

Equivariant wrapped Floer homology and symmetric periodic Reeb orbits

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Abstract. The aim of this article is to apply a Floer theory to study symmetric periodic Reeb orbits. We define positive equivariant wrapped Floer homology using a (anti-)symplectic involution on a Liouville domain and investigate its algebraic properties. By a careful analysis of index iterations, we obtain a non-trivial lower bound on the minimal number of geometrically distinct symmetric periodic Reeb orbits on a certain class of real contact manifolds. This includes non-degenerate real dynamically convex star-shaped hypersurfaces in \mathbb{R}^{2n} which are invariant under complex conjugation. As a result, we give a partial answer to the Seifert conjecture on brake orbits in the contact setting.

Key words: equivariant wrapped Floer homology, symmetric periodic Reeb orbit, Seifert conjecture, brake orbit

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Contents

1	Introduction	1709
1.1	A conjecture of Seifert	1709
1.2	The Seifert conjecture in contact geometry	1710
1.3	Equivariant wrapped Floer homology	1713
1.4	Analysis of index iterations	1714
2	Wrapped Floer homology	1715

2.1	Recollection of wrapped Floer homology	1715
2.2	Vanishing of wrapped Floer homology	1719
3	Equivariant wrapped Floer homology	1722
3.1	\mathbb{Z}_2 -complexes	1722
3.2	Equivariant Morse homology	1724
3.3	Definition of equivariant wrapped Floer homology	1729
3.4	Periodic family of Floer data	1738
3.5	Leray–Serre-type spectral sequence	1738
3.6	Positive equivariant wrapped Floer homology	1740
3.7	Morse–Bott spectral sequence	1743
3.8	Generators of $HW_*^{\mathbb{Z}_2,+}$ are Reeb chords	1745
3.9	Computing positive equivariant wrapped Floer homology	1745
4	On the minimal number of symmetric periodic Reeb orbits	1747
4.1	Index of symmetric periodic Reeb orbits	1747
4.2	Real dynamical convexity	1752
4.3	Index-increasing property	1755
4.4	The common index jump theorem	1758
4.5	Symmetric periodic Reeb orbits on displaceable hypersurfaces	1759
	Acknowledgements	1761
	References	1761

1. Introduction

1.1. *A conjecture of Seifert.* Consider a mechanical Hamiltonian system in \mathbb{R}^{2n} associated with a Hamiltonian of the form

$$E(q, p) = \sum_{i,j} g_{ij}(q) p_i p_j + U(q), \tag{1.1}$$

where the matrix $g_{ij}(q)$ is symmetric and positive-definite for each q and U is a smooth function of q . Note that this Hamiltonian is invariant under complex conjugation $\rho_0(q, p) = (q, -p)$, which is an anti-symplectic involution, i.e., ρ_0 satisfies $\rho_0^2 = \text{id}_{\mathbb{R}^{2n}}$ and $\rho_0^* \omega = -\omega$, where $\omega = \sum_{j=1}^n dq_j \wedge dp_j$ is the standard symplectic form on \mathbb{R}^{2n} .

Let $c \in \mathbb{R}$ be a regular value of E . A $2T$ -periodic orbit $x = (q, p) : [0, 2T] \rightarrow E^{-1}(c)$ of the Hamiltonian vector field of E is called a *brake orbit* if $p(0) = p(T) = 0$. See §2.1.2 for the definition of the Hamiltonian vector field. In [44], Seifert proved that if U is real analytic and the projection of $E^{-1}(c)$ to the position space \mathbb{R}^n is bounded and homeomorphic to a closed unit ball, then there exists a brake orbit on $E^{-1}(c)$. Under these assumptions, he then made the following conjecture.

There exist at least n geometrically distinct brake orbits on $E^{-1}(c)$.

We remark that the lower bound n in the Seifert conjecture is optimal. For example, the Hamiltonian system

$$H(z_1, \dots, z_n) = \sum_{j=1}^n \frac{\pi |z_j|^2}{a_j}, \quad a_i/a_j \notin \mathbb{Q} \quad \text{for } i \neq j$$

on \mathbb{C}^n has precisely n geometrically distinct brake orbits on each positive energy level.

We state some related results on brake orbits. Below, a Hamiltonian H is an arbitrary smooth function on \mathbb{R}^{2n} . It is said to be *even in p* if it is invariant under ρ_0 .

- (Rabinowitz [38]) If H is even in p , $c \in \mathbb{R}$ is a regular value and $z \cdot \nabla H(z) \neq 0$ for all $z \in H^{-1}(c)$, then there exists a brake orbit on $H^{-1}(c)$.
- (Szulkin [45]) If H satisfies the assumptions of Rabinowitz and $H^{-1}(c)$ is $\sqrt{2}$ -pinched, then the Seifert conjecture holds.
- (Long, Zhang and Zhu [33]) If H is even in p , the regular energy level $H^{-1}(c)$ is strictly convex and H is also invariant under the antipodal map $(q, p) \mapsto (-q, -p)$, then there exist at least two brake orbits on $H^{-1}(c)$.
- (Liu and Zhang [31]) Under the assumptions of Long, Zhang and Zhu, the Seifert conjecture holds.
- (Giambò, Giannoni and Piccione [14]) The Seifert conjecture holds for the case $n = 2$.
- (Frauenfelder and Kang [11]) If $H : \mathbb{R}^4 \rightarrow \mathbb{R}$ is even in p , and the regular energy level $H^{-1}(c)$ is star-shaped and dynamically convex, then there exist either two or infinitely many brake orbits (see Remark 1.4).

We also refer the reader to [3–5, 15, 16, 19, 28–30, 46, 48] for works that study the Seifert conjecture.

1.2. *The Seifert conjecture in contact geometry.* In this article, we study the Seifert conjecture in the contact setting. A *real contact manifold* is a triple (Σ, α, ρ) , where Σ is a (co-oriented) contact manifold with a contact form α and $\rho \in \text{Diff}(\Sigma)$ is an anti-contact involution, meaning that

$$\rho^2 = \text{id}_\Sigma, \quad \rho^*\alpha = -\alpha.$$

The Reeb vector field $R = R_\alpha$ is then anti-invariant under the involution ρ , i.e.,

$$\rho^*R = -R.$$

Assume that the fixed point set $\mathcal{L} = \text{Fix}(\rho)$, which is a Legendrian submanifold of Σ , is non-empty. The flow ϕ_R^t of the Reeb vector field R satisfies

$$\phi_R^t = \rho \circ \phi_R^{-t} \circ \rho. \tag{1.2}$$

A smooth integral curve $c : [0, T] \rightarrow \Sigma$ of the Reeb vector field R satisfying the boundary condition $c(0), c(T) \in \mathcal{L}$ is called a *Reeb chord*. In view of equation (1.2), associated to each Reeb chord $c : [0, T] \rightarrow \Sigma$ is a *symmetric periodic Reeb orbit*

$$c^2(t) = \begin{cases} c(t) & \text{if } t \in [0, T], \\ \rho \circ c(2T - t) & \text{if } t \in [T, 2T] \end{cases}$$

(see Figure 1). In particular, any Reeb chord comes in a pair and there is a one-to-one correspondence between the set of such pairs and the set of symmetric periodic Reeb orbits.

Example 1.1. If the regular energy level $H^{-1}(c)$ is star-shaped (with respect to the origin) which is invariant under complex conjugation ρ_0 , then the standard Liouville one-form $\lambda = \frac{1}{2} \sum_{j=1}^n (q_j dp_j - p_j dq_j)$ on \mathbb{R}^{2n} restricts to a contact form α on $H^{-1}(c)$. Moreover, the restriction $\rho := \rho_0|_{H^{-1}(c)}$ defines an anti-contact involution on $H^{-1}(c)$. Consequently,

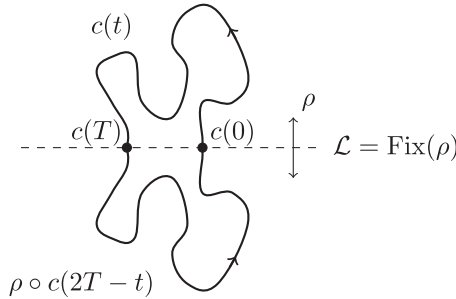


FIGURE 1. A symmetric periodic Reeb orbit.

$(H^{-1}(c), \alpha, \rho)$ is a real contact manifold. The corresponding Legendrian submanifold \mathcal{L} is given by $H^{-1}(c) \cap \{p = 0\}$, which implies that brake orbits are symmetric periodic Reeb orbits on $H^{-1}(c)$. In this case, the existence theorems of Seifert [44] and Rabinowitz [38] follow immediately from the Arnold chord conjecture proved by Mohnke [36].

We translate the Seifert conjecture into the language of contact geometry.

There exist at least n geometrically distinct symmetric periodic Reeb orbits on any real contact manifold $(\Sigma^{2n-1}, \alpha, \rho)$.

Note that the aforementioned results on brake orbits [11, 28–31, 33, 45, 48] study special cases of this conjecture.

In this article, we apply a Floer theory to this conjecture, taking up an approach given by Liu and Zhang [29, 31]. In their theory, they use strict convexity of a hypersurface twice, as pointed out in [18]. First, the Clarke dual action functional, which exists only in the strictly convex case, is used to obtain information about the interval on which the indices of brake orbits lie. Second, when the hypersurface is strictly convex, then the index of brake orbits behaves well under iterations. Borrowing an idea of Gutt and Kang [18], we show that, under a weaker assumption, the approach of Liu and Zhang works well in the framework of a Floer theory and that the index of symmetric periodic Reeb orbits behaves well under iterations, provided that the contact form is non-degenerate.

Now let $W \subset \mathbb{R}^{2n}$ be a compact star-shaped domain with the standard Liouville form λ . Assume that W admits an exact anti-symplectic involution ρ_W on W , i.e., $\rho_W^2 = \text{id}_W$ and $\rho_W^* \lambda = -\lambda$. Then the star-shaped hypersurface Σ given by the boundary of W is a real contact manifold $(\Sigma = \partial W, \alpha = \lambda|_\Sigma, \rho = \rho_W|_\Sigma)$. The contact form α is called *real dynamically convex* if the Maslov indices of all Reeb chords satisfy suitable lower bounds (see §4.2). One of our main results is the following assertion.

THEOREM 1.2. *Let the triple (Σ, α, ρ) be as above. Assume that the contact form α on Σ is non-degenerate and real dynamically convex. Then there exist at least n geometrically distinct and simple symmetric periodic Reeb orbits on Σ .*

This theorem, in particular, implies that if a compact star-shaped hypersurface in \mathbb{R}^{2n} , invariant under complex conjugation, is non-degenerate and real dynamically convex, then the contact Seifert conjecture holds. Without the non-degeneracy assumption,

Liu and Zhang [31] prove that if $H^{-1}(c)$ is strictly convex and invariant under complex conjugation and the antipodal map, then the contact Seifert conjecture holds. In Theorem 1.2, we obtain the same result under weaker assumptions, but under the additional assumption of non-degeneracy, in order to apply Floer theory.

In Theorem 4.12, we prove the same assertion for a broader class of real contact manifolds and Theorem 1.2 is obtained as a corollary. As an immediate corollary of the methods used in the proof of Theorem 1.2, we obtain the following result.

COROLLARY 1.3. *In addition to the assumptions of Theorem 1.2, assume that there exist precisely n geometrically distinct and simple symmetric periodic Reeb orbits on Σ . Then their indices are all different.*

Remark 1.4. In [29, Conjecture 1.1], Liu and Zhang conjecture that if $\Sigma \subset \mathbb{R}^{2n}$ is a strictly convex hypersurface which is invariant under complex conjugation, then there exist either precisely n or infinitely many geometrically distinct symmetric periodic Reeb orbits. Note that non-degeneracy is not assumed. The conjecture is proved for the case $n = 2$ under a weaker assumption by Frauenfelder and Kang [11, Theorem 2.5] via holomorphic curve techniques. More precisely, they prove that if (Σ, α) is a star-shaped dynamically convex hypersurface in \mathbb{C}^2 that is invariant under complex conjugation, then there are either two or infinitely many symmetric periodic Reeb orbits. Recall that convexity implies dynamical convexity (see [20]). Their theorem says, in particular, that the existence of any non-symmetric periodic Reeb orbit ensures infinitely many symmetric periodic Reeb orbits. This phenomenon can also be found in the study of symmetric periodic points of reversible maps. Indeed, any area-preserving map defined on the open unit disk in the complex plane which is reversible with respect to complex conjugation must admit a symmetric periodic point. If there is more than one periodic point, which is possibly non-symmetric, then there have to be infinitely many symmetric periodic points (see [24]).

We now further assume that W admits an exact symplectic involution ι_W , namely, $\iota_W^2 = \text{id}_W$ and $\iota_W^* \lambda = \lambda$, which commutes with ρ_W . Then its boundary Σ carries a contact involution $\iota = \iota_W|_\Sigma$. For a Reeb chord $c : [0, T] \rightarrow \Sigma$, the corresponding symmetric periodic Reeb orbit c^2 is called *doubly symmetric* if $\iota(\text{im}(c^2)) = \text{im}(c^2)$ (see Figure 2).

Our next result is a slight generalization of [29, Theorem 1.2].

THEOREM 1.5. *Let the quadruple $(\Sigma, \alpha, \rho, \iota)$ be as above. Assume that the contact form α on Σ is non-degenerate and real dynamically convex. Then there exist at least $n + \mathcal{N}(\Sigma)$ geometrically distinct and simple symmetric periodic Reeb orbits on Σ , where $2\mathcal{N}(\Sigma)$ is the number of geometrically distinct and simple symmetric periodic Reeb orbits which are not doubly symmetric.*

We prove the same assertion for more general real contact manifolds in Theorem 4.15. Then Theorem 1.5 is obtained as a corollary. Again, the proof has an immediate corollary.

COROLLARY 1.6. *In addition to the assumptions of Theorem 1.5, assume that there exist precisely n geometrically distinct and simple symmetric periodic Reeb orbits on Σ . Then they are all doubly symmetric periodic Reeb orbits and their indices are all different.*

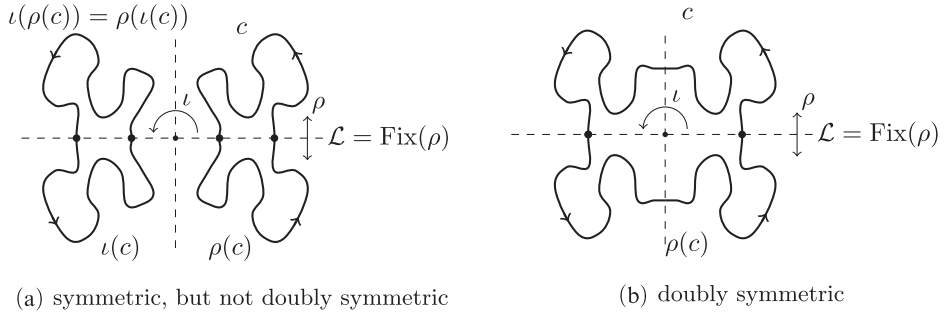


FIGURE 2. An illustration of doubly symmetric periodic Reeb orbits.

In the following, we explain two main ingredients for proving our theorems: *equivariant wrapped Floer homology* and *analysis of index iterations*.

1.3. *Equivariant wrapped Floer homology.* Our main tool is a variant of Lagrangian Floer theory, namely, *equivariant wrapped Floer homology*. Let (W, λ) be a Liouville domain with a Liouville form λ and an admissible Lagrangian L , meaning that L is exact and intersects the boundary ∂W in a Legendrian submanifold (see §2.1.1 for definitions). This is a basic set-up that one considers to define (non-equivariant) wrapped Floer homology (see, for example, [1, 39]).

For an equivariant theory, we additionally put a \mathbb{Z}_2 -symmetry on W given by a symplectic or an anti-symplectic involution $\mathcal{I} : W \rightarrow W$ under which L is invariant. We then have an induced \mathbb{Z}_2 -action on the space of paths in W with end points on the Lagrangian L (see equation (3.7)). This allows us to define a \mathbb{Z}_2 -equivariant version of wrapped Floer homology, which is an open string analogue of the S^1 -equivariant symplectic homology in [7, 17].

We are mainly interested in the case when L is given by the fixed point set of an anti-symplectic involution. In this case, we say that $(W, \lambda, \mathcal{I})$ is a *real Liouville domain*. If the anti-symplectic involution $\mathcal{I} : W \rightarrow W$ is exact, then the contact boundary $(\partial W, \alpha, \mathcal{I})$ with the induced contact form α is a real contact manifold. Examples are star-shaped domains in \mathbb{R}^{2n} invariant under complex conjugation $(q, p) \mapsto (q, -p)$.

In §3.3, we define equivariant wrapped Floer homology groups for both symplectic and anti-symplectic involutions. In view of [7], one can develop equivariant Floer theory in two different flavors. This is basically due to the fact that the Borel construction (see §3.2.1) admits several geometric structures: for example, it can be seen as a quotient space by an action or as a total space of a fiber bundle (see [7, §1]). The construction of the equivariant theory given in this paper takes the second point of view, and this matches the construction in [43] where an equivariant Lagrangian Floer theory with symplectic involutions is outlined. See also [22] for a general theory of Floer homology of families. As pointed out in [7, 43], a technical benefit of this choice is that the analysis on Floer trajectories is easier to deal with. We also provide a description of equivariant Morse homology in §3.2. Hopefully, this makes many ideas of the constructions in equivariant wrapped Floer homology transparent.

For applications to multiplicity results of symmetric periodic Reeb orbits, we need to observe that the generators of positive equivariant wrapped Floer homology correspond to (\mathbb{Z}_2 -pairs of) Reeb chords on the contact-type boundary. Even though positive equivariant wrapped Floer homology is designed to have this property, it is not directly visible from its definition. To identify generators with Reeb chords, we construct, in §3.7, a Morse–Bott spectral sequence in equivariant wrapped Floer homology, which might be of independent interest. If the contact form on the boundary is chord-non-degenerate (see §3.8), the first page of the Morse–Bott spectral sequence is generated by (\mathbb{Z}_2 -pairs of) Reeb chords and hence so is the resulting homology (see Corollary 3.35). A similar technique was used in [17] to show that positive S^1 -equivariant symplectic homology is generated by periodic Reeb orbits.

Next we compute equivariant wrapped Floer homology for the Liouville domains and the Lagrangians in question to detect sufficiently many generators. For this purpose, we establish algebraic properties of equivariant wrapped Floer homology, notably, Leray–Serre-type long exact sequences (see Corollary 3.28). A version of such a long exact sequence in symplectic homology is given in [7, Theorem 1.2]. The idea is that a special choice of Floer data, which we call a *periodic family of admissible Floer data* in §3.4, simplifies the equivariant Floer chain complex (see Lemma 3.25), so that we have a \mathbb{Z}_2 -complex structure on it (see §3.1). A \mathbb{Z}_2 -complex structure on a chain complex then algebraically produces a Leray–Serre-type spectral sequence for the resulting \mathbb{Z}_2 -equivariant homology (see Lemma 3.5).

It turns out that if the non-equivariant wrapped Floer homology vanishes, then the positive equivariant wrapped Floer homology has enough generators for our multiplicity results (see Corollary 3.40 and Corollary 3.42). Note that in Liouville domains with vanishing symplectic homology such as subcritical Stein domains, the wrapped Floer homology of any admissible Lagrangian vanishes (see Remark 2.16). In §2.2, we also prove a vanishing property of wrapped Floer homology under a displaceability condition on contact-type boundaries and Lagrangians. A similar vanishing result for the symplectic homology of displaceable contact hypersurfaces can be found in [23].

1.4. *Analysis of index iterations.* The main issue in applying Floer theory to multiplicity questions is to distinguish iterated orbits from simple ones. Even though the equivariant wrapped Floer homology groups have infinitely many generators, it is not obvious that they are represented by (geometrically distinct) symmetric periodic Reeb orbits. Indeed, iterations of a single symmetric periodic Reeb orbit might represent distinct generators of the homology groups. In order to avoid this, we have to establish a suitable condition on the Maslov index of Reeb chords. First of all, we impose a condition on the contact form on a real contact manifold to satisfy a certain lower bound of the Maslov index. Such a contact form is called *real dynamically convex* (see Definition 4.4). Our terminology is motivated by the fact that a strictly convex hypersurface in \mathbb{R}^{2n} that is invariant under complex conjugation is real dynamically convex (see Theorem 4.5). Assuming real dynamical convexity, we prove, by a careful analysis based on the properties of the Robbin–Salamon index and the formula for the Hörmander index given in [8, Formula 2.10], that the Maslov index is increasing under the iterations (see Theorem 4.6). This, together with the common

index jump theorem, which is proved by Liu and Zhang [29, Theorem 1.5], gives rise to a powerful tool for counting geometrically distinct symmetric periodic Reeb orbits in terms of the homology computations.

2. Wrapped Floer homology

2.1. *Recollection of wrapped Floer homology.* We recall the definition of wrapped Floer homology. As we will apply the index iteration theory, the homology is equipped with a \mathbb{Z} -grading which comes from the Maslov index. Basically, our conventions are those of [26]. We refer to [1, §3] and [39, §4] for more detailed descriptions.

2.1.1. *Geometric set-up.* Let (W^{2n}, λ) be a Liouville domain. By definition, W is a compact manifold with boundary such that $d\lambda$ is symplectic and the associated Liouville vector field X_λ , which is defined by the condition $d\lambda(X_\lambda, \cdot) = \lambda$, is pointing outwards along the boundary. The restricted 1-form $\alpha := \lambda|_{\partial W}$ becomes a contact form on the boundary ∂W . We denote by R_α the *Reeb vector field* of α on ∂W characterized by the conditions

$$d\alpha(R_\alpha, \cdot) = 0 \quad \text{and} \quad \alpha(R_\alpha) = 1.$$

Attaching an infinite cone, we obtain the *completion* of W as a non-compact manifold

$$\widehat{W} := W \cup ([1, \infty) \times \partial W)$$

equipped with the extended Liouville form

$$\widehat{\lambda} := \begin{cases} \lambda & \text{on } W, \\ r\alpha & \text{on } [1, \infty) \times \partial W, \end{cases}$$

where r stands for the coordinate on $[1, \infty)$. We denote by $\widehat{\omega} = d\widehat{\lambda}$ the symplectic form on \widehat{W} . Note that the Liouville vector field $X_{\widehat{\lambda}}$ restricts to $r\partial_r$ on $[1, \infty) \times \partial W$.

Definition 2.1. A Lagrangian L in W is called *admissible* if it has the following properties:

- L is transverse to the boundary ∂W and $\partial L = L \cap \partial W$ is a Legendrian, i.e., $\alpha|_{\partial L} = 0$;
- L is exact, i.e., $\lambda|_L$ is an exact 1-form; and
- the Liouville vector field of λ is tangential to L near its boundary.

The third condition allows us to complete an admissible Lagrangian L to a non-compact Lagrangian in \widehat{W} : i.e.,

$$\widehat{L} = L \cup_{\partial L} ([1, \infty) \times \partial L).$$

2.1.2. *Hamiltonian chords and Reeb chords.* Given a Hamiltonian $H : \widehat{W} \rightarrow \mathbb{R}$, the associated *Hamiltonian vector field* X_H is defined by the equation $d\widehat{\lambda}(X_H, \cdot) = dH$. We denote by $\phi_H^1 : \widehat{W} \rightarrow \widehat{W}$ the *Hamiltonian flow* of X_H . Its time-1 map ϕ_H^1 is called the *Hamiltonian diffeomorphism* of H . An orbit $c : [0, 1] \rightarrow \widehat{W}$ of the Hamiltonian vector field X_H meeting the boundary condition $c(0), c(1) \in \widehat{L}$ is called a *Hamiltonian chord* of H . The Hamiltonian chords naturally correspond to the intersection points of $L \cap \phi_H^1(L)$ via the assignment $c \mapsto c(1)$. We say that a Hamiltonian chord c is *contractible*

if the class $[c] \in \pi_1(\widehat{W}, \widehat{L})$ is trivial. We denote by $\mathcal{P}(H)$ the set of all contractible Hamiltonian chords of H . A Hamiltonian chord c is called *non-degenerate* if it satisfies

$$D\phi_H^1(T_{c(0)}\widehat{L}) \cap T_{c(1)}\widehat{L} = \{0\}.$$

A Hamiltonian $H : \widehat{W} \rightarrow \mathbb{R}$ is called *non-degenerate* if all Hamiltonian chords of H are non-degenerate: equivalently, \widehat{L} and $\phi_H^1(\widehat{L})$ intersect transversally.

Let (Σ, α) be a contact manifold and let $\mathcal{L} \subset \Sigma$ be a Legendrian. A *Reeb chord* of length $T > 0$ (with respect to \mathcal{L}) is an orbit $c : [0, T] \rightarrow \Sigma$ of the Reeb vector field R_α meeting the boundary condition $c(0), c(T) \in \mathcal{L}$. A Reeb chord $c : [0, T] \rightarrow \Sigma$ is called *non-degenerate* if it satisfies

$$D\phi_R^T(T_{c(0)}\mathcal{L}) \cap T_{c(T)}\mathcal{L} = \{0\},$$

where $\phi_R^t : \Sigma \rightarrow \Sigma$ is the *Reeb flow* of R_α . We denote by $\text{Spec}(\Sigma, \alpha, \mathcal{L})$ the set of all lengths T of Reeb chords.

Definition 2.2. A time-independent Hamiltonian $H : \widehat{W} \rightarrow \mathbb{R}$ is called *admissible* if $H < 0$ on W and H is of the form $H(r, y) = ar + b$ on the symplectization $[1, \infty) \times \partial W$ with $a \notin \text{Spec}(\partial W, \alpha, \partial L)$, $a > 0$ and $b \in \mathbb{R}$. Here a is called the *slope* of H .

We denote by \mathcal{H} the space of admissible Hamiltonians on \widehat{W} . For any admissible Hamiltonian H , its C^2 -small generic perturbation on W is an admissible Hamiltonian which is non-degenerate (see [1, Lemma 8.1]). All Hamiltonian chords are contained in the compact domain W as $a \notin \text{Spec}(\partial W, \alpha, \partial L)$.

2.1.3. *Maslov index of Hamiltonian chords.* In order to define the Maslov index of Hamiltonian chords, we additionally assume that the Maslov class $\mu_{\widehat{L}} : \pi_2(\widehat{W}, \widehat{L}) \rightarrow \mathbb{Z}$ of the Lagrangian \widehat{L} vanishes. For example, this assumption is satisfied if $c_1(W)$ vanishes on $\pi_2(W)$ and $\pi_1(L) = 0$.

Let $D^+ = \{z \in \mathbb{C} \mid |z| \leq 1, \text{im } z \geq 0\}$ be the half-disk with real part $D_{\mathbb{R}}^+ := D^+ \cap \mathbb{R}$. A Hamiltonian chord c is contractible if and only if it admits a *capping half-disk*, namely, a map $v : (D^+, D_{\mathbb{R}}^+) \rightarrow (\widehat{W}, \widehat{L})$ satisfying $v(e^{\pi i t}) = c(t)$ for $t \in [0, 1]$. A capping half-disk $v : D^+ \rightarrow \widehat{W}$ of c yields a symplectic trivialization of $v^*T\widehat{W}$,

$$\mathfrak{T}_v : (v^*T\widehat{W}, \widehat{\omega}) \longrightarrow D^+ \times (\mathbb{C}^n, \omega_0),$$

such that $\mathfrak{T}_v(T_{v(z)}\widehat{L}) = \Lambda_0$ for $z \in D_{\mathbb{R}}^+$, where $\Lambda_0 = \{z \in \mathbb{C}^n \mid \text{im } z = 0\}$ is the horizontal Lagrangian subspace in $(\mathbb{C}^n, \omega_0 := (i/2) \sum_j dz_j \wedge d\bar{z}_j)$. Such a trivialization \mathfrak{T}_v is called an *adapted symplectic trivialization* of $v^*T\widehat{W}$. We write

$$\Psi_c : [0, 1] \longrightarrow \text{Sp}(2n), \quad \Psi_c(t) := \mathfrak{T}_v(c(t)) \circ D\phi_H^t(c(0)) \circ \mathfrak{T}_v^{-1}(c(0))$$

for the linearization of the Hamiltonian flow along c with respect to the trivialization \mathfrak{T}_v .

Definition 2.3. The *Maslov index* of a contractible Hamiltonian chord $c : [0, 1] \rightarrow \widehat{W}$ is

$$\mu(c) := \mu_{\text{RS}}(\Psi_c \Lambda_0, \Lambda_0) \in \frac{1}{2}\mathbb{Z},$$

where μ_{RS} denotes the Robbin–Salamon index (see §4.1.1).

Since the Maslov class $\mu_{\widehat{L}}$ vanishes, $\mu(c)$ is independent of the choices involved (see [26, Lemma 2.1]).

Remark 2.4. One can define the Maslov index using any Lagrangian subspace Λ in \mathbb{C}^n other than Λ_0 . The Maslov index is independent of the choice of Λ .

Definition 2.5. We define the *index* of $c \in \mathcal{P}(H)$ by $|c| := -\mu(c) - (n/2)$.

Remark 2.6. If c is non-degenerate, then we have $|c| \in \mathbb{Z}$ by [40, Theorem 2.4].

2.1.4. *Action functional.* We let

$$\mathcal{P} = \{c \in W^{1,2}([0, 1], \widehat{W}) \mid c(0), c(1) \in \widehat{L}; [c] = 0 \in \pi_1(\widehat{W}, \widehat{L})\} \tag{2.1}$$

be the space of contractible paths of class $W^{1,2}$ with ends in \widehat{L} . Recall that \widehat{L} is exact, so $\widehat{\lambda}|_{\widehat{L}} = d\ell$ for some $\ell \in C^\infty(\widehat{L})$. We fix any such ℓ . The *action functional* $\mathcal{A}_H : \mathcal{P} \rightarrow \mathbb{R}$ associated to a Hamiltonian $H \in C^\infty(\widehat{W})$ is defined by

$$\begin{aligned} \mathcal{A}_H(c) &= - \int_{D^+} v^* d\widehat{\lambda} - \int_0^1 H(c(t)) dt \\ &= \ell(c(1)) - \ell(c(0)) - \int_0^1 c^* \widehat{\lambda} - \int_0^1 H(c(t)) dt, \end{aligned}$$

where $v : D^+ \rightarrow \widehat{W}$ is a capping half-disk of $c \in \mathcal{P}$. We write $\text{crit}(\mathcal{A}_H)$ for the set of critical points of \mathcal{A}_H . Then $\text{crit}(\mathcal{A}_H) = \mathcal{P}(H)$. An $\widehat{\omega}$ -compatible almost complex structure J on \widehat{W} is said to be of *contact type* if it satisfies the condition $\widehat{\lambda} \circ J = dr$ on $[1, \infty) \times \partial W$. We abbreviate

$$\mathcal{J} := \{\{J_t\}_{t \in [0,1]} \mid \exists r_0 \geq 1 \text{ such that each } J_t \text{ is of contact type for } r \geq r_0\}. \tag{2.2}$$

An element $J \in \mathcal{J}$ is called *admissible*. In particular, for an admissible J we can apply the maximum principle for solutions to Floer-type equations (see [39, Lemma 4.5]).

Now we fix a non-degenerate admissible Hamiltonian $H \in C^\infty(\widehat{W})$ and $J = \{J_t\}_{t \in [0,1]} \in \mathcal{J}$. For $c_\pm \in \mathcal{P}(H)$ with $c_- \neq c_+$, we let $\widehat{\mathcal{M}}(c_-, c_+; H, J)$ be the *space of Floer strips* consisting of $u : \mathbb{R} \times [0, 1] \rightarrow \widehat{W}$ satisfying

$$\begin{cases} \partial_s u + J_t(u)(\partial_t u - X_H(u)) = 0, \\ \lim_{s \rightarrow \pm\infty} u(s, t) = c_\pm(t), \\ u(s, 0), u(s, 1) \in \widehat{L}. \end{cases}$$

This space carries a free \mathbb{R} -action which is given by translation in the s -variable. Its quotient

$$\mathcal{M}(c_-, c_+; H, J) := \widehat{\mathcal{M}}(c_-, c_+; H, J) / \mathbb{R}$$

is called the *moduli space of Floer strips* from c_- to c_+ . The action decreases along Floer strips, and a standard argument (see [9, 41]) shows that $\mathcal{M}(c_-, c_+; H, J)$ is a smooth manifold of dimension $|c_-| - |c_+| - 1$ for a generic $J \in \mathcal{J}$.

Let $\tau \in \mathbb{R} \cup \{\infty\}$. The *filtered wrapped Floer chain complex* is defined by

$$CF_*^{<\tau}(H) = \bigoplus_{\substack{c \in \mathcal{P}(H) \text{ with } |c|=* \\ \mathcal{A}_H(c) < \tau}} \mathbb{Z}_2 \langle c \rangle.$$

The *Floer differential* is defined by counting rigid Floer strips

$$\partial : CF_*^{<\tau}(H) \rightarrow CF_{* - 1}^{<\tau}(H), \quad \partial(c_-) = \sum_{\substack{c_+ \in \mathcal{P}(H) \\ |c_-| - |c_+| = 1}} \#_2 \mathcal{M}(c_-, c_+; H, J) \cdot c_+,$$

where $\#_2$ denotes the count modulo two. For a generic choice of J , the differential is well defined and satisfies $\partial \circ \partial = 0$. We define the *filtered wrapped Floer homology of the Hamiltonian H* as the homology of the chain complex

$$HF_*^{<\tau}(H, J) = H_*(CF_*^{<\tau}(H), \partial).$$

For generic pairs (H_{\pm}, J_{\pm}) with $H_- \leq H_+$, one can define a *continuation map* $HF_*^{<\tau}(H_-, J_-) \rightarrow HF_*^{<\tau}(H_+, J_+)$ by counting rigid s -dependent Floer strips (see [39, §3.2] for the construction). Using continuation maps, one shows that $HF_*^{<\tau}(H, J)$ does not depend on the choice of J and only depends on the slope of H (see [39, Lemma 3.1]). We therefore omit J from the notation. We define the *filtered wrapped Floer homology of a Lagrangian L* by

$$HW_*^{<\tau}(L; W) := \varinjlim_H HF_*^{<\tau}(H),$$

where the direct limit is taken over admissible Hamiltonians using continuation maps. If $\tau = \infty$, we call $HW_*(L; W) := HW_*^{<\infty}(L; W)$ the *wrapped Floer homology of L* .

2.1.5. Positive wrapped Floer homology. Fix $\varepsilon > 0$, which is smaller than the minimum length of Reeb chords from ∂L to ∂L . We define the *positive wrapped chain complex* as the quotient chain complex

$$CF_*^+(H) := CF_*(H) / CF_*^{<\varepsilon}(H)$$

equipped with the induced differential. Following the usual limit procedure, the resulting homology group, denoted by $HW_*^+(L; W)$, is called *positive wrapped Floer homology* of the Lagrangian L . There exists a short exact sequence

$$0 \rightarrow CF_*^{<\varepsilon}(H) \rightarrow CF_*(H) \rightarrow CF_*^+(H) \rightarrow 0,$$

which induces a long exact sequence

$$\dots \rightarrow HF_*^{<\varepsilon}(H) \rightarrow HF_*(H) \rightarrow HF_*^+(H) \rightarrow \dots$$

Taking the direct limit with respect to H , we obtain the tautological exact sequence

$$\dots \rightarrow H_{*+n}(L, \partial L) \rightarrow HW_*(L; W) \rightarrow HW_*^+(L; W) \rightarrow \dots, \tag{2.3}$$

where we have used the fact that $HW_*^{<\varepsilon}(L; W)$ is isomorphic to $H_{*+n}(L, \partial L)$ (see [47, Proposition 1.3]).

2.2. *Vanishing of wrapped Floer homology.* In this section, we show that (non-equivariant) wrapped Floer homology vanishes under a displaceability assumption. In [23], using the notion of leaf-wise intersections, it is proved that the symplectic homology of a Liouville domain vanishes if its boundary is displaceable. We carry out this idea in wrapped Floer homology.

2.2.1. *Displaceability.* In what follows, (W, λ) is a Liouville domain, $\Sigma = \partial W$ is a contact-type boundary, L is an admissible Lagrangian and $\mathcal{L} := L \cap \Sigma$ is a Legendrian submanifold in Σ . The following notion of displaceability is taken from [35].

Definition 2.7. The hypersurface Σ is *displaceable from \widehat{L} in \widehat{W}* if there exists a compactly supported Hamiltonian diffeomorphism ϕ on \widehat{W} such that $\phi(\Sigma) \cap \widehat{L} = \emptyset$.

Example 2.8. Star-shaped hypersurfaces Σ in \mathbb{R}^{2n} are displaceable from the real Lagrangian $\widehat{L} = \{(q, p) \in \mathbb{R}^{2n} \mid p = 0\}$.

PROPOSITION 2.9. *If Σ is displaceable from \widehat{L} in \widehat{W} , then W is displaceable from \widehat{L} in \widehat{W} by the same displacing Hamiltonian diffeomorphism.*

Proof. Let $\phi : \widehat{W} \rightarrow \widehat{W}$ be a Hamiltonian diffeomorphism displacing Σ . Note that the hypersurface $\Sigma \subset \widehat{W}$ separates \widehat{W} into two connected components: the bounded piece $\text{int}(W)$ and the unbounded piece $\widehat{W} \setminus W$. The image $\phi(\Sigma)$ also separates \widehat{W} into the bounded piece $\phi(\text{int}(W))$ and the other. Since \widehat{L} is connected and unbounded, \widehat{L} cannot entirely be contained in the bounded piece $\phi(\text{int}(W))$. Since $\phi(\text{int}(W))$ is bounded by $\phi(\Sigma)$, it follows that if $\phi(\text{int}(W)) \cap \widehat{L} \neq \emptyset$, then $\phi(\Sigma) \cap \widehat{L} \neq \emptyset$. \square

2.2.2. *Filtered wrapped Floer homology and a tweaked action functional.* Let $H_\tau : \widehat{W} \rightarrow \mathbb{R}$ be an admissible Hamiltonian of slope τ and take any *compactly supported* Hamiltonian $F \in C_c^\infty([0, 1] \times \widehat{W})$. Following [35], we define a *tweaked action functional* $\mathcal{A}_{H_\tau}^F : \mathcal{P} \rightarrow \mathbb{R}$ by

$$\mathcal{A}_{H_\tau}^F(c) = \ell(c(1)) - \ell(c(0)) - \int_0^1 c^* \widehat{\lambda} - \int_0^1 \beta(t) H_\tau(c(t)) dt - \int_0^1 \dot{\chi}(t) F(\chi(t), c(t)) dt,$$

where $\beta : [0, 1] \rightarrow \mathbb{R}$ is a non-negative smooth function with $\beta(t) = 0$ for $t \in [\frac{1}{2}, 1]$ and $\int_0^1 \beta(t) dt = 1$, and $\chi : [0, 1] \rightarrow [0, 1]$ is a smooth monotone function such that $\chi(\frac{1}{2}) = 0$ and $\chi(1) = 1$. The following lemma is straightforward.

LEMMA 2.10. *The differential of $\mathcal{A}_{H_\tau}^F$ is given by*

$$d\mathcal{A}_{H_\tau}^F(c)(\zeta) = \int_0^1 \widehat{\omega}(\dot{c}(t) - X_{\beta(t)H_\tau}(c(t)) - \dot{\chi}(t)X_F(\chi(t), c(t)), \zeta(t)) dt \quad \text{for } \zeta \in T_c \mathcal{P}.$$

Consequently, $c \in \text{crit}(\mathcal{A}_{H_\tau}^F)$ if and only if $c \in \mathcal{P}$ solves the equation

$$\dot{c}(t) = \beta(t)X_{H_\tau}(c(t)) + \dot{\chi}(t)X_F(\chi(t), c(t)). \tag{2.4}$$

Remark 2.11. Critical points of the tweaked action functional $\mathcal{A}_{H_\tau}^F$ are closely related to the notion of *relative leaf-wise intersections* in [35].

The following lemma allows us to define the Floer homology of $\mathcal{A}_{H_\tau}^F$ for generic F . The proof is analogous to [2, Appendix A] (see also [35, Theorem 2.28]).

LEMMA 2.12. *For a generic F in the C^∞ -topology, the tweaked functional $\mathcal{A}_{H_\tau}^F$ is Morse.*

Denote the Floer homology of the tweaked action functional $\mathcal{A}_{H_\tau}^F$ by $HF_*(\mathcal{A}_{H_\tau}^F)$.

PROPOSITION 2.13. $HF_*(\mathcal{A}_{H_\tau}^F) \cong HW_*^{<\tau}(L; W)$.

Proof. It follows from a standard argument using continuation maps with respect to a homotopy between F and the constant function 0 that $HF_*(\mathcal{A}_{H_\tau}^F) \cong HF_*(H_\tau)$ (we refer to [23, §2.2] for details). On the other hand, since H_τ is an admissible Hamiltonian of slope τ , we have $HF_*(H_\tau) \cong HW_*^{<\tau}(L; W)$. □

2.2.3. *Vanishing of wrapped Floer homology.* We shall prove the following vanishing property.

THEOREM 2.14. *If Σ is displaceable from \widehat{L} in \widehat{W} , then the inclusion map*

$$\iota_* : HW_*^{<\tau}(L; W) \rightarrow HW_*^{<\tau+e(L)}(L; W)$$

is a zero map for all $\tau \in \mathbb{R}$, where $e(L)$ is defined as in (2.5).

An immediate corollary is the following.

COROLLARY 2.15. *If Σ is displaceable from \widehat{L} in \widehat{W} , the wrapped Floer homology $HW_*(L; W)$ vanishes.*

Remark 2.16. If Σ is displaceable from itself, then we already know that $HW_*(L; W)$ vanishes for any admissible Lagrangian L . In this case, the symplectic homology $SH_*(W)$ vanishes as shown in [23]. By the result in [39, Theorem 6.17], the wrapped Floer homology $HW_*(L; W)$ is a module over $SH_*(W)$ for any admissible Lagrangian L . It follows that the wrapped Floer homology $HW_*(L; W)$ vanishes.

For a given compactly supported Hamiltonian $F \in C_c^\infty([0, 1] \times \widehat{W})$, we define

$$f := \max_{t \in [0, 1]} \text{Supp}(r \circ F(t, \cdot)).$$

Here r denotes the cylindrical coordinate, and if F is supported only in W , then we put $f = 0$ by convention. In the rest of this section, we assume that Σ is displaceable from \widehat{L} and that ϕ_F is a displacing Hamiltonian diffeomorphism.

LEMMA 2.17. *If $c \in \text{crit}(\mathcal{A}_{H_\tau}^F)$, then we have $r \circ c(0) \in (1, f]$. In particular, if $f = 0$, the set $\text{crit}(\mathcal{A}_{H_\tau}^F)$ is empty.*

Proof. Note that c is a solution of the equation (2.4). Suppose that $r \circ c(0) > \mathfrak{f}$. Then c is a Reeb chord of period $\tau \notin \text{Spec}(\Sigma, \alpha, \mathcal{L})$, which contradicts the choice of τ . Suppose that $r \circ c(0) \leq 1$; in other words, $c(0) \in L$. Then $\phi_{H_\tau}^t(c(0)) \subset W$ for all $t \in \mathbb{R}$ since H_τ is admissible. Therefore, we have $\phi_F(\phi_{H_\tau}^t(c(0))) \in \widehat{L}$ for some $t \in \mathbb{R}$. In particular, $\phi_F(W) \cap \widehat{L} \neq \emptyset$. This contradicts Proposition 2.9. \square

$$\text{We set } \|F\|_- := \int_0^1 \min_{y \in \widehat{W}} F(t, y) dt.$$

LEMMA 2.18. *For each $a > 0$, there exists $\tau = \tau(a) > 0$ as large as one likes such that there is no critical point c of $\mathcal{A}_{H_\tau}^F$ with $\mathcal{A}_{H_\tau}^F(c) < a + \|F\|_-$. Consequently, the filtered chain complex $CF_*^{<a+\|F\|_-}(\mathcal{A}_{H_\tau}^F)$ and hence the homology $HF_*^{<a+\|F\|_-}(\mathcal{A}_{H_\tau}^F)$ vanishes.*

Proof. For notational convenience, we denote $\tilde{F}(t, \cdot) := \dot{\chi}(t)F(\chi(t), \cdot)$, and we choose, without loss of generality, $H_\tau(r, y) = \tau r - \tau$ for $r > 1$. By Lemma 2.17, if $c \in \text{crit}(\mathcal{A}_{H_\tau}^F)$, then $r \circ c(0) \in (1, \mathfrak{f}]$. In particular, c is a solution of the equation $\dot{c} = -\tau\beta(t)R(c) + X_{\tilde{F}}(t, c)$ where R denotes the Reeb vector field. Abbreviating $r_0 := r \circ c(0)$ we compute

$$\begin{aligned} \mathcal{A}_{H_\tau}^F(c) &= \ell(c(1)) - \ell(c(0)) - \int_0^1 r\alpha(-\tau\beta(t)R(c) + X_{\tilde{F}}(t, c)) dt \\ &\quad - \int_0^1 (\beta(t)H_\tau(c) + \tilde{F}(t, c)) dt \\ &= \ell(c(1)) - \ell(c(0)) + \tau \cdot r_0 - \int_0^1 r\alpha(X_{\tilde{F}}(t, c)) dt \\ &\quad - (\tau \cdot r_0 - \tau) - \int_0^1 \tilde{F}(t, c) dt \\ &= \ell(c(1)) - \ell(c(0)) + \tau - \int_0^1 (r\alpha(X_{\tilde{F}}(t, c)) + \tilde{F}(t, c)) dt. \end{aligned}$$

Since $r \circ c \leq \mathfrak{f}$, the term $\int_0^1 (r\alpha(X_{\tilde{F}}(t, c)) + \tilde{F}(t, c)) dt$ is bounded from above by a constant $C_F > 0$ which does not depend on c . It follows that

$$\mathcal{A}_{H_\tau}^F(c) \geq \ell(c(1)) - \ell(c(0)) + \tau - C_F.$$

We then take $\tau = \tau(a) > 0$ such that $\ell(c(1)) - \ell(c(0)) + \tau - C_F \geq a + \|F\|_-$. This completes the proof. \square

We now prove the vanishing property. This will basically be done by showing that the inclusion $\iota_* : HF_*^{<a}(\mathcal{A}_{H_{\tau(a)}}) \rightarrow HF_*^{<a+\|F\|_-}(\mathcal{A}_{H_{\tau(a)}})$ factors through $HF_*^{<a+\|F\|_-}(\mathcal{A}_{H_\tau}^F)$, which vanishes for suitable $\tau = \tau(a)$. Define the displacement energy of Σ from \widehat{L} in \widehat{W} by

$$e(L) := \inf\{\|F\| : F \in C_c^\infty([0, 1] \times \widehat{W}), \phi_F(\Sigma) \cap \widehat{L} = \emptyset\}. \tag{2.5}$$

Proof of Theorem 2.14. Choosing a generic homotopy F^s such that $F^0 \equiv 0$ and $F^1 = \dot{\chi}(t)F(\chi(t), \cdot)$, we have continuation maps on filtered homology groups

$$\begin{aligned} \Phi_*^{<a} : HF_*^{<a}(\mathcal{A}_{H_\tau}) &\rightarrow HF_*^{<a+\|F\|_-}(\mathcal{A}_{H_\tau}^F) \\ \Psi_*^{<a} : HF_*^{<a}(\mathcal{A}_{H_\tau}^F) &\rightarrow HF_*^{<a+\|F\|_-}(\mathcal{A}_{H_\tau}) \end{aligned}$$

for each $a > 0$. We remark that the action shift by $\|F\|_-$ (or $\|-F\|_-$) is due to the energy consumption property of parametrized Floer solutions which we count to define the continuation maps (see Remark 2.19). By composing the two maps, we have a map (note that $\|-F\|_- + \|F\|_- = \|F\|$)

$$\Psi_*^{<a+\|F\|_-} \circ \Phi_*^{<a} : HF_*^{<a}(\mathcal{A}_{H_\tau}) \rightarrow HF_*^{<a+\|F\|}(\mathcal{A}_{H_\tau}),$$

and by the usual homotopy of homotopies argument in Floer theory, we have that

$$\Psi_*^{<a+\|F\|_-} \circ \Phi_*^{<a} = \iota_*$$

where ι_* is the map induced by inclusion at the chain level.

Taking $\tau = \tau(a)$ as in Lemma 2.18, we have $HF_*^{<a+\|F\|_-}(\mathcal{A}_{H_\tau}^F) = 0$. Therefore the map

$$\iota_* : HF_*^{<a}(\mathcal{A}_{H_\tau}) \rightarrow HF_*^{<a+\|F\|}(\mathcal{A}_{H_\tau})$$

vanishes for all F and a . Finally, we take the direct limit along $\tau \rightarrow \infty$, and we conclude that the inclusion

$$\iota_* : HW_*^{<a}(L; W) \rightarrow HW_*^{<a+e(L)}(L; W)$$

also vanishes. □

Remark 2.19. For a solution of the parametrized Floer equation $w : \mathbb{R} \times [0, 1] \rightarrow \widehat{W}$, the energy

$$E(w) := \int_{-\infty}^{\infty} \|\partial_s w\|^2 ds$$

satisfies

$$E(w) \leq \mathcal{A}_{\beta \cdot H_\tau}(w_-) - \mathcal{A}_{H_\tau}^F(w_+) + \|F\|_-.$$

3. Equivariant wrapped Floer homology

3.1. \mathbb{Z}_2 -complexes. As an algebraic preliminary to developing equivariant theories, we briefly outline the notion of a \mathbb{Z}_2 -complex. The parallel notion of an S^1 -complex can be found in [7].

Definition 3.1. A \mathbb{Z} -graded chain complex (C_*, ∂) with \mathbb{Z}_2 -coefficients is called a \mathbb{Z}_2 -complex if it admits the additional datum of a sequence of maps

$$\phi = (\phi_0 = \partial, \phi_1, \phi_2, \dots)$$

such that ϕ_j has degree $j - 1$: i.e., $\phi_j : C_k \rightarrow C_{k+j-1}$, and satisfies the relations

$$\text{for all } k \geq 0, \quad \sum_{i+j=k} \phi_i \circ \phi_j = 0. \tag{3.1}$$

Remark 3.2. The first two terms of the relation (3.1) are

$$\partial \circ \partial = 0, \quad \partial \circ \phi_1 + \phi_1 \circ \partial = 0.$$

Given a \mathbb{Z}_2 -complex (C_*, ∂) , we form the \mathbb{Z}_2 -equivariant chain complex $(C_*^{\mathbb{Z}_2}, \partial^{\mathbb{Z}_2})$ by

$$C_*^{\mathbb{Z}_2} := \mathbb{Z}_2[w] \otimes_{\mathbb{Z}_2} C_*,$$

where w is a formal variable of degree one and

$$\partial^{\mathbb{Z}_2}(w^l \otimes x) := \sum_{j=0}^l w^{l-j} \phi_j(x). \tag{3.2}$$

We note that the differential formally splits into

$$\partial^{\mathbb{Z}_2} = \partial + w^{-1}\phi_1 + w^{-2}\phi_2 + \dots$$

It follows immediately from the relations (3.1) that we have $\partial^{\mathbb{Z}_2} \circ \partial^{\mathbb{Z}_2} = 0$.

Definition 3.3. The \mathbb{Z}_2 -equivariant homology of the \mathbb{Z}_2 -complex (C_*, ϕ) is the homology of the \mathbb{Z}_2 -equivariant chain complex $(C_*^{\mathbb{Z}_2}, \partial^{\mathbb{Z}_2})$: i.e.,

$$H_*^{\mathbb{Z}_2}(C_*, \phi) := H_*(C_*^{\mathbb{Z}_2}, \partial^{\mathbb{Z}_2}).$$

3.1.1. *Leray–Serre-type spectral sequence.* For each $p \in \mathbb{Z}$, we denote the \mathbb{Z}_2 -module of polynomials in w of degree less than or equal to p by $\mathbb{Z}_2[w]/\{w^{p+1} = 0\}$. We set $F_p C_*^{\mathbb{Z}_2} = 0$ for $p < 0$. Note that the tensor product

$$F_p C_*^{\mathbb{Z}_2} := \mathbb{Z}_2[w]/\{w^{p+1} = 0\} \otimes_{\mathbb{Z}_2} C_*$$

is a subgroup in $C_*^{\mathbb{Z}_2}$. The following lemma is straightforward from the definition.

LEMMA 3.4. *The subgroups $\{F_p C_*^{\mathbb{Z}_2}\}_{p \in \mathbb{Z}}$ define a filtration on $(C_*^{\mathbb{Z}_2}, \partial^{\mathbb{Z}_2})$ which is bounded from below and is exhaustive.*

By a standard theorem of homological algebra, we have a spectral sequence $\{E_{p,q}^r\}$ which converges to the \mathbb{Z}_2 -equivariant homology $H_*^{\mathbb{Z}_2}(C_*)$. Denoting the induced filtration on $C_k^{\mathbb{Z}_2}$ by $F_p C_k^{\mathbb{Z}_2}$ for each $k \geq 0$, the E^0 -term is given by

$$E_{p,q}^0 := F_p C_{p+q}^{\mathbb{Z}_2} / F_{p-1} C_{p+q}^{\mathbb{Z}_2},$$

and the differential $d_{p,q}^0$ is the one induced by $\partial^{\mathbb{Z}_2}$ on the quotient $E_{p,q}^0$.

LEMMA 3.5. *The E^1 -term of the spectral sequence is given by*

$$E_{p,q}^1 = \begin{cases} H_q(C_*) & \text{if } p \geq 0, \\ 0 & \text{if } p < 0. \end{cases}$$

The differential $d_{p,q}^1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1$ is given by $d^1 = \phi_1$.

Proof. Observe that we can identify $E_{p,q}^0$ with $\mathbb{Z}_2\langle w^p \rangle \otimes C_q$ using the fact that we are over the field \mathbb{Z}_2 . For an element $w^p \otimes x \in \mathbb{Z}_2\langle w^p \rangle \otimes C_q$,

$$\partial^{\mathbb{Z}_2}(w^p \otimes x) = \sum_{j=0}^p w^{p-j} \otimes \phi_j(x),$$

and this implies, passing to the quotient, that

$$d_{p,q}^0(w^p \otimes x) = w^p \otimes \phi_0(x) = w^p \otimes \partial(x).$$

In other words,

$$d_{p,q}^0 = \text{id} \otimes \partial,$$

so that $E_{p,q}^1 = \mathbb{Z}_2\langle w^p \rangle \otimes_{\mathbb{Z}_2} H_q(C_*) = H_q(C_*)$.

The same computation shows that

$$d_{p,q}^1(w^p \otimes x) = w^{p-1} \otimes \phi_1(x).$$

Under the identification $\mathbb{Z}_2\langle w^p \rangle \otimes_{\mathbb{Z}_2} H_q(C_*) = H_q(C_*)$, we can write $d^1 = \phi_1$, as asserted. □

Remark 3.6. Since $H_*(\mathbb{R}P^\infty) = H_*(B\mathbb{Z}_2) = \mathbb{Z}_2[w]$, one may rephrase the E^1 -term as

$$E_{p,q}^1 = H_p(B\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_q(C_*)$$

3.2. Equivariant Morse homology. Before defining equivariant wrapped Floer homology, we describe the construction of a finite-dimensional model, namely, equivariant Morse homology, following [43]. This construction is the so-called family Morse homology theory (see [22]). Throughout this paper, we use \mathbb{Z}_2 -coefficients in (equivariant) homology.

3.2.1. \mathbb{Z}_2 -spaces and the Borel constructions. A topological space endowed with a topological \mathbb{Z}_2 -action is called a \mathbb{Z}_2 -space. Note that a topological space X is a \mathbb{Z}_2 -space if and only if it admits an involution. It is well known that $B\mathbb{Z}_2 = \mathbb{R}P^\infty = \varinjlim_N \mathbb{R}P^N$

and $E\mathbb{Z}_2 = S^\infty = \varinjlim_N S^N$. For a \mathbb{Z}_2 -space X , the group \mathbb{Z}_2 acts on the product $X \times E\mathbb{Z}_2$ diagonally and its quotient space

$$X_{\text{Borel}} := X \times_{\mathbb{Z}_2} E\mathbb{Z}_2$$

is called the *Borel construction*. If X is a point, then $X_{\text{Borel}} = E\mathbb{Z}_2/\mathbb{Z}_2 = B\mathbb{Z}_2 = \mathbb{R}P^\infty$. In the Morse set-up, our approach to the equivariant theory corresponds to considering the fibration

$$X \hookrightarrow X_{\text{Borel}} \longrightarrow \mathbb{R}P^\infty,$$

where the projection map is induced by the projection onto the second factor $pr_2 : X \times E\mathbb{Z}_2 \rightarrow E\mathbb{Z}_2$. The *equivariant homology of a \mathbb{Z}_2 -space X* is defined as the singular homology of its Borel construction: i.e.,

$$H_*^{\mathbb{Z}_2}(X) := H_*(X_{\text{Borel}}).$$

If the \mathbb{Z}_2 -action is free, i.e., if the involution has no fixed points, then

$$H_*^{\mathbb{Z}_2}(X) = H_*(X/\mathbb{Z}_2).$$

In practice, we approximate $E\mathbb{Z}_2 = S^\infty$ by finite-dimensional spheres S^N on which \mathbb{Z}_2 still acts freely. It is known that

$$H_*^{\mathbb{Z}_2}(X) = \varinjlim_N H_*(X \times_{\mathbb{Z}_2} S^N),$$

where the direct limit is taken over morphisms induced by the equivariant inclusions $X \times_{\mathbb{Z}_2} S^N \hookrightarrow X \times_{\mathbb{Z}_2} S^{N+1}$.

3.2.2. \mathbb{Z}_2 -equivariant Morse chain complex. Let M be a closed manifold with a \mathbb{Z}_2 -action, induced by an involution $\iota : M \rightarrow M$. The quotient of the unit-sphere $S^N \subset \mathbb{R}^{N+1}$ by the antipodal map $\sigma(z) = -z$ is $\mathbb{R}P^N$. We take the perfect Morse function $f : \mathbb{R}P^N \rightarrow \mathbb{R}$ whose lift $\tilde{f} : S^N \rightarrow \mathbb{R}$ to the covering S^N is given by $\tilde{f}(z) = \sum_{j=0}^N j|z_j|^2$. The function f has precisely one critical point, say, $z^{(j)}$, of each index $0 \leq j \leq N$. We take a family of functions $h_z : M \rightarrow \mathbb{R}$ parametrized by $z \in S^N$, which yields a function $h : M \times S^N \rightarrow \mathbb{R}$ such that:

- h is \mathbb{Z}_2 -invariant in the sense that $h_{-z} \circ \iota = h_z$;
- $h + \tilde{f}$ is Morse, and if $(x, z) \in \text{crit}(h + \tilde{f})$, then $z \in \text{crit}(\tilde{f})$ (and hence $x \in \text{crit}(h_z)$);
- for each $z \in \text{crit}(\tilde{f})$, the corresponding function h_z is Morse; and
- h_z is independent of z near each critical point of \tilde{f} .

A construction of such an h_z is illustrated in Example 3.18 where a similar construction in Floer theory is given. The Morse case is similar but much simpler.

Remark 3.7. Note that $h_z : M \rightarrow \mathbb{R}$ itself need not be \mathbb{Z}_2 -invariant for a fixed $z \in S^N$.

Let g_{S^N} be the round metric on S^N . It is \mathbb{Z}_2 -invariant and Morse–Smale with respect to \tilde{f} . Take a \mathbb{Z}_2 -invariant family of metrics $g = \{g_z\}_{z \in S^N}$ on M , i.e., $\iota^*g_{-z} = g_z$, which is locally constant near the critical points of \tilde{f} . We then have a \mathbb{Z}_2 -invariant metric $g_z \oplus g_{S^N}$ on $M \times S^N$, which, by abuse of notation, we still denote by g .

Remark 3.8. For a fixed $z \in S^N$, the metric g_z is not necessarily \mathbb{Z}_2 -invariant.

The critical points of $h + \tilde{f}$ always come in pairs due to the \mathbb{Z}_2 -invariance: $p = (x, z) \in \text{crit}(h + \tilde{f})$ if and only if $\bar{p} := (\iota(x), -z) \in \text{crit}(h + \tilde{f})$. We denote by $Z_p = \{p, \bar{p}\}$ the \mathbb{Z}_2 -pair of $p \in \text{crit}(h)$. Its index is defined as

$$|Z_p| := \text{ind}(x; h_z) + \text{ind}(z; \tilde{f}), \quad p = (x, z).$$

We define the *equivariant Morse chain complex* by

$$CM_*^{\mathbb{Z}_2, N}(h, g) := \bigoplus_{\substack{Z_p \subset \text{crit}(h + \tilde{f}) \\ |Z_p| = *}} \mathbb{Z}_2\langle Z_p \rangle.$$

Given two critical points $p_\pm = (x_\pm, z_\pm) \in \text{crit}(h + \tilde{f})$, we denote by $\widehat{\mathcal{M}}(Z_{p_-}, Z_{p_+}; h, g)$ the space of pairs (u, v) , where $u : \mathbb{R} \rightarrow M$ and $v : \mathbb{R} \rightarrow S^N$ are solutions to the system

of ordinary differential equations

$$\begin{cases} \partial_s u(s) + \nabla h_{v(s)}(u(s)) = 0, \\ \partial_s v(s) + \nabla \tilde{f}(v(s)) = 0, \end{cases}$$

with asymptotic conditions

$$\lim_{s \rightarrow \pm\infty} (u(s), v(s)) \in Z_{p\pm}.$$

Remark 3.9. Note that the above system of differential equations is *not* the negative gradient flow equation of a Morse function $h : M \times S^N \rightarrow \mathbb{R}$.

For $Z_{p-} \neq Z_{p+}$, denote the quotient of $\widehat{\mathcal{M}}(Z_{p-}, Z_{p+}; h, g)$ by the \mathbb{R} -action by $\mathcal{M}(Z_{p-}, Z_{p+}; h, g)$. Note that this moduli space carries a free \mathbb{Z}_2 -action. We denote its \mathbb{Z}_2 -quotient by $\mathcal{M}_{\mathbb{Z}_2}(Z_{p-}, Z_{p+}; h, g)$. By a standard argument, for a generic g , the moduli space $\mathcal{M}_{\mathbb{Z}_2}(Z_{p-}, Z_{p+}; h, g)$ is a smooth manifold of dimension $|Z_{p-}| - |Z_{p+}| - 1$. In particular, if $|Z_{p-}| - |Z_{p+}| = 1$, then this space is a compact zero-dimensional manifold and hence a finite set. We define the *equivariant Morse differential* by

$$\partial^{\mathbb{Z}_2}(Z_{p-}) = \sum_{|Z_{p-}| - |Z_{p+}| = 1} \#_2 \mathcal{M}_{\mathbb{Z}_2}(Z_{p-}, Z_{p+}; h, g) \cdot Z_{p+}.$$

Denote by $HM_*^{\mathbb{Z}_2, N}(M)$ the homology of the equivariant Morse chain complex $(CM_*^{\mathbb{Z}_2, N}(h, g), \partial^{\mathbb{Z}_2})$. As the notation indicates, it can be shown that this group does not depend on the choice of (h, g) . Finally, the *equivariant Morse homology* is defined by taking the direct limit with respect to the maps induced by the equivariant inclusions $S^N \hookrightarrow S^{N+1}$: i.e.,

$$HM_*^{\mathbb{Z}_2}(M) := \varinjlim_N HM_*^{\mathbb{Z}_2, N}(M).$$

As in the non-equivariant case, the equivariant Morse homology is isomorphic to the equivariant singular homology. See [22, Theorem 5.1] and [7, Proposition 2.5] for this result in the symplectic homology set-up.

3.2.3. Periodic family and \mathbb{Z}_2 -complex structure. In this section, we re-interpret the differential $\partial^{\mathbb{Z}_2}$ as a \mathbb{Z}_2 -equivariant differential of a \mathbb{Z}_2 -complex in the sense of §3.1. The point is that we can canonically identify fibers of M_{Borel} over a critical point of f with each other. To do this, we use the canonical shift on $\mathbb{R}P^\infty$ given by the infinite shift

$$\tau : \mathbb{R}P^\infty \rightarrow \mathbb{R}P^\infty, \quad [z_0 : z_1 : \dots] \mapsto [0 : z_0 : z_1 : \dots].$$

Since $\mathbb{R}P^\infty$ is no longer a smooth manifold, let us clarify this symmetry in a finite approximation following ideas in [7, §2.3]. From now on, we write $h_N := h : M \times S^N \rightarrow \mathbb{R}$ for $N \geq 1$, where h is given in §3.2.2.

Consider the two embeddings

$$\begin{aligned} \tilde{\tau}_0 : S^N &\rightarrow S^{N+1}, & (z_0, \dots, z_N) &\mapsto (z_0, \dots, z_N, 0), \\ \tilde{\tau}_1 : S^N &\rightarrow S^{N+1}, & (z_0, \dots, z_N) &\mapsto (0, z_0, \dots, z_N). \end{aligned}$$

Note that $\tilde{\tau}_0$ is the canonical embedding and $\tilde{\tau}_1$ is an embedding given by the shift. They induce embeddings

$$\tilde{\mathcal{T}}_j : M \times S^N \rightarrow M \times S^{N+1}, \quad (x, z) \mapsto (x, \tilde{\tau}_j(z)). \tag{3.3}$$

We put an additional condition on $\{(h_N, g_N)\}_{N \geq 1}$, which we call *periodicity*: i.e.,

$$h_N = \tilde{\mathcal{T}}_0^* h_{N+1} = \tilde{\mathcal{T}}_1^* h_{N+1}, \quad g_N = \tilde{\mathcal{T}}_0^* g_{N+1} = \tilde{\mathcal{T}}_1^* g_{N+1}$$

hold for all $N \geq 1$.

Remark 3.10. A periodic family of functions $\{h_N\}$ can always be constructed from a given Morse function $h_0 : M \rightarrow \mathbb{R}$. One can use the same recipe as in Example 3.18. Indeed, if we choose a family of local slices and bump functions $\beta : S^N \rightarrow [0, 1]$ in a periodic manner, the resulting family $\{h_N\}$ is periodic. The same argument applies to construct a periodic family of metrics $\{g_N\}$.

We also observe that $\{(\tilde{f}_N, g_{S^N})\}_{N \geq 1}$ is *periodic*, which means that, for all $N \geq 1$:

- $\text{im}(\tilde{\tau}_0)$ and $\text{im}(\tilde{\tau}_1)$ are invariant under the gradient flow of \tilde{f}_{N+1} ;
- $\tilde{f}_N = \tilde{\tau}_0^* \tilde{f}_{N+1} = \tilde{\tau}_1^* \tilde{f}_{N+1} + \text{const}$, $g_{S^N} = \tilde{\tau}_0^* g_{S^{N+1}} = \tilde{\tau}_1^* g_{S^{N+1}}$.

This periodicity implies that the gradient flow of \tilde{f}_N is preserved under $\tilde{\tau}_0$ and $\tilde{\tau}_1$. A nice feature of the periodicity condition is that it simplifies our equivariant Morse chain complex.

Recall that the perfect Morse function $f_N : \mathbb{R}P^N \rightarrow \mathbb{R}$ has precisely one critical point $z^{(j)}$ of each index $0 \leq j \leq N$. Its preimage under the projection is denoted by $\{\pm z^{(j)}\} \subset S^N$. Fix a Morse function $h_0 := h_{N,z^{(0)}} : M \rightarrow \mathbb{R}$. By the periodicity, we can assume that $z^{(j)}$ are chosen such that each Morse function $h_{N,z^{(j)}} : M \rightarrow \mathbb{R}$ on the fiber $M \times \{z^{(j)}\}$ is given by h_0 . Note that $h_{N,-z^{(j)}}$ is automatically determined by the \mathbb{Z}_2 -invariance: $h_{N,-z^{(j)}} = \iota^* h_{N,z^{(j)}}$. We choose a metric g_0 on M which is Morse–Smale with respect to h_0 . We then have a canonical identification between critical points of $h_{N,z^{(j)}}$ and h_0 for all $j \geq 1$. For a critical point $x \in \text{crit}(h_0)$, we formally denote the corresponding critical point of $h_{N,z^{(j)}}$ by $w^j x$, where, as before, w denotes a formal variable of degree one. As a result, the equivariant Morse chain group is simplified under this identification to

$$CM_*^{\mathbb{Z}_2, N}(h_N, g_N) = \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes_{\mathbb{Z}_2} CM_*(h_0, g_0). \tag{3.4}$$

Indeed, every generator in $CM_*^{\mathbb{Z}_2, N}(h_N, g_N)$ is of the form Z_p for some $p = (w^j x, z^{(j)})$ with $x \in \text{crit}(h_0)$ and $j \geq 0$. Then the generator Z_p corresponds uniquely to $w^j \otimes x$ in the identification (3.4). We also simplify the moduli spaces which we count for $\partial^{\mathbb{Z}_2}$ in the following way. Since our Morse data are periodic, there is a one-to-one correspondence

$$\mathcal{M}_{\mathbb{Z}_2}(Z_{p_-}, Z_{p_+}; h_N, g_N) \cong \mathcal{M}_{\mathbb{Z}_2}(Z_{q_-}, Z_{q_+}; h_N, g_N), \tag{3.5}$$

where $p_- = (w^j x_-, z^{(j)})$, $p_+ = (w^k x_+, z^{(k)})$, $q_- = (w^{j-k} x_-, z^{(j-k)})$ and $q_+ = (x_+, z^{(0)})$ for some $j > k$. Since the periodicity condition still leaves enough freedom to achieve all necessary transversality of moduli spaces, we can define a family of maps

$\phi_\ell : CM_*(h_0, g_0) \rightarrow CM_{*+(\ell-1)}(h_0, g_0)$ by

$$\phi_\ell(x_-) := \sum_{\substack{x_+ \text{ with} \\ \text{ind}(x_-; h_0) - \text{ind}(x_+; h_0) + \ell = 1}} \#_2 \mathcal{M}_{\mathbb{Z}_2}(w^\ell \otimes x_-, x_+; h_N, g_N) \cdot x_+. \tag{3.6}$$

By analyzing the boundary of the one-dimensional moduli spaces, we get the relations (3.1) for $0 \leq k \leq N$.

COROLLARY 3.11. *Under the identification (3.4) the equivariant Morse differential $\partial^{\mathbb{Z}_2}$ is given by (3.2).*

Proof. Choose Z_{p_-} (respectively, Z_{p_+}) which corresponds to $w^j \otimes x_-$ (respectively, $w^k \otimes x_+$). In view of the correspondence (3.5), we can identify the moduli space $\mathcal{M}_{\mathbb{Z}_2}(Z_{p_-}, Z_{p_+}; h_N, g_N)$ with $\mathcal{M}_{\mathbb{Z}_2}(w^{j-k} \otimes x_-, x_+; h_N, g_N)$. Moreover, the identity $|Z_{p_-}| - |Z_{p_+}| = 1$ can be rephrased as

$$\begin{aligned} 1 &= |Z_{p_-}| - |Z_{p_+}| \\ &= (\text{ind}(x_-; h_0) + \text{ind}(z^{(j-k)}; \tilde{f}_N)) - (\text{ind}(x_+; h_0) + \text{ind}(z^{(0)}; \tilde{f}_N)) \\ &= \text{ind}(x_-; h_0) - \text{ind}(x_+; h_0) + (j - k). \end{aligned}$$

Then the differential can be written as

$$\partial^{\mathbb{Z}_2}(w^j \otimes x_-) = \sum_{k=0}^j \sum_{\substack{x_+ \text{ with} \\ \text{ind}(x_-; h_0) - \text{ind}(x_+; h_0) + j - k = 1}} \#_2 \mathcal{M}_{\mathbb{Z}_2}(w^{j-k} \otimes x_-, x_+; h_N, g_N) \cdot w^k \otimes x_+.$$

The differential $\partial^{\mathbb{Z}_2}$ is, therefore, given by

$$\partial^{\mathbb{Z}_2}(w^j \otimes x) = \sum_{k=0}^j w^k \otimes \phi_{j-k}(x) = \sum_{k=0}^j w^{j-k} \otimes \phi_k(x).$$

This completes the proof of the corollary. □

Since for a fixed j , the maps ϕ_j defined in (3.6) for varying $N \geq j$ coincide, we can take a direct limit of chain complexes $\{(CM_*^{\mathbb{Z}_2, N}(h_N, g_N), \{\phi_j\}_{0 \leq j \leq N})\}$ as $N \rightarrow \infty$ with respect to the maps induced by the equivariant inclusions $S^N \hookrightarrow S^{N+1}$. We denote by

$$CM_*^{\mathbb{Z}_2}(h, g) = \mathbb{Z}_2[w] \otimes_{\mathbb{Z}_2} CM_*(h_0, g_0)$$

the direct limit of chain complexes equipped with maps $\{\phi_j\}_{j \geq 0}$. Then the Morse chain complex $CM_*(h_0, g_0)$ is a \mathbb{Z}_2 -complex with the collection of maps $\{\phi_j\}$. Therefore, by definition, the (limit) chain complex $(CM_*^{\mathbb{Z}_2}(h, g), \partial^{\mathbb{Z}_2})$ is the equivariant chain complex of the \mathbb{Z}_2 -complex $(CM_*(h_0, g_0), \partial, \{\phi_j\})$. Its homology is isomorphic to $HM_*^{\mathbb{Z}_2}(M)$ as the homology functor commutes with direct limits. By Lemma 3.5, we immediately obtain the following assertion which gives the classical Leray–Serre spectral sequence.

COROLLARY 3.12. *There is a spectral sequence $\{E^r\}_{r \geq 0}$ converging to $HM_*^{\mathbb{Z}_2}(M)$ with the first page given by*

$$E_{p,q}^1 = \begin{cases} HM_q(M) & \text{if } p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

3.3. *Definition of equivariant wrapped Floer homology.* In this section, we introduce the equivariant wrapped Floer homology of an admissible Lagrangian L in a Liouville domain (W, λ) when it admits an involution $\mathcal{I} : W \rightarrow W$. We assume that one of the following holds.

Symplectic involution

- \mathcal{I} is exact symplectic, i.e., $\mathcal{I}^*\lambda = \lambda$; and
- L is \mathcal{I} -invariant.

Anti-symplectic involution

- \mathcal{I} is exact anti-symplectic, i.e., $\mathcal{I}^*\lambda = -\lambda$; and
- $L = \text{Fix}(\mathcal{I})$ and $\partial L = \text{Fix}(\mathcal{I}|_{\partial W})$ are non-empty.

In the latter case, the triple $(W, \lambda, \mathcal{I})$ is called a real Liouville domain and $L = \text{Fix}(\mathcal{I})$ is called a *real Lagrangian*. Note that real Lagrangians are admissible and \mathcal{I} -invariant. We mainly follow ideas in [43], where equivariant Lagrangian Floer homology (with a symplectic involution) is constructed using family Morse homology theory. See also [7] for the construction of S^1 -equivariant symplectic homology and [10] for the case of an anti-symplectic involution.

3.3.1. *Geometric set-up.* We observe that \mathcal{I} restricts to an involution $\mathcal{I}|_{\partial W}$ on the contact manifold $(\partial W, \alpha := \lambda|_{\partial W})$ which is:

- a (strict) *contact involution*, i.e., $\mathcal{I}|_{\partial W}^* \alpha = \alpha$ if \mathcal{I} is exact symplectic; or
- an (strict) *anti-contact involution*, i.e., $\mathcal{I}|_{\partial W}^* \alpha = -\alpha$ if \mathcal{I} is exact anti-symplectic.

Note that a Legendrian ∂L is $\mathcal{I}|_{\partial W}$ -invariant. We then extend \mathcal{I} to an exact (anti-)symplectic involution $\widehat{\mathcal{I}}$ defined on the completion \widehat{W} : i.e.,

$$\widehat{\mathcal{I}}(x) = \begin{cases} \mathcal{I}(x) & \text{for } x \in W, \\ (r, \mathcal{I}|_{\partial W}(y)) & \text{for } x = (r, y) \in [1, \infty) \times \partial W. \end{cases}$$

One can see that \widehat{L} is $\widehat{\mathcal{I}}$ -invariant. The path space \mathcal{P} defined in (2.1) admits an involution

$$\mathcal{I}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}, \quad \mathcal{I}_{\mathcal{P}}(c(t)) := \begin{cases} \widehat{\mathcal{I}}(c(t)) & \text{if } \mathcal{I} \text{ is symplectic,} \\ \widehat{\mathcal{I}}(c(1-t)) & \text{if } \mathcal{I} \text{ is anti-symplectic.} \end{cases} \tag{3.7}$$

By abuse of notation, we abbreviate by \mathcal{I} the various involutions on \widehat{W} , ∂W , and \mathcal{P} .

3.3.2. *A family of action functionals.* Fix $N \in \mathbb{N}$. For a family of Hamiltonians

$$H : \widehat{W} \times S^N \longrightarrow \mathbb{R}$$

we denote $H_z := H(\cdot, z)$ for $z \in S^N$. We say that a family of Hamiltonians $H : \widehat{W} \times S^N \rightarrow \mathbb{R}$ is \mathbb{Z}_2 -invariant if it is invariant under the diagonal \mathbb{Z}_2 -action on $\widehat{W} \times S^N$: i.e.,

$$H_{-z}(\mathcal{I}x) = H_z(x) \quad \text{for all } x \in \widehat{W} \text{ and } z \in S^N. \tag{3.8}$$

Since \widehat{L} is exact, there exists $\ell \in C^\infty(\widehat{L})$ such that $\widehat{\lambda}|_{\widehat{L}} = d\ell$ and ℓ is \mathcal{I} -invariant up to a constant. Given a \mathbb{Z}_2 -invariant Hamiltonian family $H : \widehat{W} \times S^N \rightarrow \mathbb{R}$, we consider the corresponding family of action functionals $\mathcal{A} : \mathcal{P} \times S^N \rightarrow \mathbb{R}$ given by

$$\mathcal{A}(c, z) := \mathcal{A}_{H_z}(c) = \ell(c(1)) - \ell(c(0)) - \int_0^1 c^* \widehat{\lambda} - \int_0^1 H_z(c(t)) dt.$$

One sees that \mathcal{A} is \mathbb{Z}_2 -invariant with respect to the diagonal \mathbb{Z}_2 -action on $\mathcal{P} \times S^N$. The differential of \mathcal{A} is given by

$$d\mathcal{A}(c, z) \cdot (\zeta, \ell) = \int_0^1 \widehat{\omega}(\dot{c}(t) - X_{H_z}(c(t)), \zeta(t)) dt - \int_0^1 \frac{\partial H}{\partial z}(c(t), z) dt \cdot \ell,$$

where $\zeta \in T_c \mathcal{P}$ and $\ell \in T_z S^N$. Hence, $(c, z) \in \mathcal{P} \times S^N$ is a critical point of \mathcal{A} if and only if

$$c \in \mathcal{P}(H_z) \quad \text{and} \quad \int_0^1 \frac{\partial H}{\partial z}(c(t), z) dt = 0. \tag{3.9}$$

We write $\mathcal{P}(H) := \text{crit}(\mathcal{A})$ for the set of critical points of \mathcal{A} .

3.3.3. A family of almost complex structures. A family of $\widehat{\omega}$ -compatible almost complex structures $J = \{J_z^t\}_{t \in [0,1], z \in S^N}$ on \widehat{W} is called \mathbb{Z}_2 -invariant if the following hold.

- If \mathcal{I} is symplectic, it satisfies

$$J_{-z}^t = \mathcal{I}^* J_z^t := \mathcal{I}_* \circ J_z^t \circ \mathcal{I}_* \quad \text{for all } t, z.$$

- If \mathcal{I} is anti-symplectic, it satisfies

$$-J_{-z}^{1-t} = \mathcal{I}^* J_z^t \quad \text{for all } t, z.$$

Such a J is called *admissible* if $J_z = \{J_z^t\}_{t \in [0,1]} \in \mathcal{J}$ for all $z \in S^N$, where \mathcal{J} is defined in (2.2). We denote by $\mathcal{J}_N^{\mathbb{Z}_2}$ the set of \mathbb{Z}_2 -invariant admissible families of $\widehat{\omega}$ -compatible almost complex structures. The set of $\mathcal{J}_N^{\mathbb{Z}_2}$ is non-empty and contractible (see [34]). For an admissible J , we consider the \mathbb{Z}_2 -invariant S^N -family of L^2 -metrics on \mathcal{P} given by

$$\langle \zeta_1, \zeta_2 \rangle_z := \int_0^1 \widehat{\omega}(\zeta_1(t), J_z^t \zeta_2(t)) dt \quad \text{for } z \in S^N \text{ and } \zeta_1, \zeta_2 \in T_c \mathcal{P}.$$

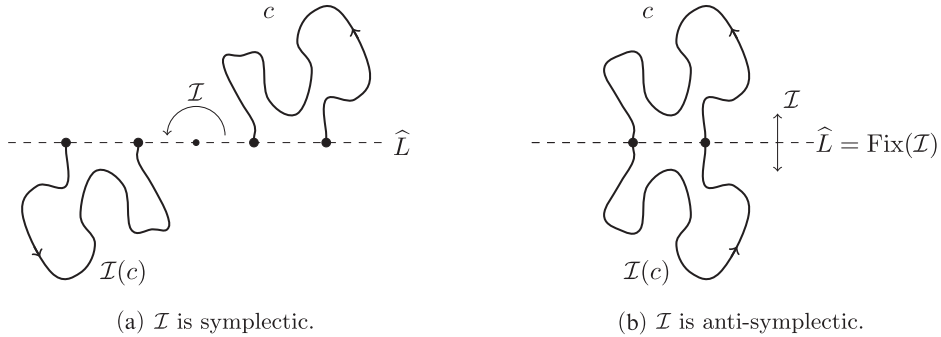


FIGURE 3. An illustration of \mathbb{Z}_2 -pairs: the antipodal involution and complex conjugation on \mathbb{C} .

As in §3.2.2, this, together with a \mathbb{Z}_2 -invariant metric g on S^N , gives rise to a \mathbb{Z}_2 -invariant metric on $\mathcal{P} \times S^N$. The L^2 -gradient of \mathcal{A} at $(c, z) \in \mathcal{P} \times S^N$ is given by

$$\begin{aligned} \nabla \mathcal{A}(c, z) &= J_z^t(\dot{c} - X_{H_z}(c)) - \int_0^1 \frac{\partial H}{\partial z}(c(t), z) dt \\ &= \nabla \mathcal{A}_{H_z}(c) - \int_0^1 \frac{\partial H}{\partial z}(c(t), z) dt, \end{aligned} \tag{3.10}$$

where $\nabla \mathcal{A}_{H_z}$ denotes the L^2 -gradient of the functional \mathcal{A}_{H_z} with respect to the metric $\langle \cdot, \cdot \rangle_z$. Since \mathcal{A} is \mathbb{Z}_2 -invariant, the set $\mathcal{P}(H)$ is \mathbb{Z}_2 -invariant. Given $p \in \mathcal{P}(H)$, we denote by Z_p the critical \mathbb{Z}_2 -pair of p . In particular, we have $Z_p = Z_{\mathcal{I}p}$ (see Figure 3).

Definition 3.13. A \mathbb{Z}_2 -invariant Hamiltonian family $H : \widehat{W} \times S^N \rightarrow \mathbb{R}$ is called weakly admissible if $H_z \in \mathcal{H}$ for all $z \in S^N$ with constant slope not depending on z .

We denote by $\mathcal{H}_N^{\mathbb{Z}_2}$ the space of \mathbb{Z}_2 -invariant weakly admissible families of Hamiltonians. Note that the set $\mathcal{H}_N^{\mathbb{Z}_2}$ is non-empty.

3.3.4. *Hessian and non-degeneracy.* Let ∇ be the t -dependent Levi-Civita connection with respect to the metric $\widehat{\omega}(\cdot, J_z^t \cdot) \oplus g$ on $\widehat{W} \times S^N$.

LEMMA 3.14. *The Hessian of \mathcal{A} at a critical point $p = (c, z) \in \mathcal{P}(H)$ is given by*

$$\begin{aligned} \text{Hess}_{(c,z)} \mathcal{A} : T_c \mathcal{P} \oplus T_z S^N &\longrightarrow L^2(c^* T \widehat{W}) \oplus T_z S^N \\ (\zeta, \ell) &\longmapsto \begin{pmatrix} J_z^t(\nabla_t \zeta - \nabla_\zeta X_{H_z} - (D_z X_{H_z}) \cdot \ell) \\ - \int_0^1 \nabla_\zeta \frac{\partial H}{\partial z} dt - \int_0^1 \nabla_\ell \frac{\partial H}{\partial z} dt \end{pmatrix}. \end{aligned} \tag{3.11}$$

Proof. Take $c_s \in \mathcal{P}$ and $z_s \in S^N$, $s \in (-\varepsilon, \varepsilon)$ such that $(d/ds)c_s = \zeta$ and $(d/ds)z_s = \ell$. Using (3.10), we verify that

$$\begin{aligned} \text{Hess}_{(c,z)} \mathcal{A}(\zeta, \ell) &= \nabla_s \nabla \mathcal{A}(c_s, z_s)|_{s=0} \\ &= \nabla_s \left[\nabla \mathcal{A}_{H_{z_s}}(c_s) - \int_0^1 \frac{\partial H}{\partial z}(c_s, z_s) dt \right] \Big|_{s=0} \end{aligned}$$

$$\begin{aligned}
 &= \nabla_\zeta \nabla \mathcal{A}_{H_{z_s}}(c_s)|_{s=0} + \nabla_\ell \nabla \mathcal{A}_{H_{z_s}}(c_s)|_{s=0} - \nabla_s \int_0^1 \frac{\partial H}{\partial z}(c_s, z_s) dt|_{s=0} \\
 &= \text{Hess}_c \mathcal{A}_{H_z}(\zeta) + \nabla_\ell (J'_{z_s}(c)(\dot{c} - X_{H_{z_s}}(c)))|_{s=0} \\
 &\quad - \int_0^1 \nabla_\zeta \frac{\partial H}{\partial z}(c_s, z_s)|_{s=0} dt - \int_0^1 \nabla_\ell \frac{\partial H}{\partial z}(c_s, z_s)|_{s=0} dt \\
 &= \text{Hess}_c \mathcal{A}_{H_z}(\zeta) + \underbrace{(\nabla_\ell J'_z(c)(\dot{c} - X_{H_z}(c)))}_{=0} - J'_z(c) \nabla_\ell X_{H_z}(c) \\
 &\quad - \int_0^1 \nabla_\zeta \frac{\partial H}{\partial z}(c, z) dt - \int_0^1 \nabla_\ell \frac{\partial H}{\partial z}(c, z) dt \\
 &= \text{Hess}_c \mathcal{A}_{H_z}(\zeta) - J'_z(c) D_z X_{H_z}(\ell) - \int_0^1 \nabla_\zeta \frac{\partial H}{\partial z}(c, z) dt \\
 &\quad - \int_0^1 \nabla_\ell \frac{\partial H}{\partial z}(c, z) dt.
 \end{aligned}$$

Here, $\text{Hess}_c \mathcal{A}_{H_z}$ denotes the Hessian of \mathcal{A}_{H_z} at c . This completes the proof. □

Definition 3.15. A \mathbb{Z}_2 -pair of critical points $Z_p \subset \mathcal{P}(H)$ is called *non-degenerate* if the kernel of the Hessian $\text{Hess}_{(c,z)} \mathcal{A}$ is trivial for some (and hence any) $(c, z) \in Z_p$.

3.3.5. *Admissible family of Hamiltonians.* Let f and \tilde{f} be defined as in §3.2.2. From now on we fix a local slice for each critical \mathbb{Z}_2 -pair of \tilde{f} and assume that all local slices are disjoint. Since \mathbb{Z}_2 is finite, local slices are nothing but \mathbb{Z}_2 -invariant open neighborhoods where the \mathbb{Z}_2 -action is given by the antipodal map σ on S^N .

Let U be the union of all local slices, which is an open neighborhood of $\text{crit}(\tilde{f})$. Choose an open set U_0 in S^N such that U_0 and $\sigma(U_0)$ are disjoint and $U = U_0 \sqcup \sigma(U_0)$. Intuitively, one may think of U_0 as a neighborhood of $\text{crit}(f)$ in $\mathbb{R}P^N$.

Definition 3.16. A family of Hamiltonians $H : \widehat{W} \times S^N \rightarrow \mathbb{R}$ is called *admissible* if it satisfies the following.

- (1) $H \in \mathcal{H}_N^{\mathbb{Z}_2}$.
- (2) Critical points of $\mathcal{A}_{H+\tilde{f}}$ lie over critical points of \tilde{f} ; more precisely, if $(c, z) \in \text{crit}(\mathcal{A}_{H+\tilde{f}})$, then we have $z \in \text{crit}(\tilde{f})$.
- (3) For each $z \in \text{crit}(\tilde{f})$, the corresponding Hamiltonian H_z is non-degenerate.
- (4) All critical \mathbb{Z}_2 -pairs in $\text{crit}(\mathcal{A}_{H+\tilde{f}})$ are non-degenerate.
- (5) H_z is independent of z along each local slice.
- (6) The inequality

$$\|\partial_z H\| < \min_{S^N \setminus U} \|\nabla \tilde{f}\| \tag{3.12}$$

holds.

We denote by $\mathcal{H}_f \subset \mathcal{H}_N^{\mathbb{Z}_2}$ the subclass of admissible families of Hamiltonians.

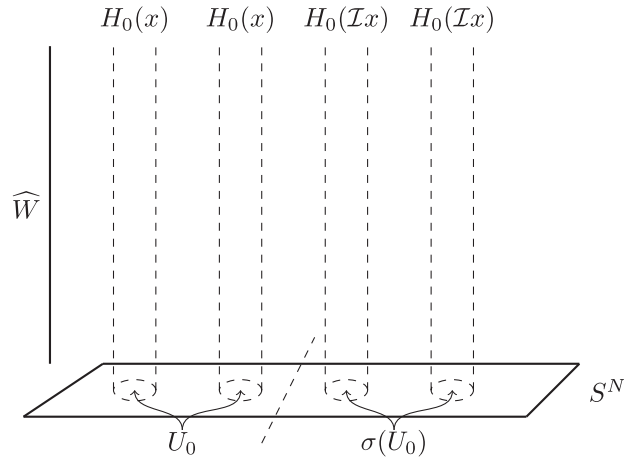


FIGURE 4. An illustration of H' .

Remark 3.17. Condition (2) above is equivalent to the condition

$$\text{crit}(\mathcal{A}_{H+\tilde{f}}) = \bigcup_{z \in \text{crit}(\tilde{f})} \mathcal{P}(H_z) \times \{z\}.$$

Condition (3) actually follows from the other conditions, but, for simplicity, we included it. Condition (6) implies the action-decreasing property of solutions that is used to define the \mathbb{Z}_2 -equivariant differential (see Lemma 3.29).

The space \mathcal{H}_f is non-empty in view of the following explicit example.

Example 3.18. We construct an admissible family of Hamiltonians $H \in \mathcal{H}_f$. Fix an admissible non-degenerate Hamiltonian $H_0 : \widehat{W} \rightarrow \mathbb{R}$. We first define the \mathbb{Z}_2 -invariant Hamiltonian $H' : \widehat{W} \times U \rightarrow \mathbb{R}$ by

$$H'(x, z) := \begin{cases} H_0(x) & \text{if } z \in U_0, \\ H_0(\mathcal{I}x) & \text{if } z \in \sigma(U_0). \end{cases}$$

See Figure 4. To be able to extend H' to a globally defined family H with constant slope, we choose a \mathbb{Z}_2 -invariant cut-off function $\beta : S^N \rightarrow \mathbb{R}$ such that

$$\beta(z) = \begin{cases} 1 & \text{on } U', \\ 0 & \text{outside of } U, \end{cases}$$

where $U' \subset U$ is a \mathbb{Z}_2 -invariant neighborhood of $\text{crit}(\tilde{f})$ such that $\overline{U'} \subset U$. Also choose a cut-off function $\eta : \widehat{W} \rightarrow \mathbb{R}$ such that

$$\eta(x) = \begin{cases} 1 & \text{on } [1, \infty) \times \partial W, \\ 0 & \text{on } W \setminus ((1 - \delta, 1] \times \partial W), \end{cases}$$

for sufficiently small $\delta > 0$. We then define the \mathbb{Z}_2 -invariant Hamiltonian $H : \widehat{W} \times S^N \rightarrow \mathbb{R}$ by

$$H(x, z) := \varepsilon[\beta(z)H'(x, z) + (1 - \beta(z))\eta(x)H_0(x)],$$

where $\varepsilon > 0$ is chosen so small that the inequality (3.12) holds (with U' instead of U). By construction, we have $H \in \mathcal{H}_N^{\mathbb{Z}_2}$ and it satisfies condition (5) for U' . Since H_0 is non-degenerate, condition (3) is also fulfilled.

Claim 1. Condition (2) holds.

Let $\tilde{H} = H + \tilde{f}$. We observe that $\partial_z \tilde{H} = \nabla \tilde{f}$ on $\widehat{W} \times U'$. By (3.9), we obtain

$$\text{crit}(\mathcal{A}_{\tilde{H}}) \supset \bigcup_{z \in \text{crit}(\tilde{f})} \mathcal{P}(H_z) \times \{z\}.$$

To see the reverse, let $z \in S^N \setminus U'$. Using (3.12), we verify that

$$\int_0^1 \frac{\partial \tilde{H}}{\partial z}(c(t), z) dt = \int_0^1 \frac{\partial H}{\partial z}(c(t), z) + \nabla \tilde{f}(z) dt \neq 0 \quad \text{for } c \in \mathcal{P}(H_z).$$

Using (3.9) again, we obtain Claim 1.

Claim 2. Condition (4) holds.

Fix $(c, z) \in \text{crit}(\mathcal{A}_{\tilde{H}})$. Let $(\zeta, \ell) \in T_c \mathcal{P} \oplus T_z S^N$ such that

$$\text{Hess}_{(c,z)} \mathcal{A}(\zeta, \ell) = \begin{pmatrix} J_z(\nabla_t \zeta - \nabla_\zeta X_{\tilde{H}_z} - (D_z X_{\tilde{H}_z}) \cdot \ell) \\ - \int_0^1 \nabla_\zeta \frac{\partial \tilde{H}}{\partial z} dt - \int_0^1 \nabla_\ell \frac{\partial \tilde{H}}{\partial z} dt \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We need to show that $(\zeta, \ell) = 0$. We first verify that

$$\begin{aligned} \int_0^1 \nabla_\zeta \frac{\partial \tilde{H}}{\partial z} dt + \int_0^1 \nabla_\ell \frac{\partial \tilde{H}}{\partial z} dt &= \int_0^1 \underbrace{\nabla_\zeta \frac{\partial \tilde{f}}{\partial z}}_{=0} dt + \int_0^1 \nabla_\ell \frac{\partial \tilde{f}}{\partial z} dt \\ &= \nabla_\ell \nabla \tilde{f} \\ &= \text{Hess}_z \tilde{f}(\ell), \end{aligned}$$

where $\text{Hess}_z \tilde{f}$ is the Hessian of \tilde{f} at z . Since \tilde{f} is non-degenerate, $\ell = 0$. Further,

$$J_z(\nabla_t \zeta - \nabla_\zeta X_{\tilde{H}_z} - (D_z X_{\tilde{H}_z}) \cdot \ell) = J_z(\nabla_t \zeta - \nabla_\zeta X_{\tilde{H}_z}) = \text{Hess}_c \mathcal{A}_{\tilde{H}_z}(\zeta).$$

Since $\tilde{H}_z = H_z + \text{const}$ is non-degenerate, we have $\zeta = 0$. This completes the proof of Claim 2.

Definition 3.19. Let $H \in \mathcal{H}_f$. The Maslov index of $p = (c, z) \in \text{crit}(\mathcal{A}_{H+\tilde{f}})$ is defined by

$$\mu(p) := \mu(c) - \text{ind}(z; -\tilde{f}),$$

where $\text{ind}(z; -\tilde{f})$ denotes the Morse index of z with respect to $-\tilde{f}$. The Maslov index of a critical \mathbb{Z}_2 -pair Z_p is defined by

$$\mu(Z_p) := \mu(p') \quad \text{for } p' \in Z_p.$$

3.3.6. *Moduli spaces.* Fix $H \in \mathcal{H}_f$. Let $J \in \mathcal{J}_N^{\mathbb{Z}_2}$ be such that J_z is independent of z along each local slice and let g be a \mathbb{Z}_2 -invariant metric on S^N that is Morse–Smale with respect to \tilde{f} . Given $p_{\pm} = (c_{\pm}, z_{\pm}) \in \text{crit}(\mathcal{A}_{H+\tilde{f}})$, we denote by $\widetilde{\mathcal{M}}(Z_{p_-}, Z_{p_+}; H, J)$ the space of pairs (u, z) , where $u : \mathbb{R} \times [0, 1] \rightarrow \widehat{W}$ and $z : \mathbb{R} \rightarrow S^N$ are solutions to the system of equations

$$\begin{cases} \partial_s u + J_{z(s)}^t(u)(\partial_t u - X_{H_{z(s)}}(u)) = 0, \\ \dot{z} - \nabla \tilde{f}(z) = 0, \end{cases} \tag{3.13}$$

with asymptotic and boundary conditions

$$\lim_{s \rightarrow \pm\infty} (u(s, \cdot), z(s)) \in Z_{p_{\pm}}, \quad u(s, 0), u(s, 1) \in \widehat{L}.$$

If $Z_{p_-} \neq Z_{p_+}$ let $\mathcal{M}(Z_{p_-}, Z_{p_+}; H, J) := \widetilde{\mathcal{M}}(Z_{p_-}, Z_{p_+}; H, J)/\mathbb{R}$ be the quotient by the \mathbb{R} -translation in the s -variable. This space admits a free \mathbb{Z}_2 -action

$$(u(s, t), z(s)) \mapsto \begin{cases} (\mathcal{I}(u(s, t)), -z(s)) & \text{if } \mathcal{I} \text{ is symplectic,} \\ (\mathcal{I}(u(s, 1-t)), -z(s)) & \text{if } \mathcal{I} \text{ is anti-symplectic,} \end{cases}$$

which is induced by the involution (3.7) on the path space equipped with the antipodal map on S^N . The \mathbb{Z}_2 -quotient of $\mathcal{M}(Z_{p_-}, Z_{p_+}; H, J)$ is denoted by

$$\mathcal{M}_{\mathbb{Z}_2}(Z_{p_-}, Z_{p_+}; H, J).$$

Remark 3.20. Since the system of differential equations (3.13) is not the negative gradient flow equation of $\mathcal{A}_{H+\tilde{f}}$, it is not obvious that the action decreases along solutions to equation (3.13). However, in §3.6.1, it will turn out that, with condition (3.12), we have the action-decreasing property.

3.3.7. *Transversality issues.* This section is devoted to proving the following proposition.

PROPOSITION 3.21. *For generic $J \in \mathcal{J}_N^{\mathbb{Z}_2}$, the moduli space $\mathcal{M}_{\mathbb{Z}_2}(Z_{p_-}, Z_{p_+}; H, J)$ is a smooth manifold of dimension $\mu(Z_{p_+}) - \mu(Z_{p_-}) - 1$.*

For the proof, we shall translate the system (3.13) to a parametrized continuation equation for which the analysis is well known. See [7, §2.2] for a similar situation in the symplectic homology set-up.

Let $\widehat{\mathcal{M}}(Z_{z_-}, Z_{z_+}; -\tilde{f}, g)$ be the space of parametrized gradient flow lines on S^N for $-\tilde{f}$ from Z_{z_-} to Z_{z_+} , where $Z_{z_{\pm}}$ are critical \mathbb{Z}_2 -pairs of $-\tilde{f}$. Since g is Morse–Smale, the space $\widehat{\mathcal{M}}(Z_{z_-}, Z_{z_+}; -\tilde{f}, g)$ is a smooth manifold of dimension $\text{ind}(Z_{z_-}; -\tilde{f}) - \text{ind}(Z_{z_+}; -\tilde{f})$, where the index is given by $\text{ind}(Z_z; -\tilde{f}) := \text{ind}(z'; -\tilde{f})$ for any $z' \in Z_z$. From now on, we think of

$$\Gamma := \widehat{\mathcal{M}}(Z_{z_-}, Z_{z_+}; -\tilde{f}, g)$$

as a parameter space. Consider parametrized data consisting of pairs given by

$$(H_v^s, J_v^s) := (H_{v(s)}, J_{v(s)}) \quad \text{for } s \in \mathbb{R} \text{ and } v \in \Gamma. \tag{3.14}$$

Note that (H_v^s, J_v^s) is independent of s for $|s| \gg 1$. We define the *parametrized moduli space of continuation trajectories*

$$\mathcal{P}(Z_{p-}, Z_{p+}; \{H_v^s\}, \{J_v^s\})$$

as the space consisting of pairs (u, v) with $u : \mathbb{R} \times [0, 1] \rightarrow \widehat{W}$ and $v \in \Gamma$ satisfying

$$\begin{cases} \partial_s u + J_v^{s,t}(u)(\partial_t u - X_{H_v^s}(u)) = 0, \\ u(s, j) \in \widehat{L} \quad \text{for } j = 0, 1, \\ \lim_{s \rightarrow \pm\infty} (u(s, t), v(s)) \in Z_{p\pm}. \end{cases}$$

It follows immediately from (3.14) that $\mathcal{P}(Z_{p-}, Z_{p+}; \{H_v^s\}, \{J_v^s\}) = \widetilde{\mathcal{M}}(Z_{p-}, Z_{p+}; H, J)$.

THEOREM 3.22. *For generic $J \in \mathcal{J}_N^{\mathbb{Z}_2}$, the space $\mathcal{P}(Z_{p-}, Z_{p+}; \{H_v^s\}, \{J_v^s\})$ is a smooth manifold of dimension $\mu(Z_{p+}) - \mu(Z_{p-})$.*

Proof. We follow a standard argument in [9] and point out the different parts. Although J is \mathbb{Z}_2 -invariant, we have much freedom to perturb it as the parameter space Γ is additionally involved. We denote by $\mathcal{J}_N^{\mathbb{Z}_2, \ell}$ the completion of $\mathcal{J}_N^{\mathbb{Z}_2}$ with respect to the C^ℓ -topology. Let $J \in \mathcal{J}_N^{\mathbb{Z}_2, \ell}$. We consider the section of a suitable Banach bundle with the factor \mathcal{B} of the base a $W^{1,p}$ -space and the fibers of \mathcal{E} an L^p -space with $p > 2$: i.e.,

$$\begin{aligned} \mathcal{F} : \mathcal{B} \times \Gamma &\longrightarrow \mathcal{E} \\ (u, v) &\longmapsto \mathcal{F}_v(u) := \partial_s u + J_v^{s,t}(u)(\partial_t u - X_{H_v^s}(u)). \end{aligned}$$

Then

$$\mathcal{F}^{-1}(0) = \mathcal{P}(Z_{p-}, Z_{p+}; \{H_v^s\}, \{J_v^s\}).$$

For the construction of such a Banach bundle we refer to [6, §4.1], where the same construction appears in Morse–Bott symplectic homology. Our case is much simpler as the asymptotics are \mathbb{Z}_2 -pairs which are zero-dimensional. In particular, we do not need to work with a weighted Sobolev space. We now consider the linearized operator of \mathcal{F} at $(u, v) \in \mathcal{F}^{-1}(0)$ given by

$$\begin{aligned} D_{(u,v)}\mathcal{F} : T_u\mathcal{B} \oplus T_v\Gamma &\longrightarrow L^p(u^*T\widehat{W}) \\ D_{(u,v)}\mathcal{F}(\zeta, \ell) &= \underbrace{\nabla_s \zeta + J_v^s(\nabla_t \zeta - \nabla_t X_{H_v^s}) + \nabla_\zeta J_v^s(\partial_t u - X_{H_v^s})}_{=D_u\mathcal{F}_v(\zeta)} + \nabla_\ell(J_v^s(\partial_t u - X_{H_v^s})), \end{aligned} \tag{3.15}$$

where $D_u\mathcal{F}_v$ is the usual Floer operator. Note that $D_{(u,v)}\mathcal{F}$ is a Fredholm operator with index

$$\begin{aligned} \text{ind } D_{(u,v)}\mathcal{F} &= \text{ind } D_u\mathcal{F}_v + \dim \Gamma \\ &= \mu(c_+) - \mu(c_-) + \text{ind}(z_-; -\tilde{f}) - \text{ind}(z_+; -\tilde{f}) \\ &= \mu(p_+) - \mu(p_-). \end{aligned}$$

Now consider the section

$$\begin{aligned} \mathcal{F}_{\text{univ}} : \mathcal{B} \times \Gamma \times \mathcal{J}_N^{\mathbb{Z}_2, \ell} &\longrightarrow \mathcal{E} \\ (u, v, J) &\longmapsto \partial_s u + J_v^s(u)(\partial_t u - X_{H_v^s}) \end{aligned}$$

and define the universal moduli space to be the zero locus

$$\mathcal{P}_{\text{univ}} := \mathcal{F}_{\text{univ}}^{-1}(0).$$

We claim that $\mathcal{P}_{\text{univ}}$ is a Banach manifold of class C^ℓ . Given this, we can apply the parametric transversality theorem and Taubes’ trick to complete the proof.

To see the claim, consider the linearization of $\mathcal{F}_{\text{univ}}$ given by

$$\begin{aligned} D_{(u,v,J)}\mathcal{F}_{\text{univ}} : T_u\mathcal{B} \oplus T_v\Gamma \oplus T_J\mathcal{J}_N^{\mathbb{Z}_2, \ell} &\longrightarrow L^p(u^*T\widehat{W}) \\ D_{(u,v,J)}\mathcal{F}_{\text{univ}}(\zeta, V, Y) &= D_{(u,v)}\mathcal{F}(\zeta, V) + D_{(u,v,J)}(J_v^s(u)(\partial_t u - X_{H_v^s}))(Y). \end{aligned} \tag{3.16}$$

Since $D_{(u,v)}\mathcal{F}$ is Fredholm, the cokernel of $D_{(u,v,J)}\mathcal{F}_{\text{univ}}$ has a finite dimension. Hence, the image of $D_{(u,v,J)}\mathcal{F}_{\text{univ}}$ is closed. We claim that the image is dense. It suffices to show that the annihilator of $\text{im}(D_{(u,v,J)}\mathcal{F}_{\text{univ}})$ is trivial. Suppose that there exists $\eta \in L^q(u^*T\widehat{W})$ with $1/p + 1/q = 1$, which annihilates the image of $D_{(u,v,J)}\mathcal{F}_{\text{univ}}$. Then η is of class C^ℓ and

$$\int_{\mathbb{R} \times [0,1]} \langle \eta, Y_t(u)(\partial_t u - X_{H_v^s}(u)) \rangle ds \wedge dt = 0 \quad \text{for all } Y \in T_J\mathcal{J}_N^{\mathbb{Z}_2, \ell}. \tag{3.17}$$

Assume, by contradiction, that there is a point $z_0 = (s_0, t_0) \in R(u)$ with $\eta(z_0) \neq 0$ where $R(u)$ is the set of regular points defined in [9, §4]. Choose $Y_0 \in \text{End}(T_{u(z_0)}\widehat{W}, J_{t_0}, \widehat{\omega})$ such that $\langle \eta, Y_{t_0}(u)(\partial_t u - X_{H_v^s}(u)) \rangle > 0$ (see [42, §8]). Now we want to extend Y_0 to $\tilde{Y} \in T_J\mathcal{J}_N^{\mathbb{Z}_2, \ell}$ such that the left-hand side of (3.17) is strictly positive, which gives the contradiction. This is the place that requires a specific argument. This \mathbb{Z}_2 -invariant extension can be obtained by a construction as in Example 3.18. To show that the extension \tilde{Y} is independent of z along each local slice, we refer to [9, Remark 5.2]. By the maximum principle, a Floer strip u lies in a compact region in \widehat{W} so that we can find the extension \tilde{Y} even under the requirement that J is admissible. This proves that the universal moduli space $\mathcal{P}_{\text{univ}}$ is a Banach manifold of class C^ℓ , as claimed, and proves the theorem. \square

3.3.8. *Chain complex.* We define the \mathbb{Z}_2 -equivariant chain groups by

$$CF_*^{\mathbb{Z}_2, N}(H, J) := \bigoplus_{Z_p \subset \mathcal{P}(H+\tilde{f})} \mathbb{Z}_2\langle Z_p \rangle$$

with grading given by $|Z_p| := -\mu(Z_p) - n/2 \in \mathbb{Z}$. For generic $J \in \mathcal{J}_N^{\mathbb{Z}_2}$, we define the \mathbb{Z}_2 -equivariant differential $\partial^{\mathbb{Z}_2} : CF_*^{\mathbb{Z}_2, N}(H, J) \rightarrow CF_{*-1}^{\mathbb{Z}_2, N}(H, J)$ by the formula

$$\partial^{\mathbb{Z}_2}(Z_{p-}) := \sum_{|Z_{p-}| - |Z_{p+}| = 1} \#_2 \mathcal{M}_{\mathbb{Z}_2}(Z_{p-}, Z_{p+}; H, J) \cdot Z_{p+}.$$

By a standard argument, we have $\partial^{\mathbb{Z}_2} \circ \partial^{\mathbb{Z}_2} = 0$. We denote by

$$HF_*^{\mathbb{Z}_2, N}(H, J)$$

the homology group associated to the chain complex $(CF_*^{\mathbb{Z}_2, N}(H, J), \partial^{\mathbb{Z}_2})$. By the usual direct limit procedure, we obtain

$$HW_*^{\mathbb{Z}_2}(L; W) := \varinjlim_N \varinjlim_H HF_*^{\mathbb{Z}_2, N}(H, J),$$

where the limit \varinjlim_H is taken over admissible families of Hamiltonians using continuation maps, and the limit \varinjlim_N is taken using morphisms induced by the equivariant inclusions $S^N \hookrightarrow S^{N+1}$. We call $HW_*^{\mathbb{Z}_2}(L; W)$ the \mathbb{Z}_2 -equivariant wrapped Floer homology group of L in W . A standard continuation argument shows that this homology is independent of the choice of data (H, f, J, g) .

3.4. Periodic family of Floer data. In this section, we choose a special kind of \mathbb{Z}_2 -invariant admissible family of Hamiltonians, parallel to §3.2.3. We write $\tilde{\mathcal{T}}_j : \widehat{W} \times S^N \rightarrow \widehat{W} \times S^{N+1}$ for the embeddings defined in (3.3), with M being replaced by \widehat{W} .

Definition 3.23. A sequence $\{(H_N, J_N)\}$ of pairs with $H_N \in \mathcal{H}_N^{\mathbb{Z}_2}$ and $J_N \in \mathcal{J}_N^{\mathbb{Z}_2}$ is called *periodic* if the following conditions hold for all $N \geq 1$:

- $H_N = \tilde{\mathcal{T}}_0^* H_{N+1} = \tilde{\mathcal{T}}_1^* H_{N+1}$; and
- $J_N = \tilde{\mathcal{T}}_0^* J_{N+1} = \tilde{\mathcal{T}}_1^* J_{N+1}$.

Remark 3.24. To construct a periodic sequence $\{(H_N, J_N)\}$, we follow the ideas in Example 3.18 (see also Remark 3.10).

Suppose that a periodic sequence $\{(H_N, J_N)\}$ is given. The same procedure as in §3.2.2 leads to the following canonical identification, which is analogous to (3.4). One should be aware that the role of \tilde{f} is replaced by $-\tilde{f}$ as we used the (positive) gradient flow of \tilde{f} in the definition of the moduli spaces (3.13). Take $H_0 := H_{N, z^{(0)}} \in \mathcal{H}$, where $z^{(0)}$ is a critical point of \tilde{f} of index zero, and a generic $J_0 \in \mathcal{J}$.

LEMMA 3.25. *There is a canonical identification*

$$CF_*^{\mathbb{Z}_2, N}(H_N, J_N) = \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes_{\mathbb{Z}_2} CF_*(H_0, J_0),$$

where w is a formal variable with $|w| = 1$.

Proof. Let Z_p be a generator of $CF_*^{\mathbb{Z}_2, N}(H_N, J_N)$. As a representative of Z_p , we take $p = (w^j c, z^{(j)})$, where $z^{(j)}$ is a critical point of $-\tilde{f}$ with index j . Define the map

$$\begin{aligned} CF_*^{\mathbb{Z}_2, N}(H_N, J_N) &\rightarrow \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes_{\mathbb{Z}_2} CF_*(H_0, J_0) \\ Z_p &= [(w^j c, z^{(j)})] \mapsto w^j \otimes c. \end{aligned}$$

By the canonical choice of the representative of Z_p , this map is well defined. The inverse is given by $w^j \otimes c \mapsto Z_p$, where $p = (w^j c, z^{(j)})$, which completes the proof. □

3.5. Leray–Serre-type spectral sequence. As in Lemma 3.5, the periodicity of $\{(H_N, J_N)\}$ gives rise to the following lemma.

LEMMA 3.26. *There is a one-to-one correspondence*

$$\mathcal{M}_{\mathbb{Z}_2}(Z_{p_-}, Z_{p_+}; H_N, J_N) \leftrightarrow \mathcal{M}_{\mathbb{Z}_2}(Z_{q_-}, Z_{q_+}; H_N, J_N),$$

where $p_- = (w^j c_-, z^{(j)})$, $p_+ = (w^k c_+, z^{(k)})$, and $q_- = (w^{j-k} c_-, z^{(j-k)})$, $q_+ = (c_+, z^{(0)})$.

Under the identification in Lemma 3.25, the differential $\partial^{\mathbb{Z}_2}$ reads

$$\begin{aligned} \partial^{\mathbb{Z}_2}(Z_{p_-}) &= \sum_{Z_{p_+}} \#_2 \mathcal{M}_{\mathbb{Z}_2}(Z_{p_-}, Z_{p_+}; H_N, J_N) \cdot Z_{p_+} \\ &= \sum_{k=0}^j \sum_{Z_{p_+}} \#_2 \mathcal{M}_{\mathbb{Z}_2}(w^{j-k} \otimes c_-, c_+; H_N, J_N) \cdot w^k \otimes c_+. \end{aligned}$$

We introduce the following maps on $CF_*(H_0, J_0)$: i.e.,

$$\phi_j(c_-) = \sum_{c_+} \#_2 \mathcal{M}_{\mathbb{Z}_2}(w^j \otimes c_-, c_+; H_N, J_N) \cdot c_+. \tag{3.18}$$

Then the \mathbb{Z}_2 -equivariant differential $\partial^{\mathbb{Z}_2}$ satisfies, under the identification in Lemma 3.25,

$$\partial^{\mathbb{Z}_2}(w^\ell \otimes c) = \sum_{j=0}^{\ell} w^j \phi_{\ell-j}(c) = \sum_{j=0}^{\ell} w^{\ell-j} \phi_j(c).$$

As in the Morse case (§3.2.3), we take the direct limit of chain complexes

$$\{(CF_*^{\mathbb{Z}_2, N}(H_N, J_N), \{\phi_j\}_{0 \leq j \leq N})\}$$

as $N \rightarrow \infty$ using the morphisms induced by the equivariant inclusions $S^N \hookrightarrow S^{N+1}$, and we write

$$CF_*^{\mathbb{Z}_2}(H, J)$$

for the limit chain complex equipped with maps $\{\phi_j\}_{j \geq 0}$. Summarizing, we have the following proposition.

PROPOSITION 3.27. *The wrapped Floer chain complex $CF_*(H_0, J_0)$ admits a \mathbb{Z}_2 -complex structure given by $\{\phi_j\}$ defined in (3.18), and the resulting \mathbb{Z}_2 -equivariant chain complex is exactly the complex $CF_*^{\mathbb{Z}_2}(H, J)$.*

A direct consequence of Lemma 3.5, after taking direct limits over H , is the following corollary.

COROLLARY 3.28. (Leray–Serre-type spectral sequence in $HW_*^{\mathbb{Z}_2}$) *There is a spectral sequence $\{E^r\}_{r \geq 0}$ converging to $HW_*^{\mathbb{Z}_2}(L; W)$ with first page given by*

$$E_{p,q}^1 = \begin{cases} HW_q(L; W) & \text{if } p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

3.6. *Positive equivariant wrapped Floer homology.* Since the equivariant differential $\partial^{\mathbb{Z}_2}$, in principle, counts parametrized Floer solutions, it is not obvious that the action decreases along Floer solutions. However, it turns out that we can obtain the action-decreasing property under additional assumptions on the Hamiltonians (see Lemma 3.29). We shall then define the positive equivariant wrapped Floer homology $HW_*^{\mathbb{Z}_2,+}(L; W)$, which plays a crucial role for our applications in §4.

3.6.1. *Action filtration.* Let (H, J) be an admissible \mathbb{Z}_2 -invariant family of Floer data. By definition, H satisfies the inequality (3.12). Let (u, z) be a solution of the Floer equation (3.13) from Z_{p_-} to Z_{p_+} with $p_{\pm} = (c_{\pm}, z_{\pm}) \in \mathcal{P}(H)$. Let U be the union of local slices given in §3.3.5.

LEMMA 3.29. (Action-decreasing property) *The action decreases along the solution (u, z) , i.e.,*

$$\mathcal{A}_{H+\tilde{f}_N}(c_-, z_-) \geq \mathcal{A}_{H+\tilde{f}_N}(c_+, z_+).$$

Proof. Recall that

$$\mathcal{A}_{H+\tilde{f}_N}(c, z) = \ell(c(1)) - \ell(c(0)) - \int_0^1 c^* \widehat{\lambda} - \int_0^1 (H + \tilde{f}_N)(c(t), z) dt.$$

We shall prove the lemma by showing that

$$\frac{\partial}{\partial s} \mathcal{A}_{H+\tilde{f}_N}(u(s, \cdot), z(s)) \leq 0$$

for all $s \in \mathbb{R}$. We first compute that

$$\begin{aligned} & \frac{\partial}{\partial s} \left\{ \ell(u(s, 1)) - \ell(u(s, 0)) - \int_0^1 u(s, \cdot)^* \widehat{\lambda} \right\} \\ &= \widehat{\lambda}(\partial_s u(s, 1)) - \widehat{\lambda}(\partial_s u(s, 0)) - \int_0^1 d(\widehat{\lambda}(\partial_s u)) - \int_0^1 \widehat{\omega}(\partial_s u, \partial_t u) dt \\ &= \widehat{\lambda}(\partial_s u(s, 1)) - \widehat{\lambda}(\partial_s u(s, 0)) - \{ \widehat{\lambda}(\partial_s u(s, 1)) - \widehat{\lambda}(\partial_s u(s, 0)) \} - \int_0^1 \widehat{\omega}(\partial_s u, \partial_t u) dt \\ &= - \int_0^1 \widehat{\omega}(\partial_s u, \partial_t u) dt. \end{aligned}$$

We then compute the remaining term as follows:

$$\begin{aligned} & \frac{\partial}{\partial s} \int_0^1 (H + \tilde{f}_N)(u(s, t), z(s)) dt \\ &= \frac{\partial}{\partial s} \int_0^1 H_{z(s)}(u(s, t)) + \tilde{f}_N(z(s)) dt \\ &= \frac{\partial}{\partial s} \int_0^1 H_{z(s)}(u(s, t)) dt + \int_0^1 d\tilde{f}_N(z(s)) \cdot \nabla \tilde{f}_N(z(s)) dt \end{aligned}$$

$$\begin{aligned} &= \int_0^1 dH_{z(s)}(\partial_s u(s, t)) dt + \int_0^1 \partial_z H_z(u(s, t)) \cdot \nabla \tilde{f}_N dt + \|\nabla \tilde{f}_N\|^2 \\ &= - \int_0^1 \widehat{\omega}(\partial_s u, X_{H_{z(s)}}(u)) dt + \int_0^1 \partial_z H_z(u(s, t)) \cdot \nabla \tilde{f}_N dt + \|\nabla \tilde{f}_N\|^2. \end{aligned}$$

Combining these two identities we obtain

$$\begin{aligned} &\frac{\partial}{\partial s} \mathcal{A}_{H+\tilde{f}_N}(u(s, \cdot), z(s)) \\ &= - \int_0^1 \widehat{\omega}(\partial_s u, \partial_t u - X_{H_{z(s)}}) dt - \int_0^1 \partial_z H_z(u(s, t)) \cdot \nabla \tilde{f}_N dt - \|\nabla \tilde{f}_N\|^2 \\ &= - \int_0^1 \|\partial_s u\|^2 dt - \int_0^1 \partial_z H_z(u(s, t)) \cdot \nabla \tilde{f}_N dt - \|\nabla \tilde{f}_N\|^2. \end{aligned}$$

Now the claim follows from the following observations.

- In U , since H_z is constant in z , we have $\partial_z H_z = 0$.
- By the inequality (3.12), on $S^N \setminus U$ we have

$$- \int_0^1 \partial_z H_z(u(s, t)) \cdot \nabla \tilde{f}_N dt - \|\nabla \tilde{f}_N\|^2 \leq 0.$$

This completes the proof. □

3.6.2. *Positive equivariant chain complex.* Let (H, J) be an admissible \mathbb{Z}_2 -invariant Floer datum. For $\tau \in \mathbb{R}$, define the subgroup

$$CF_*^{\mathbb{Z}_2, N, < \tau}(H, J) \subset CF_*^{\mathbb{Z}_2, N}(H, J),$$

which is generated by elements with action value less than τ . By Lemma 3.29, this subgroup forms a subcomplex of $(CF_*^{\mathbb{Z}_2, N}(H, J), \partial^{\mathbb{Z}_2})$.

Let $\varepsilon > 0$ be smaller than the lengths of any Reeb chord on the boundary ∂W . We define the *positive equivariant chain complex* as the quotient

$$CF_*^{\mathbb{Z}_2, +, N}(H, J) := CF_*^{\mathbb{Z}_2, N}(H, J) / CF_*^{\mathbb{Z}_2, N, < \varepsilon}(H, J)$$

with the induced differential, still denoted by $\partial^{\mathbb{Z}_2}$. Its homology is denoted by $HW_*^{\mathbb{Z}_2, +, N}(H, J)$. We take the direct limit over N and H , and we call the resulting homology the *positive equivariant wrapped Floer homology*, denoted by $HW_*^{\mathbb{Z}_2, +}(L; W)$.

3.6.3. *A tautological exact sequence in $HW_*^{\mathbb{Z}_2}$.* The action filtration on the equivariant chain complex $CF_*^{\mathbb{Z}_2}(H, J)$ yields the short exact sequence of chain complexes

$$0 \rightarrow CF_*^{\mathbb{Z}_2, N, < \varepsilon}(H, J) \rightarrow CF_*^{\mathbb{Z}_2, N}(H, J) \rightarrow CF_*^{\mathbb{Z}_2, +, N}(H, J) \rightarrow 0.$$

The induced long exact sequence, after taking the direct limit, is

$$\dots \rightarrow HW_*^{\mathbb{Z}_2, < \varepsilon}(L; W) \rightarrow HW_*^{\mathbb{Z}_2}(L; W) \rightarrow HW_*^{\mathbb{Z}_2, +}(L; W) \rightarrow \dots$$

Assuming that $H(\cdot, z)$ is C^2 -small in the interior of W for all $z \in S^N$, a standard argument [47, Proposition 1.3] yields

$$HW_*^{\mathbb{Z}_2, < \varepsilon}(L; W) \cong H_{*+n}^{\mathbb{Z}_2}(L, \partial L),$$

where the \mathbb{Z}_2 -action on L is induced by the involution \mathcal{I} on W . We thus obtain the following tautological exact sequence for $HW_*^{\mathbb{Z}_2}(L; W)$.

PROPOSITION 3.30. *There exists a long exact sequence*

$$\dots \rightarrow H_{*+n}^{\mathbb{Z}_2}(L, \partial L) \rightarrow HW_*^{\mathbb{Z}_2}(L; W) \rightarrow HW_*^{\mathbb{Z}_2, +}(L; W) \rightarrow H_{*+n-1}^{\mathbb{Z}_2}(L, \partial L) \rightarrow \dots$$

3.6.4. *Action filtration via \mathcal{A}_{H_0} .* Recall that (H, J) has the properties described in §3.4. The positive part $CF_*^{\mathbb{Z}_2, +, N}(H, J)$ can also be simplified as in Lemma 3.25 using a periodic family. However, the identification may not restrict to the positive part since positivity of the action $\mathcal{A}_{H+\tilde{f}}(Z_p)$ does not, in general, imply positivity of $\mathcal{A}_{H_0}(c)$, where $w^j \otimes c$ is the generator corresponding to Z_p . Notice that our previous constructions and results also hold for \tilde{f} replaced by $\delta \tilde{f}$ with any $\delta > 0$.

LEMMA 3.31. *Under the assumptions and notation in Lemma 3.25, the action $\mathcal{A}_{H+\delta \tilde{f}}(w^j c, z^{(j)})$ is arbitrarily close to $\mathcal{A}_{H_0}(c)$, provided that $\delta > 0$ is sufficiently small.*

Proof. Note that the action $\mathcal{A}_{H+\delta \tilde{f}}$ is defined by

$$\mathcal{A}_{H+\delta \tilde{f}}(w^j c, z^{(j)}) = \ell(c(1)) - \ell(c(0)) - \int_0^1 c^* \widehat{\lambda} - \int_0^1 (H + \delta \tilde{f})(c(t), z^{(j)}) dt.$$

Since the family H is periodic, we may replace H by H_0 as

$$\mathcal{A}_{H_0}(c) = \ell(c(1)) - \ell(c(0)) - \int_0^1 c^* \widehat{\lambda} - \int_0^1 H_0(c(t)) dt.$$

Now the difference between $\mathcal{A}_{H+\delta \tilde{f}}(w^j c, z^{(j)})$ and $\mathcal{A}_{H_0}(c)$ is bounded by

$$\left| \int_0^1 \delta \tilde{f}(z^{(j)}) dt \right| \leq |\delta \tilde{f}(z^{(j)})| \leq \delta N.$$

Since N is fixed, the lemma follows. □

Remark 3.32. We have made several assumptions on the family H and $\delta \tilde{f}$ so far. Namely:

- the family H is admissible;
- $\{H_N\}, \{f_N\}$ are periodic; and
- $\delta > 0$ is sufficiently small in the sense of Lemma 3.31.

By abuse of the notation, we write \tilde{f} instead of $\delta \tilde{f}$. All these conditions can be achieved compatibly by the construction in Example 3.18. First, choose $\delta > 0$ sufficiently small in the sense of Lemma 3.31. Then, take H_0 which is sufficiently small in the region where Hamiltonian chords occur. Finally, construct a family $H = \{H_z\}$ that satisfies inequality (3.12) following the scheme in Example 3.18.

From now on, unless stated otherwise, we assume all the properties listed in Remark 3.32. Then the identification in Lemma 3.25 restricts to the positive part. Moreover, we can follow the same recipe as in §3.4 to provide a \mathbb{Z}_2 -complex structure for the positive parts $HW_*^{\mathbb{Z}_2,+}(L; W)$ and $HW_*^+(L; W)$. As a result, we have an analogous Leray–Serre-type spectral sequence in positive equivariant wrapped Floer homology.

PROPOSITION 3.33. Leray–Serre-type spectral sequence in $HW_*^{\mathbb{Z}_2,+}$ *There is a spectral sequence $\{E^r\}_{r \geq 0}$, converging to $HW_*^{\mathbb{Z}_2,+}(L; W)$, with first page*

$$E_{p,q}^1 = \begin{cases} HW_q^+(L; W) & \text{if } p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

3.7. Morse–Bott spectral sequence. For our applications, we need to know that the positive equivariant wrapped Floer homology is generated by \mathbb{Z}_2 -pairs of Reeb chords; this is not entirely clear from the definition. We do this by constructing a Morse–Bott spectral sequence in $HW_*^{\mathbb{Z}_2,+}$. A similar technique was used in [17] for positive equivariant symplectic homology.

Suppose that the contact form α on the boundary ∂W is of Morse–Bott type (for the definition, we refer to [26]). To have the well-defined Maslov indices of Reeb chords we assume that the Maslov class $\mu_{\mathcal{L}} : \pi_2(\Sigma, \mathcal{L}) \rightarrow \mathbb{Z}$ of the Legendrian $\mathcal{L} = \text{Fix}(\rho)$ vanishes and that $\pi_1(\Sigma, \mathcal{L}) = 0$.

We arrange the action spectrum of Reeb chords as

$$\text{Spec}(\Sigma, \alpha, \mathcal{L}) = \{0 < T_1 < T_2 < \dots\}. \tag{3.19}$$

For each $T \in \text{Spec}(\Sigma, \alpha, \mathcal{L})$, denote the Morse–Bott submanifold of length T by

$$\mathcal{L}_T := \{y \in \mathcal{L} \mid \phi_R^T(y) \in \mathcal{L}\},$$

where ϕ_R^t is the flow of the Reeb vector field associated to the contact form α . This forms a closed submanifold of \mathcal{L} by definition of the Morse–Bott condition. For each Morse–Bott submanifold \mathcal{L}_T , we define the Maslov index $\mu(\mathcal{L}_T)$ as the Maslov index of a Reeb chord starting at \mathcal{L}_T . Note that the Maslov index can be different at different components of \mathcal{L}_T . Starting from a Morse function $h_0 : \mathcal{L}_T \rightarrow \mathbb{R}$, we form a periodic family of Morse functions $h = \{h_z\}$ in the sense of §3.2.3 following the recipe given in Example 3.18. This allows us to consider the \mathbb{Z}_2 -complex structure on the Morse chain complex $(CM_*(h_0, g_0), \phi_j)$, together with the identification (3.4): i.e.,

$$CM_*^{\mathbb{Z}_2,N}(h, g) = \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes_{\mathbb{Z}_2} CM_*(h_0, g_0).$$

Denote the corresponding homology groups by $HM_*(\mathcal{L}_T)$ and $HM_*^{\mathbb{Z}_2,N}(\mathcal{L}_T)$. We do the same procedure for all Morse–Bott submanifolds: take a Morse function, form a periodic family, and so on.

Let $H_0 : \widehat{W} \rightarrow \mathbb{R}$ be an admissible Hamiltonian such that H_0 is C^2 -small on W and depends only on the variable r for $(r, y) \in (1 - \varepsilon, \infty)$. Since α is of Morse–Bott type, H_0 is automatically Morse–Bott. Perturb H_0 in a standard way using the Morse function h_0 on

each submanifold \mathcal{L}_T ; for an explicit description of the perturbation we refer to [27]. For notational convenience, we denote the perturbed function still by H_0 .

Following the recipe in Example 3.18, we construct a periodic family $\{(H_N, J_N)\}_{N \geq 1}$ starting from H_0 , which satisfies all the conditions in Remark 3.32. This yields the identification

$$CF_*^{\mathbb{Z}_2, +, N}(H_N, J_N) = \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes_{\mathbb{Z}_2} CF_*^+(H_0, J_0)$$

as in Lemma 3.25 and Lemma 3.31. We also have the corresponding \mathbb{Z}_2 -complex structure, say, $(CF_*^+(H_0, J_0), \partial, \phi_j)$. By Lemma 3.29 and Lemma 3.31, we can filter the complex $CF_*^{\mathbb{Z}_2, +, N}(H_N, J_N)$ by action values according to the spectrum (3.19). The E_{p*}^0 -terms of the resulting spectral sequence are freely generated by those generators of $CF_*^{\mathbb{Z}_2, +, N}(H_N, J_N)$ whose action is (approximately) equal to T_p .

Note that, by Lemma 3.31, the generators in $CF_*^{\mathbb{Z}_2, +, N}(H_N, J_N)$ of action (approximately) T_p correspond to the generators in $CF_*^+(H_0, J_0)$ of action (approximately) T_p . Consequently,

$$E_{p*}^0 = \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes_{\mathbb{Z}_2} CF_*^{loc}(\mathcal{L}_{T_p}),$$

where $CF_*^{loc}(\mathcal{L}_{T_p}; H_0, J_0)$ denotes the local chain complex near \mathcal{L}_{T_p} . It is well known that the generators and Floer trajectories for $CF_*^{loc}(\mathcal{L}_{T_p}; H_0, J_0)$ are in one-to-one correspondence with those for the Morse chain complex $CM_*(h_0, g_0)$ up to a degree shift (see [37], for example). As a result, we have the identifications

$$\begin{aligned} E_{p*}^0 &= \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes CF_*^{loc}(\mathcal{L}_{T_p}) = \mathbb{Z}_2[w]/\{w^{N+1} = 0\} \otimes CM_*(h_0, g_0) \\ &= CM_*^{\mathbb{Z}_2, N}(h_N, g_N). \end{aligned}$$

Actually, the correspondence between Morse and Floer trajectories is also valid for parametrized ones. This allows us to identify the \mathbb{Z}_2 -complex structures on $CF_*^{loc}(\mathcal{L}_{T_p})$ and $CM_*(h_0, g_0)$. Therefore, the differential $d^0 : E^0 \rightarrow E^0$ of the 0-page is identified with the \mathbb{Z}_2 -equivariant differential $\partial^{\mathbb{Z}_2}$ of $CM_*^{\mathbb{Z}_2, N}(h_N, g_N)$. It follows that, up to a degree shift,

$$E_{p,*}^1 = HM_*^{\mathbb{Z}_2, N}(\mathcal{L}_{T_p}).$$

The degree shift here is the same as that of the non-equivariant part; we refer to the Morse–Bott spectral sequence for the non-equivariant case in [26]. After taking the direct limits, we conclude the following.

THEOREM 3.34. (Morse–Bott spectral sequence in $HW_*^{\mathbb{Z}_2, +}$) *There is a spectral sequence $\{E^r\}_{r \geq 0}$ converging to $HW_*^{\mathbb{Z}_2, +}(L; W)$ whose first page is given by*

$$E_{p,q}^1 = \begin{cases} H_{p+q-\text{shift}(\mathcal{L}_{T_p})+(1/2)\dim L}^{\mathbb{Z}_2}(\mathcal{L}_{T_p}) & \text{if } p > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.20}$$

where $\text{shift}(\mathcal{L}_{T_p}) := \mu(\mathcal{L}_{T_p}) - \frac{1}{2}(\dim \mathcal{L}_{T_p} - 1)$.

3.8. *Generators of $HW_*^{\mathbb{Z}_2,+}$ are Reeb chords.* At the beginning of the construction in §3.7, we may assume that α is *chord-non-degenerate*, i.e., every Reeb chord is non-degenerate. In this case, because of the \mathbb{Z}_2 -symmetry, the non-degeneracy means that each Morse–Bott component of \mathcal{L}_{T_p} consists of two points. The corresponding Reeb chords form a \mathbb{Z}_2 -pair. The \mathbb{Z}_2 -equivariant homology of the Morse–Bott submanifold \mathcal{L}_{T_p} is therefore given by

$$H_*^{\mathbb{Z}_2}(\mathcal{L}_{T_p}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2^r & \text{for } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here r denotes the number of \mathbb{Z}_2 -pairs in \mathcal{L}_{T_p} . It follows that the E^1 -page of the Morse–Bott spectral sequence (3.20) is expressed by \mathbb{Z}_2 -pairs of Reeb chords. We conclude the following corollary.

COROLLARY 3.35. *Suppose that the contact form α on ∂W is chord-non-degenerate. If $HW_k^{\mathbb{Z}_2,+}(L; W) \neq 0$ for some $k \in \mathbb{Z}$, then there exists a \mathbb{Z}_2 -pair of Reeb chords, say, $\{c, \mathcal{I}(c)\}$, whose Maslov index is given by*

$$\mu(c) - \frac{n}{2} + \frac{1}{2} = k. \tag{3.21}$$

When \mathcal{I} is an anti-symplectic involution, a \mathbb{Z}_2 -pair of Reeb chords corresponds to a symmetric periodic Reeb orbit (see Figure 1). Hence, $HW_*^{\mathbb{Z}_2,+}(L; W) \neq 0$ implies the existence of a symmetric periodic Reeb orbit.

3.9. *Computing positive equivariant wrapped Floer homology.* The vanishing property of wrapped Floer homology in §2.2.3 allows us to compute the positive equivariant wrapped Floer homology in many cases. We use the algebraic tools developed in §3.

PROPOSITION 3.36. *Suppose that $HW_*(L; W)$ vanishes. Then the positive equivariant wrapped Floer homology group is given by*

$$HW_*^{\mathbb{Z}_2,+}(L; W) \cong H_{*+n-1}^{\mathbb{Z}_2}(L, \partial L). \tag{3.22}$$

Proof. By the Leray–Serre-type spectral sequence in Corollary 3.28, if $HW_*(L; W)$ vanishes, then $HW_*^{\mathbb{Z}_2}(L; W)$ also vanishes. Recall, from Proposition 3.30, that we have a long exact sequence

$$\dots \rightarrow H_{*+n}^{\mathbb{Z}_2}(L, \partial L) \rightarrow HW_*^{\mathbb{Z}_2}(L; W) \rightarrow HW_*^{\mathbb{Z}_2,+}(L; W) \rightarrow \dots$$

from which it follows that

$$HW_*^{\mathbb{Z}_2,+}(L; W) \cong H_{*+n-1}^{\mathbb{Z}_2}(L, \partial L).$$

This completes the proof. □

Since $HW_*(L; W)$ vanishes in the displaceable case (Corollary 2.15), an immediate corollary is the following.

COROLLARY 3.37. *If Σ is displaceable from \widehat{L} in \widehat{W} , then $HW_*^{\mathbb{Z}_2,+}(L; W)$ is given by (3.22).*

Note that the \mathbb{Z}_2 -action on $(L, \partial L)$ is induced by the involution \mathcal{I} on the ambient space W . So the further computation of the equivariant homology $H_*^{\mathbb{Z}_2}(L, \partial L)$ depends on \mathcal{I} .

3.9.1. *Case of symplectic involutions.* We derive a simple version of a localization property on equivariant homology theory which is enough for our applications.

PROPOSITION 3.38. *Suppose that the \mathbb{Z}_2 -action by \mathcal{I} is free on the boundary ∂L and that the fixed point set $\text{Fix}(\mathcal{I}|_L)$ is non-empty. Then $H_k^{\mathbb{Z}_2}(L, \partial L) \neq 0$ for $k \geq n$.*

This follows directly from the following, more general, observation.

LEMMA 3.39. *Let X^n be a compact manifold with boundary equipped with an involution I . Suppose that I acts freely on ∂X and that $\text{Fix}(I)$ is non-empty. Then $H_k^{\mathbb{Z}_2}(X, \partial X) \neq 0$ for $k \geq n$*

Proof. We first claim that $H_k^{\mathbb{Z}_2}(X) \neq 0$ for $k \geq 0$. Take a fixed point $x \in \text{Fix}(I) \subset X \setminus \partial X$. We consider the inclusion $i : \{x\} \hookrightarrow X$. This map is clearly \mathbb{Z}_2 -equivariant, so we obtain a section $\bar{i} : \{x\}_{\text{Borel}} = \mathbb{R}P^\infty \rightarrow X_{\text{Borel}}$ of the bundle $\pi : X_{\text{Borel}} \rightarrow \mathbb{R}P^\infty$. Since $\pi \circ \bar{i} = \text{id}$, it follows that the induced map

$$\bar{i}_* : H_*(\mathbb{R}P^\infty) \rightarrow H_*(X_{\text{Borel}}) = H_*^{\mathbb{Z}_2}(X)$$

is injective. In particular, $H_k^{\mathbb{Z}_2}(X) \neq 0$ for $k \geq 0$, as we claimed.

Consider the long exact sequence in homology for the pair $(X_{\text{Borel}}, \partial X_{\text{Borel}})$, namely,

$$\cdots \rightarrow H_k(\partial X_{\text{Borel}}) \rightarrow H_k(X_{\text{Borel}}) \rightarrow H_k(X_{\text{Borel}}, \partial X_{\text{Borel}}) \rightarrow H_{k-1}(\partial X_{\text{Borel}}) \rightarrow \cdots$$

Since the \mathbb{Z}_2 -action is free on the boundary ∂X ,

$$H_k^{\mathbb{Z}_2}(\partial X) = H_k(\partial X_{\text{Borel}}) = H_k(\partial X/\mathbb{Z}_2).$$

In particular, since $\dim(\partial X/\mathbb{Z}_2) = n - 1$, we have $H_k(\partial X_{\text{Borel}}) = 0$ for $k \geq n$. Applying this together with the previous claim to the long exact sequence, we find that $H_k(X_{\text{Borel}}, \partial X_{\text{Borel}}) \cong H_k(X_{\text{Borel}}) \neq 0$ for $k \geq n + 1$, and the inclusion $0 \neq H_n(X_{\text{Borel}}) \hookrightarrow H_n(X_{\text{Borel}}, \partial X_{\text{Borel}})$ is injective so that also $H_n(X_{\text{Borel}}, \partial X_{\text{Borel}}) \neq 0$. This completes the proof. \square

Applying Proposition 3.38 to the displaceable case computed in Corollary 3.37, we conclude the following theorem.

THEOREM 3.40. *Suppose that Σ is displaceable from \widehat{L} in \widehat{W} and that \mathcal{I} is an exact symplectic involution on W . If the involution $\mathcal{I}|_L$ acts freely on the boundary and the fixed point set $\text{Fix}(\mathcal{I}|_L)$ is non-empty, then $HW_k^{\mathbb{Z}_2,+}(L; W) \neq 0$ for all $k \geq 1$.*

Example 3.41. Consider the real Lagrangian \mathbb{R}^n , which is the fixed point set of complex conjugation on \mathbb{C}^n . Consider the antipodal map \mathcal{I} as an exact symplectic involution on \mathbb{C}^n .

Note that $S^{2n-1} = \partial B^{2n}$ is displaceable from \mathbb{R}^n in \mathbb{C}^n , and that the fixed point set $\text{Fix}(\mathcal{I}|_{B^n})$ of the antipodal map on B^n is a singleton set (the origin). It follows from Theorem 3.40 that $HW_k^{\mathbb{Z}_2,+}(\mathbb{R}^n; \mathbb{C}^n) \neq 0$ for all $k \geq 1$.

3.9.2. *Case of anti-symplectic involutions.* Since the anti-symplectic involution \mathcal{I} on W restricts to the identity map on L ,

$$H_*^{\mathbb{Z}_2}(L, \partial L) \cong H_*(L, \partial L) \otimes_{\mathbb{Z}_2} H_*(\mathbb{R}P^\infty).$$

By Corollary 3.37 and Lefschetz duality, we obtain the following theorem.

THEOREM 3.42. *Suppose that Σ is displaceable from \widehat{L} in \widehat{W} and that L is a real Lagrangian, i.e., \mathcal{I} is exact anti-symplectic. Then*

$$HW_*^{\mathbb{Z}_2,+}(L; W) \cong (H_*(L, \partial L) \otimes_{\mathbb{Z}_2} H_*(\mathbb{R}P^\infty))[n - 1],$$

where $[n - 1]$ means the downward shift in grading by $n - 1$. In particular, if L is orientable, then $HW_k^{\mathbb{Z}_2,+}(L; W) \neq 0$ for all $k \geq 1$.

Example 3.43. Consider again the real Lagrangian \mathbb{R}^n in \mathbb{C}^n and the complex conjugation \mathcal{I} which is an anti-symplectic involution on \mathbb{C}^n . Then one checks that

$$H_*(L, \partial L) \cong \tilde{H}_*(S^n) = \begin{cases} \mathbb{Z}_2 & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.42, $HW_k^{\mathbb{Z}_2,+}(\mathbb{R}^n; \mathbb{C}^n) \neq 0$ for all $k \geq 1$.

4. On the minimal number of symmetric periodic Reeb orbits

Let (Σ, α, ρ) be a real contact manifold. Assume that $\mathcal{L} := \text{Fix}(\rho)$ is non-empty and that the Maslov class $\mu_{\mathcal{L}} : \pi_2(\Sigma, \mathcal{L}) \rightarrow \mathbb{Z}$ of the Legendrian \mathcal{L} vanishes. Abbreviate by $R = R_\alpha$ the associated Reeb vector field and by $\xi = \ker(\alpha)$ the contact structure. Since the involution ρ is anti-contact, one can associate to each Reeb chord $c : [0, T] \rightarrow \Sigma$ a symmetric periodic Reeb orbit

$$c^2(t) := \begin{cases} c(t) & \text{if } t \in [0, T], \\ \rho \circ c(2T - t) & \text{if } t \in [T, 2T]. \end{cases}$$

We apply the equivariant wrapped Floer homology to the study of the minimal number of symmetric periodic Reeb orbits on the real contact manifold (Σ, α, ρ) . In order to do this, we will develop an index theory on iterations of symmetric periodic Reeb orbits.

4.1. Index of symmetric periodic Reeb orbits.

4.1.1. *The Robbin–Salamon index.* We briefly recall the definition of the Robbin–Salamon index of a path of Lagrangians. For more details, we refer to [40] and [13, Ch. 10].

Let $\mathcal{L}(n)$ be the Lagrangian Grassmannian, i.e., the space of Lagrangian subspaces of (\mathbb{C}^n, ω_0) , where $\omega_0 = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ is the standard symplectic form. In coordinates $z_j = q_j + ip_j$, the form ω_0 equals $\sum_{j=1}^n dq_j \wedge dp_j$. We fix $\Lambda_0 \in \mathcal{L}(n)$. Without loss of

generality, we may assume that $\Lambda_0 = \mathbb{R}^n \times \{0\} \subset \mathbb{C}^n$. Any Lagrangian complement Λ_1 , i.e., $\Lambda_1 \in \mathcal{L}(n)$ such that $\Lambda_0 \cap \Lambda_1 = \{0\}$, can then be written as

$$\Lambda_1 = \{(Bx, x) : x \in \mathbb{R}^n\} \tag{4.1}$$

for some real symmetric $n \times n$ matrix B . Choose a smooth path $\Lambda : (-\varepsilon, \varepsilon) \rightarrow \mathcal{L}(n)$ such that

$$\Lambda(t) = \{(x, A(t)x) : x \in \mathbb{R}^n\},$$

where A is a path of symmetric $n \times n$ matrices satisfying $A(0) = 0$ so that $\Lambda(0) = \Lambda_0$. Abbreviate $\widehat{\Lambda} = \dot{\Lambda}(0)$. Since $\Lambda_0 \cap \Lambda_1 = \{0\}$, for each $(x, 0) \in \Lambda_0$ there exists a unique $w_x(t) \in \Lambda_1$ such that $(x, 0) + w_x(t) \in \Lambda(t)$, provided that $\varepsilon > 0$ is small enough. We then define the quadratic form

$$Q_{\Lambda(t)}^{\Lambda_0, \Lambda_1}((x, 0)) := \omega_0((x, 0), w_x(t)).$$

By means of (4.1), for each $t \in (-\varepsilon, \varepsilon)$ we find a unique $y(t) \in \mathbb{R}^n$ such that $w_x(t) = (By(t), y(t))$. It follows that

$$(x, 0) + w_x(t) = (x + By(t), y(t)) \in \Lambda(t) \implies y(t) = A(t)(x + By(t)). \tag{4.2}$$

Then the quadratic form $Q_{\Lambda(t)}^{\Lambda_0, \Lambda_1}$ becomes

$$Q_{\Lambda(t)}^{\Lambda_0, \Lambda_1}((x, 0)) = \omega_0((x, 0), (By(t), y(t))) = \langle x, y(t) \rangle.$$

Differentiating with respect to t at $t = 0$ gives rise to

$$Q_{\widehat{\Lambda}}^{\Lambda_0, \Lambda_1}((x, 0)) := \left. \frac{d}{dt} \right|_{t=0} Q_{\Lambda(t)}^{\Lambda_0, \Lambda_1}((x, 0)) = \langle x, \dot{y}(0) \rangle = \langle x, \dot{A}(0)x \rangle,$$

where the last equality follows from (4.2) together with $y(0) = 0$. Note that the expression of $Q_{\widehat{\Lambda}}^{\Lambda_0, \Lambda_1}$ does not depend on the choice of B or, equivalently, Λ_1 . Thus, we can write $Q_{\widehat{\Lambda}} = Q_{\widehat{\Lambda}}^{\Lambda_0, \Lambda_1}$.

We are ready to define the Robbin–Salamon index. Fix $V \in \mathcal{L}(n)$ and a path $\Lambda : [0, T] \rightarrow \mathcal{L}(n)$. For each $t \in [0, T]$, the *crossing form* associated to Λ and V is defined to be

$$C(\Lambda, V, t) := Q_{\dot{\Lambda}(t)}|_{V \cap \Lambda(t)}.$$

If $C(\Lambda, V, t)$ is non-singular, then t is called a *regular crossing*. Suppose that the path Λ admits only regular crossings. Then the *Robbin–Salamon index* of Λ with respect to V is defined to be

$$\mu_{RS}(\Lambda, V) := \frac{1}{2} \operatorname{sgn} C(\Lambda, V, 0) + \sum_{\substack{t: \text{crossing} \\ 0 < t < T}} \operatorname{sgn} C(\Lambda, V, t) + \frac{1}{2} \operatorname{sgn} C(\Lambda, V, T).$$

Since regular crossings are isolated, the sum is finite. This index has the following properties.

- (Homotopy) Two paths Λ_0, Λ_1 are homotopic with same endpoints if and only if they have the same Robbin–Salamon index.
- (Catenation) For $a < c < b$, it holds that $\mu_{RS}(\Lambda|_{[a,c]}, V) + \mu_{RS}(\Lambda|_{[c,b]}, V) = \mu_{RS}(\Lambda|_{[a,b]}, V)$.

- (Reversal) If $\Lambda : [0, 1] \rightarrow \mathcal{L}(n)$ is a path of Lagrangians, then $\mu_{RS}(\Lambda, V) = -\mu_{RS}(\overline{\Lambda}, V)$, where $\overline{\Lambda}(t) := \Lambda(1 - t)$.
- (Naturality) If Φ is a path of symplectic matrices and if $V_0, V_1 \in \mathcal{L}(n)$, then $\mu_{RS}(\Phi V_0, V_1) = -\mu_{RS}(\Phi^{-1} V_1, V_0)$. If Φ is anti-symplectic, we have $\mu_{RS}(\Phi V_0, V_1) = \mu_{RS}(\Phi^{-1} V_1, V_0)$.

4.1.2. *The spectral flow.* Let $\mathcal{S} = C^\infty([0, T], \text{Sym}(2n))$ be the space of smooth paths of symmetric $2n \times 2n$ matrices endowed with the metric

$$d(S_0, S_1) = \int_0^T \|S_0(t) - S_1(t)\| dt, \quad S_0, S_1 \in \mathcal{S}.$$

We introduce the Hilbert space

$$W_{\mathbb{R}^n}^{1,2}([0, T], \mathbb{C}^n) = \{v \in W^{1,2}([0, T], \mathbb{C}^n) : v(0), v(T) \in \mathbb{R}^n\}.$$

For $S \in \mathcal{S}$, consider the bounded linear operator between Hilbert spaces

$$L_S : W_{\mathbb{R}^n}^{1,2}([0, T], \mathbb{C}^n) \rightarrow L^2([0, T], \mathbb{C}^n), \quad v \mapsto -J_0 \partial_t v - S v. \tag{4.3}$$

It is well known that the operator L_S is a symmetric Fredholm operator of index zero. Moreover, its spectrum

$$\mathfrak{S}(L_S) = \{\lambda \in \mathbb{C} : L_S - \lambda \text{Id is non-invertible}\}$$

has the following properties.

- It consists precisely of the eigenvalues.
- It is real and discrete.
- The eigenvalues can accumulate only at ∞ .

Note that, if $S \equiv 0$, then the operator L_0 has $\mathfrak{S}(L_0) = \{\pi k : k \in \mathbb{Z}\}$ and each eigenvalue πk has multiplicity n . An important fact is that, by Kato’s perturbation theory [25], the eigenvalues of L_S vary continuously under perturbations of S . It follows that there exist continuous mappings

$$\lambda_k : \mathcal{S} \rightarrow \mathfrak{S} := \{(S, \lambda) \in \mathcal{S} \times \mathbb{R} : \lambda \in \mathfrak{S}(L_S)\}, \quad k \in \mathbb{Z}$$

having the following properties.

- (Continuity) For every $k \in \mathbb{Z}$, $S \mapsto \lambda_k(S)$ is a continuous map.
- (Monotonicity) For every $k \in \mathbb{Z}$, $\lambda_k(S) \leq \lambda_{k+1}(S)$.
- (Eigenvalue) For each S , $\mathfrak{S}(L_S) = \{\lambda_k(S) : k \in \mathbb{Z}\}$.
- (Multiplicity) If $\lambda \in \mathfrak{S}(L_S)$, then $\#\{k \in \mathbb{Z} : \lambda_k(S) = \lambda\} = \text{multiplicity of } \lambda$.
- (Normalization) For $k \in \{1, 2, \dots, n\}$, $\lambda_k(0) = 0$.

Associated to $S \in \mathcal{S}$ is the path of symplectic matrices $\Phi_S : [0, T] \rightarrow \text{Sp}(2n)$, i.e., the solution of the Cauchy problem $\partial_t \Phi_S(t) = J_0 S(t) \Phi_S(t)$, $\Phi_S(0) = \text{id}$. The following theorem relates the Robbin–Salamon index of the path Φ_S with respect to the real Lagrangian \mathbb{R}^n to the spectrum of L_S in the non-degenerate case. For the proof, we refer the reader to [13, Theorem 11.2.3].

THEOREM 4.1. *Assume that $\Phi_S(T)\mathbb{R}^n \cap \mathbb{R}^n = \{0\}$. Then*

$$\mu_{RS}(\Phi_S\mathbb{R}^n, \mathbb{R}^n) = \max\{k \in \mathbb{Z} : \lambda_k(S) < 0\} - \frac{n}{2}.$$

Similarly, one can consider the Hilbert space

$$W_{i\mathbb{R}^n}^{1,2}([0, T], \mathbb{C}^n) = \{v \in W^{1,2}([0, T], \mathbb{C}^n) : v(0), v(T) \in i\mathbb{R}^n\}$$

and the bounded linear operator $\tilde{L}_S : W_{i\mathbb{R}^n}^{1,2}([0, T], \mathbb{C}^n) \rightarrow L^2([0, T], \mathbb{C}^n)$ defined by the same formula as L_S . This operator satisfies the same properties as L_S , and hence there exist continuous mappings $\tilde{\lambda}_k, k \in \mathbb{Z}$ having the same properties as λ_k . Consequently, if $\Phi_S(T)i\mathbb{R}^n \cap i\mathbb{R}^n = \{0\}$, then

$$\mu_{RS}(\Phi_S i\mathbb{R}^n, i\mathbb{R}^n) = \max\{k \in \mathbb{Z} : \tilde{\lambda}_k(S) < 0\} - \frac{n}{2}. \tag{4.4}$$

4.1.3. Index of Reeb chords and iterations. Let $c : [0, T] \rightarrow \Sigma$ be a Reeb chord and let c^2 be the associated symmetric periodic Reeb orbit. We define further iterations of c by

$$c^{2\ell-1}(t) := \begin{cases} c(t) & \text{if } t \in [0, T], \\ \rho \circ c(2T - t) & \text{if } t \in [T, 2T], \\ c(t - 2T) & \text{if } t \in [2T, 3T], \\ \rho \circ c(4T - t) & \text{if } t \in [3T, 4T], \\ \vdots & \\ c(t - (2\ell - 2)T) & \text{if } t \in [(2\ell - 2)T, (2\ell - 1)T], \end{cases}$$

and

$$c^{2\ell}(t) := \begin{cases} c^{2\ell-1}(t) & \text{if } t \in [0, (2\ell - 1)T], \\ \rho \circ c(2\ell T - t) & \text{if } t \in [(2\ell - 1)T, 2\ell T]. \end{cases}$$

Choose a $d\alpha|_\xi$ -compatible almost complex structure J on ξ which is ρ -anti-invariant: i.e.,

$$\rho^*J := (D\rho|_\xi)^{-1} \circ J \circ D\rho|_\xi = -J.$$

Given a contractible Reeb chord $c : [0, T] \rightarrow \Sigma$, let $u : D \rightarrow \Sigma$ be a capping disk for the symmetric periodic Reeb orbit c^2 such that $u \circ I = \rho \circ u$, where

$$I = \begin{pmatrix} \text{id}_{\mathbb{R}^{n-1}} & 0 \\ 0 & -\text{id}_{\mathbb{R}^{n-1}} \end{pmatrix}.$$

We choose a symmetric unitary trivialization

$$\tilde{\Psi} : D \times \mathbb{C}^{n-1} \rightarrow u^*\xi$$

satisfying:

- $\Psi(z)^*d\alpha|_\xi = \omega_0$;
- $J(u(z))\Psi(z) = \Psi(z)J_0$; and
- $D\rho|_{\xi_{u(z)}} \circ \Psi(z) = \Psi(\bar{z}) \circ I$,

where $\Psi(z) := \tilde{\Psi}(z, \cdot)$,

$$J_0 = \begin{pmatrix} 0 & -\text{id}_{\mathbb{R}^{n-1}} \\ \text{id}_{\mathbb{R}^{n-1}} & 0 \end{pmatrix}.$$

In particular, the Legendrian \mathcal{L} becomes $L_0 = \mathbb{R}^{n-1} \times \{0\}$, which is a Lagrangian subspace of $(\mathbb{C}^{n-1}, \omega_0)$. We fix $L_1 := \{0\} \times \mathbb{R}^{n-1} = J_0 L_0$, which is a Lagrangian complement of L_0 in \mathbb{C}^{n-1} . For the existence of such a trivialization, we refer to [11, Lemma 3.10]. We write $\Psi(t) := \Psi(e^{2\pi i t/2T})$. Associated to the symmetric periodic Reeb orbit c^2 is the path of symplectic matrices $\Phi^2 : [0, 2T] \rightarrow \text{Sp}(2n - 2)$

$$\Phi^2(t) := \Psi(t)^{-1} \circ D\phi_R^t(c^2(0))|_{\xi} \circ \Psi(0).$$

Abbreviating $\Phi = \Phi^2|_{[0, T]}$, one represents the paths of symplectic matrices corresponding to the iterations: i.e.,

$$\Phi^{2\ell-1}(t) = \begin{cases} \Phi(t) & \text{if } t \in [0, T], \\ I\Phi(2T - t)\Phi(T)^{-1}I\Phi(T) & \text{if } t \in [T, 2T], \\ \vdots & \\ I\Phi(2(\ell - 1)T - t)I\Phi(2T)^{\ell-1} & \text{if } t \in [(2\ell - 3)T, (2\ell - 2)T], \\ \Phi(t - 2(\ell - 1)T)\Phi(2T)^{\ell-1} & \text{if } t \in [(2\ell - 2)T, (2\ell - 1)T], \end{cases} \tag{4.5}$$

and

$$\Phi^{2\ell}(t) = \begin{cases} \Phi^{2\ell-1}(t) & \text{if } t \in [0, (2\ell - 1)T], \\ I\Phi(2\ell T - t)I\Phi(2T)^\ell & \text{if } t \in [(2\ell - 1)T, 2\ell T], \end{cases} \tag{4.6}$$

where $\Phi(2T) = I\Phi(T)^{-1}I\Phi(T)$.

The Maslov indices of a non-degenerate Reeb chord c are defined by

$$\mu_I(c) := \mu_{\text{RS}}(\Phi L_0, L_0), \quad \mu_{-I}(c) := \mu_{\text{RS}}(\Phi L_1, L_1)^\dagger.$$

Recall that c is non-degenerate (as a chord) if and only if $\Phi(T)L_0 \cap L_0 = \{0\}$. If we write Φ as a block matrix

$$\Phi = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

with respect to the Lagrangian splitting $\mathbb{C}^{n-1} = L_0 \oplus L_1$, then this condition is equivalent to the block $Z(T)$ being invertible. Similarly, $\Phi(T)L_1 \cap L_1 = \{0\}$ is equivalent to the block $Y(T)$ being invertible.

We have defined the Maslov indices μ_I, μ_{-I} of non-degenerate Reeb chords via the Robbin–Salamon indices. We now extend the definitions to degenerate Reeb chords. If we extend them by the same formula, i.e., via the Robbin–Salamon index, then the indices are neither lower nor upper semi-continuous. In other words, under a small perturbation, the indices of a degenerate Reeb chord may jump up or down. Following [20], we extend the definitions to degenerate Reeb chords via the spectral flow.

† We borrow the notation μ_I, μ_{-I} from [11, §3].

Definition 4.2. Let c be a contractible Reeb chord on a real contact manifold $(\Sigma^{2n-1}, \alpha, \rho)$. Abbreviate by $\Phi(t) \in \text{Sp}(2n - 2)$, $t \in [0, T]$ the corresponding path of symplectic matrices. Then its Maslov indices μ_I, μ_{-I} are defined to be

$$\begin{aligned} \mu_I(c) &:= \max\{k \in \mathbb{Z} : \lambda_k(S) < 0\} - \frac{n - 1}{2}, \\ \mu_{-I}(c) &:= \max\{k \in \mathbb{Z} : \tilde{\lambda}_k(S) < 0\} - \frac{n - 1}{2}, \end{aligned}$$

where $\lambda_k, \tilde{\lambda}_k$ are the continuous mappings defined in §4.1.2 and $S(t) := -J_0\dot{\Phi}(t)\Phi(t)^{-1}$, $t \in [0, T]$ is the path of $(2n - 2) \times (2n - 2)$ symmetric matrices.

It is worth mentioning that with these definitions the indices become lower semi-continuous, i.e., under a small perturbation, they may jump up, but not down. Moreover, in view of Theorem 4.1, in the non-degenerate case,

$$\mu_I(c) = \mu(c),$$

where $\mu(c)$ is defined as in Definition 2.3.

In the following, we say that a contact form α is:

- non-degenerate if all periodic Reeb orbits are non-degenerate; and
- L_i -non-degenerate if, for each Reeb chord $c : [0, T] \rightarrow \Sigma$, the corresponding path of symplectic matrices Φ satisfies $\Phi(T)L_i \cap L_i = \{0\}$, $i = 0, 1$.

PROPOSITION 4.3. ([33, Proposition C], [11, Propositions 3.6 and 3.7]) *Let $c : [0, T] \rightarrow \Sigma$ be a Reeb chord, as above. Then the iteration c^2 is non-degenerate (as a periodic orbit) if and only if c is both L_0 - and L_1 -non-degenerate. In particular, in this case,*

$$\mu_{\text{CZ}}(c^2) = \mu_I(c) + \mu_{-I}(c),$$

where μ_{CZ} denotes the Conley–Zehnder index of a periodic orbit.

4.2. Real dynamical convexity. In this section, we introduce the notion of real dynamical convexity.

Definition 4.4. A real contact manifold $(\Sigma^{2n-1}, \alpha, \rho)$ is said to be *real dynamically convex* if the Maslov class $\mu_{\mathcal{L}} : \pi_2(\Sigma, \mathcal{L}) \rightarrow \mathbb{Z}$, where $\mathcal{L} = \text{Fix}(\rho)$, vanishes and every contractible Reeb chord c satisfies the lower bound

$$\mu_I(c), \mu_{-I}(c) \geq \frac{n + 1}{2}.$$

In order to explain the reason for this terminology, let Σ be a strictly convex hypersurface in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ equipped with the standard contact form

$$\alpha_0 = \frac{1}{2} \sum_{j=1}^n q_j dp_j - p_j dq_j \Big|_{\Sigma}.$$

The corresponding Reeb vector field and the contact structure are denoted by $R = R_{\alpha_0}$ and $\xi = \ker(\alpha_0)$, respectively. We assume that Σ is invariant under complex conjugation

$$I = \begin{pmatrix} \text{id}_{\mathbb{R}^n} & 0 \\ 0 & -\text{id}_{\mathbb{R}^n} \end{pmatrix}$$

of (\mathbb{C}^n, ω_0) , where $\omega_0 = \sum_{j=1}^n dq_j \wedge dp_j$.

THEOREM 4.5. *Let the triple (Σ, α_0, I) be described as above. Then it is real dynamically convex.*

Proof. We imitate the strategy of the proofs of [20, Theorem 3.4] and [13, Theorem 12.2.1]. Without loss of generality, we may assume that Σ encircles the origin. Choose a Hamiltonian $H : \mathbb{C}^n \rightarrow \mathbb{R}$ that satisfies the following.

- (Hypersurface) $H^{-1}(1) = \Sigma$ and $H(z) < 1$ for $z \in \text{int}(K)$, where K is a strictly convex and compact subset of \mathbb{C}^n with $\partial K = \Sigma$.
- (Invariance) $H(Iz) = H(z)$, $z \in \mathbb{C}^n$.
- (Homogeneity) $H(rz) = r^2 H(z)$, $z \in \mathbb{C}^n \setminus \{0\}$, $r \geq 0$.

Let X_H denote the Hamiltonian vector field associated to H defined implicitly by the relation $\iota_{X_H} \omega_0 = -dH$. Using homogeneity of H , one sees that

$$R(z) = X_H(z), \quad z \in \Sigma.$$

In particular, for a Reeb chord $c : [0, T] \rightarrow \Sigma$, it holds that $\phi_H^t(c(0)) = c(t)$, where ϕ_H^t denotes the Hamiltonian flow of H . Recall that the path of symplectic matrices $\Phi(t) := D\phi_H^t(c(0)) \in \text{Sp}(2n)$, $t \in [0, T]$, satisfies

$$\dot{\Phi}(t) = J_0 \mathcal{H}(c(t)) \Phi(t),$$

where \mathcal{H} is the Hessian of H , which is positive-definite due to the strict convexity of the hypersurface Σ . By considering $\frac{1}{T}H$, from now on, we may assume that $T = 1$.

Let $u : D \rightarrow \Sigma$ be a capping disk of a symmetric periodic Reeb orbit c^2 . Choose a symmetric unitary trivialization $\Psi_\xi : u^*\xi \rightarrow D \times \mathbb{C}^{n-1}$. Abbreviating $\eta_z = \text{span}\{z, X_H(z)\}$, $z \in \mathbb{C}^n$, we obtain the symplectic splitting $u^*\mathbb{C}^n = u^*\eta \oplus u^*\xi$ of the vector bundle $u^*\mathbb{C}^n \rightarrow D$. Note that $u^*\eta$ admits a canonical unitary trivialization $\Psi_\eta : u^*\eta \rightarrow D \times \mathbb{C}$ given by

$$u(z) \mapsto 1 \quad \text{and} \quad X_H(u(z)) \mapsto i.$$

This gives rise to the unitary trivialization

$$\Psi := \Psi_\eta \oplus \Psi_\xi : u^*\mathbb{C}^n \rightarrow D \times \mathbb{C}^n.$$

In this trivialization, the path $\Phi : [0, 1] \rightarrow \text{Sp}(2n)$ is given by the path of $2n \times 2n$ symplectic matrices $\widehat{\Phi}(t) = \widehat{\Phi}_\eta \oplus \widehat{\Phi}_\xi(t)$, where $\widehat{\Phi}_\eta := \Psi_\eta(c(t)) \circ \Phi(t)|_\eta \circ \Psi_\eta(c(0))^{-1} = \text{id}_{\mathbb{C}}$. The catenation property of the Robbin–Salamon index then implies that $\mu_{\text{RS}}(\widehat{\Phi} \Lambda_0, \Lambda_0) = \mu_{\text{RS}}(\widehat{\Phi}_\xi L_0, L_0)$, where $\Lambda_0 = \mathbb{R}^n \times \{0\}$. On the other hand, since D is contractible, the trivialization $\Psi : u^*\mathbb{C}^n = D \times \mathbb{C}^n \rightarrow D \times \mathbb{C}^n$ is homotopic as a bundle map to the identity map of $D \times \mathbb{C}^n$. It thus follows from homotopy invariance of

the Robbin–Salamon index that $\mu_{RS}(\widehat{\Phi}\Lambda_0, \Lambda_0) = \mu_{RS}(\Phi\Lambda_0, \Lambda_0)$ and, similarly, $\mu_{RS}(\widehat{\Phi}\Lambda_1, \Lambda_1) = \mu_{RS}(\Phi\Lambda_1, \Lambda_1)$, where $\Lambda_1 = \{0\} \times \mathbb{R}^n$.

CLAIM. Consider the path of Lagrangian subspaces $Gr(\Phi)$ of $(\mathbb{C}^n \times \mathbb{C}^n, \Omega = (-\omega) \times \omega)$. For each $t_0 \in [0, 1]$, the quadratic form $Q = Q_{\partial_t Gr(\Phi)|_{t=t_0}}$ is given by

$$Q(x, \Phi(t_0)x) = \langle \Phi(t_0)x, \mathcal{H}(c(t_0))\Phi(t_0)x \rangle.$$

Proof of the claim. A typical Lagrangian complement to the graph Lagrangian $Gr(\Phi(t_0))$ is given by $Gr(-\Phi(t_0))$. By definition, the quadratic form Q is given by

$$Q(x, \Phi(t_0)x) = \Omega((x, \Phi(t_0)x), (\dot{y}(t_0), -\Phi(t_0)\dot{y}(t_0))),$$

where $y(t)$ is obtained as follows: for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, we find $y(t) \in \mathbb{C}^n$ satisfying that

$$(x, \Phi(t_0)x) + (y(t), -\Phi(t_0)y(t)) \in Gr(\Phi(t)).$$

It follows that

$$\Phi(t_0)(x - y(t)) = \Phi(t)(x + y(t))$$

and hence we obtain that

$$\dot{y}(t_0) = -\frac{1}{2}\Phi(t_0)^{-1}\dot{\Phi}(t_0)x,$$

where we have used the fact that $y(t_0) = 0$. We then conclude that

$$\begin{aligned} Q(x, \Phi(t_0)x) &= -\omega(x, \dot{y}(t_0)) - \omega(\Phi(t_0)x, \Phi(t_0)\dot{y}(t_0)) \\ &= -2\omega(x, \dot{y}(t_0)) \\ &= \omega(x, \Phi(t_0)^{-1}\dot{\Phi}(t_0)x) \\ &= \langle \Phi(t_0)x, \mathcal{H}\Phi(t_0)x \rangle, \end{aligned}$$

from which the claim follows. □

The claim implies that the crossing form $C(Gr(\Phi), \Lambda_j \times \Lambda_j, t) = Q|_{Gr(\Phi(t)) \cap (\Lambda_j \times \Lambda_j)}$ is positive-definite, provided that $Gr(\Phi(t)) \cap (\Lambda_j \times \Lambda_j) \neq \{0\}$, $j = 0, 1$. Hence, we obtain that

$$\begin{aligned} \mu_{RS}(Gr(\Phi), \Lambda_j \times \Lambda_j) &= \frac{1}{2}n + \sum_{0 < t < 1} \dim(\Phi(t)\Lambda_j \cap \Lambda_j) \\ &\quad + \frac{1}{2} \dim(\Phi(1)\Lambda_j \cap \Lambda_j), \quad j = 0, 1. \end{aligned}$$

In view of [40, Theorem 3.2],

$$\mu_{RS}(\Phi\Lambda_0, \Lambda_0) = \mu_{RS}(Gr(\Phi), \Lambda_0 \times \Lambda_0), \quad \mu_{RS}(\Phi\Lambda_1, \Lambda_1) = \mu_{RS}(Gr(\Phi), \Lambda_1 \times \Lambda_1).$$

We compute that

$$D\phi_H^1(c(0))R(c(0)) = D\phi_H^1(c(0))X_H(c(0)) = X_H(\phi_H^1(c(0))) = R(c(1))$$

and

$$D\phi_H^1(c(0))c(0) = \phi_H^1(c(0)) = c(1),$$

where, in the first equality, we used $D\phi_H^t(z)z = \phi_H^t(z)$ for all z , which follows from homogeneity of H . These identities show that

$$\dim(\Phi(1)\Lambda_j \cap \Lambda_j) = \dim(\widehat{\Phi}_\xi(1)L_j \cap L_j) + 1.$$

Combining the discussion so far, we obtain that

$$\mu_{RS}(\widehat{\Phi}_\xi L_j, L_j) \geq \frac{n+1}{2} + \frac{1}{2} \dim(\widehat{\Phi}_\xi(1)L_j \cap L_j), \quad j = 0, 1. \tag{4.7}$$

Case 1. The Reeb chord c is non-degenerate. In view of (4.7), it follows from the definitions of the Maslov indices that

$$\begin{aligned} \mu_I(c) &\geq \frac{n+1}{2} + \frac{1}{2} \dim(\widehat{\Phi}_\xi(1)L_0 \cap L_0), \\ \mu_{-I}(c) &\geq \frac{n+1}{2} + \frac{1}{2} \dim(\widehat{\Phi}_\xi(1)L_1 \cap L_1). \end{aligned}$$

Since $\dim(\widehat{\Phi}_\xi(1)L_j \cap L_j) \geq 0$ for $j = 0, 1$, these give rise to the desired lower bounds.

Case 2. The Reeb chord c is degenerate. For $\varepsilon > 0$ small enough, we choose a path of symplectic matrices $\Phi_\varepsilon^j : [0, 1 + \varepsilon] \rightarrow \text{Sp}(2n - 2)$ such that $\Phi_\varepsilon^j(t) = \widehat{\Phi}_\xi(t)$ for $t \in [0, 1]$ and there exist no further crossings with respect to L_j in $t \in (1, 1 + \varepsilon]$, for $j = 0, 1$. In particular, the path Φ_ε^j is L_j -non-degenerate for $j = 0, 1$. We find that

$$\begin{aligned} \mu_{RS}(\Phi_\varepsilon^j L_j, L_j) &= \mu_{RS}(\widehat{\Phi}_\xi L_j, L_j) + \frac{1}{2} \text{sgn}C(\widehat{\Phi}_\xi L_j, L_j, 1) \\ &= \mu_{RS}(\widehat{\Phi}_\xi L_j, L_j) + \frac{1}{2} \dim(\widehat{\Phi}_\xi(1)L_j \cap L_j) \\ &\geq \frac{n+1}{2} + \dim(\widehat{\Phi}_\xi(1)L_j \cap L_j). \end{aligned}$$

Abbreviate by $S(t), t \in [0, 1]$ the paths of symmetric matrices associated with $\widehat{\Phi}_\xi(t)$. Note that

$$\dim \ker L_S = \dim(\widehat{\Phi}_\xi(1)L_0 \cap L_0), \quad \dim \ker \widetilde{L}_S = \dim(\widehat{\Phi}_\xi(1)L_1 \cap L_1),$$

where L_S, \widetilde{L}_S are the bounded linear operators defined as in §4.1.2. It then follows from the continuity of eigenvalues that

$$\mu_I(c) \geq \lim_{\varepsilon \rightarrow 0} \mu_{RS}(\Phi_\varepsilon^0 L_0, L_0) - \dim(\widehat{\Phi}_\xi(1)L_0 \cap L_0) \geq \frac{n+1}{2}$$

and, similarly,

$$\mu_{-I}(c) \geq \frac{n+1}{2}.$$

This completes the proof of the theorem. □

4.3. *Index-increasing property.* We prove the following increasing property of the Robbin–Salamon index which is one of the main tools in counting symmetric periodic Reeb orbits via equivariant wrapped Floer homology.

THEOREM 4.6. *Assume that a real contact manifold $(\Sigma^{2n-1}, \alpha, \rho)$ is non-degenerate. Let c be a Reeb chord. If $\mu_I(c) \geq k$ for some $k \in \frac{1}{2}\mathbb{Z}$, then $\mu_I(c^{\ell+1}) \geq \mu_I(c^\ell) + k - ((n - 1)/2)$ for every $\ell \in \mathbb{N}$. In particular, if α is real dynamically convex, then $\mu_I(c^{\ell+1}) > \mu_I(c^\ell)$ for every $\ell \in \mathbb{N}$.*

Let $c^2 : [0, 2T] \rightarrow \Sigma$ be a non-degenerate symmetric periodic Reeb orbit. Denote by $\Phi : [0, T] \rightarrow \text{Sp}(2n - 2)$ the path of symplectic matrices corresponding to the Reeb chord c . By abuse of notation, we write $\Phi(2kT) = \Phi^{2k}(2kT)$ for $k \in \mathbb{N}$.

With respect to the Lagrangian splitting $\mathbb{C}^{n-1} = L_0 \oplus L_1$, we write

$$\Phi(T) = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \quad \Phi(2T) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It follows from [12, Lemma 3.1] that

$$\Phi(2T)^k = \begin{pmatrix} T_k(A) & U_{k-1}(A)B \\ CU_{k-1}(A) & T_k(A^T) \end{pmatrix},$$

where T_k and U_{k-1} are Chebyshev polynomials of the first and second kind, respectively. For example, $T_0(x) = 1, T_1(x) = x, U_0(x) = x$ and $U_1(x) = 2x$. The following lemma will be used in the proof of Theorem 4.6.

LEMMA 4.7. *The blocks $X, Y, Z, W, T_k(A)$ and $CU_{k-1}(A)$ are invertible for all $k \geq 1$.*

Proof. In [12, Lemma 3.2] it is shown that the block $CU_{k-1}(A)$ is invertible for all $k \geq 1$. In order to show that $T_k(A)$ is also invertible, we consider the k th iteration of c^2 , i.e., $c^{2k} : [0, 2kT] \rightarrow \Sigma$. With respect to the Lagrangian splitting above, we write

$$\Phi(2kT) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

and then we have

$$\Phi(2kT)^2 = \begin{pmatrix} T_2(K) & U_1(K)L \\ MU_1(K) & T_2(K^T) \end{pmatrix} = \begin{pmatrix} T_2(K) & 2KL \\ 2MK & T_2(K^T) \end{pmatrix}.$$

By [12, Lemma 3.2], both M and MK are invertible and hence so is K . However, in view of the relation $\Phi(2kT) = \Phi(2T)^k$, we have $K = T_k(A)$. This proves the claim.

Because of the non-degeneracy, Y and Z are invertible. In view of the relation $\Phi(2T) = I\Phi(T)^{-1}I\Phi(T)$, we see that $C = 2X^T Z$ and $B = 2W^T Y$. The invertibility of C and Z implies that X is invertible as well. Using the argument given in the proof of [12, Lemma 3.2], one can see that B is invertible, implying the invertibility of W . This completes the proof. □

Proof of Theorem 4.6. The catenation property of the Robbin–Salamon index implies that

$$\mu_{\text{RS}}(\Phi^{\ell+1}L_0, L_0) = \mu_{\text{RS}}(\Phi^\ell L_0, L_0) + \mu_{\text{RS}}(\Phi^{\ell+1}|_{[\ell T, (\ell+1)T]}L_0, L_0).$$

We distinguish the following two cases.

Case 1. $\ell = 2k$ is even. In view of the iteration formulas (4.5) and (4.6) we have

$$\begin{aligned} &\mu_{\text{RS}}(\Phi^{\ell+1}|_{[\ell T, (\ell+1)T]}L_0, L_0) \\ &= \mu_{\text{RS}}(\Phi^{2k+1}|_{[2kT, (2k+1)T]}L_0, L_0) = \mu_{\text{RS}}(\Phi\Phi(2T)^kL_0, L_0). \end{aligned}$$

By the naturality property of the Robbin–Salamon index we obtain

$$\mu_{\text{RS}}(\Phi\Phi(2T)^kL_0, L_0) = -\mu_{\text{RS}}(\Phi^{-1}L_0, \Phi(2T)^kL_0)$$

from which we see that

$$\mu_{\text{RS}}(\Phi^{2k+1}L_0, L_0) - \mu_{\text{RS}}(\Phi^{2k}L_0, L_0) = -\mu_{\text{RS}}(\Phi^{-1}L_0, \Phi(2T)^kL_0).$$

Following [40, Theorem 3.5] we obtain that

$$\mu_{\text{RS}}(\Phi^{-1}L_0, L_0) - \mu_{\text{RS}}(\Phi^{-1}L_0, \Phi(2T)^kL_0) = s(\Phi(2T)^kL_0, L_0; L_0, \Phi(T)^{-1}L_0),$$

where s denotes the Hörmander index [8, Eq. (2.10)], [21, Section 3.3]. Choose $A_0 = CU_{k-1}(A)T_k(A)^{-1}$, $A_1 = B_0 = O$ and $B_1 = -Z^T(W^T)^{-1}$. The same theorem then implies that

$$\begin{aligned} s(\Phi(2T)^kL_0, L_0; L_0, \Phi(T)^{-1}L_0) &= \frac{1}{2}\text{sgn}(-Z^T(W^T)^{-1}) - \frac{1}{2}\text{sgn}(CU_{k-1}(A)T_k(A)^{-1}) \\ &\quad - \frac{1}{2}\text{sgn}(-Z^T(W^T)^{-1} - CU_{k-1}(A)T_k(A)^{-1}) \\ &= \frac{1}{2}\text{sgn}(-W^T(Z^T)^{-1} - T_k(A)U_{k-1}(A)^{-1}C^{-1}), \end{aligned}$$

where in the last identity we used [40, Lemma 5.2]. For dimensional reasons we then have

$$|s(\Phi(2T)^kL_0, L_0; L_0, \Phi(T)^{-1}L_0)| \leq \frac{n-1}{2}$$

from which we conclude that

$$\begin{aligned} -\mu_{\text{RS}}(\Phi^{-1}L_0, \Phi(2T)^kL_0) &= -\mu_{\text{RS}}(\Phi^{-1}L_0, L_0) + s(\Phi(2T)^kL_0, L_0; L_0, \Phi(T)^{-1}L_0) \\ &= \mu_{\text{RS}}(\Phi L_0, L_0) + s(\Phi(2T)^kL_0, L_0; L_0, \Phi(T)^{-1}L_0) \\ &\geq \mu_{\text{RS}}(\Phi L_0, L_0) - \frac{n-1}{2}. \end{aligned}$$

Case 2. $\ell = 2k - 1$ is odd. In a similar way, using the naturality and reversal properties of the Robbin–Salamon index, we find

$$\mu_{\text{RS}}(\Phi^{2k}L_0, L_0) - \mu_{\text{RS}}(\Phi^{2k-1}L_0, L_0) = -\mu_{\text{RS}}(\Phi^{-1}L_0, I\Phi(2T)^kL_0)$$

and

$$\mu_{\text{RS}}(\Phi^{-1}L_0, L_0) - \mu_{\text{RS}}(\Phi^{-1}L_0, I\Phi(2T)^kL_0) = s(I\Phi(2T)^kL_0, L_0; L_0, \Phi(T)^{-1}L_0).$$

Choosing $A_0 = -CU_{k-1}(A)T_k(A)^{-1}$, $A_1 = B_0 = O$ and $B_1 = -Z^T(W^T)^{-1}$, the same argument as above shows that

$$-\mu_{\text{RS}}(\Phi^{-1}L_0, I\Phi(2T)^kL_0) \geq \mu_{\text{RS}}(\Phi L_0, L_0) - \frac{n-1}{2}.$$

This proves the theorem. □

Remark 4.8. In the strictly convex case, a more general version of Theorem 4.6 was proved by Liu and Zhang [29, Lemma 7.3].

4.4. *The common index jump theorem.* In the following, we only consider paths of symplectic $(2n - 2) \times (2n - 2)$ matrices for which all iterations are both L_0 - and L_1 -non-degenerate. Abbreviate by $\mathcal{P}^*(2n - 2)$ the set of such paths. In view of Proposition 4.3, one can define the Conley–Zehnder index of any even iteration of $\Phi \in \mathcal{P}^*(2n - 2)$.

For $\Phi \in \mathcal{P}^*(2n - 2)$, the mean Robbin–Salamon index with respect to L_0 is defined by

$$\widehat{\mu}_{\text{RS}}(\Phi L_0, L_0) = \lim_{\ell \rightarrow \infty} \frac{\mu_{\text{RS}}(\Phi^\ell L_0, L_0)}{\ell}.$$

LEMMA 4.9. *Let $\Phi : [0, T] \rightarrow \text{Sp}(2n - 2)$ be an element in $\mathcal{P}^*(2n - 2)$. Assume that*

$$\mu_{\text{RS}}(\Phi L_0, L_0), \mu_{\text{RS}}(\Phi L_1, L_1) \geq \frac{n}{2}.$$

Then the mean Robbin–Salamon index satisfies

$$\widehat{\mu}_{\text{RS}}(\Phi L_0, L_0) \geq \frac{1}{2}.$$

Proof. By [29, Corollary 5.1],

$$\widehat{\mu}_{\text{RS}}(\Phi L_0, L_0) = \frac{1}{2} \mu_{\text{CZ}}(\Phi^2) + \sum_{\vartheta \in (0, 2\pi)} \left(\frac{\vartheta}{2\pi} - \frac{1}{2} \right) S_{\Phi^2(2T)}^-(e^{i\vartheta}),$$

where $S_{\Phi^2(2T)}^-(e^{i\vartheta})$ is the splitting number of the matrix $\Phi^2(2T)$ defined in [32]. Recall that it satisfies

$$0 \leq S_{\Phi^2(2T)}^-(e^{i\vartheta}) \leq \dim_{\mathbb{C}} \ker(\Phi^2(2T) - e^{i\vartheta} \text{id}).$$

We estimate that

$$\begin{aligned} \widehat{\mu}_{\text{RS}}(\Phi L_0, L_0) &= \frac{1}{2} \mu_{\text{CZ}}(\Phi^2) + \sum_{\vartheta \in (0, 2\pi)} \left(\frac{\vartheta}{2\pi} - \frac{1}{2} \right) S_{\Phi^2(2T)}^-(e^{i\vartheta}) \\ &\geq \frac{n}{2} + \sum_{\vartheta \in (0, \pi)} \left(\frac{\vartheta}{2\pi} - \frac{1}{2} \right) S_{\Phi^2(2T)}^-(e^{i\vartheta}) \\ &\geq \frac{n}{2} - \frac{1}{2} \sum_{\vartheta \in (0, \pi)} S_{\Phi^2(2T)}^-(e^{i\vartheta}) \\ &\geq \frac{n}{2} - \frac{n-1}{2} \\ &= \frac{1}{2}, \end{aligned}$$

where the first (respectively, last) inequality follows from Proposition 4.3 (respectively, the fact that along the half circle $\{e^{i\vartheta} \in \mathbb{C} : \vartheta \in (0, \pi)\}$ the $(n - 1) \times (n - 1)$ matrix $\Phi^2(2T)$ has at most $(n - 1)$ eigenvalues). This finishes the proof of the lemma. □

The next main tool is the so-called *common index jump theorem* which is proved in [29, Theorem 1.5] for different types of Maslov indices i_{L_0} and i_{L_1} . In the non-degenerate case, one can show that these indices are related to ours by

$$i_{L_j}(\Phi) = \mu_{RS}(\Phi L_j, L_j) - \frac{n-1}{2} \in \mathbb{Z}, \quad j = 0, 1.$$

In the following, we only treat the non-degenerate case.

THEOREM 4.10. [29, Theorem 1.5] *Let $\Phi_j \in \mathcal{P}^*(2n-2)$, $j = 1, 2, \dots, k$, satisfy $\widehat{\mu}_{RS}(\Phi_j L_0, L_0) > 0$ for all j . Then there exist infinitely many vectors $(K, m_1, m_2, \dots, m_k) \in \mathbb{N}^{k+1}$ such that*

$$\begin{aligned} \mu_{RS}(\Phi_j^{2m_j-1} L_0, L_0) &= K - \mu_{RS}(\Phi_j L_1, L_1), \\ \mu_{RS}(\Phi_j^{2m_j+1} L_0, L_0) &= K + \mu_{RS}(\Phi_j L_0, L_0). \end{aligned}$$

PROPOSITION 4.11. [29, Remark 5.1] *Let $\Phi_j \in \mathcal{P}^*(2n-2)$, $j = 1, 2, \dots, k$, be as in Theorem 4.10. Then, on the infinite set of vectors (K, m_1, \dots, m_k) in the assertion of that theorem, the first component K is unbounded.*

Proof. Assume, by contradiction, that all vectors (K, m_1, \dots, m_k) have bounded first components. Since there exist infinitely many such vectors, without loss of generality, we may assume that the m_1 -component is unbounded. By Theorem 4.10, we see that the $\mu_{RS}(\Phi_1^{2m_1+1} L_0, L_0)$ are bounded, which implies, by definition, that $\widehat{\mu}_{RS}(\Phi_1 L_0, L_0) = 0$. Since we have assumed in particular, $\widehat{\mu}_{RS}(\Phi_1 L_0, L_0) > 0$, this yields a contradiction. \square

4.5. Symmetric periodic Reeb orbits on displaceable hypersurfaces. To describe the main results, we recall the set-up. Let (W^{2n}, λ, ρ) be a real Liouville domain such that $L = \text{Fix}(\rho)$ and $\mathcal{L} = \partial L$ are not empty. Assume that the Maslov classes $\mu_L : \pi_2(W, L) \rightarrow \mathbb{Z}$ and $\mu_{\mathcal{L}} : \pi_2(\Sigma, \mathcal{L}) \rightarrow \mathbb{Z}$ vanish so that the Maslov indices of Hamiltonian and Reeb chords are well defined.

THEOREM 4.12. *Assume that the real contact manifold (Σ, α, ρ) is non-degenerate real dynamically convex, and displaceable from \widehat{L} in \widehat{W} . We further assume that the Lagrangian L is orientable. Then there exist at least n geometrically distinct, contractible and simple symmetric periodic Reeb orbits on Σ .*

Proof. We follow the strategy of the proof of [18, Theorem 1.1]. Without loss of generality, we may assume that there are only finitely many contractible and simple Reeb chords, say, c_1, \dots, c_k . By a simple Reeb chord we mean a Reeb chord c such that the corresponding symmetric periodic Reeb orbit c^2 is simple. Since the contact form is real dynamically convex, in view of Lemma 4.9, we obtain that $\widehat{\mu}(c_j) := \widehat{\mu}_{RS}(\Phi_j L_0, L_0) > 0$ for $j = 1, 2, \dots, k$, where Φ_j are the corresponding paths of symplectic matrices. We can therefore apply Theorem 4.10 and Proposition 4.11. We find a vector $(K, m_1, m_2, \dots, m_k) \in \mathbb{N}^{k+1}$ with K as large as we like such that, for each j , by Proposition 4.3,

$$\mu_I(c_j^{2m_j-1}) = K - \mu_{-I}(c_j), \quad \mu_I(c_j^{2m_j+1}) = K + \mu_I(c_j).$$

In view of Theorem 4.6 and real dynamical convexity, for each j , only $c_j^{2m_j}$ can have the index

$$|c_j^{2m_j}| := \mu_I(c_j^{2m_j}) - \frac{n-1}{2}$$

in the interval $[K - n + 1, K]$. Therefore, if $K \geq n + 2$, then, by the relation (3.21) and Theorem 3.42, $2m_j$ -iteration of c_j is a generator of the positive equivariant wrapped Floer homology from which we obtain $n \leq k$. This completes the proof of the theorem. \square

Recall that a star-shaped hypersurface in $\mathbb{R}^{2n} \cong \mathbb{C}^n$, which is invariant under complex conjugation $(q, p) \mapsto (q, -p)$, is displaceable from the real Lagrangian $\mathbb{R}^n = \mathbb{R}^n \times \{0\}$ (see Example 2.8). Theorem 1.2 then follows immediately. Another corollary of the proof of Theorem 4.12 is the following result.

COROLLARY 4.13. *In addition to the assumptions of Theorem 4.12, we further assume that there exist precisely n geometrically distinct simple symmetric periodic Reeb orbits on Σ . Then they are contractible and their indices are all different.*

In what follows, we assume that a real Liouville domain (W^{2n}, λ, ρ) admits an exact symplectic involution ι under which L is ι -invariant and ρ and ι commute. In particular, $\rho \circ \iota$ is an additional exact anti-symplectic involution. By abuse of notation, we also use the symbols ρ and ι for their restrictions to $\Sigma = \partial W$. Note that $\rho^*\alpha = -\alpha$ and $\iota^*\alpha = \alpha$. Given a Reeb chord $c : [0, T] \rightarrow \Sigma$, the corresponding symmetric periodic Reeb orbit c^2 is called *doubly symmetric* if $\iota(\text{im}(c^2)) = \text{im}(c^2)$.

Example 4.14. A typical example of a real Liouville domain that admits commuting ρ and ι is a compact star-shaped hypersurface $\Sigma \subset \mathbb{R}^{2n}$ which is invariant under the complex conjugate and the antipodal map.

THEOREM 4.15. *Under the assumptions of Theorem 4.12, we further assume that $\iota|_L$ acts freely on the boundary ∂L . Then there exist at least $n + \mathcal{N}(\Sigma)$ geometrically distinct, contractible and simple symmetric periodic Reeb orbits on Σ , where $2\mathcal{N}(\Sigma)$ is the number of geometrically distinct simple symmetric periodic Reeb orbits which are not doubly symmetric.*

Proof. Following the strategy of the proof of [29, Theorem 1.2], without loss of generality, we may assume that there are only finitely many contractible and simple Reeb chords, say, $c_1, \dots, c_k, \pm c_{k+1}, \dots, \pm c_{k+l}$. Here, $c_j^2, j = 1, 2, \dots, k$, are doubly symmetric, and for $k + 1 \leq j \leq k + l$, the orbits c_j^2 are not doubly symmetric. By $\pm c_j$, we mean the \mathbb{Z}_2 -pair of $c_j, j = k + 1, \dots, k + l$.

Set $r = k + l$. As in the proof of Theorem 4.12, we obtain $n \leq r$. We then conclude that

$$k + 2l = r + l \geq n + l = n + \mathcal{N}(\Sigma).$$

This completes the proof of the theorem. \square

Parallel to Corollary 4.13, we have the following result.

COROLLARY 4.16. *In addition to the assumptions of Theorem 4.15, we further assume that there exist precisely n geometrically distinct simple symmetric periodic Reeb orbits on Σ . Then they are contractible and doubly symmetric periodic Reeb orbits and their indices are different.*

Remark 4.17. As the proofs show, Theorems 4.12 and 4.15 still hold if we assume that $HW_*(L; W)$ vanishes, instead of the displaceability condition. In particular, if the symplectic homology of the domain W vanishes, then $HW_*(L; W)$ vanishes for any admissible Lagrangian L in W (see Remark 2.16).

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