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p-DIVISIBILITY OF CO-DEGREES OF IRREDUCIBLE CHARACTERS

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Abstract

For a character χ of a finite group G, the co-degree of χ is $\chi^c(1) = [G : \ker\chi]/\chi(1)$. We study finite groups whose co-degrees of nonprincipal (complex) irreducible characters are divisible by a given prime p.

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1. Introduction and preliminaries

In this paper, G is a finite group and p is a prime number. Let $\mathrm{Irr}(G)$ denote the set of (complex) irreducible characters of G. For a normal subgroup N of G and $\theta \in \mathrm{Irr}(N)$, let $I_G(\theta)$ denote the inertia group of θ in G and let $\mathrm{Irr}(G|\theta)$ be the set of the irreducible constituents of the induced character θ^G. If n is a positive integer, n_p denotes the p-part of n. For a character χ of G, the number $\chi^c(1) = [G : \ker\chi]/\chi(1)$ is called the co-degree of χ (see [11]). Set $\mathrm{Codeg}(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$. In [1, 4, 11], some properties of the co-degrees of irreducible characters of finite groups have been studied. In 1970, Thompson [14] proved that if the degree of every nonlinear irreducible character of G is divisible by p, then G has a normal p-complement. Our first result is the following theorem.

THEOREM 1.1. *Suppose that p is neither 2 nor a Mersenne prime. Let G be p-solvable. Then $p \mid \chi^c(1)$ for every nonprincipal character $\chi \in \mathrm{Irr}(G)$ if and only if G is a p-group.*

In Examples 2.4, 2.5 and 2.6, we show that the hypotheses 'p-solvability of G' and 'p is neither 2 nor a Mersenne prime' in Theorem 1.1 are essential. We also prove a second result.

THEOREM 1.2. *Let e be a positive integer. Then $\chi^c(1)_p = p^e$ for every nonprincipal irreducible character χ of G if and only if $e = 1$ and G is an elementary abelian p-group.*

2. Proofs of the main results

We first state a lemma that will be used frequently in the proofs of the main results without explicit reference.

Lemma 2.1 [11, Lemma 2.1]. *Let N be a normal subgroup of G. Then* $\mathrm{Codeg}(G/N) \subseteq \mathrm{Codeg}(G)$. *Also, if $\psi \in \mathrm{Irr}(N)$, then $\psi^c(1) \mid \chi^c(1)$ for every $\chi \in \mathrm{Irr}(G|\psi)$.*

Lemma 2.2. *Let $N = S_1 \times \cdots \times S_t$ be a minimal normal subgroup of G, where $S_i \cong S$ is a nonabelian simple group. Then there exists $\varphi \in \mathrm{Irr}(N)$ that extends to G and* $\ker\varphi = \{1\}$.

Proof. Lemma 2.2 follows immediately from [12, 13] and [2, Theorems 3–4 and Lemma 5]. □

Proposition 2.3. *Let G be a p-solvable group. If $p \mid \chi^c(1)$ for every nonprincipal character $\chi \in \mathrm{Irr}(G)$, then G is solvable.*

Proof. Since every 2-solvable group is solvable, from now on we can assume that $p \neq 2$. Assume by contradiction that G is nonsolvable. Let M be a normal subgroup of G of maximal order such that G/M is nonsolvable. Set $\bar{G} = G/M$ and, for every $x \in G$, let $\bar{x}$ be the image of x in $\bar{G}$. Since $\mathrm{Codeg}(\bar{G}) \subseteq \mathrm{Codeg}(G)$, p divides the nontrivial elements of $\mathrm{Codeg}(\bar{G})$. Consequently, $p \mid |\bar{G}|$. Let $\bar{N} = N/M$ be a minimal normal subgroup of $\bar{G}$. Note that $\bar{G}/\bar{L} \cong G/L$ for every nontrivial normal subgroup $\bar{L} = L/M$ of $\bar{G}$, so by our assumption on M, $\bar{G}/\bar{L}$ is solvable. Since $\bar{N}/(\bar{N} \cap \bar{L}) \lesssim \bar{G}/\bar{L}$, our assumption on M forces $\bar{N}$ to be the unique minimal normal subgroup of $\bar{G}$. Also, $\bar{G}/\bar{N} \cong G/N$ is solvable and $\bar{N} = S_1 \times \cdots \times S_t$, where $S_1, \ldots, S_t$ are isomorphic nonabelian simple groups. Therefore, $C_{\bar{G}}(\bar{N}) = \{1\}$ and $\bar{N} \trianglelefteq \bar{G} \lesssim \mathrm{Aut}(\bar{N})$.

Suppose that $t = 1$. Since $\bar{G}$ is p-solvable and $p \mid |\bar{G}|$, it follows that $p \nmid |\bar{N}|$ and $p \mid |\bar{G}/\bar{N}|$. Thus, there exists a nonprincipal character $\theta \in \mathrm{Irr}(\bar{N})$ such that $I_{\bar{G}}(\theta)$ is a p'-group, by [10, Corollary 2.3]. Let $\varphi \in \mathrm{Irr}(\bar{G}|\theta)$. By Clifford's theorem, $[\bar{G} : I_{\bar{G}}(\theta)] \mid \varphi(1)$. Therefore, $\varphi^c(1) \mid |I_{\bar{G}}(\theta)|$ and, consequently, $p \nmid \varphi^c(1)$, which is a contradiction.

Next suppose that $t \geq 2$. Then $\bar{G}$ acts transitively on the set $\Omega = \{S_1, \ldots, S_t\}$. Let $\bar{K}$ be the kernel of the action of $\bar{G}$ on Ω. Note that $\bar{N} \leq \bar{K}$, so $\bar{G}/\bar{K}$ is solvable. By [3, Corollary 4], there are disjoint (possibly empty) subsets Ω_1 and Ω_2 of Ω such that $\mathrm{Stab}_{\bar{G}/\bar{K}}(\Omega_1, \Omega_2)$ (the subgroup of $\bar{G}/\bar{K}$ containing all elements of $\bar{G}/\bar{K}$ that fix the sets Ω_1 and Ω_2, setwise) is a 2-group. By [10, Corollary 2.3], for $i \in \{1, \ldots, t\}$, there exist nonprincipal characters $\theta_i, \varphi_i \in \mathrm{Irr}(S_i)$ of different degrees such that $p \nmid |I_{\mathrm{Aut}(S_i)}(\theta_i)|, |I_{\mathrm{Aut}(S_i)}(\varphi_i)|$. Thus, $I_{\bar{K}}(\theta_i)/C_{\bar{K}}(S_i)$ and $I_{\bar{K}}(\varphi_i)/C_{\bar{K}}(S_i)$ are p'-groups. It follows that $C_{\bar{K}}(S_i)$ contains the Sylow p-subgroups of $I_{\bar{K}}(\theta_i)$ and $I_{\bar{K}}(\varphi_i)$ for every $i \in \{1, \ldots, t\}$. Set $\psi = \Pi_{S_i \in \Omega_1} \theta_i \Pi_{S_i \in \Omega_2} \varphi_i$. Then $\psi \in \mathrm{Irr}(\bar{N})$, $\ker\psi = \{1\}$ and $I_{\bar{K}}(\psi) = (\bigcap_{S_i \in \Omega_1} I_{\bar{K}}(\theta_i)) \cap (\bigcap_{S_i \in \Omega_2} I_{\bar{K}}(\varphi_i))$. This shows that the Sylow p-subgroups of $I_{\bar{K}}(\psi)$ are subgroups of $\bigcap_{i=1}^{t} C_{\bar{K}}(S_i) \leq C_{\bar{G}}(\bar{N}) = \{1\}$. Therefore, $I_{\bar{K}}(\psi)$ is a p'-group. Also, $\theta_i(1) \neq \varphi_j(1)$ for $i, j \in \{1, \ldots, t\}$. It follows that $I_{\bar{G}}(\psi)\bar{K}/\bar{K} \leq \mathrm{Stab}_{\bar{G}/\bar{K}}(\Omega_1, \Omega_2)$. So, $I_{\bar{G}}(\psi)\bar{K}/\bar{K} \cong I_{\bar{G}}(\psi)/I_{\bar{K}}(\psi)$ is a 2-group and so $I_{\bar{G}}(\psi)$ is a p'-group, because $p \neq 2$.

Let $\chi \in \mathrm{Irr}(\bar{G}|\psi)$. By Clifford's theorem, $[\bar{G} : I_{\bar{G}}(\psi)] \mid \chi(1)$. Thus, $\chi^c(1) \mid |I_{\bar{G}}(\psi)|$, so $p \nmid \chi^c(1)$, which is a contradiction. Therefore, G is solvable. □

Proof of Theorem 1.1. If G is a p-group, then clearly $p \mid \chi^c(1)$ for every nonprincipal character $\chi \in \mathrm{Irr}(G)$, as desired.

Now suppose that $p \mid \chi^c(1)$ for every nonprincipal character $\chi \in \mathrm{Irr}(G)$. Then G is solvable, by Proposition 2.3. Working towards a contradiction, suppose that G is not a p-group. Let M be a normal subgroup of G of the maximal order such that G/M is not a p-group. Set $\bar{G} = G/M$ and, for $x \in G$, let $\bar{x}$ be the image of x in $\bar{G}$. Since $\mathrm{Codeg}(\bar{G}) \subseteq \mathrm{Codeg}(G)$, for every nonprincipal character $\chi \in \mathrm{Irr}(\bar{G})$, we have $p \mid \chi^c(1)$ and so $p \mid |\bar{G}|$. Let $\bar{N} = N/M$ be a minimal normal subgroup of $\bar{G}$. Note that $\bar{G}/\bar{L} \cong G/L$ for every nontrivial normal subgroup $\bar{L} = L/M$ of $\bar{G}$, so by our assumption on M, $\bar{G}/\bar{L}$ is a p-group (possibly trivial). Since $\bar{N}/(\bar{N} \cap \bar{L}) \lesssim \bar{G}/\bar{L}$, our assumption on M forces $\bar{N}$ to be the unique minimal normal subgroup of $\bar{G}$, which is an elementary abelian r-group for some prime r with $r \neq p$. Since $p \mid |\bar{G}|$, it follows that $\bar{G}/\bar{N}$ is a nontrivial p-group. Moreover, $\bar{G}/\bar{N}$ acts faithfully and irreducibly on $\bar{N}$. By [9, Theorem 1.1], there exists $\bar{x} \in \bar{N}$ such that $\mathrm{Stab}_{\bar{G}/\bar{N}}(\bar{x}) = \{1\}$, where $\mathrm{Stab}_{\bar{G}/\bar{N}}(\bar{x})$ denotes the stabiliser of $\bar{x}$ in $\bar{G}/\bar{N}$ under the relevant action. However, $\gcd(|\bar{N}|, |\bar{G}/\bar{N}|) = 1$ and hence the actions of $\bar{G}/\bar{N}$ on $\mathrm{Irr}(\bar{N})$ and on the conjugacy classes of $\bar{N}$ are permutation isomorphic. So, there exists $\theta \in \mathrm{Irr}(\bar{N})$ such that $I_{\bar{G}}(\theta) = \bar{N}$. Let $\chi \in \mathrm{Irr}(\bar{G}|\theta)$. By Clifford's theorem, $[\bar{G} : I_{\bar{G}}(\theta)] \mid \chi(1)$. Therefore, $\chi^c(1) \mid |I_{\bar{G}}(\theta)|$, so $p \nmid \chi^c(1)$, which is a contradiction. This shows that G is a p-group, as desired. □

In Examples 2.4, 2.5 and 2.6, we show that the hypotheses 'p-solvability of G' and 'p is neither 2 nor a Mersenne prime' in Theorem 1.1 are essential.

Example 2.4. Let G be a finite simple group and suppose that $\chi \in \mathrm{Irr}(G) - \{1_G\}$. Then $p \mid \chi^c(1)$ if and only if $\chi(1)_p < |G|_p$, that is, G has no irreducible character of p-defect zero (a character of degree divisible by $|G|_p$). By [5, Corollary 2], the co-degrees of nonprincipal irreducible characters of G are divisible by p if and only if either $p = 2$ and (up to isomorphism) $G \in \{\mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{24}, J_2, \mathrm{HS}, \mathrm{Suz}, \mathrm{Ru}, \mathrm{co}_1, \mathrm{Co}_3, \mathrm{BM}, \mathrm{Alt}_n\}$ (for some particular integer n) or $p = 3$ and (up to isomorphism) $G \in \{\mathrm{Co}_3, \mathrm{Suz}, \mathrm{Alt}_n\}$ (for some particular integer n).

Example 2.5. Let S be a simple group of Lie type over a field with p^k elements such that $p \mid k$. Then there exists an almost simple group G with socle S such that $|G/S| = p$. Let χ be a nonprincipal element of $\mathrm{Irr}(G)$. If $\chi \in \mathrm{Irr}(G/S)$, then obviously $p \mid \chi^c(1)$. Otherwise, $\ker\chi = \{1\}$ and there exists a nonprincipal constituent $\theta \in \mathrm{Irr}(S)$ of χ_S. By [8, Theorem 1.1], either $p \mid \theta^c(1)$ or θ is the Steinberg character of S. In the former case, $p \mid \chi^c(1)$ and, in the latter one, $\chi_S = \theta$. Thus, $p = |G/S| \mid \chi^c(1)$. This shows that there are many finite almost simple groups whose co-degrees of nonprincipal irreducible characters are divisible by p.

Example 2.6. By [9, Theorem 2.1(i)], there exists a finite group G of order 72 with the unique minimal normal subgroup N whose order is nine and the action of G/N on N has no regular orbit ($\mathtt{G = SmallGroup(72, 40)}$ in the GAP library of small groups).

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Let $\chi \in \mathrm{Irr}(G) - \{1_G\}$ and $p = 2$. If $\chi \in \mathrm{Irr}(G/N)$, then $p \mid \chi^c(1)$, because G/N is a nontrivial p-group. Otherwise, $\ker\chi = 1$ and there exists a nonprincipal constituent $\theta \in \mathrm{Irr}(N)$ of χ_N. Since $\gcd(|N|, |G/N|) = 1$, the actions of G/N on $\mathrm{Irr}(N)$ and on the conjugacy classes of N are permutation isomorphic. Hence, $p \mid |I_G(\theta)|$. By Clifford's theorem, $\chi(1) = [G : I_G(\theta)]e$, where $e^2 \leq |I_G(\theta) : N|$, so $p \mid \chi^c(1)$. Now let $p = 2^a - 1$ be a Mersenne prime. By [9, Theorem 2.1(ii)], there exists a finite group G of order $2^{ap}p^{p+1}$ with a unique minimal normal subgroup N whose order is 2^{ap} and the action of G/N on N has no regular orbit. Now applying the same argument given for $p = 2$ shows that $p \mid \chi^c(1)$ for every $\chi \in \mathrm{Irr}(G) - \{1_G\}$. Thus, if either $p = 2$ or p is a Mersenne prime, then there exists a non-nilpotent solvable group G such that $p \mid \chi^c(1)$ for every nonprincipal character $\chi \in \mathrm{Irr}(G)$.

Proof of Theorem 1.2. If G is an elementary abelian p-group, then [4, Lemma 2.4] shows that $\mathrm{Codeg}(G) = \{1, p\}$, as desired. From now on, we suppose that $\chi^c(1)_p = p^e$ for every nonprincipal irreducible character χ of G. We are going to prove that $e = 1$ and G is an elementary abelian p-group.

First, let G be a p-group. By the assumption of the theorem, $\mathrm{Codeg}(G) = \{1, p^e\}$. Now [1, Lemma 3.1] and [4, Lemma 2.4] force G to be an elementary abelian p-group and $e = 1$. So, in order to complete the proof, it is enough to show that G is a p-group.

Assume by contradiction that G is not a p-group. Let M be a normal subgroup of G of the maximal order such that G/M is not a p-group. Set $\bar{G} = G/M$ and let $\bar{x}$ be the image of $x \in G$ in $\bar{G}$. By the argument used in the proof of Theorem 1.1, $p \mid |\bar{G}|$ and $\bar{G}$ has a unique minimal normal subgroup $\bar{N}$. In particular, $\bar{G}/\bar{N}$ is a p-group (possibly trivial). We claim that $\bar{G}$ is p-solvable. Working towards a contradiction, suppose that $\bar{G}$ is not p-solvable. Thus, $\bar{N} = S_1 \times \cdots \times S_t$, where $S_1, \ldots, S_t$ are isomorphic nonabelian simple groups with orders divisible by p. Then, for $1 \leq i \leq t$, there exist $\theta_i, \phi_i \in \mathrm{Irr}(S_i) - \{1_{S_i}\}$ such that $p \mid \theta_i(1)$ and $p \nmid \phi_i(1)$. Set $\psi_1 = \theta_1 \times \phi_2 \times \cdots \times \phi_t$ and $\psi_2 = \phi_1 \times \phi_2 \times \cdots \times \phi_t$. Hence, $\psi_1, \psi_2 \in \mathrm{Irr}(\bar{N}) - \{1_{\bar{N}}\}$ and $\ker\psi_1 = \ker\psi_2 = \{1\}$. If $|\bar{G}/\bar{N}| = 1$, then $\psi_1^c(1)_p = |\bar{G}|_p/\theta_1(1)_p \neq |\bar{G}|_p = \psi_2^c(1)_p$, which is a contradiction. Next let $|\bar{G}/\bar{N}| \neq 1$. For every $\chi \in \mathrm{Irr}(\bar{G}|\psi_2)$, we have $\psi_2^c(1) \mid \chi^c(1)$ and so

$$(|S_1|_p)^t = \psi_2^c(1)_p \leq \chi^c(1)_p = p^e. \tag{2.1}$$

On the other hand, since $\bar{G}/\bar{N}$ is a p-group and $\mathrm{Codeg}(\bar{G}/\bar{N}) \subseteq \mathrm{Codeg}(G)$, we have $\mathrm{Codeg}(\bar{G}/\bar{N}) = \{1, p^e\}$. Thus, $e = 1$ and $p \neq 2$, by [1, Lemma 3.1] and [4, Lemma 2.4]. Hence, (2.1) forces $t = 1$ and $|S_1|_p = p$. So, $\bar{N}$ is a simple group, which is the unique minimal normal subgroup of $\bar{G}$. Thus, we can easily check that $C_{\bar{G}}(\bar{N}) = \{1\}$ and consequently $\bar{N} \trianglelefteq \bar{G} \lesssim \mathrm{Aut}(\bar{N})$. Hence, by Lemma 2.2, there exists $\psi \in \mathrm{Irr}(\bar{N})$ such that $\ker\psi = \{1\}$ and ψ extends to $\bar{G}$, that is, there exists $\chi \in \mathrm{Irr}(\bar{G})$ such that $\chi_{\bar{N}} = \psi$. It is obvious that $\ker\chi = \{1\}$ and so $|\bar{G}/\bar{N}| \mid \chi^c(1)$. Hence, $|\bar{G}/\bar{N}| = p$. This forces $|\bar{G}|_p = p^2$. Also, $|\bar{G}|_p/|\ker\chi|_p\chi(1)_p = \chi^c(1)_p = p$ for every $\chi \in \mathrm{Irr}(\bar{G}) - \mathrm{Irr}(\bar{G}/\bar{N})$. Thus, $\chi(1)_p \leq p$ and, by [7, Lemma 3.2], $\bar{N} \simeq \mathrm{Alt}_7$ and $p = 3$. Therefore, $|\bar{G}/\bar{N}| \mid |\mathrm{Out}(\mathrm{Alt}_7)|$. It follows that $p = 3 \mid |\mathrm{Out}(\mathrm{Alt}_7)| = 2$, which is a contradiction. This shows that $p \nmid |\bar{N}|$.

Consequently, $\bar{G}$ is p-solvable and $p \mid |\bar{G}/\bar{N}|$. Now Proposition 2.3 guarantees that $\bar{G}$ is solvable. Also, since $\bar{G}/\bar{N}$ is a p-group, $\mathrm{Codeg}(\bar{G}/\bar{N}) = \{1, p^e\}$. Thus, [1,

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Lemma 3.1] forces $\bar{G}/\bar{N}$ to be abelian. Therefore, $(\bar{G})' = \bar{N}$ is the unique minimal normal subgroup of $\bar{G}$. Hence, [6, Lemma 12.3] implies that $\bar{G}$ is a Frobenius group with the Frobenius kernel $\bar{N}$. So, there exists $\chi \in \mathrm{Irr}(\bar{G})$ such that $\chi^c(1) = |\bar{N}|$ and $p \nmid \chi^c(1)$. However, this contradicts the assumption of the theorem, because $\mathrm{Codeg}(\bar{G}) \subseteq \mathrm{Codeg}(G)$. Therefore, G must be an elementary abelian p-group. □

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