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COMPACT WEIGHTED COMPOSITION OPERATORS BETWEEN L^p-SPACES

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Abstract

We provide complete characterisations for the compactness of weighted composition operators between two distinct L^p-spaces, where $1 \leq p \leq \infty$. As a corollary, when the underlying measure space is nonatomic, the only compact weighted composition map between L^p-spaces is the zero operator.

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1. Introduction

Let (X, Σ, μ) and (Y, Γ, ν) be two σ-finite and complete measure spaces and suppose that $1 \leq p < \infty$. The Lebesgue space consisting of all (equivalence classes of) p-integrable complex-valued Σ-measurable (respectively Γ-measurable) functions on X (respectively on Y) is denoted by $L^p(\mu)$ (respectively by $L^p(\nu)$). The functions in $L^\infty(\mu)$ and $L^\infty(\nu)$ are essentially bounded. The norm of a function in $L^p(\mu)$ or $L^p(\nu)$ is written as $\|\cdot\|_{L^p(\mu)}$ or $\|\cdot\|_{L^p(\nu)}$, respectively.

Let $w := \{w_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. If we take $X = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and $\mu(E) = \sum_{n\in E} w_n$ for every $E \in \mathcal{P}(\mathbb{N})$, then $L^p(\mu)$ is just the *weighted* sequence space $l^p(w)$ for $1 \leq p < \infty$. When $p = \infty$, we define $l^\infty(w)$ (or simply l^∞) as the space of all bounded sequences of complex numbers.

Let $u : Y \to \mathbb{C}$ be a Γ-measurable function and $\varphi\colon Y \to X$ be a point mapping such that $\varphi^{-1}(E) \in \Gamma$ for all $E \in \Sigma$. Assume that φ is also nonsingular, which means that the measure defined by $\nu\varphi^{-1}(E) := \nu(\varphi^{-1}(E))$, for $E \in \Sigma$, is absolutely continuous with respect to μ.

The functions u and φ induce the *weighted composition operator* uC_φ from $L^p(\mu)$, $1 \leq p \leq \infty$, into the linear space of all Γ-measurable functions on Y by

$$uC_\varphi(f)(y) := u(y)f(\varphi(y)) \quad \text{for every } f \in L^p(\mu) \text{ and } y \in Y.$$

The nonsingularity of φ guarantees that uC_φ is a well-defined mapping of equivalence classes of functions. When $u \equiv 1$, the corresponding operator C_φ is called a *composition operator* and, when $(X, \Sigma, \mu) = (Y, \Gamma, \nu)$ and $\varphi(x) = x$ for all $x \in X$, the corresponding operator M_u is called a *multiplication operator*.

If uC_φ maps $L^p(\mu)$ into $L^q(\nu)$, where $1 \le p, q \le \infty$, it follows from the closed graph theorem that uC_φ is bounded.

Operator-theoretic properties of weighted composition operators from $L^p(\mu)$ into itself were studied in [1, 8, 9] and such operators acting between distinct L^p-spaces in [3–5]. However, some results and proofs in [5] are rather sketchy or incomplete. In this paper, we completely characterise compact weighted composition operators between L^p-spaces and illustrate the results with examples.

2. Preliminaries

Let B_1 and B_2 be two normed spaces over $\mathbb{C}$. A linear operator $T : B_1 \to B_2$ is *compact* if it maps bounded subsets of B_1 into relatively compact subsets of B_2. In other words, T is compact if and only if it maps every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in B_1 onto a sequence $\{Tx_n\}_{n\in\mathbb{N}}$ in B_2 which has a convergent subsequence.

We adopt the following decomposition of (X, Σ, μ):

$$X = \left(\bigcup_{i=1}^{\infty} A_i\right) \cup B,$$

where $\{A_i\}_{i=1}^{\infty}$ is a countable collection of pairwise-disjoint atoms and B, being disjoint from each A_i, is nonatomic. This decomposition is unique in the sense that equality of two Σ-measurable sets interpreted as their symmetric difference is of zero μ-measure. The σ-finiteness of (X, Σ, μ) ensures that $\mu(A_i) < \infty$ for every $i \in \mathbb{N}$. Moreover, if $X = \bigcup_{i=1}^{\infty} A_i$ (respectively $X = B$), then (X, Σ, μ) is said to be atomic (respectively nonatomic). The limits in the theorems of this paper are assumed to take the value zero when the number of atoms in X is finite. The following facts will be useful.

(a) Let E be a nonatomic set in Σ with $\mu(E) > 0$.

  (i) There is a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise-disjoint (or decreasing) Σ-measurable subsets of E with $\mu(E_n) > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(E_n) = 0$.

  (ii) For every real number α satisfying $0 < \alpha < \mu(E)$, there is a set $E_\alpha \in \Sigma$ with $E_\alpha \subset E$ and $\mu(E_\alpha) = \alpha$.

(b) A Σ-measurable function $f : X \to \mathbb{C}$ is constant μ-almost everywhere on an atom M in (X, Σ, μ). Consequently, we view an atom M as a 'point' at which f takes a constant value denoted by $f(M)$.

For $1 \le q < \infty$, the measure μ_q defined by

$$\mu_q(E) := \int_{\varphi^{-1}(E)} |u|^q \, d\nu \quad \text{for every } E \in \Sigma$$

is absolutely continuous with respect to μ. The corresponding Radon–Nikodym derivative, denoted by $[d\mu_q/d\mu]$, has the following important property.

LEMMA 2.1. *Suppose that* $1 \le p, q < \infty$. *If* uC_φ *is a weighted composition operator from* $L^p(\mu)$ *into* $L^q(\nu)$, *then*

$$\|uC_\varphi f\|^q_{L^q(\nu)} = \int_X \Big[\frac{d\mu_q}{d\mu}\Big]|f|^q \, d\mu \quad \textit{for every } f \in L^p(\mu).$$

PROOF. We first establish the equality

$$\int_Y |u|^q f \circ \varphi \, d\nu = \int_X \Big[\frac{d\mu_q}{d\mu}\Big] f \, d\mu \tag{2.1}$$

for every nonnegative Σ-measurable function f. For each $E \in \Sigma$,

$$\int_Y |u|^q \chi_E \circ \varphi \, d\nu = \int_Y |u|^q \chi_{\varphi^{-1}(E)} \, d\nu = \int_{\varphi^{-1}(E)} |u|^q d\nu = \mu_q(E) = \int_X \Big[\frac{d\mu_q}{d\mu}\Big] \chi_E \, d\mu.$$

Thus, (2.1) holds for characteristic functions. The linearity of the integral and the map $f \mapsto f \circ \varphi$ implies that (2.1) also holds for all simple functions. Let f be any nonnegative Σ-measurable function. There exists a sequence of simple functions $\{s_n\}_{n=1}^\infty$ such that $0 \le s_1 \le s_2 \le \cdots$ and $\lim_{n\to\infty} s_n(x) = f(x)$ on X. It follows from the monotone convergence theorem that

$$\int_Y |u|^q f \circ \varphi \, d\nu = \lim_{n\to\infty} \int_Y |u|^q s_n \circ \varphi \, d\nu = \lim_{n\to\infty} \int_X \Big[\frac{d\mu_q}{d\mu}\Big] s_n \, d\mu = \int_X \Big[\frac{d\mu_q}{d\mu}\Big] f \, d\mu.$$

Consequently, if $f \in L^p(\mu)$, then

$$\|uC_\varphi f\|^q_{L^q(\nu)} = \int_Y |u|^q |f \circ \varphi|^q \, d\nu = \int_Y |u|^q |f|^q \circ \varphi \, d\nu = \int_X \Big[\frac{d\mu_q}{d\mu}\Big] |f|^q \, d\mu. \qquad \square$$

Let $g : Y \to \mathbb{C}$ be a Γ-measurable function. Its co-zero set, written as $\operatorname{coz} g$, is defined by $\operatorname{coz} g := \{y \in Y : g(y) \ne 0\}$ (this set is also denoted by $\operatorname{supp} g$, the support of g, by other authors).

3. Main results

Singh and Dharmadhikari [8] studied compact weighted composition operators on $L^2(\mu)$ by assuming that the corresponding composition operators had dense ranges. A characterisation for a general compact weighted composition operator on $L^p(\mu)$ was obtained by Chan [2] and independently around the same time by Takagi [9]. Narita and Takagi [7] characterised compact composition operators between two distinct L^p-spaces. This work motivated the present study.

For the sake of completeness, we include the following result which characterises compact weighted composition operators from $L^p(\mu)$ into $L^p(\nu)$. (The original theorem was stated in the setting $(X, \Sigma, \mu) = (Y, \Gamma, \nu)$, but the result still holds if the two underlying measure spaces are different.)

THEOREM 3.1 [9, Theorem 1]. *Suppose that* $1 \le p < \infty$. *Then* uC_φ *is a compact weighted composition operator from* $L^p(\mu)$ *into* $L^p(\nu)$ *if and only if:*

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(i) $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$; *and*

(ii) $\lim_{i\to\infty} \frac{1}{\mu(A_i)} \int_{\varphi^{-1}(A_i)} |u|^p \, d\nu = 0.$

We now consider compact weighted composition operators between different L^p-spaces.

Theorem 3.2. *Suppose that $1 \le q < p < \infty$. Then uC_φ is a compact weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$ if and only if:*

(i) $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$; *and*

(ii) $\sum_{i\in\mathbb{N}} \frac{1}{\mu(A_i)^{q/(p-q)}} \Big(\int_{\varphi^{-1}(A_i)} |u|^q \, d\nu \Big)^{p/(p-q)} < \infty.$

Proof. Note that the conditions (i) and (ii) are equivalent to (i)′ $[d\mu_q/d\mu] = 0$ μ-almost everywhere on B and (ii)′ $\sum_{i\in\mathbb{N}}([d\mu_q/d\mu](A_i))^{p/(p-q)}\mu(A_i) < \infty$, respectively. We shall show that (i)′ and (ii)′ characterise the compactness of uC_φ.

We first prove the necessity of (i)′ by contradiction. Suppose, on the contrary, that $\mu(\{x \in B : [d\mu_q/d\mu](x) > 0\}) > 0$. Then there exists some $\delta > 0$ such that the set $S := \{x \in B : [d\mu_q/d\mu](x) \ge \delta\}$ has positive μ-measure. We may also assume that $\mu(S) < \infty$. Construct a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $L^p(\mu)$ as follows.

Since S is nonatomic, there is some $S_1^1 \in \Sigma$ with $S_1^1 \subset S$ and $\mu(S_1^1) = \frac{1}{2}\mu(S)$. Put $S_2^1 := S \setminus S_1^1$ so that $\mu(S_2^1) = \frac{1}{2}\mu(S)$. Define $f_1 := \chi_{S_1^1}$.

We now choose some $S_1^2 \in \Sigma$ with $S_1^2 \subset S_1^1$ and $\mu(S_1^2) = \frac{1}{2}\mu(S_1^1) = \frac{1}{4}\mu(S)$ and $S_3^2 \in \Sigma$ with $S_3^2 \subset S_2^1$ and $\mu(S_3^2) = \frac{1}{2}\mu(S_2^1) = \frac{1}{4}\mu(S)$. Put $S_2^2 := S_1^1 \setminus S_1^2$ and $S_4^2 := S_2^1 \setminus S_3^2$. Define $f_2 := \chi_{S_1^2} + \chi_{S_3^2}$.

Continuing recursively in this manner, we obtain a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n := \sum_{i=1}^{2^{n-1}} \chi_{S_{2i-1}^n}$, where $S_{2i-1}^{n+1} \subset S_i^n$ and $\mu(S_{2i-1}^{n+1}) = \frac{1}{2}\mu(S_i^n)$ for $i, n \in \mathbb{N}$ with $i = 1, 2, \ldots, 2^n$. Note that $f_n \in L^p(\mu)$ with $\|f_n\|_{L^p(\mu)}^p = \frac{1}{2}\mu(S)$ for all $n \in \mathbb{N}$. Moreover, whenever $j \ne k$,

$$\mu(\{x \in S : |f_j(x) - f_k(x)| = 1\}) = \mu(\{x \in S : |f_j(x) - f_k(x)| = 0\}) = \frac{\mu(S)}{2}.$$

Hence,

$$\|uC_\varphi f_j - uC_\varphi f_k\|_{L^q(\nu)}^q = \int_S \Big[\frac{d\mu_q}{d\mu}\Big] |f_j - f_k|^q \, d\mu = \int_{\{x\in S : |f_j(x) - f_k(x)|=1\}} \Big[\frac{d\mu_q}{d\mu}\Big] \, d\mu \ge \frac{\delta\mu(S)}{2}.$$

This shows that uC_φ is not compact.

It remains to prove the necessity of (ii)′. Assume that uC_φ is compact. Then uC_φ is bounded and so $\int_X [d\mu_q/d\mu]^{p/(p-q)} \, d\mu < \infty$ by [3, Theorem 2.2]. Since $[d\mu_q/d\mu]$ vanishes μ-almost everywhere on B, we have $\sum_{i\in\mathbb{N}}([d\mu_q/d\mu](A_i))^{p/(p-q)}\mu(A_i) < \infty$.

We shall prove the sufficiency of (i)′ and (ii)′. Since

$$\|uC_\varphi f\|_{L^q(\nu)}^q = \int_X \Big[\frac{d\mu_q}{d\mu}\Big] |f|^q \, d\mu \quad \text{for every } f \in L^p(\mu),$$

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it suffices to show the compactness of the multiplication operator $M : L^p(\mu) \to L^q(\mu)$ defined by $Mf := [d\mu_q/d\mu]^{1/q} f$ for each $f \in L^p(\mu)$ (the conditions (i)′ and (ii)′ guarantee that M is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$).

From (ii)′, for every $n \in \mathbb{N}$, there exists some $k_n \in \mathbb{N}$ such that

$$\sum_{i>k_n} \left(\left[\frac{d\mu_q}{d\mu} \right](A_i) \right)^{p/(p-q)} \mu(A_i) < \frac{1}{n}.$$

Define $M_n : L^p(\mu) \to L^q(\mu)$ by $M_n f := u_n f$ for $f \in L^p(\mu)$, where

$$u_n := \sum_{i=1}^{k_n} \left(\left[\frac{d\mu_q}{d\mu} \right](A_i) \right)^{1/q} \chi_{A_i}.$$

Each M_n is a finite-rank operator. By Hölder's inequality,

$$\begin{aligned} \|M_n f - Mf\|^q_{L^q(\mu)} &= \int_{\bigcup_{i\in\mathbb{N}} A_i} \left| u_n - \left[\frac{d\mu_q}{d\mu} \right]^{1/q} \right|^q |f|^q \, d\mu \\ &= \sum_{i>k_n} \left[\frac{d\mu_q}{d\mu} \right](A_i) |f(A_i)|^q \mu(A_i)^{q/p+(p-q)/p} \\ &\le \left[\sum_{i>k_n} \left(\left[\frac{d\mu_q}{d\mu} \right](A_i) \right)^{p/(p-q)} \mu(A_i) \right]^{(p-q)/p} \left(\sum_{i>k_n} |f(A_i)|^p \mu(A_i) \right)^{q/p} \\ &\le \left(\frac{1}{n} \right)^{(p-q)/p} \|f\|^q_{L^p(\mu)} \end{aligned}$$

for all $f \in L^p(\mu)$. Thus,

$$\|M_n - M\| = \sup_{0\neq f\in L^p(\mu)} \frac{\|M_n f - Mf\|_{L^q(\mu)}}{\|f\|_{L^p(\mu)}} \le \left(\frac{1}{n} \right)^{(p-q)/(pq)} \to 0 \quad \text{as } n \to \infty.$$

Hence, M is compact. □

THEOREM 3.3. *Suppose that* $1 \le p < q < \infty$. *Then* uC_φ *is a compact weighted composition operator from* $L^p(\mu)$ *into* $L^q(\nu)$ *if and only if:*

(i) $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$; *and*

(ii) $\displaystyle \lim_{i\to\infty} \frac{1}{\mu(A_i)^{q/p}} \int_{\varphi^{-1}(A_i)} |u|^q \, d\nu = 0.$

PROOF. Note that the conditions (i) and (ii) are equivalent to (i)′ $[d\mu_q/d\mu] = 0$ μ-almost everywhere on B and (ii)′ $\lim_{i\to\infty} [d\mu_q/d\mu](A_i)/\mu(A_i)^{(q-p)/p} = 0$, respectively. We shall show that both (i)′ and (ii)′ are necessary and sufficient for the compactness of uC_φ.

The necessity of (i)′ is a direct consequence of [3, Theorem 2.5] and the fact that the compact operator uC_φ is bounded. We now prove that (ii)′ also holds.

Assuming otherwise, we can find a constant $\epsilon_0 > 0$ and a subsequence of natural numbers $\{i_k\}_{k\in\mathbb{N}}$ such that

$$\frac{[d\mu_q/d\mu](A_{i_k})}{\mu(A_{i_k})^{(q-p)/p}} \ge \epsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

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Define $f_k := \mu(A_{i_k})^{-1/p}\chi_{A_{i_k}}$ for every $k \in \mathbb{N}$. Then $f_k \in L^p(\mu)$ and $\|f_k\|_{L^p(\mu)} = 1$. If $j, k \in \mathbb{N}$ with $j \neq k$, then $uC_\varphi f_j$ and $uC_\varphi f_k$ have disjoint co-zero sets. Moreover,

$$\begin{aligned}\|uC_\varphi f_j - uC_\varphi f_k\|^q_{L^q(\nu)} &= \|uC_\varphi f_j\|^q_{L^q(\nu)} + \|uC_\varphi f_k\|^q_{L^q(\nu)} \\ &= \frac{[d\mu_q/d\mu](A_{i_j})}{\mu(A_{i_j})^{(q-p)/p}} + \frac{[d\mu_q/d\mu](A_{i_k})}{\mu(A_{i_k})^{(q-p)/p}} \geq 2\epsilon_0.\end{aligned}$$

Therefore, uC_φ is not compact.

It remains to prove the sufficiency of (i)′ and (ii)′. As in the proof of Theorem 3.2, we only need to establish compactness of the multiplication operator defined by $Mf := [d\mu_q/d\mu]^{1/q} f$ for $f \in L^p(\mu)$.

From (ii)′, for every $n \in \mathbb{N}$, there exists some $k_n \in \mathbb{N}$ such that

$$\frac{[d\mu_q/d\mu](A_i)}{\mu(A_i)^{(q-p)/p}} < \frac{1}{n} \quad \text{for all } i > k_n.$$

Define $M_n : L^p(\mu) \to L^q(\mu)$ by $M_n f := u_n f$ for $f \in L^p(\mu)$, where

$$u_n := \sum_{i=1}^{k_n} \Big(\Big[\frac{d\mu_q}{d\mu}\Big](A_i)\Big)^{1/q} \chi_{A_i}.$$

Then each M_n is a finite-rank operator. For every $f \in L^p(\mu)$,

$$\begin{aligned}\|M_n f - Mf\|^q_{L^q(\mu)} &= \sum_{i>k_n} \Big[\frac{d\mu_q}{d\mu}\Big](A_i)|f(A_i)|^q \mu(A_i) \\ &= \sum_{i>k_n} \frac{[d\mu_q/d\mu](A_i)}{\mu(A_i)^{(q-p)/p}} (|f(A_i)|^p \mu(A_i))^{q/p} \\ &\leq \frac{1}{n} \sum_{i>k_n} (|f(A_i)|^p \mu(A_i))^{q/p}.\end{aligned}$$

Suppose that $f \in L^p(\mu)$ and $\|f\|_{L^p(\mu)} = 1$. Then

$$|f(A_i)|^p \mu(A_i) = \int_{A_i} |f|^p \, d\mu \leq \|f\|^p_{L^p(\mu)} = 1$$

for all $i \in \mathbb{N}$. Since $q/p > 1$, we have $(|f(A_i)|^p\mu(A_i))^{q/p} \leq |f(A_i)|^p\mu(A_i)$, so that

$$\begin{aligned}\sum_{i>k_n} (|f(A_i)|^p \mu(A_i))^{q/p} &\leq \sum_{i\in\mathbb{N}} (|f(A_i)|^p \mu(A_i))^{q/p} \\ &\leq \sum_{i\in\mathbb{N}} |f(A_i)|^p \mu(A_i) \\ &= \int_{\bigcup_{i\in\mathbb{N}} A_i} |f|^p \, d\mu \leq \|f\|^p_{L^p(\mu)} = 1.\end{aligned}$$

Thus,

$$\|M_n - M\| = \sup_{\|f\|_{L^p(\mu)}=1} \|M_n f - Mf\|_{L^q(\mu)} \leq \Big(\frac{1}{n}\Big)^{1/q} \to 0 \quad \text{as } n \to \infty.$$

This shows that M is compact. □

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EXAMPLE 3.4. Let $X = (\bigcup_{n\in\mathbb{N}}(n-1,n)) \cup \mathbb{N}$. Let μ be the Lebesgue measure on $\bigcup_{n\in\mathbb{N}}(n-1,n)$ and $\mu(\{n\}) = n$ for each $n \in \mathbb{N}$. Define $\varphi : X \to X$ by

$$\varphi(x) := \begin{cases} n & \text{if } n-1 < x < n \text{ (for } n \in \mathbb{N}), \\ x+1 & \text{if } x \text{ is an odd positive integer,} \\ x^2 & \text{if } x \text{ is an even positive integer.} \end{cases}$$

Then $\mu\varphi^{-1}(\bigcup_{n\in\mathbb{N}}(n-1,n)) = 0$ and

$$\mu\varphi^{-1}(\{n\}) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ n+\sqrt{n} & \text{if } n \text{ is even and a perfect square,} \\ n & \text{if } n \text{ is even but not a perfect square.} \end{cases}$$

By Theorem 3.1, C_φ is not a compact composition operator on $L^p(\mu)$. However, according to Theorem 3.3, C_φ is a compact operator from $L^p(\mu)$ into $L^q(\mu)$, where $p < q$.

It remains to consider compact weighted composition operators from $L^p(\mu)$ into $L^q(\nu)$, where either $p = \infty$ or $q = \infty$. The first result was proved by the authors in [6].

THEOREM 3.5 [6, Theorem 3.1]. *Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^\infty(\nu)$. Then uC_φ is compact if and only if:*

(i) $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$; *and*
(ii) *the set* $\{i \in \mathbb{N} : \nu(\{y \in \varphi^{-1}(A_i) : |u(y)| > \epsilon\}) > 0\}$ *is finite for every* $\epsilon > 0$.

The proof of the next result is analogous to that of Theorem 3.2 and is omitted.

THEOREM 3.6. *Suppose that $1 \le q < \infty$. Then uC_φ is a compact weighted composition operator from $L^\infty(\mu)$ into $L^q(\nu)$ if and only if:*

(i) $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$; *and*
(ii) $\displaystyle\sum_{i\in\mathbb{N}} \int_{\varphi^{-1}(A_i)} |u|^q \, d\nu < \infty$.

THEOREM 3.7. *Suppose that $1 \le p < \infty$. Then uC_φ is a compact weighted composition operator from $L^p(\mu)$ into $L^\infty(\nu)$ if and only if:*

(i) $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$;
(ii) *u is essentially bounded on $\varphi^{-1}(A_i)$ for each $i \in \mathbb{N}$; and*
(iii) $\displaystyle\lim_{i\to\infty} \|u\|_{i,\infty}^p / \mu(A_i) = 0$, *where*

$$\|u\|_{i,\infty} := \inf\{M > 0 : \nu(\{y \in \varphi^{-1}(A_i) : |u(y)| > M\}) = 0\}.$$

PROOF. We shall first prove the necessity of these three conditions. Suppose, on the contrary, that $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) > 0$. Then there is some constant $\delta > 0$ such that $\nu(\{y \in \varphi^{-1}(B) : |u(y)| > \delta\}) > 0$. With $\nu\varphi^{-1}(B) > 0$, the absolute continuity of $\nu\varphi^{-1}$ with respect to μ implies that $\mu(B) > 0$. By the σ-finiteness of (X, Σ, μ), we may assume that

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$\mu(B) < \infty$. Since B is nonatomic, we can find a decreasing sequence of Σ-measurable subsets of B, say $\{G_n\}_{n\in\mathbb{N}}$, such that

$$\mu(G_n) = \frac{\mu(B)}{2^n} \quad \text{and} \quad \nu(\{y \in \varphi^{-1}(G_n) : |u(y)| > \delta\}) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Let $S_n := \{y \in \varphi^{-1}(G_n) : |u(y)| > \delta\}$. Then $|uC_\varphi \chi_{G_n}| = |u\chi_{\varphi^{-1}(G_n)}| > \delta$ on S_n and so $\|uC_\varphi \chi_{G_n}\|_{L^\infty(\nu)} > \delta$. Now,

$$\frac{\|uC_\varphi \chi_{G_n}\|_{L^\infty(\nu)}}{\|\chi_{G_n}\|_{L^p(\mu)}} > \frac{2^{n/p}\delta}{\mu(B)^{1/p}} \to \infty \quad \text{as } n \to \infty.$$

This shows that uC_φ is unbounded and is thus not compact.

By the boundedness of uC_φ,

$$\|uC_\varphi \chi_{A_i}\|_{L^\infty(\nu)} \le \|uC_\varphi\| \|\chi_{A_i}\|_{L^p(\mu)}$$

or

$$|u| \le \|uC_\varphi\| \mu(A_i)^{1/p} \quad \nu\text{-almost everywhere on } \varphi^{-1}(A_i)$$

for $i \in \mathbb{N}$. That is, u is essentially bounded on $\varphi^{-1}(A_i)$ (with $\|u\|_{i,\infty}^p \le \|uC_\varphi\|^p \mu(A_i)$).

It remains to prove that (iii) also holds. Assume on the contrary that there exist a constant $\epsilon_0 > 0$ and a subsequence of natural numbers $\{i_k\}_{k\in\mathbb{N}}$ such that

$$\frac{\|u\|_{i_k,\infty}^p}{\mu(A_{i_k})} \ge \epsilon_0.$$

Let $S_k := \{y \in \varphi^{-1}(A_{i_k}) : |u(y)| > \|u\|_{i_k,\infty}/2\}$, which has positive ν-measure. Moreover, let $f_k := \mu(A_{i_k})^{-1/p}\chi_{A_{i_k}}$. Then $f_k \in L^p(\mu)$ with $\|f_k\|_{L^p(\mu)} = 1$. The f_k also have disjoint co-zero sets. Since

$$\begin{aligned} |(uC_\varphi f_k)(y)|^p &= |\mu(A_{i_k})^{-1/p} u\chi_{\varphi^{-1}(A_{i_k})}(y)|^p \\ &= \mu(A_{i_k})^{-1}|u(y)|^p \\ &> \frac{\|u\|_{i_k,\infty}^p}{2^p \mu(A_{i_k})} \ge \frac{\epsilon_0}{2^p} \end{aligned}$$

on S_k, it follows that

$$\|uC_\varphi f_j - uC_\varphi f_k\|_{L^\infty(\nu)} > \frac{\epsilon_0^{1/p}}{2} \quad \text{for all } j, k \in \mathbb{N} \text{ with } j \ne k.$$

Hence, uC_φ cannot be compact.

We now assume that (i)–(iii) hold. For every $n \in \mathbb{N}$, it follows from (iii) that there is some $j_n \in \mathbb{N}$ such that

$$\frac{\|u\|_{i,\infty}^p}{\mu(A_i)} < \frac{1}{n} \quad \text{for each } i > j_n.$$

Let $u_n : Y \to \mathbb{C}$ be a Γ-measurable function defined by

$$u_n(y) := \begin{cases} u(y) & \text{if } y \in \bigcup_{i=1}^{j_n} \varphi^{-1}(A_i), \\ 0 & \text{otherwise.} \end{cases}$$

Choose any $f \in L^p(\mu)$. Each $u_n C_\varphi$ is a finite-rank operator because

$$u_n C_\varphi f = \sum_{i=1}^{j_n} f(A_i) u C_\varphi \chi_{A_i} \quad \nu\text{-almost everywhere on } Y.$$

From (i) and the definition of u_n,

$$u_n C_\varphi f - u C_\varphi f = 0 \quad \nu\text{-almost everywhere on } \varphi^{-1}(B) \cup \Big(\bigcup_{i=1}^{j_n} \varphi^{-1}(A_i) \Big).$$

If $y \in \bigcup_{i > j_n} \varphi^{-1}(A_i)$, there is a unique $j > j_n$ for which $\varphi(y) \in A_j$. Then

$$\begin{aligned} |(u_n C_\varphi f)(y) - (u C_\varphi f)(y)|^p &= \frac{|u(y)|^p |f(A_j)|^p \mu(A_j)}{\mu(A_j)} \\ &\le \frac{\|u\|_{j,\infty}^p}{\mu(A_j)} \|f\|_{L^p(\mu)}^p \le \frac{1}{n} \|f\|_{L^p(\mu)}^p. \end{aligned}$$

Hence,

$$\|u_n C_\varphi f - u C_\varphi f\|_{L^\infty(\nu)} \le \Big(\frac{1}{n}\Big)^{1/p} \|f\|_{L^p(\mu)}.$$

Since it is the limit of a sequence of finite-rank operators, uC_φ is compact. □

From Theorems 3.1 to 3.7, a necessary condition for the compactness of uC_φ is $\nu(\varphi^{-1}(B) \cap \operatorname{coz} u) = 0$, that is, φ maps $\operatorname{coz} u$ essentially into $\bigcup_{i \in \mathbb{N}} A_i$. Two important consequences of this observation are stated below.

Corollary 3.8. *Suppose that* $1 \le p, q \le \infty$.

(a) *If* uC_φ *is a compact weighted composition operator from* $L^p(\mu)$ *into* $L^q(\nu)$*, then* $\operatorname{coz} u \subset \varphi^{-1}(\bigcup_{i \in \mathbb{N}} A_i)$.

(b) *When* (X, Σ, μ) *is nonatomic, no compact weighted composition operator from* $L^p(\mu)$ *into* $L^q(\nu)$ *exists except the zero operator.*

The operator uC_φ maps $L^p(\mu)$ into $L^q(\nu)$, where $1 \le q < p < \infty$, if and only if $[d\mu_q/d\mu] \in L^{p/(p-q)}(\mu)$ [3, Theorem 2.2]. It is also clear that a necessary and sufficient condition for uC_φ to map $L^\infty(\mu)$ into $L^q(\nu)$, with $1 \le q < \infty$, is $u \in L^q(\nu)$. These results, together with Theorems 3.2 and 3.6, yield the following corollary.

Corollary 3.9. *Suppose that* $1 \le q < p \le \infty$. *If* (X, Σ, μ) *is atomic, then every weighted composition operator from* $L^p(\mu)$ *into* $L^q(\nu)$ *is compact.*

Example 3.10. Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a fixed sequence of complex numbers. The *unilateral weighted right* and *left shift operators*, T_R and T_L, on $l^p(w)$ are defined by

$$T_R(x_1, x_2, \ldots) := (0, \alpha_1 x_1, \alpha_2 x_2, \ldots) \quad \text{and} \quad T_L(x_1, x_2, \ldots) := (\alpha_2 x_2, \alpha_3 x_3, \ldots)$$

for every $\{x_n\}_{n \in \mathbb{N}} \in l^p(w)$. Compactness of T_R and T_L can be deduced using the theorems in this section. We provide the details for T_R only (T_L can be dealt with by a similar argument).

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TABLE 1. Compactness of weighted shift operators.

Case	*Characterisation for* T_R
$1 \leq p = q < \infty$	$\lim_{n\to\infty} \frac{\lvert\alpha_n\rvert^p w_{n+1}}{w_n} = 0$
$1 \leq q < p < \infty$	$\sum_{n\in\mathbb{N}} \left(\frac{\lvert\alpha_n\rvert^{pq} w_{n+1}^p}{w_n^q}\right)^{1/(p-q)} < \infty$
$1 \leq p < q < \infty$	$\lim_{n\to\infty} \frac{\lvert\alpha_n\rvert^q w_{n+1}}{w_n^{q/p}} = 0$
$p = q = \infty$	$\lim_{n\to\infty} \alpha_n = 0$
$p = \infty$ and $1 \leq q < \infty$	$\sum_{n\in\mathbb{N}} \lvert\alpha_n\rvert^q w_{n+1} < \infty$
$1 \leq p < \infty$ and $q = \infty$	$\lim_{n\to\infty} \frac{\lvert\alpha_n\rvert^p}{w_n} = 0$
Case	*Characterisation for* T_L
$1 \leq p = q < \infty$	$\lim_{n\to\infty} \frac{\lvert\alpha_{n+1}\rvert^p w_n}{w_{n+1}} = 0$
$1 \leq q < p < \infty$	$\sum_{n\in\mathbb{N}} \left(\frac{\lvert\alpha_{n+1}\rvert^{pq} w_n^p}{w_{n+1}^q}\right)^{1/(p-q)} < \infty$
$1 \leq p < q < \infty$	$\lim_{n\to\infty} \frac{\lvert\alpha_{n+1}\rvert^q w_n}{w_{n+1}^{q/p}} = 0$
$p = q = \infty$	$\lim_{n\to\infty} \alpha_n = 0$
$p = \infty$ and $1 \leq q < \infty$	$\sum_{n\in\mathbb{N}} \lvert\alpha_{n+1}\rvert^q w_n < \infty$
$1 \leq p < \infty$ and $q = \infty$	$\lim_{n\to\infty} \frac{\lvert\alpha_n\rvert^p}{w_n} = 0$

Note that T_R can be realised as a weighted composition operator uC_φ for $u : \mathbb{N} \to \mathbb{C}$ and $\varphi : \mathbb{N} \to \mathbb{N}$ defined by

$$u(n) := \begin{cases} \alpha_{n-1} & \text{if } n > 1, \\ 0 & \text{if } n = 1 \end{cases} \quad \text{and} \quad \varphi(n) := \begin{cases} n-1 & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

Then

$$\int_{\varphi^{-1}(\{n\})} \lvert u\rvert^p \, d\mu = \lvert\alpha_n\rvert^p w_{n+1} \quad \text{for each } n \in \mathbb{N}.$$

Characterisations for the compactness of T_R from $l^p(w)$ into $l^q(w)$ are shown in Table 1.

In particular, the usual right and left shift operators between l^p-spaces, that is, $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ and $\mapsto (x_2, x_3, \ldots)$, are not compact.

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References

[1] J. T. Campbell and J. E. Jamison, 'On some classes of weighted composition operators', *Glasg. Math. J.* **32** (1990), 87–94.

[2] J. T. Chan, 'A note on compact weighted composition operators on $L^p(\mu)$', *Acta Sci. Math. (Szeged)* **56** (1992), 165–168.

[3] M. R. Jabbarzadeh, 'Weighted composition operators between L^p-spaces', *Bull. Korean Math. Soc.* **42** (2005), 369–378.

[4] M. R. Jabbarzadeh and E. Pourreza, 'A note on weighted composition operators on L^p-spaces', *Bull. Iranian Math. Soc.* **29** (2003), 47–54.

[5] R. Kumar, 'Weighted composition operators between two L^p-spaces', *Mat. Vesnik* **61** (2009), 111–118.

[6] C. O. Lo and A. W. K. Loh, 'Compact weighted composition operators between L^∞-spaces', *Int. J. Contemp. Math. Sci.* **13** (2018), 169–176.

[7] K. Narita and H. Takagi, 'Compact composition operators between L^p-spaces. Harmonic/analytic function spaces and linear operators', *Kyoto Univ. Res. Inst. Math. Sci. Kokyuroku* **1049** (1998), 129–136 (in Japanese).

[8] R. K. Singh and N. S. Dharmadhikari, 'Compact and Fredholm composite multiplication operators', *Acta Sci. Math. (Szeged)* **52** (1988), 437–441.

[9] H. Takagi, 'Compact weighted composition operators on L^p', *Proc. Amer. Math. Soc.* **116** (1992), 505–511.

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