the manifold

$$\tilde{U} = \{(z, l) \in U \times \mathbb{CP}^{n-1} \mid z \in l\}$$

by using the isomorphism

$$\tilde{U} \setminus \{z = 0\} \cong U \setminus \{p\}$$

given by the projection $(z, l) \to z$. In this way, there is a natural projection $\pi\colon \tilde{M}_p \to M$ extending the identity on $M \setminus \{p\}$ and the exceptional divisor $\pi^{-1}(p)$ of the blow-up is naturally isomorphic to the complex projective space $\mathbb{CP}^{n-1}$.

The 2-form $\pi^*\Omega$ is a $(1, 1)$ form on $\tilde{M}_p$ since π is holomorphic, but it is not positive definite on $\pi^{-1}(M \setminus \{p\})$. As in the Kähler case, let h be a $\mathcal{C}^\infty$-function having support in U, i.e. $0 \leqslant h \leqslant 1$ and $h = 1$ in a neighbourhood of p. On $U \times (\mathbb{C}^n \setminus \{0\})$ consider the 2-form

$$\gamma = \mathrm{i}\partial\bar{\partial}((p_1^* h) p_2^* \log \|\cdot\|^2),$$

where p_1 and p_2 denote the two projections of $U \times (\mathbb{C}^n \setminus \{0\})$ on U and $\mathbb{C}^n \setminus \{0\}$, respectively.

Let ψ be the restriction of γ to $\tilde{M}_p$. Then there exists a small enough real number ϵ such that the 2-form $\tilde{\Omega} = \epsilon\psi + \pi^*\Omega$ is positive definite.

Then

$$\partial\bar{\partial}\tilde{\Omega} \wedge \tilde{\Omega}^{n-2} = \partial\bar{\partial}\pi^*\Omega \wedge (\pi^*\Omega + \epsilon\psi)^{n-2}.$$

Since $\mathrm{i}\partial\bar{\partial}\pi^*\Omega \wedge \pi^*\Omega^{n-2} > 0$ (or < 0) by assumption we have that we can choose ϵ small enough in order to have $\mathrm{i}\partial\bar{\partial}\tilde{\Omega} \wedge \tilde{\Omega}^{n-2} > 0$ if $\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega^{n-2} > 0$ (or < 0 if $\mathrm{i}\partial\bar{\partial}\Omega \wedge \Omega^{n-2} < 0$).

For the case of the blow-up of (M, J) along a compact submanifold one proceeds in a similar way. Let $Y \subset M$ be a compact complex submanifold of M. Then consider the blow-up $\tilde{M}_Y$ of M along Y. If we denote by $\pi\colon \tilde{M}_Y \to M$ the holomorphic projection, by construction $\pi\colon \tilde{M} \setminus \pi^{-1}(Y) \to M \setminus Y$ is a biholomorphism and $\pi^{-1}(Y) \cong \mathbb{P}(\mathcal{N}_{Y|M})$, where $\mathbb{P}(\mathcal{N}_{Y|M})$ is the projectified normal bundle of Y. As in [**8**], we can show that there exists a holomorphic line bundle L on $\tilde{M}_Y$ such that L is trivial on $\tilde{M}_Y \setminus \pi^{-1}(Y)$ and such that its restriction to $\pi^{-1}(Y)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y|M})}(1)$.

Let h be a Hermitian structure on $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y|M})}(1)$ and let ω be the corresponding Chern form. As in [**8**], one can show that the metric h can be extended to a metric structure $\hat{h}$ on L in such a way that $\hat{h}$ is the flat metric structure on the complement of a compact neighbourhood W of Y induced by the trivialization of L on $\tilde{M}_Y \setminus \pi^{-1}(Y)$. Therefore, the Chern curvature $\hat{\omega}$ of L vanishes on $M \setminus W$ and $\hat{\omega}|_{\mathbb{P}(\mathcal{N}_{Y|M})} = \omega$.

Hence, since Y is compact, there exists $\epsilon \in \mathbb{R}$, $\epsilon > 0$, small enough such that

$$\tilde{\Omega} = \pi^* \Omega + \epsilon \hat{\omega}$$

is positive definite. As for the previous case of blow-up at a point, we can choose ϵ small enough in order to have $i\partial\bar{\partial}\tilde{\Omega} \wedge \tilde{\Omega}^{n-2} > 0$ if $i\partial\bar{\partial}\Omega \wedge \Omega^{n-2} > 0$ (or < 0 if $i\partial\bar{\partial}\Omega \wedge \Omega^{n-2} < 0$). □

If one applies the previous proposition to a compact complex manifold of complex dimension 3 endowed with a Hermitian structure (J, g, Ω) such that $\gamma_1(\Omega) < 0$, then by using Proposition 2.3 and Corollary 10 in [**11**] there exists a first Gauduchon metric on the complex blow-up at a point or along a compact submanifold.

6. Twists

In this section we will show that by applying the twist construction of [**28**, Proposition 4.5] one can get new simply connected six-dimensional complex manifolds which admit first Gauduchon (non-SKT) metrics.

We recall that, in general, given a manifold M with a T_M-torus action and a principal T_P-torus bundle P with connection θ, if the torus action of T_M lifts to P commuting with the principal action of T_P, then one may construct the twist W of the manifold, as the quotient of P/T_M by the torus action (see [**28**]). Moreover, if the lifted torus action preserves the principal connection θ, then tensors on M can be transferred to tensors on W if their pullbacks to P coincide on $\mathcal{H} = \operatorname{Ker}\theta$. A differential form α on M is *$\mathcal{H}$-related* to a differential form α_W on W, $\alpha \sim_{\mathcal{H}} \alpha_W$, if their pull-backs to P coincide on $\mathcal{H}$.

Now, let $A_M \cong A^m$ be a connected abelian group acting on M in such a way that there is a smooth twist W given via curvature $F \in \Omega^2(M, \mathfrak{a}_P)$ and invertible lifting function $a \in \Omega^0(M, \mathfrak{a}_M \otimes \mathfrak{a}_P^*)$, where $\mathfrak{a}_M$ (respectively, $\mathfrak{a}_P$) denotes the Lie algebra of A_M (respectively, $A_P \cong A^m$).

If the twist W has a Hermitian structure (J_W, g_W, Ω_W) induced by the one (J, g, Ω) on M, one has for the fundamental 2-forms and for the torsion 3-forms of the Bismut connections the following relations

$$\Omega_W \sim_{\mathcal{H}} \Omega, \quad c_W \sim_{\mathcal{H}} c - a^{-1} J_M F \wedge \xi^b,$$

where $\xi\colon \mathfrak{a}_M \to \chi(M)$ is the infinitesimal action and ξ^b is the dual of ξ by using the metric. In the case when F is of instanton type, i.e. if F is of type $(1,1)$, one has that

$$dc_W \sim_{\mathcal{H}} dc - a^{-1} F \wedge [i_\xi c + d\xi^b - g(\xi, \xi) a^{-1} F].$$

With the instanton construction, Swann obtained new examples of simply connected SKT manifolds. One can adapt the previous construction in the case of complex manifolds endowed with a first Gauduchon metric.

Let (N^{2n}, J) be a simply connected compact complex manifold of complex dimension $n \geqslant 2$ with a J-Hermitian metric $g_{N^{2n}}$ that is first Gauduchon. Consider the product $M^{2n+2} = N^{2n} \times \mathbb{T}^2$, where $\mathbb{T}^2$ is a 2-torus with an invariant Kähler structure. Then

M^{2n+2} has a first Gauduchon metric $g_{M^{2n+2}} = g_{N^{2n}} + g_{\mathbb{T}^2}$ with torsion 3-form c supported on N^{2n}.

Let ξ be the torus action on the $\mathbb{T}^2$ factor. We have then $\mathfrak{a}_M \cong \mathfrak{a}_P \cong \mathbb{R}^2$,

$$i_\xi c = 0, \quad d\xi^b = 0$$

and a is a constant isomorphism $\mathfrak{a}_M \to \mathfrak{a}_P$. Since

$$\Omega_W^{n-1} \sim_{\mathcal{H}} (\Omega_{N^{2n}} + \Omega_{\mathbb{T}^2})^{n-1} = \Omega_{N^{2n}}^{n-1} + (n-1)\Omega_{N^{2n}}^{n-2} \wedge \Omega_{\mathbb{T}^2},$$

we get

$$(dc_W \wedge \Omega_W^{n-1}) \sim_{\mathcal{H}} [dc + g(\xi,\xi)a^{-1}F \wedge a^{-1}F] \wedge [\Omega_{N^{2n}}^{n-1} + (n-1)\Omega_{N^{2n-2}}^{n-2} \wedge \Omega_{\mathbb{T}^2}]$$

and thus, by using the assumption that $g_{N^{2n}}$ is first Gauduchon, we obtain

$$(dc_W \wedge \Omega_W^{n-1}) \sim_{\mathcal{H}} g(\xi,\xi)a^{-1}F \wedge a^{-1}F \wedge (n-2)\Omega_{N^{2n}}^{n-2} \wedge \Omega_{\mathbb{T}^2}.$$

Assume that there are two linearly independent integral closed $(1,1)$-forms $F_i \in \Lambda_{\mathbb{Z}}^{1,1}(N^{2n})$, $i = 1, 2$, with $[F] \in H^2(N^{2n}, \mathbb{Z})$. If, for $n = 2$,

$$\left(\sum_{i,j=1}^{2} \gamma_{ij} F_i \wedge F_j\right) < 0$$

or, for $n > 3$,

$$\left(\sum_{i,j=1}^{2} \gamma_{ij} F_i \wedge F_j\right) \wedge \Omega_{N^{2n}}^{n-2} = 0,$$

for some positive definite matrix $(\gamma_{ij}) \in M_2(\mathbb{R})$, then by [**28**, Proposition 4.5] there is a compact simply connected $\mathbb{T}^2$-bundle $\tilde{W}$ over N^{2n} whose total space admits a first Gauduchon metric. Note that the condition

$$\sum_{i,j=1}^{2} \gamma_{ij} F_i \wedge F_j = 0$$

is equivalent to the condition that the first Gauduchon metric on $\tilde{W}$ is SKT.

The manifold $\tilde{W}$ is the universal covering of the twist W of $N^{2n} \times \mathbb{T}^2$, where the Kähler flat metric over $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ is given by the matrix (γ_{ij}) with respect to the standard generators with a compatible complex structure, and topologically W is a principal torus bundle over N^{2n} with Chern classes $[F_i]$.

Example 6.1. Consider as in [**28**, Example 4.6] a simply connected projective Kähler manifold N_0 of real dimension 4 and fix an imbedding of N_0 in $\mathbb{CP}^r$. Then, a generic linear subspace L of complex dimension $r-2$ in $\mathbb{CP}^r$ intersects N_0 transversely at a finite number of points $p_1, \dots, p_d$. One may choose homogeneous coordinates $[z_0, \dots, z_r]$ on $\mathbb{CP}^r$ in such a way that L is defined by $z_0 = z_1 = 0$. Now, let $\widetilde{\mathbb{CP}^r} \subset \mathbb{CP}^r \times \mathbb{CP}^1$ be the

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complex blow-up of $\mathbb{CP}^r$ along L, and denote by π_1 and π_2, respectively, the projections on the first and second factors of $\mathbb{CP}^r \times \mathbb{CP}^1$. As shown in [**28**, Example 4.6], if we define $N_1 = \pi^{-1}(N_0)$, then $f|_{N_1}$ defines a $(1,1)$-form F_1 on N_1 such that the cohomology class $[F_1]$ is non-zero and $F_1^2 = 0$. If one iterates this construction, one gets another non-zero cohomology class $[F_2]$ such that $F_2^2 = 0$. Now, since $F_1 \wedge F_2 \neq 0$, one can choose a positive matrix $(\gamma_{ij}) \in M_2(\mathbb{R})$ such that $\sum_{i,j=1}^{2} \gamma_{ij} F_i \wedge F_j < 0$. Then, by applying Proposition 2.3 and [**11**, Corollary 10], one gets a first Gauduchon metric on the universal covering of the twist of $N_1 \times \mathbb{T}^2$.

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