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Invariant means on weakly almost periodic functionals with application to quantum groups

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Abstract. Let $\mathcal{A}$ be a Banach algebra, and let φ be a nonzero character on $\mathcal{A}$. For a closed ideal I of $\mathcal{A}$ with $I \nsubseteq \ker\varphi$ such that I has a bounded approximate identity, we show that $\mathrm{WAP}(\mathcal{A})$, the space of weakly almost periodic functionals on $\mathcal{A}$, admits a right (left) invariant φ-mean if and only if $\mathrm{WAP}(I)$ admits a right (left) invariant $\varphi|_I$-mean. This generalizes a result due to Neufang for the group algebra $L^1(G)$ as an ideal in the measure algebra $M(G)$, for a locally compact group G. Then we apply this result to the quantum group algebra $L^1(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$. Finally, we study the existence of left and right invariant 1-means on $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}^*$ is canonically a Banach $\mathcal{A}$-bimodule with the actions

$$\langle x\cdot a, b\rangle = \langle x, ab\rangle, \quad \langle a\cdot x, b\rangle = \langle x, ba\rangle$$

for all $a, b \in \mathcal{A}$ and $x \in \mathcal{A}^*$. There are two naturally defined products, which we denote by $\square$ and $\diamond$ on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$, each extending the product on $\mathcal{A}$. For $m, n \in \mathcal{A}^{**}$ and $x \in \mathcal{A}^*$, the first Arens product $\square$ in $\mathcal{A}^{**}$ is given as follows:

$$\langle m \,\square\, n, x\rangle = \langle m, n\cdot x\rangle,$$

where $n\cdot x \in \mathcal{A}^*$ is defined by $\langle n\cdot x, a\rangle = \langle n, x\cdot a\rangle$ for all $a\in\mathcal{A}$. Similarly, the second Arens product $\diamond$ in $\mathcal{A}^{**}$ satisfies

$$\langle m \diamond n, x\rangle = \langle n, x\cdot m\rangle,$$

where $x\cdot m \in \mathcal{A}^*$ is given by $\langle x\cdot m, a\rangle = \langle m, a\cdot x\rangle$ for all $a\in\mathcal{A}$. The Banach algebra $\mathcal{A}$ is called Arens regular if $\square$ and $\diamond$ coincide on $\mathcal{A}^{**}$.

We denote the spectrum of $\mathcal{A}$ by $\mathrm{sp}(\mathcal{A})$. Let $\varphi\in\mathrm{sp}(\mathcal{A})$, and let X be a Banach right $\mathcal{A}$-submodule of $\mathcal{A}^*$ with $\varphi \in X$. Then a left invariant φ-mean on X is a functional

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$m \in X^*$ satisfying

$$\langle m, \varphi \rangle = 1, \quad \langle m, x \cdot a \rangle = \varphi(a) \langle m, x \rangle \quad (a \in \mathcal{A}, x \in X).$$

Right and (two-sided) invariant φ-means are defined similarly. The Banach algebra $\mathcal{A}$ is called left φ-amenable if there exists a left invariant φ-mean on $\mathcal{A}^*$ (see [7]). This notion generalizes the concept of left amenability for Lau algebras, a class of Banach algebras including all convolution quantum group algebras, which was first introduced and studied in [10].

A Banach right (resp. left) $\mathcal{A}$-submodule X of $\mathcal{A}^*$ is called left (resp. right) introverted if $X^* \cdot X \subseteq X$ (resp. $X \cdot X^* \subseteq X$). In this case, X^* is a Banach algebra with the multiplication induced by the first (resp. second) Arens product $\square$ (resp. $\diamond$) inherited from $\mathcal{A}^{**}$. A Banach $\mathcal{A}$-subbimodule X of $\mathcal{A}^*$ is called introverted if it is both left and right introverted (see [2, Chapter 5] for details).

An element x of $\mathcal{A}^*$ is weakly almost periodic if the map $\lambda_x : a \mapsto a \cdot x$ from $\mathcal{A}$ into $\mathcal{A}^*$ is a weakly compact operator. Let $\mathrm{WAP}(\mathcal{A})$ denote the closed subspace of $\mathcal{A}^*$ consisting of the weakly almost periodic functionals on $\mathcal{A}$. Then $\mathrm{WAP}(\mathcal{A})$ is an introverted subspace of $\mathcal{A}^*$ containing $\mathrm{sp}(\mathcal{A})$. We would like to recall from [2, Proposition 3.11] that $m \square n = m \diamond n$ for all $m, n \in \mathrm{WAP}(\mathcal{A})^*$. Now suppose that I is a closed ideal in $\mathcal{A}$ with a bounded approximate identity. Then, by [2, Proposition 3.12] $\mathrm{WAP}(I)$ is a neo-unital Banach I-bimodule; that is, $\mathrm{WAP}(I) = I \cdot \mathrm{WAP}(I) = \mathrm{WAP}(I) \cdot I$. Moreover, $\mathrm{WAP}(I)$ becomes a Banach $\mathcal{A}$-bimodule (see [14, Proposition 2.1.6]).

In the case that A is the group algebra $L^1(G)$ of a locally compact group G, it is known that $\mathrm{WAP}(L^1(G))$ admits an invariant mean, which is unique, that is, a norm one functional $m \in L^1(G)^{**}$ with $\langle m, 1 \rangle = 1$ and

$$\langle m, f \cdot x \rangle = \langle m, x \cdot f \rangle = f(1) \langle m, x \rangle$$

for all $x \in \mathrm{WAP}(L^1(G))$ and $f \in L^1(G)$ (see [17]).

Furthermore, it is known from [3, Proposition 5.16] that if G is discrete or amenable, then $\mathrm{WAP}(M(G))$ admits an invariant mean, which is unique, where $M(G)$ denotes the measure algebra of G. Recently, Neufang in [12] generalized this latter result to arbitrary locally compact groups, thereby answering a question posed in [3].

In this article, we generalize the main result of [12] to an arbitrary Banach algebra $\mathcal{A}$. More precisely, for $\varphi \in \mathrm{sp}(\mathcal{A})$, we show that if I is a closed ideal of $\mathcal{A}$ with a bounded approximate identity such that $I \nsubseteq \ker \varphi$, then $\mathrm{WAP}(\mathcal{A})$ admits a right (left) invariant φ-mean if and only if $\mathrm{WAP}(I)$ admits a right (left) invariant $\varphi|_I$-mean. Applying our results to algebras over locally compact (quantum) groups, we show that, if I is a closed ideal of $L^1(G)$ with a bounded approximate identity such that $I \nsubseteq \ker 1$, then I is Arens regular if and only if it is reflexive.

Finally, for a locally compact quantum group $\mathbb{G}$, we characterize the existence of left and right invariant 1-means on $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$, where $\mathcal{T}_\triangleright(\mathbb{G})$ denotes the trace class operators on $L^2(\mathbb{G})$, but equipped with a product different from composition (see [6].

2 Preliminaries

The class of locally compact quantum groups was first introduced and studied by Kustermans and Vaes [8, 9]. Recall that a (*von Neumann algebraic*) *locally compact quantum group* is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta, \phi, \psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra with identity element 1 and a co-multiplication $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$. Moreover, ϕ and ψ are normal faithful semifinite left and right Haar weights on $L^\infty(\mathbb{G})$, respectively. Here, $\bar{\otimes}$ denotes the von Neumann algebra tensor product.

The predual of $L^\infty(\mathbb{G})$ is denoted by $L^1(\mathbb{G})$ which is called *quantum group algebra* of $\mathbb{G}$. Then the pre-adjoint of the co-multiplication Δ induces on $L^1(\mathbb{G})$ an associative completely contractive multiplication $\Delta_* : L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G}) \to L^1(\mathbb{G})$, where $\widehat{\otimes}$ is the operator space projective tensor product. Therefore, $L^1(\mathbb{G})$ is a Banach algebra under the product $*$ given by $f * g := \Delta_*(f \otimes g) \in L^1(\mathbb{G})$ for all $f, g \in L^1(\mathbb{G})$. Moreover, the module actions of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$ are given by

$$f \cdot x := (\mathrm{id} \otimes f)(\Delta(x)), \quad x \cdot f := (f \otimes \mathrm{id})(\Delta(x))$$

for all $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$.

For every locally compact quantum group $\mathbb{G}$, there is a left fundamental unitary operator $W \in L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\widehat{\mathbb{G}})$ and a right fundamental unitary operator $V \in L^\infty(\widehat{\mathbb{G}})'\bar{\otimes}L^\infty(\mathbb{G})$ which the co-multiplication Δ can be given in terms of W and V by the formula

$$\Delta(x) = W^*(1 \otimes x)W = V(x \otimes 1)V^* \quad (x \in L^\infty(\mathbb{G})),$$

where $L^\infty(\widehat{\mathbb{G}}) := \{(f \otimes \mathrm{id})(W) \ : \ f \in L^1(\mathbb{G})\}''$. The Gelfand–Naimark–Segal (GNS) representation space for the left Haar weight will be denoted by $L^2(\mathbb{G})$. Put $\widehat{W} = \sigma W^* \sigma$, where σ denotes the flip operator on $B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$, and define

$$\widehat{\Delta} : L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})\bar{\otimes}L^\infty(\widehat{\mathbb{G}}), \quad x \mapsto \widehat{W}^*(1 \otimes x)\widehat{W},$$

which is a co-multiplication. One can also define a left Haar weight $\hat{\varphi}$ and a right Haar weight $\hat{\psi}$ on $L^\infty(\widehat{\mathbb{G}})$ that $\widehat{\mathbb{G}} = (L^\infty(\widehat{\mathbb{G}}), \widehat{\Gamma}, \hat{\varphi}, \hat{\psi})$, the *dual quantum group* of $\mathbb{G}$, turn it into a locally compact quantum group. Moreover, a Pontryagin duality theorem holds, that is, $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$ (for more details, see [8, 9]). The *reduced quantum group C^*-algebra* of $L^\infty(\mathbb{G})$ is defined as

$$C_0(\mathbb{G}) := \overline{\{(\mathrm{id} \otimes \omega)(W); \ \omega \in B(L^2(\mathbb{G}))_*\}}^{\|\cdot\|}.$$

We say that $\mathbb{G}$ is *compact* if $C_0(\mathbb{G})$ is a unital C^*-algebra. The co-multiplication Δ maps $C_0(\mathbb{G})$ into the multiplier algebra $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ of the minimal C^*-algebra tensor product $C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$. Thus, we can define the completely contractive product $*$ on $C_0(\mathbb{G})^* = M(\mathbb{G})$ by

$$\langle \omega * \nu, x\rangle = (\omega \otimes \nu)(\Delta x) \quad (x \in C_0(\mathbb{G}), \omega, \nu \in M(\mathbb{G})),$$

whence $(M(\mathbb{G}), *)$ is a completely contractive Banach algebra and contains $L^1(\mathbb{G})$ as a norm closed two-sided ideal. If X is a Banach right $L^1(\mathbb{G})$-submodule of $L^\infty(\mathbb{G})$

with $1 \in X$, then a left invariant mean on X, is a functional $m \in X^*$ satisfying

$$\|m\| = \langle m, 1\rangle = 1, \quad \langle m, x \cdot f\rangle = \langle f, 1\rangle\langle m, x\rangle \quad (f \in L^1(\mathbb{G}), x \in X).$$

Right and (two-sided) invariant means are defined similarly. A locally compact quantum group $\mathbb{G}$ is said to be amenable if there exists a left (equivalently, right, or two-sided) invariant mean on $L^\infty(\mathbb{G})$ (see [4, Proposition 3]). A standard argument, used in the proof of [10, Theorem 4.1] on Lau algebras shows that $\mathbb{G}$ is amenable if and only if $L^1(\mathbb{G})$ is left 1-amenable. We also recall that, $\mathbb{G}$ is called *co-amenable* if $L^1(\mathbb{G})$ has a bounded approximate identity.

The right fundamental unitary V of $\mathbb{G}$ induces a co-associative co-multiplication

$$\Delta^r : \mathcal{B}\left(L^2(\mathbb{G})\right) \ni x \mapsto V(x \otimes 1)V^* \in \mathcal{B}\left(L^2(\mathbb{G})\right)\bar{\otimes}\mathcal{B}\left(L^2(\mathbb{G})\right),$$

and the restriction of Δ^r to $L^\infty(\mathbb{G})$ yields the original co-multiplication Δ on $L^\infty(\mathbb{G})$. The pre-adjoint of Δ^r induces an associative completely contractive multiplication on space $\mathcal{T}\left(L^2(\mathbb{G})\right)$ of trace class operators on $L^2(\mathbb{G})$, defined by

$$\triangleright : \mathcal{T}\left(L^2(\mathbb{G})\right)\widehat{\otimes}\mathcal{T}\left(L^2(\mathbb{G})\right) \ni \omega \otimes \tau \mapsto \omega \triangleright \tau = \Delta^r_*(\omega \otimes \tau) \in \mathcal{T}\left(L^2(\mathbb{G})\right),$$

where $\widehat{\otimes}$ denotes the operator space projective tensor product.

It was shown in [6, Lemma 5.2], that the pre-annihilator $L^\infty(\mathbb{G})_\perp$ of $L^\infty(\mathbb{G})$ in $\mathcal{T}\left(L^2(\mathbb{G})\right)$ is a norm closed two-sided ideal in $\left(\mathcal{T}\left(L^2(\mathbb{G})\right), \triangleright\right)$ and the complete quotient map

$$\pi : \mathcal{T}\left(L^2(\mathbb{G})\right) \ni \omega \mapsto f = \omega|_{L^\infty(\mathbb{G})} \in L^1(\mathbb{G})$$

is a completely contractive algebra homomorphism from $\mathcal{T}_\triangleright(\mathbb{G}) := \left(\mathcal{T}\left(L^2(\mathbb{G})\right), \triangleright\right)$ onto $L^1(\mathbb{G})$. The multiplication $\triangleright$ defines a canonical $\mathcal{T}_\triangleright(\mathbb{G})$-bimodule structure on $\mathcal{B}\left(L^2(\mathbb{G})\right)$. Note that since $V \in L^\infty(\widehat{\mathbb{G}}')\bar{\otimes}L^\infty(\mathbb{G})$, the bimodule action on $L^\infty(\widehat{\mathbb{G}})$ becomes rather trivial. Indeed, for $\hat{x} \in L^\infty(\widehat{\mathbb{G}})$ and $\omega \in \mathcal{T}_\triangleright(\mathbb{G})$, we have

$$\hat{x} \triangleright \omega = (\omega \otimes \iota)V(\hat{x} \otimes 1)V^* = \langle \omega, \hat{x}\rangle 1, \quad \omega \triangleright \hat{x} = (\iota \otimes \omega)V(\hat{x} \otimes 1)V^* = \langle \omega, 1\rangle \hat{x}.$$

This implies that $L^\infty(\widehat{\mathbb{G}}) \subseteq \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$. It is also known from [6, Proposition 5.3] that $B(L^2(\mathbb{G})) \triangleright \mathcal{T}_\triangleright(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$. In particular, the actions of $\mathcal{T}_\triangleright(\mathbb{G})$ on $L^\infty(\mathbb{G})$ satisfies

$$\omega \triangleright x = \pi(\omega) \cdot x, \quad x \triangleright \omega = x \cdot \pi(\omega)$$

for all $\omega \in \mathcal{T}_\triangleright(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$.

3 Invariant means on weakly almost periodic functionals

Let I be a closed ideal of the Banach algebra $\mathcal{A}$. Then for every $b \in I$ and $x \in I^*$, define $x \bullet b, b \bullet x \in \mathcal{A}^*$ as follows:

$$\langle x \bullet b, a\rangle = \langle x, ba\rangle, \quad \langle b \bullet x, a\rangle = \langle x, ab\rangle \quad (a \in \mathcal{A}).$$

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We note that, given $a \in \mathcal{A}, b_1, b_2 \in I$, and $x \in I^*$, for $a' \in \mathcal{A}$, we have

$$\begin{aligned}\langle a \cdot ((b_1 \cdot x) \bullet b_2), a' \rangle &= \langle (b_1 \cdot x) \bullet b_2, a'a \rangle = \langle b_1 \cdot x, b_2 a' a \rangle = \langle x, b_2 a' a b_1 \rangle \\ &= \langle ab_1 \cdot x, b_2 a' \rangle = \langle (ab_1 \cdot x) \bullet b_2, a' \rangle,\end{aligned}$$

so that, $a \cdot ((b_1 \cdot x) \bullet b_2) = (ab_1 \cdot x) \bullet b_2$.

Lemma 3.1 *Let $\mathcal{A}$ be a Banach algebra, and let I be a closed ideal of $\mathcal{A}$ with a bounded approximate identity. Then*

$$\mathrm{WAP}(I) \bullet I \subseteq \mathrm{WAP}(\mathcal{A}), \quad I \bullet \mathrm{WAP}(I) \subseteq \mathrm{WAP}(\mathcal{A}).$$

Proof Let $x \in \mathrm{WAP}(I)$ and $b_1, b_2 \in I$. Suppose that (a_n) is a bounded sequence in $\mathcal{A}$. Then $(a_n b_1)$ is a bounded sequence in I and so by weak compactness of the map $\lambda_x : I \to I^*$, there is a subsequence $(a_{n_j} b_1)$ of $(a_n b_1)$ such that $(a_{n_j} b_1 \cdot x)$ converges weakly in I^* to some $y \in I^*$. Now, for each $m \in \mathcal{A}^{**}$, define the functional $b_2 \bullet m \in I^{**}$ as follows:

$$\langle b_2 \bullet m, z \rangle = \langle m, z \bullet b_2 \rangle \quad (z \in I^*).$$

It follows that

$$\begin{aligned}\langle m, a_{n_j} \cdot ((b_1 \cdot x) \bullet b_2) \rangle &= \langle m, (a_{n_j} b_1 \cdot x) \bullet b_2 \rangle \\ &= \langle b_2 \bullet m, a_{n_j} b_1 \cdot x \rangle \to \langle b_2 \bullet m, y \rangle \\ &= \langle m, y \bullet b_2 \rangle\end{aligned}$$

for all $m \in \mathcal{A}^{**}$. That is, $(b_1 \cdot x) \bullet b_2 \in \mathrm{WAP}(\mathcal{A})$. Since I has a bounded right approximate identity, it follows from [2, Proposition 3.12] that $I \cdot \mathrm{WAP}(I) = \mathrm{WAP}(I)$. This shows that $\mathrm{WAP}(I) \bullet I \subseteq \mathrm{WAP}(\mathcal{A})$. The inclusion $I \bullet \mathrm{WAP}(I) \subseteq \mathrm{WAP}(\mathcal{A})$ can be proved similarly. ∎

Theorem 3.2 *Let $\mathcal{A}$ be a Banach algebra with $\varphi \in \mathrm{sp}(\mathcal{A})$, and let I be a closed ideal of $\mathcal{A}$ with a bounded approximate identity such that $I \nsubseteq \ker \varphi$. Then the following statements are equivalent:*

(i) $\mathrm{WAP}(I)$ *has a right (left) invariant $\varphi|_I$-mean.*
(ii) $\mathrm{WAP}(A)$ *has a right (left) invariant φ-mean.*

Proof We only prove the right version of the theorem. Similar arguments will establish the left side version.

(i) ⇒ (ii). Let m be a right invariant $\varphi|_I$-mean on $\mathrm{WAP}(I)$. This means that for every $x \in \mathrm{WAP}(I)$ and $b \in I$, we have

$$\langle m, b \cdot x \rangle = \varphi(b) \langle m, x \rangle.$$

We denote by $\imath : I \to \mathcal{A}$ the canonical embedding map. By [18, Corollary to Lemma 1], the map $R := \imath^* : \mathcal{A}^* \to I^*$ maps $\mathrm{WAP}(\mathcal{A})$ to $\mathrm{WAP}(I)$. Define $\widetilde{m} := m \circ R \in \mathcal{A}^{**}$. It is easy to see that $\langle \widetilde{m}, \varphi \rangle = 1$. Let (e_α) be a bounded approximate identity for I. By [2, Proposition 3.12], we have $I \cdot \mathrm{WAP}(I) = \mathrm{WAP}(I) \cdot I = \mathrm{WAP}(I)$. Thus, $\lim_\alpha e_\alpha \cdot R(y) = R(y)$ for all $y \in \mathrm{WAP}(\mathcal{A})$. Moreover, by [14, Proposition 2.1.6], $\mathrm{WAP}(I)$

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becomes a Banach $\mathcal{A}$-bimodule and since I is an ideal in $\mathcal{A}$, it is not hard check that $R(a \cdot y) = a \cdot R(y)$ for all $a \in \mathcal{A}$ and $y \in \mathrm{WAP}(\mathcal{A})$. Therefore, for every $a \in \mathcal{A}$ and $y \in \mathrm{WAP}(\mathcal{A})$, we have

$$\begin{aligned}\langle \widetilde{m}, a \cdot y\rangle &= \langle m, R(a \cdot y)\rangle = \langle m, a \cdot R(y)\rangle \\ &= \lim_{\alpha}\langle m, a \cdot (e_\alpha \cdot R(y))\rangle = \lim_{\alpha}\langle m, ae_\alpha \cdot R(y)\rangle \\ &= \lim_{\alpha} \varphi(ae_\alpha)\langle m, R(y)\rangle = \varphi(a)\varphi(e_\alpha)\langle \widetilde{m}, y\rangle = \varphi(a)\langle \widetilde{m}, y\rangle.\end{aligned}$$

Thus, $\widetilde{m}$ is a right invariant φ-mean on $\mathrm{WAP}(\mathcal{A})$.

(ii) $\Rightarrow$ (i). Let $m \in \mathcal{A}^{**}$ be a right invariant φ-mean on $\mathrm{WAP}(\mathcal{A})$. Fix $b_0 \in I$ with $\varphi(b_0) = 1$. Since $\mathrm{WAP}(I) \bullet b_0 \subseteq \mathrm{WAP}(\mathcal{A})$, by Lemma 3.1, we can define $\tilde{m} \in \mathrm{WAP}(I)^*$ as follows:

$$\langle \tilde{m}, x\rangle = \langle m, x \bullet b_0\rangle \quad (x \in \mathrm{WAP}(I)).$$

It is easily verified that

$$\langle \tilde{m}, \varphi|_I\rangle = \langle m, \varphi|_I \bullet b_0\rangle = \langle m, \varphi\rangle = 1.$$

Moreover, for every $b \in I$ and $x \in \mathrm{WAP}(I)$, we have

$$\begin{aligned}\langle \tilde{m}, b \cdot x\rangle = \langle m, (b \cdot x) \bullet b_0\rangle &= \langle m, b \cdot (x \bullet b_0)\rangle \\ &= \varphi|_I(b)\langle m, x \bullet b_0\rangle \\ &= \varphi|_I(b)\langle \tilde{m}, x\rangle.\end{aligned}$$

Therefore, $\tilde{m}$ is a right $\varphi|_I$-mean on $\mathrm{WAP}(I)$. ■

Remark 3.3 We would like to point out the following fact related to right and left invariant φ-means on $\mathrm{WAP}(\mathcal{A})$. Suppose that m is a left invariant φ-mean and n is a right invariant φ-mean on $\mathrm{WAP}(\mathcal{A})$. Using weak*-continuity of the maps $p \mapsto p \,\square\, m$ and $p \mapsto n \diamond p$ on $\mathrm{WAP}(\mathcal{A})^*$, we obtain that $m = n(\varphi)m = n \,\square\, m = n \diamond m = m(\varphi)n = n$. In particular, if there is an invariant φ-mean on $\mathrm{WAP}(\mathcal{A})$, then it is unique.

We now consider some special cases. Suppose that $\mathbb{G}$ is a locally compact quantum group. Then $\mathbb{G}$ has a canonical co-involution $\mathcal{R}$, called the unitary antipode of $\mathbb{G}$. That is, $\mathcal{R} : L^\infty(\mathbb{G}) \longrightarrow L^\infty(\mathbb{G})$ is a *-anti-homomorphism satisfying $\mathcal{R}^2 = \mathrm{id}$ and $\Delta \circ \mathcal{R} = \sigma(\mathcal{R} \otimes \mathcal{R}) \circ \Delta$, where σ is the flip map on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$. Then $\mathcal{R}$ induces a completely isometric involution on $L^1(\mathbb{G})$ defined by

$$\langle x, f'\rangle = \overline{\langle f, \mathcal{R}(x^*)\rangle} \quad (x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G})).$$

Hence, $L^1(\mathbb{G})$ becomes an involutive Banach algebra.

Now, assume that m is a left (resp. right) invariant 1-mean on $\mathrm{WAP}(L^1(\mathbb{G}))$, and let $\widetilde{m} \in L^\infty(\mathbb{G})^*$ be a Hahn–Banach extension of m. It is not hard to see that $n := \widetilde{m}^\circ|_{\mathrm{WAP}(L^1(\mathbb{G})}$ is a right (resp. left) invariant 1-mean on $\mathrm{WAP}(L^1(\mathbb{G}))$, where $\circ : L^\infty(\mathbb{G})^* \to L^\infty(\mathbb{G})^*, m \mapsto m^\circ$ is the unique weak*-weak* continuous extension of the involution on $L^1(\mathbb{G})$ which is called the linear involution (see [2, Chapter 2,

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p. 18]. Thus, by Remark 3.3, we obtain that any left (resp. right) invariant 1-mean on $\mathrm{WAP}(L^1(\mathbb{G}))$ is unique and (two-sided) invariant.

Our next result yields a generalization of [12, Theorem 2.3] which is concerned with the group algebra $L^1(G)$ as an ideal in the measure algebra $M(G)$, for a locally compact group G.

Corollary 3.4 *Let $\mathbb{G}$ be a co-amenable locally compact quantum group. Then $\mathrm{WAP}(L^1(\mathbb{G}))$ has a right invariant 1-mean or equivalently has an invariant 1-mean if and only if $\mathrm{WAP}(M(\mathbb{G}))$ has an invariant 1-mean.*

Proposition 3.5 *Let $\mathcal{A}$ is a Banach algebra, and let I is a closed ideal in $\mathcal{A}$. Let $\varphi \in \mathrm{sp}(\mathcal{A})$ be such that $I \nsubseteq \ker\varphi$. Then $\mathcal{A}^*$ admits a right invariant φ-mean if and only if I^* admits a right invariant $\varphi|_I$-mean.*

Proof To see this, first note that, since we can identify I^{**} with $I^{\perp\perp}$, it follows that I^{**} is a closed ideal in $\mathcal{A}^{**}$ (see [2, p. 17]). Fix $b_0 \in I$ with $\varphi(b_0) = 1$. Now, suppose that $m \in \mathcal{A}^{**}$ is a right invariant φ-mean on $\mathcal{A}^*$. Since I^{**} is an ideal in $\mathcal{A}^{**}$, we obtain that $b_0 \square m \in I^{**}$. Furthermore, $\langle b_0 \square m, \varphi\rangle = 1$ and

$$(b_0 \square m) \square b = \varphi(b) b_0 \square m$$

for all $b \in I$. Thus, $b_0 \square m$ is a right invariant $\varphi|_I$-mean on I^*. For the converse, suppose that $m \in I^{**}$ is a right invariant $\varphi|_I$-mean on I^*. Then

$$m \square a = (m \square b_0) \square a = m \square (b_0 a) = \varphi(b_0 a) m = \varphi(a) m$$

for all $a \in \mathcal{A}$. This shows that m is a right invariant φ-mean on $\mathcal{A}^*$. ∎

Before giving the next result, we recall that a Banach algebra $\mathcal{A}$ is weakly sequentially complete if every weakly Cauchy sequence in $\mathcal{A}$ is weakly convergent in $\mathcal{A}$. For example, preduals of von Neumann algebras are weakly sequentially complete (see [15]).

Proposition 3.6 *Let $\mathbb{G}$ be a locally compact quantum group such that $\mathrm{WAP}(L^1(\mathbb{G}))$ has an invariant 1-mean, and let I be a closed ideal of $L^1(\mathbb{G})$ with a bounded approximate identity such that $I \nsubseteq \ker 1$. If I is Arens regular, then $\mathbb{G}$ is compact.*

Proof By assumption and Theorem 3.2, we conclude that $\mathrm{WAP}(I)$ has a right invariant 1-mean. Since I is Arens regular, we have that $\mathrm{WAP}(I) = I^*$. This implies that I is right 1-amenable. Now, by Proposition 3.5, we obtain that $L^1(\mathbb{G})$ is right 1-amenable or equivalently, $\mathbb{G}$ is amenable. Thus, there is an invariant 1-mean on $L^\infty(\mathbb{G})$. Again by two-sided version of Proposition 3.5, we conclude that there is an invariant 1-mean m on I^*. Since I is Arens regular and weakly sequentially complete, it follows from [7, Theorem 3.9] that $m \in I$. Therefore, for every $f \in L^1(\mathbb{G})$, we have

$$f * m = f * (m * m) = (f * m) * m = \langle f * m, 1\rangle m = \langle f, 1\rangle m.$$

Thus, m is a left invariant 1-mean belonging to $L^1(\mathbb{G})$, and equivalently $\mathbb{G}$ is compact (see [1, Proposition 3.1]). ∎

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Theorem 3.7 *Let $\mathbb{G}$ be a locally compact quantum group such that* $\mathrm{WAP}(L^1(\mathbb{G}))$ *has an invariant* 1*-mean, and let I be a closed ideal of* $L^1(\mathbb{G})$ *with a bounded approximate identity such that* $I \nsubseteq \ker 1$*. Then I is Arens regular if and only if it is reflexive.*

Proof If I is reflexive, then I is clearly Arens regular. Conversely, suppose that I is Arens regular. Then $\mathbb{G}$ is compact by Proposition 3.6 and so by [13, Theorem 3.8], $L^1(\mathbb{G})$ is an ideal in its bidual. Since I has a bounded approximate identity, Cohen's Factorization theorem implies that $I * I = \{a * b : a, b \in I\} = I$. Hence, we drive that

$$I \square I^{**} = (I * I) \square I^{**} \subseteq I \square (I \square L^1(\mathbb{G})^{**}) \subseteq I * L^1(\mathbb{G}) \subseteq I.$$

This shows that I is a right ideal in its bidual. Thus, by [16, Corollar ies 3.7 and 3.9], we obtain that I is reflexive. ∎

Dually to [5, Proposition 3.14], we obtain the result below for the group algebra $L^1(G)$ of a locally compact group G. We would like to recall that $\mathrm{WAP}(L^1(G))$ admits an invariant mean.

Corollary 3.8 *Let G be a locally compact group, and let I be a closed ideal of* $L^1(G)$ *with a bounded approximate identity such that* $I \nsubseteq \ker 1$*. Then I is Arens regular if and only if it is reflexive.*

4 Convolution trace class operators

We recall from [10] that a Lau algebra $\mathcal{A}$ is a Banach algebra such that $\mathcal{A}^*$ is a von Neumann algebra whose unit 1 lies in the spectrum of $\mathcal{A}$. Let $\mathbb{G}$ be a locally compact quantum group. Then it is easy to see that $1 = 1 \circ \pi \in \mathrm{sp}(\mathcal{T}_\triangleright(\mathbb{G}))$. Now, since $B(L^2(\mathbb{G}))$ is a von Neumann algebra, it follows that $\mathcal{T}_\triangleright(\mathbb{G})$ is a Lau algebra. In this section, we are interested to study the relation between the existence of left or right invariant 1-means on $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$ and on $\mathrm{WAP}(L^1(\mathbb{G}))$.

Lemma 4.1 *Let $\mathbb{G}$ be a locally compact quantum group. Then*

$$\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G})) \triangleright \mathcal{T}_\triangleright(\mathbb{G}) \subseteq \mathrm{WAP}(L^1(\mathbb{G})).$$

Proof Suppose that $x \in \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$ and $w \in \mathcal{T}_\triangleright(\mathbb{G})$. Let $(f_k)_k$ be a bounded sequence in $L^1(\mathbb{G})$. For each k, let $w_k \in \mathcal{T}_\triangleright(\mathbb{G})$ be a normal extension of f_k. By weak compactness of the map $\lambda_x : \mathcal{T}_\triangleright(\mathbb{G}) \to B(L^2(\mathbb{G}))$, there is a subsequence (w_{k_j}) of (w_k) such that $(w_{k_j} \triangleright x)$ converges weakly in $B(L^2(\mathbb{G}))$ to some $y \in B(L^2(\mathbb{G}))$. It is easy to check that $(w_{k_j} \triangleright x \triangleright w)$ converges weakly in $B(L^2(\mathbb{G}))$ to $y \triangleright w$. Now, let $m \in L^\infty(\mathbb{G})^*$, and let $\widetilde{m} \in B(L^2(\mathbb{G}))^*$ be a Hahn–Banach extension of m. Since $B(L^2(\mathbb{G})) \triangleright \mathcal{T}_\triangleright(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$, we have

$$\langle m, f_{k_j} \cdot (x \triangleright w) \rangle = \langle \widetilde{m}, w_{k_j} \triangleright x \triangleright w \rangle \to \langle \widetilde{m}, y \triangleright w \rangle = \langle m, y \triangleright w \rangle.$$

This shows that $x \triangleright w \in \mathrm{WAP}(L^1(\mathbb{G}))$. ∎

Theorem 4.2 *Let $\mathbb{G}$ be a locally compact quantum group. Then* $\mathrm{WAP}(L^1(\mathbb{G}))$ *has a right invariant* 1*-mean if and only if* $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$ *has a right invariant* 1*-mean.*

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Proof Let m be a right invariant 1-mean on $\mathrm{WAP}(L^1(\mathbb{G}))$. Define $\widetilde{m} \in \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))^*$ by $\langle \widetilde{m}, x\rangle = \langle m, x \triangleright w_0\rangle$ for all $x \in \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$, where $w_0 \in \mathcal{T}_\triangleright(\mathbb{G})$ with $\|w_0\| = \langle w_0, 1\rangle = 1$. Then it is easy to check that $\langle \widetilde{m}, 1\rangle = 1$. Moreover, we have

$$\begin{aligned}\langle \widetilde{m}, w \triangleright x\rangle &= \langle m, w \triangleright (x \triangleright w_0)\rangle \\ &= \langle m, \pi(w)\cdot(x \triangleright w_0)\rangle \\ &= \langle w, 1\rangle\langle m, x \triangleright w_0\rangle \\ &= \langle w, 1\rangle\langle \widetilde{m}, x\rangle\end{aligned}$$

for all $w \in \mathcal{T}_\triangleright(\mathbb{G})$ and $x \in \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$, proving that $\widetilde{m}$ is a right invariant 1-mean on $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$.

Conversely, suppose that n is a right invariant 1-mean on $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$. Since $\pi : \mathcal{T}_\triangleright(\mathbb{G}) \to L^1(\mathbb{G})$ is a continuous algebra homomorphism, it follows from [18, Corollary to Lemma 1] that the map π^* maps $\mathrm{WAP}(L^1(\mathbb{G}))$ to $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$. Thus, we can define $\widetilde{n} \in \mathrm{WAP}(L^1(\mathbb{G}))^*$ by $\widetilde{n} := n \circ \pi^*$. It is easily verified that $\langle \widetilde{n}, 1\rangle = 1$. For every $f \in L^1(\mathbb{G})$ and $x \in \mathrm{WAP}(L^1(\mathbb{G}))$, let $w \in \mathcal{T}_\triangleright(\mathbb{G})$ be a normal extension of f. Then we have

$$\begin{aligned}\langle \pi^*(f\cdot x), w'\rangle = \langle f\cdot x, \pi(w')\rangle &= \langle x, \pi(w') * \pi(w)\rangle \\ &= \langle \pi^*(x), w' \triangleright w\rangle \\ &= \langle w \triangleright \pi^*(x), w'\rangle,\end{aligned}$$

for all $w' \in \mathcal{T}_\triangleright(\mathbb{G})$. Therefore,

$$\begin{aligned}\langle \widetilde{n}, f\cdot x\rangle &= \langle n, \pi^*(f\cdot x)\rangle \\ &= \langle n, w \triangleright \pi^*(x)\rangle \\ &= \langle w, 1\rangle\langle n, \pi^*(x)\rangle \\ &= \langle f, 1\rangle\langle \widetilde{n}, x\rangle.\end{aligned}$$

That is, $\widetilde{n}$ is a right invariant 1-mean on $\mathrm{WAP}(L^1(\mathbb{G}))$. ∎

Before giving the next result, recall that if $\mathbb{G} = L^\infty(G)$ for a locally compact group G, then $\mathcal{T}_\triangleright(\mathbb{G})$ is the convolution algebra introduced by Neufang in [11].

Corollary 4.3 *Let G be a locally compact group, and let $\mathbb{G} = L^\infty(G)$. Then $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$ admits a right invariant 1-mean.*

Theorem 4.4 *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$ has a left invariant 1-mean if and only if $\mathbb{G}$ is trivial.*

Proof Let m be a left invariant 1-mean on $\mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$. Then for every $x \in \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$, we have $m\cdot x = \langle m, x\rangle 1$, by left invariance. Now, consider the map

$$E : \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G})) \to \mathrm{WAP}(\mathcal{T}_\triangleright(\mathbb{G}))$$

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defined by $E(x) = m \cdot x = \langle m, x\rangle 1$ for all $x \in \mathrm{WAP}(\mathcal{T}_{\triangleright}(\mathbb{G}))$. Then for every $\hat{x} \in L^{\infty}(\hat{\mathbb{G}})$, we have

$$E(\hat{x}) = m \cdot \hat{x} = \langle m, 1\rangle \hat{x} = \hat{x}.$$

These prove that $L^{\infty}(\widehat{\mathbb{G}}) = E(L^{\infty}(\widehat{\mathbb{G}})) \subseteq \mathbb{C}1$. Therefore, $L^{\infty}(\widehat{\mathbb{G}}) = \mathbb{C}1$ and so $\mathbb{G}$ is trivial. ∎

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