ANY GROUP IS A MAXIMAL SUBGROUP OF THE SEMIGROUP OF BINARY RELATIONS ON SOME SET†

by G. B. PRESTON

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We show that the theorem stated in the title is a corollary to a result of K. A. Zaretskii [5] and a theorem of G. Birkhoff [1]. The construction we use further shows that all groups with cardinal less than or equal to the cardinal of the given group are simultaneously realised as maximal subgroups of the same semigroup of binary relations $\mathscr{B}_X$. For finite or countable groups, when X may be taken to be finite or countable, respectively, and for an entirely different method of proof, the paper of J. S. Montague and R. J. Plemmons [3] should be consulted. For two further proofs of the theorem of the title to this note, this time for any X, see also R. J. Plemmons and B. M. Schein [4] and A. H. Clifford [2].

A lattice M is said to be an *X-lattice of unions*, if its elements are subsets of X, if containment is its partial order, and if the (set-theoretic) union of every set of elements of M also belongs to M. In particular this implies that the empty set, $\square$, belongs to M and that M is a complete lattice with (set-theoretic) union as its least upper bound operation. Dually, we say that L is an *X-lattice of intersections*, if its elements are subsets of X, if containment is its partial order and if the (set-theoretic) intersection of every subset of L also belongs to L. Then X is an element of L and L is a complete lattice with set-theoretic intersection as its greatest lower bound operation.

We shall reserve the symbols $\cap$ and $\cup$ for (set-theoretic) intersection and union, respectively, and use the symbols $\wedge$ and $\vee$ to denote greatest lower bound and least upper bound operations that are not necessarily intersection and union.

$\mathscr{B}_X$ denotes the semigroup of (binary) relations on X under the operation of composition: if $\alpha, \beta \in \mathscr{B}_X$, then the composite of α and β is $\alpha\beta = \{(x, y) \mid (x, z) \in \alpha \text{ and } (z, y) \in \beta \text{ for some } z\}$.

If $A \subseteq X$, then we write

$$A\alpha = \{x \in X \mid (a, x) \in \alpha \text{ for some } a \text{ in } A\}.$$

If $\alpha \in \mathscr{B}_X$, then we write

$$\alpha^{-1} = \{(x, y) \mid (y, x) \in \alpha\}.$$

Each element α of $\mathscr{B}_X$ determines an ordered pair of lattices (L_α, M_α), which we shall call *its pair of lattices*, defined thus:

$$M_\alpha = \{A\alpha \mid A \subseteq X\};$$

$$L_\alpha = \{X \backslash A\alpha^{-1} \mid A \subseteq X\}.$$

† This work was done while at the University of Stirling and the author wishes to record his gratitude to the Department of Mathematics for his stimulating year there. The results were presented in a talk at the University of Glasgow in February 1970.

It is easily verified that M_α is an X-lattice of unions, that L_α is an X-lattice of intersections, and (Zaretskii [5, §1.7]) that the mapping

$$f_\alpha : A \mapsto A\alpha,\ A \in L_\alpha,$$

is a lattice isomorphism of L_α onto M_α. Conversely, if $f: L \to M$ is an isomorphism of an X-lattice of intersections L onto an X-lattice of unions M, then there is a unique α in $\mathscr{B}_X$ such that $L_\alpha = L$, $M_\alpha = M$ and $f_\alpha = f$ (Zaretskii [5, §1.8]).

Let ε be an idempotent of $\mathscr{B}_X$. Then M_ε is fully distributive and H_ε, the maximal subgroup of $\mathscr{B}_X$ having ε as its identity element, is isomorphic to $\mathscr{A}(M_\varepsilon)$, the automorphism group of the lattice M_ε (Zaretskii [5, Theorem 3.9]).

Conversely, Zaretskii proves the following theorem [5, Corollary 3.10].

THEOREM 1 (Zaretskii). *Let M be a fully distributive X-lattice of unions such that there exists an isomorphic X-lattice of intersections. Let $G = \mathscr{A}(M)$, the automorphism group of M. Then there exists a maximal subgroup of $\mathscr{B}_X$ isomorphic to G.*

In fact there is an idempotent ε, say, in $\mathscr{B}_X$ such that $M_\varepsilon = M$; so that $H_\varepsilon \cong G$.

To show that Zaretskii's result implies that any group can be realised as a maximal subgroup of some $\mathscr{B}_X$ we merely have to show that, given a group G, then a fully distributive lattice M can be found, satisfying the conditions of Theorem 1, and such that $\mathscr{A}(M) \cong G$.

For this we turn to Birkhoff's paper [1]. Birkhoff proved that, given a group G, there exists a distributive lattice M, say, such that $\mathscr{A}(M) \cong G$. In fact, as we now show, the lattice constructed by Birkhoff also satisfies the further conditions required by Zaretskii for M in Theorem 1.

Birkhoff's construction is as follows. Let G be a group and set $Y = G \cup (G \times G)$. Let the set G be well-ordered, by $\leqq$, say, with 1, the identity of G, as least element. Then order Y partially by the rules:

$$g > (gh, h),$$

$$(g, h) > (g, k), \text{ if } h < k, \text{ in } G,$$

for all g, h, k in G. Then Y is partially ordered by $\geqq$ and G is anti-isomorphic to $\mathscr{A}(Y)$, the automorphism group of the partially ordered set Y. In fact, $a \mapsto \alpha_a$, $a \in G$, where $g\alpha_a = ag$ and $(g, h)\alpha_a = (ag, h)$, is an anti-isomorphism of G onto $\mathscr{A}(Y)$.

Now let N consist of all order-preserving mappings of Y into $\{0, 1\}$, where $0 < 1$. Define a partial order on N by agreeing that $f \leqq g$ if and only if $yf \leqq yg$, for all y in Y. Then Birkhoff showed that N is a distributive lattice and that G is isomorphic to $\mathscr{A}(N)$. In fact $a \mapsto \theta_a$, $a \in G$, where θ_a is the mapping $f \mapsto f_a$, $f \in N$, and where $yf_a = (y\alpha_a)f$, is an isomorphism of G onto $\mathscr{A}(N)$.

Denote by M the set of all subsets of Y which contain with any element of Y all elements of Y greater than that element:

$$M = \{U \subseteqq Y \mid u \in U \text{ and } v > u \quad \text{implies} \quad v \in U\}.$$

Denote the characteristic function of a subset U of Y by χ_U. Then

$$f \in N \Leftrightarrow f = \chi_U \text{ and } U \in M; \tag{1}$$

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for, if $f = \chi_U \in N$, then $uf = 1$ if $u \in U$, so that, since f is order-preserving, $v > u$ implies that $vf = 1$, i.e., that $v \in U$; and, conversely, if $f = \chi_U$, where $U \in M$, then, if $x, y \in Y$ and $x \geqq y$, $y\chi_U = 1$ implies $x\chi_U = 1$, so that $f = \chi_U$ is order-preserving.

From (1) it immediately follows that M is a lattice of subsets of Y and that M is isomorphic to N. Since M is complete and fully distributive, so also is N. In fact, more strongly, we have the following lemma.

LEMMA 1. *M is a complete lattice of subsets of Y in which the least upper bound operation and the greatest lower bound operation are set-theoretic union and set-theoretic intersection respectively. Moreover, Y and $\square$ both belong to M.*

Proof. Since M is a lattice of subsets of Y with containment as its partial order relation, it suffices to show that arbitrary set-theoretic intersections and arbitrary set-theoretic unions of elements of M belong to M.

Let U_i, $i \in I$, belong to M. Let $U = \bigcap U_i$ and $V = \bigcup U_i$. Let $u \in U$ and $v > u$. Then $u \in U_i$ and so, since $U_i \in M$, $v \in U_i$, for all i in I; hence $v \in U$ and so $U \in M$. Similarly, if $u \in V$ and $v > u$, then there exists an i in I such that $u \in U_i$, so that $v \in U_i \subseteqq V$; hence $V \in M$.

That Y and $\square$ belong to M is clear from the definition of M.

COROLLARY. *If X is any set containing Y, then M is an X-lattice of unions and there exists an X-lattice of intersections L, say, isomorphic to M.*

Proof. By the lemma, M is an X-lattice of unions. Set $X \backslash Y = Z$ and define L thus:

$$L = \{Z \cup U \mid U \in M\}.$$

Then

$$Z \cup U \mapsto U, \; U \in M, \tag{2}$$

is an isomorphism of L onto M. Since $Y \in M$, $X = Y \cup Z$ belongs to L. By the lemma, M is a Y-lattice of intersections and hence L is an X-lattice of intersections.

We can now apply Zaretskii's theorem to prove our main result.

THEOREM 2. *Let H be a set. Set $W = H \cup (H \times H)$ and let X be any set containing W. Then any group of cardinal less than or equal to that of H is isomorphic to a maximal subgroup of $\mathscr{B}_X$.*

Proof. Let G be a group such that $|G| \leqq |H|$. Set $Y = G \cup (G \times G)$ and construct Birkhoff's lattice M, of subsets of Y, such that $G \cong \mathscr{A}(M)$. This isomorphism still holds if M is the corresponding set of subsets of any set of the same cardinal as Y. So we may suppose that Y is a subset of X and apply the corollary to the preceding lemma. This means that M satisfies precisely the conditions necessary to apply Theorem 1. Hence G is isomorphic to a maximal subgroup of $\mathscr{B}_X$.

It is perhaps worth while, as suggested by A. H. Clifford, rephrasing the above result in the following form.

THEOREM 3. *If X is an infinite set, the set of maximal subgroups of $\mathcal{B}_X$ includes all groups of cardinal less than or equal to that of X. If X is finite, the set of maximal subgroups of $\mathcal{B}_X$ includes all groups of cardinal m, for m such that $m(m+1) \leqq |X|$.*

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MONASH UNIVERSITY
CLAYTON, AUSTRALIA 3168

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