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GEGENBAUER COLLOCATION INTEGRATION METHODS: ADVANCES IN COMPUTATIONAL OPTIMAL CONTROL THEORY

KAREEM ELGINDY

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Analytic solutions of simple optimal control problems may be found using classical tools such as the calculus of variations, dynamic programming or the minimum principle. However, in practice, a closed-form expression for the optimal control is difficult or even impossible to determine for general nonlinear optimal control problems. Therefore such intricate optimal control problems must be solved numerically.

The research goals of this dissertation are to furnish an efficient, accurate, rapid and robust optimal control solver, and to produce a significantly small-scale nonlinear programming problem using few collocation points. To this end, we introduce a direct optimisation method based on a novel Gegenbauer collocation integration scheme which draws upon the power of well-developed nonlinear programming techniques and computer codes, and the well-conditioning of the numerical integration operators.

Chapter 1 is an introductory chapter highlighting the strengths and the weaknesses of various solution methods for optimal control problems, and provides the motivation for the present work. The chapter concludes with a general framework for using Gegenbauer expansions to solve optimal control problems and an overview of the remainder of the dissertation.

Chapter 2 presents some preliminary mathematical background and basic concepts relevant to the solution of optimal control problems. In particular, the chapter introduces some key concepts of the calculus of variations, optimal control theory,

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direct optimisation methods, Gegenbauer polynomials and Gegenbauer collocation, in addition to some other essential topics.

Chapter 3 presents the paper [2]. We introduce a novel optimal Gegenbauer quadrature to efficiently approximate definite integrations numerically. The novel numerical scheme introduces the idea of exploiting the strengths of the Chebyshev, Legendre, and Gegenbauer polynomials through a unified approach, and using a unique numerical quadrature. The Gegenbauer quadrature developed can be used for approximating integrals with any arbitrary sets of integration nodes. Moreover, exact integrations are obtained for polynomials of any arbitrary degree n if the number of columns in the Gegenbauer integration matrix developed is greater than or equal to n. The error formula for the Gegenbauer quadrature is derived. Moreover, a study of the error bounds and convergence rate shows that the optimal Gegenbauer quadrature exhibits very rapid convergence rates faster than any finite power of the number of Gegenbauer expansion terms. Two efficient computational algorithms are presented for optimally constructing the Gegenbauer quadrature, and to ideally maintain the robustness and the rapid convergence of the discrete approximations.

Chapter 4 (see [4]) is focused on the intriguing question: what value of the Gegenbauer parameter α is optimal for a Gegenbauer integration matrix to best approximate the solution of various dynamical systems and optimal control problems? The chapter highlights those methods presented in the literature which recast the aforementioned problems into unconstrained/constrained optimisation problems, and then add the Gegenbauer parameter α associated with the Gegenbauer polynomials as an extra unknown variable to be optimised. The theoretical arguments presented in this chapter prove that this naive policy is invalid since it violates the discrete Gegenbauer orthonormality relation, and may in turn produce false optimisation problems analogous to the original problems with poor solution approximations.

Chapter 5 (see [1]) resolves the issues raised in the previous chapter through the introduction of a hybrid Gegenbauer collocation integration method for solving various dynamical systems such as boundary value problems, integral and integro-differential equations. The resulting linear systems are generally well-conditioned and can be easily solved using standard linear system solvers. A study on the error bounds of the proposed method is presented, and the spectral convergence is proven for two-point boundary-value problems.

Chapter 6 (see [5]) presents a novel direct orthogonal collocation method using Gegenbauer–Gauss collocation for solving continuous-time optimal control problems with nonlinear dynamics, state and control constraints, where the admissible controls are continuous functions.

In Chapter 7 (see [3]) we extend the Gegenbauer transcription method introduced in the preceding chapter to deal further with continuous-time optimal control problems including time derivatives, of different orders, of the states, by solving the continuous-time optimal control problem directly for the control $u(t)$ and the highest-order time derivative $x^{(N)}(t), N \in \mathbb{Z}^+$. The state vector and its derivatives up to the $(N-1)$th-order derivative can then be stably recovered by successive integration. Moreover, we present our solution method for solving linear quadratic regulator problems as we aim to cover a wider collection of continuous-time optimal control problems with the concrete aim of comparing the efficiency of the current work with other classical discretisation methods in the literature. The advantages of the proposed direct Gegenbauer transcription method over other traditional discretisation methods are shown through four well-studied optimal control test examples. The present work is a major breakthrough in the area of computational optimal control theory as it delivers significantly accurate solutions using considerably smaller numbers of collocation points, states and control expansion terms. Moreover, the Gegenbauer transcription method produces very small-scale nonlinear programming problems, which can be solved very quickly using modern nonlinear programming software.

Chapter 8 presents some concluding remarks, including some suggestions for future research.

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KAREEM ELGINDY, School of Mathematical Sciences,
Monash University, Clayton, Victoria 3800, Australia
e-mail: kareem.elgindy@monash.edu, kareem.elgindy@gmail.com