Proceedings of the Edinburgh Mathematical Society (2015) **58**, 421–439
DOI:10.1017/S0013091514000108

ON NONLINEAR BOUNDARY CONDITIONS INVOLVING DECOMPOSABLE LINEAR FUNCTIONALS

CHRISTOPHER S. GOODRICH[1,2]

[1] *Department of Mathematics, Creighton Preparatory School, Omaha, NE 68114, USA*
[2] *Department of Mathematics, University of Rhode Island, Kingston, RI 02881, USA* (cgood@prep.creighton.edu)

(Received 8 May 2013)

Abstract In this paper we consider the existence of a positive solution to boundary-value problems with non-local nonlinear boundary conditions, the archetypical example being $-y''(t) = \lambda f(t, y(t))$, $t \in (0, 1)$, $y(0) = H(\varphi(y))$, $y(1) = 0$. Here, H is a nonlinear function, $\lambda > 0$ is a parameter and φ is a linear functional that is realized as a Lebesgue–Stieltjes integral with signed measure. By requiring φ to decompose in a certain way, we show that this problem has at least one positive solution for each $\lambda \in (0, \lambda_0)$, for a number $\lambda_0 > 0$ that is explicitly computable. We also give a separate result that holds for all $\lambda > 0$.

Keywords: positive solution; nonlinear boundary condition; non-local boundary condition; fixed point index; eigenvalue

2010 *Mathematics subject classification:* Primary 34B09; 34B10; 34B15; 34B18; 47H07
Secondary 47B40; 47G10; 47H11

1. Introduction

In this paper we consider the boundary-value problem (BVP)

$$\begin{aligned} -y''(t) &= \lambda f(t, y(t)), \quad t \in (0, 1), \\ y(0) &= H(\varphi(y)), \\ y(1) &= 0, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $H : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\varphi : \mathcal{C}([0, 1]) \to \mathbb{R}$ is a linear functional, which we realize in the form

$$\varphi(y) = \int_{[0,1]} y(t)\,\mathrm{d}\alpha(t), \tag{1.2}$$

where $\alpha \in \mathrm{BV}([0, 1])$. As is explained later, our results easily extend to more general problems such as the perturbed Hammerstein integral equation

$$y(t) = \xi(t)H(\varphi(y)) + \int_0^1 K(t, s)f(s, y(s))\,\mathrm{d}s$$

for a suitable function ξ and kernel K, which means that we can easily adapt our results to treat a variety of nonlinear, non-local boundary conditions in (1.1). In any case, since α need not be monotone in (1.2), we allow for the possibility that the measure associated with the integrator α is signed. Due to the presence of the nonlinear boundary condition (BC) as well as the fact that the measure associated with α is signed, it is unclear whether (1.1) can admit a non-trivial, positive solution. In addition to this abstract mathematical interest, BVPs equipped with non-local BCs are of interest in, for example, modelling the heat flow through a bar when the bar is equipped with heat sources that respond to the temperature at different locations along the bar (see, for example, [**17**]). In this context, the non-local boundary conditions can be interpreted as representing this temperature control and regulation.

The literature on BVPs with non-local, *linear* BCs is extensive and culminates with a paper by Infante and Webb [**33**], which provides a unified theory for BVPs with *linear* non-local boundary conditions wherein the BCs are realized as Lebesgue–Stieltjes integrals of the form of (1.2). The archetypical BVP considered in [**33**] can be realized as

$$\begin{aligned} -y''(t) &= f(t, y(t)), \quad t \in (0,1), \\ y(0) &= \int_{[0,1]} y(s)\,\mathrm{d}A(s), \\ y(1) &= \int_{[0,1]} y(s)\,\mathrm{d}B(s), \end{aligned} \tag{1.3}$$

where the integrators A and B are of class $\mathrm{BV}([0,1])$ and are not necessarily monotone. Consequently, the well-studied multipoint BCs, as well as linear, integral BCs, are then included as special cases. In any case, the particularly clever insight provided by Infante and Webb [**33**] was to consider the cone

$$\left\{ y \in \mathcal{C}([0,1]) \colon y(t) \geqslant 0,\ \min_{t \in [a,b]} y(t) \geqslant \gamma \|y\|,\ \int_{[0,1]} y(s)\,\mathrm{d}A(s) \geqslant 0,\ \int_{[0,1]} y(s)\,\mathrm{d}B(s) \geqslant 0 \right\},$$

where $\gamma \in (0,1]$ is a constant and $0 < a < b < 1$. Proving the existence of a solution to problem (1.3) by means of the above cone is then quite easy and standard. It turns out that the introduction of this new cone is useful in a wide variety of non-local BVPs. Indeed, since the appearance of this result, the ideas of Infante and Webb have been extended in a number of different directions by numerous authors; see, for example, [**1**–**7**, **11**,**14**,**15**,**19**,**20**,**23**–**25**,**29**–**32**,**34**,**35**] and references therein for some of these extensions as well as other recent advances in non-local BVPs.

Related more closely to problem (1.1), there have been some attempts at combining the concept of the very general non-locality captured by (1.2) with a nonlinear boundary condition. Some early attempts were provided by Kang and Wei [**26**] as well as by Yang [**36**–**39**] and some more recent ones by Infante [**16**], and Infante and Pietramala [**17**, **18**, **21**, **22**]. Often in these papers, however, somewhat restrictive growth is imposed on H, e.g. there exist $0 \leqslant \eta_1 < \eta_2$ such that $\eta_1 z \leqslant H(z) \leqslant \eta_2 z$ for all $z \in [0, +\infty)$; note that this does not even allow the map $z \mapsto z^\rho$ for any $\rho \neq 1$. It would be better if the growth assumption on H was allowed, for instance, to be captured asymptotically, much as is frequently imposed on the nonlinearity f. In addition, each of these results requires that the integrator α appearing in (1.2) be monotone increasing.

In this paper we wish to provide some extensions of the ideas we initiated in [**10**], which we subsequently partially extended to the case of systems in [**8**, **9**]. In particular, a principal assumption we make is that

$$\lim_{z\to\infty}\frac{|H(z)-C_2z|}{z}=0 \tag{1.4}$$

for some constant $C_2 \geqslant 0$, a condition that we used in [**10**]. It is perhaps worth noting that (1.4) implies that if $\varphi(y)$ is sufficiently large, then the BC at $t=0$ in (1.1) essentially 'looks like' the linear condition $y(0)=C^*\varphi(y)$ for a constant $C^* \geqslant 0$. In fact, we can use this sort-of-asymptotic linearity to give a very simple yet general existence proof (see Theorem 3.3).

A significant problem with the asymptotic relatedness condition of (1.4), however, is that, in general, (1.4) is meaningless in the context of problem (1.1) because we cannot deduce that $\varphi(y)$ is large whenever, say, $\|y\|$ is large. Indeed, consider, for example, the functional

$$\varphi(y) := \tfrac{1}{3}y(\tfrac{2}{5}) - \tfrac{1}{5}y(\tfrac{1}{2}). \tag{1.5}$$

Then $\varphi(y)$ can be 0 even if $\|y\|$ is large; in fact, this is true even if we additionally impose the typical restriction that y satisfies the Harnack-like inequality $\min_{t\in[a,b]}y(t) \geqslant \gamma\|y\|$ for some $0<a<2/5<1/2<b<1$.

In order to solve this dilemma we use a novel insight whose origins can be traced to [**10**] but which must be modified suitably in order to work in the context presented in this paper. This insight is to require that φ decompose in a special way, namely, that there are linear functionals φ_1 and φ_2 such that

$$\varphi(y)=\varphi_1(y)+\varphi_2(y), \tag{1.6}$$

where φ_2 satisfies $\varphi_2(y) \geqslant C_0\|y\|$ for some constant $C_0>0$ and all y in a suitable cone $\mathcal{K}$, and φ_1 essentially captures the signed part of φ. Since this cone $\mathcal{K}$, much as the one introduced by Infante and Webb, permits us to assume that $\varphi_1(y)$, $\varphi_2(y) \geqslant 0$, we obtain from (1.6) that $\varphi(y) \geqslant C_0\|y\|$, which then affords us the control necessary to make the asymptotic relatedness condition (1.4) meaningful in the context of problem (1.1).

This decomposition is easily applicable (see the examples at the end of § 3) and it is worthwhile to note that it need not be the case that φ is originally written in the decomposed form (1.6). Indeed, (1.5) is clearly not in the decomposed form. However, as will be shown in Example 3.10 at the end of this paper, if we merely write

$$\varphi(y)=\underbrace{\tfrac{99}{300}y(\tfrac{2}{5})-\tfrac{1}{5}y(\tfrac{1}{2})}_{:=\varphi_1(y)}+\underbrace{\tfrac{1}{300}y(\tfrac{2}{5})}_{:=\varphi_2(y)}, \tag{1.7}$$

then we can show that φ is realized in an admissible decomposition form. This simple insight then allows us to prove the existence of solutions of problem (1.1) under much

https://doi.org/10.1017/S0013091514000108 Published online by Cambridge University Press

weaker assumptions on H than previously used. Furthermore, it allows the measure associated with α to be signed.

We comment next on the relationship of our results here to those we recently gave in [**10**]. One of the improvements we give here is to show that problem (1.1) can admit at least one positive solution for $\lambda > 0$ small. By contrast, in [**10**] we only considered the case where either $\lambda = 1$ or λ was large. Furthermore, in [**10**] the admissible values of λ were somewhat technical to calculate. In contrast, this calculation is simpler here. In addition, we also weaken the requirements on the nonlinearity f. In fact, we only require f to satisfy a growth condition for either small y or large y but *not* both. Even in the case when $\lambda = 1$, our results improve those of [**10**] by greatly weakening the growth hypotheses. We also believe that our method of proof (i.e. computing an appropriate Fréchet derivative at $+\infty$) is interesting. In this latter case, we even improve, in certain cases, the results of [**12**] (see Remark 3.5). Finally, none of the results in [**16**–**18**, **21**, **36**–**39**] can treat the type of problems here, for each either assumes very restrictive growth on the equivalent of our H and/or does not allow the non-locality to be signed. Furthermore, these works, in general, suppose more complicated and/or restrictive growth assumptions on the nonlinearity f than we do.

Let us conclude this section by remarking that although we treat problem (1.1) with relatively simple boundary conditions, it is clear from the proofs of our existence theorems that our results can be easily modified to accommodate much more complicated boundary conditions. Moreover, we believe that the techniques used here could be applied successfully in other types of BVPs such as the semipositone problem and higher-order problems. In particular, although we illustrate our techniques with the simple BCs in problem (1.1), their use is not limited to this special case. Nonetheless, we do not explicitly write the numerous results that follow.

2. Preliminaries

We begin this section by observing that the operator $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ defined by

$$(Ty)(t) := (1-t)H(\varphi(y)) + \lambda \int_0^1 G(t,s) f(s,y(s))\,\mathrm{d}s \tag{2.1}$$

may be studied as a means of deducing the existence of positive solutions to problem (1.1). Indeed, it is the case that a fixed point of T is simultaneously a solution to (1.1). Note (see [**27**]) that the function $G\colon [0,1]\times[0,1] \to \mathbb{R}$ in (2.1) is the Green function associated with the conjugate problem, namely,

$$G(t,s) := \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.2}$$

We assume here and throughout that $[a,b]$ is a given fixed subinterval of $(0,1)$. Then there exists a constant $\gamma := \min\{a, 1-b\}$ such that

$$\min_{t\in[a,b]} G(t,s) \geqslant \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(s,s) \tag{2.3}$$

for each $s \in [0,1]$; observe that γ also satisfies $1-t \geqslant \gamma$ for each $t \in [a,b]$. Finally, recall both that $\max_{t\in[0,1]} G(t,s) = G(s,s)$, for each $s \in [0,1]$, and that $\gamma \in (0,1)$.

In contrast to the author's earlier works [**10**, **12**, **13**] we do not appeal to the well-known Krasnosel′skiĭ fixed-point theorem. Rather, we shall use two different approaches, one involving classical degree theory and a second involving the concept of Fréchet differentiability at $+\infty$, and demonstrate that we can achieve more general results with these approaches. We first recall the Leray–Schauder degree; the reader is referred to Zeidler [**40**] for more information on classical degree theory.

Lemma 2.1 (Zeidler [40, Definition 12.3]). *Let $\mathcal{V}(G,\mathcal{X})$ denote the set of all mappings f of the following form.*

(1) *$f\colon \bar{G} \subset \mathcal{X} \to \mathcal{X}$ is compact, where G is an open bounded set in a Banach space $\mathcal{X}$.*

(2) *f has no fixed points on ∂G.*

Then, to each $f \in \mathcal{V}(G,\mathcal{X})$, an integer $i(f,G)$, called the fixed-point index of f on G, may be assigned so that the following hold.

(1) *If $f(x) = x_0$ for all $x \in \bar{G}$ and some fixed $x_0 \in G$, then $i(f,G) = 1$.*

(2) *If $i(f,G) \neq 0$, then there exists $x \in G$ such that $f(x) = x$.*

(3) *It holds that*

$$i(f,G) = \sum_{j=1}^{n} i(f,G_j)$$

whenever $f \in \mathcal{V}(G,\mathcal{X})$ and $f \in \mathcal{V}(G_j,\mathcal{X})$ for each $1 \leqslant j \leqslant n$, where $\{G_j\}_{j=1}^{n}$ is a regular partition of G.

(4) *If $\partial G\colon f \cong g$, then $i(f,G) = i(g,G)$.*

Note that in (4) the notation $\partial G\colon f \cong g$ means that f and g are compactly homotopic on ∂G in the sense that there is a compact map $H\colon \bar{G} \times [0,1] \to \mathcal{X}$ such that $H(x,t) \neq x$ for each $(x,t) \in \partial G \times [0,1]$ and both $H(x,0) = f(x)$ and $H(x,1) = g(x)$ on $\bar{G}$.

Next we recall a perhaps less frequently used fixed-point theorem for asymptotically linear operators. In order to state this result we recall first a definition of the existence of the Fréchet derivative at $+\infty$ of a suitable operator T. Since we will use this result in the context of an order cone $\mathcal{K}$, we state the result in that form; see [**40**, § 7.9] for more details.

Definition 2.2 (Zeidler [40, Definition 7.32 b]). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{R}$. Set

$$U(+\infty, r) := \{x \in \mathcal{X}\colon \|x\| \geqslant r\},$$

where $r > 0$. Let $T\colon \mathcal{X} \to \mathcal{Y}$ be an operator. If $\mathcal{X}$ has an order cone $\mathcal{K}$, then the operator $T'(+\infty) \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, where $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is the collection of all linear transformations between

https://doi.org/10.1017/S0013091514000108 Published online by Cambridge University Press

$\mathcal{X}$ and $\mathcal{Y}$, is called the *positive Fréchet derivative of* T *at* $+\infty$ *along the cone* if and only if there is a fixed $r > 0$ such that, for all $x \in U(+\infty, r) \cap \mathcal{K}$, it holds that

$$Tx = T'(+\infty)x + o(\|x\|), \quad \|x\| \to +\infty,$$

that is,

$$\frac{\|Tx - T'(+\infty)x\|}{\|x\|} \to 0 \quad \text{as } \|x\| \to +\infty.$$

Lemma 2.3 (Zeidler [40, Corollary 7.34]). *Suppose that the following hold.*

(1) $T\colon \mathcal{K} \subseteq \mathcal{X} \to \mathcal{K}$ *is a compact operator on the Banach space* $\mathcal{X}$ *with order cone* $\mathcal{K}$.

(2) $T'(+\infty)$ *exists as a positive Fréchet derivative of* T *at* $+\infty$ *and its spectral radius, denoted by* $r(T'(+\infty))$, *satisfies* $r(T'(+\infty)) < 1$.

Then T *has a fixed point.*

In § 3 we shall make use of a cone $\mathcal{K} \subseteq \mathcal{C}([0,1])$ defined by

$$\mathcal{K} := \Big\{ y \in \mathcal{C}([0,1])\colon y(t) \geqslant 0,\ \min_{t\in[a,b]} y(t) \geqslant \gamma\|y\|,\ \varphi_1(y) \geqslant 0,\ \varphi_2(y) \geqslant 0 \Big\}. \tag{2.4}$$

We have previously used this cone in some of our recent work on nonlinear, non-local BVPs (see [**10**, **13**]). The cone $\mathcal{K}$ is a slight modification of a cone originally introduced by Infante and Webb [**33**], as mentioned in § 1. Note that $\mathcal{K} \neq \varnothing$ since, due to condition (H4), it holds that $1 - t \in \mathcal{K}$.

Let us conclude this section by stating the conditions that we impose on problem (1.1). We first provide the conditions that we impose on the non-locality φ and the nonlinearity H. These structure assumptions, which we label (H0)–(H4), enable us to obtain the correct control over φ, as discussed in § 1. We point out that while we use (H0), (H1), (H3) and (H4) in each existence theorem, condition (H2) is only used in the first existence theorem. Note that we shall provide examples at the end of § 3 illustrating the use of each of these conditions in the context of non-local BVPs.

(H0) Assume that there are two linear functionals φ_1, $\varphi_2\colon \mathcal{C}([0,1]) \to \mathbb{R}$ such that

$$\varphi(y) = \varphi_1(y) + \varphi_2(y).$$

Moreover, assume that there exists a constant $C_0 \geqslant 0$ such that

$$\varphi_2(y) \geqslant C_0\|y\|$$

holds for each $y \in \mathcal{K}$. Note that since φ is continuous, there is another constant, say $C_1 \geqslant 0$, such that

$$|\varphi(y)| \leqslant C_1\|y\|$$

for each $y \in \mathcal{C}([0,1])$. Henceforth, C_1 shall denote this constant.

(H1) The functionals $\varphi(y)$, $\varphi_1(y)$ and $\varphi_2(y)$ are linear and, in particular, have the form

$$\begin{aligned}\varphi(y) &:= \int_{[0,1]} y(t)\,\mathrm{d}\alpha(t),\\ \varphi_1(y) &:= \int_{[0,1]} y(t)\,\mathrm{d}\alpha_1(t),\\ \varphi_2(y) &:= \int_{[0,1]} y(t)\,\mathrm{d}\alpha_2(t),\end{aligned}$$

where α, α_1, $\alpha_2\colon [0,1]\to\mathbb{R}$ satisfy α, α_1, $\alpha_2 \in \mathrm{BV}([0,1])$.

(H2) The function $H\colon \mathbb{R}\to\mathbb{R}$ is continuous, satisfies $H([0,+\infty))\subseteq[0,+\infty)$, and either

(1) there exists $+\infty > M \geqslant 1$ such that the number M satisfies that $M > 1/\gamma C_0$, $\lim_{z\to 0^+}(H(z)/z)$ exists, $\lim_{z\to 0^+}(H(z)/z) > M$, or

(2) $\lim_{z\to 0^+}(H(z)/z) = +\infty$.

(H3) There is a $C_2 \geqslant 0$ such that

$$\lim_{z\to+\infty} \frac{|H(z)-C_2 z|}{z} = 0 \tag{2.5}$$

holds.

(H4) Both

$$\int_{[0,1]} (1-t)\,\mathrm{d}\alpha_1(t),\quad \int_{[0,1]} (1-t)\,\mathrm{d}\alpha_2(t) \geqslant 0 \tag{2.6}$$

and

$$\int_{[0,1]} G(t,s)\,\mathrm{d}\alpha_1(t),\quad \int_{[0,1]} G(t,s)\,\mathrm{d}\alpha_2(t) \geqslant 0 \tag{2.7}$$

hold, where the latter holds for each $s\in[0,1]$.

We next list the conditions on the nonlinearity f. Once again, condition (H5) is used only in the first existence theorem, whereas condition (H6) is used only in the second existence theorem. We remark that, throughout, we assume that $f\colon [0,1]\times\mathbb{R}\to[0,+\infty)$ is continuous and $\vartheta\colon [0,1]\to[0,+\infty)$.

(H5) There exists a number $r_2>0$ and a measurable function $\vartheta\colon [0,1]\to\mathbb{R}$ such that

$$f(t,y)\leqslant r_2\vartheta(t)$$

for almost every $t\in[0,1]$ and each $y\in[0,r_2/C_0]$, where r_2 must satisfy the following condition: there exists a $\varepsilon>0$ such that ε and r_2 satisfy each of

$$1 > (C_2+\varepsilon)\int_0^1 (1-t)\,\mathrm{d}\alpha(t), \tag{2.8}$$

https://doi.org/10.1017/S0013091514000108 Published online by Cambridge University Press

$$r_2 \geqslant \frac{1}{C_1} \sup \left\{ z \in [0,1) \colon \frac{H(x)}{x} > M \text{ for all } x \in (0,z] \right\}$$

and

$$r_2 \geqslant \inf \left\{ z \in [0,+\infty) \colon \frac{|H(x) - C_2 x|}{x} \leqslant \varepsilon \text{ for all } x \in [z,+\infty) \right\}.$$

Here, M is the constant from condition (H2). If $\lim_{z\to 0^+}(H(z)/z) = +\infty$, then M can be any number satisfying $M > 1/\gamma C_0$.

(H6) Assume that

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = 0$$

uniformly for $t \in [0,1]$.

Remark 2.4. Note that, as discussed in some sense in § 1, condition (H3) implies that, for $\|y\|$ large, it holds that the boundary condition $y(0) = H(\varphi(y))$ essentially 'looks like' the simpler condition $y(0) = C^*\varphi(y)$ for a constant $C^* \geqslant 0$. This is precisely the sort of 'asymptotic linearity' that was referred to in § 1 and is used in our second existence theorem.

Remark 2.5. Instead of imposing condition (H2), it is possible to require that, for each $t \in [a,b]$, it holds that $f(t,y) \geqslant \rho\psi(t)$, for some measurable function ψ and some number $\rho > 0$, on a set of the form $[\rho\gamma/C_1, \rho/C_1]$. But we do not explicitly write out this alternative formulation in the existence theorems of § 3.

We conclude by recalling that the following lemma was already proved in [**10**] so we do not repeat the proof here.

Lemma 2.6 (Goodrich [10, Lemma 3.4]). *Let T be the operator defined in (2.1). Provided that conditions (H0), (H1) and (H4) hold, $T(\mathcal{K}) \subseteq \mathcal{K}$.*

3. Proofs of theorems and discussion

We begin by presenting our first existence result, which uses Lemma 2.1. In order to facilitate the proof of Theorem 3.1 we introduce the following notation, where $r > 0$ is assumed:

$$\begin{aligned} \Omega_r &:= \{y \in \mathcal{K} \colon \|y\| < r\}, \\ V_r &:= \left\{ y \in \mathcal{K} \colon \min_{t\in[a,b]} y(t) < r \right\}. \end{aligned} \tag{3.1}$$

Note that the idea of using sets of the form V_r rather than merely Ω_r appears to have first been introduced by Lan [**28**]. Furthermore, it is trivial to show that

$$\Omega_r \subset V_r \subset \Omega_{r/\gamma}.$$

Theorem 3.1. *Suppose that conditions (H0)–(H5) hold. If, for each $i = 1,2$, it holds that*

$$\int_{[0,1]} \mathrm{d}\alpha_i(t) \geqslant 0, \tag{3.2}$$

https://doi.org/10.1017/S0013091514000108 Published online by Cambridge University Press

then problem (1.1) has at least one positive solution for each λ satisfying

$$\begin{aligned} 0 &< \lambda \\ &< \left(1-(C_2+\varepsilon)\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)\right)\left[\int_0^1\int_{[0,1]}G(t,s)\vartheta(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]^{-1}, \end{aligned} \tag{3.3}$$

where we assume that

$$\int_0^1\int_{[0,1]}G(t,s)\vartheta(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s>0$$

and that the number ε appearing in (3.3) is from condition (H5).

Proof. Recall that $T\colon\mathcal{K}\to\mathcal{K}$ is a compact operator. By condition (H2) we may select a number r_1 sufficiently small such that $H(z)>Mz$ whenever $0<z\leqslant r_1$; without loss of generality, it may be assumed that $r_1<1$. In the case in which it holds that $\lim_{z\to 0^+}(H(z)/z)=+\infty$, we may choose some finite M sufficiently large such that $M>1/\gamma C_0$ and proceed as in the previous sentence. In any case, we note that by condition (H0) it follows that $\varphi(y)\leqslant r_1$ whenever $\|y\|\leqslant r_1/C_1$. Since it holds that $\partial V_{r_1\gamma/C_1}\subseteq\bar{\Omega}_{r_1/C_1}$, it follows that $H(\varphi(y))>M\varphi(y)$ whenever $y\in\partial V_{r_1\gamma/C_1}$.

We claim that

$$i_{\mathcal{K}}(T,V_{r_1\gamma/C_1})=0. \tag{3.4}$$

To prove (3.4), we shall show that $y\neq Ty+\mu e$ for all $y\in\partial V_{r_1\gamma/C_1}$ and $\mu>0$, where $e(t)\equiv 1$. Note that $e\in\mathcal{K}$, due, in part, to condition (3.2). To this end, suppose for contradiction that there exists $y\in\partial V_{r_1\gamma/C_1}$ and $\mu>0$ such that $y=Ty+\mu e$. Since $y\in\partial V_{r_1\gamma/C_1}\cap\bar{\Omega}_{r_1/C_1}$, it holds that $r_1\gamma/C_1\leqslant y(t)\leqslant r_1/C_1$ for each $t\in[a,b]$. Then, for each $t\in[a,b]$, we estimate

$$\begin{aligned} y(t) &= (Ty)(t)+\mu e(t) \\ &= (1-t)H(\varphi(y))+\lambda\int_0^1 G(t,s)f(s,y(s))\,\mathrm{d}s+\mu \\ &\geqslant M\gamma\varphi(y)+\mu \\ &\geqslant M\gamma\varphi_2(y)+\mu \\ &\geqslant M\gamma C_0\|y\|+\mu \\ &\geqslant M\gamma C_0\frac{\gamma r_1}{C_1}+\mu \\ &> \frac{\gamma r_1}{C_1}, \end{aligned} \tag{3.5}$$

which contradicts the fact that $y\in\partial V_{r_1\gamma/C_1}$; here we have used the fact that, since $y\in\mathcal{K}$, $\varphi(y)=\varphi_1(y)+\varphi_2(y)\geqslant\varphi_2(y)$, as well as the fact that $M\gamma C_0>1$. Note, furthermore, that (3.5) actually demonstrates that $Ty\neq y$ whenever $y\in\partial V_{r_1\gamma/C_1}$. Consequently, (3.4) holds, as desired.

On the other hand, we next show that for the number r_2 of condition (H5) it holds that

$$i_{\mathcal{K}}(T,\Omega_{r_2/C_0})=1. \tag{3.6}$$

https://doi.org/10.1017/S0013091514000108 Published online by Cambridge University Press

To prove (3.6) we shall show that $Ty \neq \mu y$ for each $\mu \geqslant 1$ and each $y \in \partial\Omega_{r_2/C_0}$. To this end, fix a number λ satisfying inequality (3.3), letting ε be the fixed number from condition (H5). Then (3.3) implies that

$$\frac{1-(C_2+\varepsilon)\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)}{\int_0^1\int_{[0,1]}G(t,s)\vartheta(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s} > \lambda > 0. \tag{3.7}$$

Condition (H3) implies the existence of a number $r_2^* > 0$ such that $H(z) - C_2 z \leqslant \varepsilon z$ whenever $z \geqslant r_2^*$. From condition (H5) it holds that

$$r_2 \geqslant \frac{1}{C_1}\sup\left\{z \in [0,1) \colon \frac{H(x)}{x} > M \text{ for all } x \in (0,z]\right\} \geqslant \frac{r_1}{C_1}. \tag{3.8}$$

Note that the supremum on the right-hand side of (3.8) must be finite by virtue of condition (H2). In addition, (H5) implies that

$$r_2 \geqslant \inf\left\{z \in [0,+\infty) \colon \frac{|H(x)-C_2x|}{x} \leqslant \varepsilon \text{ for all } x \in [z,+\infty)\right\} = r_2^* \tag{3.9}$$

since we may assume without loss of generality that r_2^* is equal to the infimum in (3.9). By condition (H0) it follows that

$$\varphi(y) \geqslant \varphi_2(y) \geqslant C_0\|y\|, \tag{3.10}$$

from which it follows that, whenever $\|y\| \geqslant r_2/C_0$, it holds that $\varphi(y) \geqslant r_2 \geqslant r_2^*$. Thus, for each $y \in \partial\Omega_{r_2/C_0}$, it holds that

$$H(\varphi(y)) - C_2\varphi(y) \leqslant \varepsilon\varphi(y). \tag{3.11}$$

Moreover, note that, for each $y \in \partial\Omega_{r_2/C_0}$, it holds that

$$0 \leqslant y(t) \leqslant \frac{r_2}{C_0}, \tag{3.12}$$

whence condition (H5) implies that $f(t,y) \leqslant r_2\vartheta(t)$ for almost every $t \in [0,1]$.

Now, suppose for contradiction that there exists a number $\mu > 1$ and a function $y \in \partial\Omega_{r_2/C_0}$, such that $Ty = \mu y$. Then it holds that

$$\mu y(t) = (1-t)H(\varphi(y)) + \lambda\int_0^1 G(t,s)f(s,y(s))\,\mathrm{d}s \tag{3.13}$$

for each $t \in [0,1]$. Integrating both sides of (3.13) yields

$$\mu\varphi(y) = H(\varphi(y))\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t) + \lambda\int_0^1\int_{[0,1]}G(t,s)f(s,y(s))\,\mathrm{d}\alpha(t)\,\mathrm{d}s, \tag{3.14}$$

where Fubini's theorem has been invoked since $\alpha_1, \alpha_2, \alpha \in \mathrm{BV}([0,1])$. Using (3.11), we deduce from (3.14) that

$$\mu\varphi(y) \leqslant [C_2\varphi(y)+\varepsilon\varphi(y)]\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)+\lambda\int_0^1\int_{[0,1]}G(t,s)f(s,y(s))\,\mathrm{d}\alpha(t)\,\mathrm{d}s. \tag{3.15}$$

Finally, solving inequality (3.15) for λ and using the fact that $\varphi(y) \geqslant r_2$, we obtain

$$\begin{aligned}
\lambda &\geqslant \frac{[\mu - (C_2+\varepsilon)\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)]\varphi(y)}{\int_0^1\int_{[0,1]} G(t,s)f(s,y(s))\,\mathrm{d}\alpha(t)\,\mathrm{d}s} \\
&> \frac{[1 - (C_2+\varepsilon)\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)]\varphi(y)}{\int_0^1\int_{[0,1]} G(t,s)f(s,y(s))\,\mathrm{d}\alpha(t)\,\mathrm{d}s} \\
&\geqslant \frac{[1 - (C_2+\varepsilon)\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)]r_2}{\int_0^1\int_{[0,1]} G(t,s)r_2\vartheta(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s} \\
&= \frac{1 - (C_2+\varepsilon)\int_{[0,1]}(1-t)\,\mathrm{d}\alpha(t)}{\int_0^1\int_{[0,1]} G(t,s)\vartheta(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s},
\end{aligned} \tag{3.16}$$

but this contradicts inequality (3.7). Since, in fact, we have shown that $Ty \neq y$ whenever $y \in \partial\Omega_{r_2/C_0}$, it follows that (3.6) holds, as desired. It should be noted that in inequality (3.16) we have used the fact that, because condition (2.7) holds for each $s \in [0,1]$, we may estimate

$$\begin{aligned}
0 &< \int_0^1\int_{[0,1]} G(t,s)f(s,y(s))\,\mathrm{d}\alpha(t)\,\mathrm{d}s \\
&= \int_0^1\left[\int_{[0,1]} G(t,s)\,\mathrm{d}\alpha(t)\right]f(s,y(s))\,\mathrm{d}s \\
&\leqslant \int_0^1\left[\int_{[0,1]} G(t,s)\,\mathrm{d}\alpha(t)\right]r_2\vartheta(s)\,\mathrm{d}s.
\end{aligned}$$

In addition, note that it is not possible that

$$\int_0^1\int_{[0,1]} G(t,s)f(s,y(s))\,\mathrm{d}\alpha(t)\,\mathrm{d}s < 0$$

since condition (H4) is in force; in particular, inequality (2.7), together with the non-negativity of the function f, obviates this possibility.

Finally, we have shown that $i_{\mathcal{K}}(T, V_{r_1\gamma/C_1}) = 0$ and that $i_{\mathcal{K}}(T, \Omega_{r_2/C_0}) = 1$. Clearly, we have that $\bar{V}_{r_1\gamma/C_1} \subset \Omega_{r_1/C_1} \subset \Omega_{r_2/C_0}$ and that, in fact, $\Omega_{r_2/C_0} \setminus \bar{V}_{r_1\gamma/C_1} \neq \varnothing$. The properties of the fixed-point index then imply that

$$\begin{aligned}
1 &= i_{\mathcal{K}}(T, \Omega_{r_2/C_0}) \\
&= i_{\mathcal{K}}(T, \Omega_{r_2/C_0} \setminus \bar{V}_{r_1\gamma/C_1}) + i_{\mathcal{K}}(T, V_{r_1\gamma/C_1}) \\
&= i_{\mathcal{K}}(T, \Omega_{r_2/C_0} \setminus \bar{V}_{r_1\gamma/C_1}),
\end{aligned} \tag{3.17}$$

from which it follows that there exists $y_0 \in \Omega_{r_2/C_0} \setminus \bar{V}_{r_1\gamma/C_1}$ such that $Ty_0 = y_0$. Since this function y_0 is a non-trivial, positive solution of problem (1.1), the proof is thus complete. □

We now present the second existence theorem by using Lemma 2.3 instead of Lemma 2.1. First we need a preliminary technical lemma. Its proof is simple, but we state it for completeness since the result is very important in the proof of Theorem 3.3. Let us also note that Lemma 3.2 can be easily extended to the case in which we replace g by a continuous function $f\colon [0,1]\times[0,+\infty)\to[0,+\infty)$ satisfying $\lim_{y\to+\infty}(f(t,y)/y)=0$ uniformly for $t\in[0,1]$; it is this latter form that we shall use in Theorem 3.3 even though we prove Lemma 3.2 in the slightly simpler version below.

Lemma 3.2. *Suppose that $g\colon [0,+\infty)\to[0,+\infty)$ is a continuous function satisfying*

$$\lim_{y\to+\infty}\frac{g(y)}{y}=0.$$

Let $N\colon [0,+\infty)\to[0,+\infty)$ be the function defined by

$$N(r):=\max_{y\in[0,r]} g(y). \tag{3.18}$$

It then holds that

$$\lim_{r\to+\infty}\frac{N(r)}{r}=0. \tag{3.19}$$

Proof. Suppose for contradiction that (3.19) fails. Then there would exist a sequence of numbers $\{r_i\}_{i=1}^{\infty}\subseteq[0,+\infty)$ satisfying $r_i\to+\infty$ as $i\to+\infty$ such that

$$\frac{N(r_i)}{r_i}\geqslant\eta>0 \tag{3.20}$$

for some constant η and each $i\in\mathbb{N}$. By the definition of N, there exists a sequence $\{y_i\}_{i=1}^{\infty}\subseteq[0,+\infty)$, which satisfies $y_i\in[0,r_i]$ for each $i\in\mathbb{N}$, such that, for each $i\in\mathbb{N}$, it holds that

$$N(r_i)=g(y_i). \tag{3.21}$$

That is, g is maximal on $[0,r_i]$ at y_i. Without loss of generality we may assume that g is unbounded at $+\infty$ or else the result is trivial. Under this assumption it must hold that

$$\lim_{i\to\infty} y_i=+\infty. \tag{3.22}$$

Finally, since it is evident that $r_i\geqslant y_i$, we estimate

$$\frac{g(y_i)}{y_i}=\frac{N(r_i)}{y_i}\geqslant\frac{N(r_i)}{r_i}\geqslant\eta>0 \tag{3.23}$$

for each $i\in\mathbb{N}$. But since (3.23) implies that $\lim_{y\to+\infty}(g(y)/y)\neq 0$, we have arrived at a contradiction. This completes the proof. □

Theorem 3.3. *Suppose that conditions (H0), (H1), (H3), (H4) and (H6) hold. Furthermore, assume that the constants C_1 and C_2 from, respectively, conditions (H0) and (H3) satisfy*

$$C_1C_2<1. \tag{3.24}$$

If it holds that $H(0)\neq 0$, then problem (1.1) has at least one positive solution for each $\lambda>0$.

Proof. Define the operators $T\colon \mathcal{K} \to \mathcal{K}$ and $T'(+\infty)\colon \mathcal{K} \to \mathcal{C}([0,1])$ by

$$(Ty)(t) := (1-t)H(\varphi(y)) + \lambda \int_0^1 G(t,s) f(s, y(s))\, \mathrm{d}s \tag{3.25}$$

and

$$(T'(+\infty)y)(t) := C_2(1-t)\varphi(y), \tag{3.26}$$

respectively. Clearly, the operator $T'(+\infty)$ is linear since φ is linear. We claim first that $T'(+\infty)$ is the Fréchet derivative of T at $+\infty$. To prove this claim, we show that Definition 2.2 is satisfied. In particular, we show that $\|Ty - Ty'(+\infty)\|$ is $o(\|y\|)$ as $\|y\| \to +\infty$, that is, that $\|Ty - T'(+\infty)y\|/\|y\| \to 0$ as $\|y\| \to +\infty$.

To this end, fix a $\lambda > 0$ and let $\varepsilon > 0$ be given. Henceforth, let N be the function defined by $N(r) := \max_{(t,y)\in[0,1]\times[0,r]} f(t,y)$; this is a simple extension of the function stated in Lemma 3.2, and the lemma still applies to this slightly more general function. In any case, choose $r_0 > 0$ sufficiently large such that

$$N(r_0) \leqslant \frac{\varepsilon}{3\lambda \int_0^1 G(s,s)\, \mathrm{d}s} r_0. \tag{3.27}$$

Furthermore, by choosing r_0 even larger if necessary, in addition to (3.27) we can ensure that

$$|H(\varphi(y)) - C_2\varphi(y)| \leqslant \frac{\varepsilon}{3C_1}\varphi(y) \tag{3.28}$$

holds whenever $\|y\| \geqslant r_0$ and that

$$f(t,y) \leqslant \frac{\varepsilon}{3\lambda \int_0^1 G(s,s)\, \mathrm{d}s} y \tag{3.29}$$

holds whenever $y \geqslant r_0$, where inequality (3.29) is true for each $t \in [0,1]$. Note that (3.29) is possible due to condition (H6), whereas (3.27) follows from the conclusion of Lemma 3.2. On the other hand, inequality (3.28) holds due to condition (H0), just as in the proof of Theorem 3.1; consequently, we do not repeat the argument here. Note that, by the definition of N, it holds that

$$0 \leqslant f(t,y) \leqslant N(r_0) \tag{3.30}$$

for each $(t,y) \in [0,1] \times [0, r_0]$. For the remainder of the proof r_0 is now fixed such that (3.27)–(3.29) hold.

Now, let $y \in U(+\infty, r_0) := \{y \in \mathcal{K}\colon \|y\| \geqslant r_0\}$ be fixed but otherwise arbitrary. Define the measurable sets $\mathcal{N}$ and $[0,1] \setminus \mathcal{N}$ by

$$\mathcal{N} := \{s \in [0,1]\colon 0 \leqslant y(s) \leqslant r_0\} \tag{3.31}$$

and

$$[0,1] \setminus \mathcal{N} := \{s \in [0,1]\colon y(s) > r_0\}. \tag{3.32}$$

https://doi.org/10.1017/S0013091514000108 Published online by Cambridge University Press

Then, for any $t \in [0,1]$, we estimate

$$\begin{aligned}
&|(Ty)(t) - (T'(+\infty)y)(t)| \\
&\quad \leqslant |H(\varphi(y)) - C_2\varphi(y)| + \lambda \int_0^1 G(s,s) f(s,y(s))\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{3C_1}\varphi(y) + \lambda \int_{\mathcal{N}} G(s,s) f(s,y(s))\,\mathrm{d}s + \lambda \int_{[0,1]\setminus\mathcal{N}} G(s,s) f(s,y(s))\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{3}\|y\| + \lambda \int_{\mathcal{N}} G(s,s) N(r_0)\,\mathrm{d}s + \frac{\varepsilon\lambda}{3\lambda\int_0^1 G(s,s)\,\mathrm{d}s} \int_{[0,1]\setminus\mathcal{N}} G(s,s) y(s)\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{3}\|y\| + \lambda \int_0^1 G(s,s) N(r_0)\,\mathrm{d}s + \frac{\varepsilon\lambda}{3\lambda\int_0^1 G(s,s)\,\mathrm{d}s}\|y\| \int_0^1 G(s,s)\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{3}\|y\| + \left(N(r_0) + \frac{\varepsilon}{3\lambda\int_0^1 G(s,s)\,\mathrm{d}s}\|y\|\right)\lambda \int_0^1 G(s,s)\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{3}\|y\| + \left(\frac{\varepsilon}{3\lambda\int_0^1 G(s,s)\,\mathrm{d}s} r_0 + \frac{\varepsilon}{3\lambda\int_0^1 G(s,s)\,\mathrm{d}s}\|y\|\right)\lambda \int_0^1 G(s,s)\,\mathrm{d}s \\
&\quad \leqslant \left[\frac{\varepsilon}{3} + \frac{2\varepsilon}{3\int_0^1 G(s,s)\,\mathrm{d}s} \int_0^1 G(s,s)\,\mathrm{d}s\right]\|y\| \\
&\quad = \varepsilon\|y\|,
\end{aligned} \tag{3.33}$$

where in the second-to-last inequality we have used the fact that $\|y\| \geqslant r_0$. But then (3.33) implies that, whenever $y \in U(+\infty, r_0)$,

$$\frac{\|Ty - T'(+\infty)y\|}{\|y\|} \leqslant \varepsilon, \tag{3.34}$$

which, by the arbitrariness of $\varepsilon > 0$, implies that

$$Ty = T'(+\infty)y + o(\|y\|), \quad \|y\| \to +\infty. \tag{3.35}$$

So, by Definition 2.2, we conclude that $T'(+\infty)$ is the Fréchet derivative of T at $+\infty$, as claimed.

Now we apply the fixed-point theorem provided by Lemma 2.3. As remarked in the proof of Theorem 3.1, it is standard to argue that $T\colon \mathcal{K} \to \mathcal{K}$ is a compact operator on $\mathcal{K}$. Consequently, we shall show, in particular, that the spectral radius of the Fréchet derivative $T'(+\infty)$, namely $r(T'(+\infty))$, is less than unity. In fact, we shall show that this operator cannot possess an eigenvalue greater than or equal to unity.

Indeed, suppose for contradiction that there exists an eigenvalue $\mu \in [1, +\infty)$ and an associated eigenvector $y \in \mathcal{K}$ such that $T'(+\infty)y = \mu y$. Since y is an eigenvector, it satisfies $\|y\| > 0$. For each $t \in [0,1]$ we thus have the equality

$$\mu y(t) = C_2(1-t)\varphi(y). \tag{3.36}$$

Let $t_0 \in [0,1]$ be a point at which y attains its maximum, i.e. $y(t_0) = \|y\|$. Suppose that $t_0 \neq 1$. Then, by condition (H0), we estimate from (3.36) that

$$0 < \|y\| \leqslant \mu\|y\| = \mu y(t_0) = C_2(1-t_0)\varphi(y) \leqslant C_1C_2(1-t_0)\|y\|, \tag{3.37}$$

from which it follows that

$$1 \leqslant C_1C_2(1-t_0), \tag{3.38}$$

since $\|y\| \neq 0$. But then, since (3.38) implies that

$$C_1C_2 \geqslant 1, \tag{3.39}$$

we obtain a contradiction to (3.24). On the other hand, if $t_0 = 1$, then (3.37) implies that $\|y\| \leqslant 0$, which contradicts the strict positivity of $\|y\|$. Thus, in either case we obtain a contradiction and so it holds that the operator $T'(+\infty)$ has no eigenvalue greater than or equal to unity. Hence, Lemma 2.3 implies the existence of a function $y_0 \in \mathcal{K}$ such that $Ty_0 = y_0$.

Finally, it cannot hold that y_0 is the zero element of $\mathcal{K}$. Indeed, if $y_0 \equiv 0$ then, since y_0 satisfies problem (1.1), it must hold that $0 = y(0) = H(\varphi(y)) = H(0)$, contradicting the fact that $H(0) > 0$. Consequently, we conclude that $\|y_0\| > 0$ and so y_0 is a non-trivial positive solution of BVP (1.1). And this completes the proof. □

Having presented two different existence theorems for problem (1.1), we would like to point out that it is easy to extend our results to the more general setting of the perturbed Hammerstein integral equation

$$y(t) = \xi_1(t)H_1(\varphi_1(y)) + \xi_2(t)H_2(\varphi_2(y)) + \int_0^1 K(t,s)f(s,y(s))\,\mathrm{d}s,$$

where ξ_1, $\xi_2 \colon [0,1] \to [0,+\infty)$ and $K\colon [0,1]\times[0,1] \to [0,+\infty)$ satisfy, roughly speaking, the same general properties as the maps $t \mapsto 1-t$ and $(t,s) \mapsto G(t,s)$ above, respectively. This generalization permits us to equip the differential equation appearing in (1.1) with a variety of nonlinear, non-local boundary conditions. Indeed, by suitably choosing ξ_1 and ξ_2 above we could consider the boundary conditions $y'(0) = H_1(\varphi_1(y))$, $y(1) = H_2(\varphi_2(y))$, for instance. But we do not state the multitude of individual results that thus follow.

We conclude with some discussion regarding these results, as well as two numerical examples in order to clarify the application of the results.

Remark 3.4. Regarding Theorem 3.1 we point out that this result provides improvements to the results presented recently by Goodrich [**10**], as already intimated in § 1. In particular, Theorem 3.1 gives conditions under which a non-trivial positive solution of (1.1) may be guaranteed for $\lambda > 0$ small, which contrasts with the results of [**10**]. Moreover, the upper bound on λ is easy to compute. Finally, it is worth noting that Theorem 3.1 does not require any growth assumption on f at $+\infty$ and, furthermore, the growth assumption we do impose (i.e. (H5)) is weak. This also distinguishes this result from those presented in [**10**].

Remark 3.5. Regarding Theorem 3.3, we point out that a direct comparison of this result to the results given recently by Goodrich [**10**] indicates that we need not make any growth assumptions on the map $(t, y) \mapsto f(t, y)$ for y small. This is a direct improvement of [**10**, Theorem 3.7]. In fact, it is also a direct improvement of [**12**, Theorem 3.4] in the special case where the nonlinear non-locality is realized in the form $H \circ \varphi$. Finally, as mentioned in § 1, previous results of other authors are not applicable in this setting due to the fact that the integrator α in (1.2) does not have to be monotone here. The same, of course, may be said about Theorem 3.1.

Remark 3.6. Observe that for the case in which $C_2 = 0$, as will be the case if H is sublinear at $+\infty$, it follows that condition (3.24) of Theorem 3.3 is trivially satisfied. Moreover, in this case Theorem 3.3 guarantees existence even if C_1 is very large, for (H4) need not hold in Theorem 3.3. This is rather more general than the corresponding results of [**10**], wherein there is always a restriction on the size of C_1. In fact, Theorem 3.3 shows that this is unnecessary, even under very weak growth assumptions on f and H. Once again, previous results of other authors are not applicable in this setting due to the fact that the integrator α in (1.2) does not have to be monotone here.

Remark 3.7. It is possible to allow for $H(z) < 0$ for some $z \geqslant 0$; see [**6**] for how this might be pursued.

Remark 3.8. It is worth noting that our results reveal that it is possible for H to be linear at $+\infty$, whereas f is sublinear (see [**11**, Remark 3.3]). In particular, these functions need not have the same asymptotic behaviour at $+\infty$.

Example 3.9. Let us consider the functional

$$\varphi(y) := \underbrace{\tfrac{2}{5}y(\tfrac{1}{2}) - \tfrac{1}{10}y(\tfrac{1}{3})}_{:=\varphi_1(y)} + \underbrace{\tfrac{1}{20}y(\tfrac{2}{3})}_{:=\varphi_2(y)}. \tag{3.40}$$

Let us also set $[a, b] := [\frac{1}{4}, \frac{3}{4}]$ so that $\gamma = \frac{1}{4}$. In particular, here we may select $C_0 := \frac{1}{20}\gamma = \frac{1}{80}$ and $C_1 := \frac{11}{20}$. Note that C_0 may be so selected since, due to the cone property, we calculate

$$\tfrac{1}{20}y(\tfrac{2}{3}) \geqslant \tfrac{1}{20}\gamma\|y\| = \tfrac{1}{80}\|y\|,$$

since $2/3 \in [a, b]$. In any case, straightforward computations indicate that each of conditions (H0), (H1), (H4) and (H5) are satisfied. It is also easy to check that condition (3.2) holds.

On the other hand, suppose that $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(z) := z^{1/3} + z. \tag{3.42}$$

Note that $H([0, +\infty)) \subseteq [0, +\infty)$. Furthermore, it holds that both

$$\lim_{z \to 0^+} \frac{H(z)}{z} = +\infty \quad \text{and} \quad \lim_{z \to +\infty} \frac{|H(z) - z|}{z} = 0. \tag{3.43}$$

Thus, (3.43) implies that we may select M as large as we like, in particular, sufficiently large so that $M > 1/\gamma C_0$. Furthermore, (3.43) demonstrates that (2.5) holds with $C_2 = 1$, say.

Finally, suppose that we select $\varepsilon = 5$ in (2.8), which is easily checked and seen to be admissible. If we assume, for the sake of simplicity, that $\vartheta \equiv \vartheta_0 \in (0, +\infty)$, then we calculate that

$$0 < \lambda < \frac{2.571}{\vartheta_0} \tag{3.44}$$

must hold to three decimal places of accuracy. Furthermore, it must hold both that

$$r_2 \geqslant \inf_{z\in[0,+\infty)} \{x^{-2/3} \leqslant 5 \text{ for all } x \in [z, +\infty)\} \tag{3.45}$$

and that

$$r_2 \geqslant \frac{1}{C_1} \sup_{z\in[0,1)} \left\{\frac{H(x)}{x} > 20 \text{ for all } x \in (0, z]\right\}, \tag{3.46}$$

say, where we have chosen the minimal admissible value of M. An easy calculation shows that (3.45) and (3.46) imply that r_2 must satisfy the inequality $r_2 \geqslant 0.089$, again to three decimal places of accuracy.

For example, we can conclude from Theorem 3.1 that problem (1.1) has at least one positive solution provided that λ satisfies (3.44) and $f(t, y) \leqslant 0.089\vartheta_0$ for $(t, y) \in [0, 1] \times [0, 7.152]$. In particular, if, say, $\vartheta_0 = 100$, then problem (1.1) has at least one positive solution for each

$$0 < \lambda < 0.0257$$

with $f(t, y) \leqslant 8.9$ for $(t, y) \in [0, 1] \times [0, 7.152]$.

Example 3.10. Let us consider the functional

$$\varphi(y) := \tfrac{1}{3}y(\tfrac{2}{5}) - \tfrac{1}{5}y(\tfrac{1}{2}), \tag{3.47}$$

which was mentioned in § 1 (see (1.5)). Let us suppose, once again, that $[a, b] = [\frac{1}{4}, \frac{3}{4}]$. This functional is not initially written in an admissible decomposition form. However, as indicated in § 1, if we instead write

$$\varphi(y) = \underbrace{\tfrac{99}{300}y(\tfrac{2}{5}) - \tfrac{1}{5}y(\tfrac{1}{2})}_{:=\varphi_1(y)} + \underbrace{\tfrac{1}{300}y(\tfrac{2}{5})}_{:=\varphi_2(y)}, \tag{3.48}$$

then we can now show that this functional satisfies the required structure conditions. Indeed, it is easy to check that each of conditions (H0), (H1) and (H5) is satisfied; here we select $C_0 = \frac{1}{300}\gamma = \frac{1}{200}$ and $C_1 = \frac{8}{15}$.

Define $H : \mathbb{R} \to [0, +\infty)$ by

$$H(z) := \frac{1}{1 + \mathrm{e}^{-z}}. \tag{3.49}$$

Then $H(0) > 0$ and condition (2.5) is satisfied with $C_2 = 0$. Consequently, inequality (3.24) is trivially satisfied. Hence, for any function f that satisfies (H6), we conclude by Theorem 3.3 that problem (1.1) has at least one positive solution for all $\lambda > 0$.

Acknowledgements. The author thanks Dr Lynn Erbe for helpful discussions during the preparation of this manuscript.

References

1. A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, *Bound. Value Probl.* **2011** (2011), 893753.
2. C. S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, *Computers Math. Applic.* **61** (2011), 191–202.
3. C. S. Goodrich, Existence of a positive solution to systems of differential equations of fractional order, *Computers Math. Applic.* **62** (2011), 1251–1268.
4. C. S. Goodrich, Existence of a positive solution to a first-order p-Laplacian BVP on a time scale, *Nonlin. Analysis* **74** (2011), 1926–1936.
5. C. S. Goodrich, The existence of a positive solution to a second-order p-Laplacian BVP on a time scale, *Appl. Math. Lett.* **25** (2012), 157–162.
6. C. S. Goodrich, Positive solutions to boundary value problems with nonlinear boundary conditions, *Nonlin. Analysis* **75** (2012), 417–432.
7. C. S. Goodrich, On a discrete fractional three-point boundary value problem, *J. Diff. Eqns Applic.* **18** (2012), 397–415.
8. C. S. Goodrich, Nonlocal systems of BVPs with asymptotically superlinear boundary conditions, *Commentat. Math. Univ. Carolinae* **53** (2012), 79–97.
9. C. S. Goodrich, Nonlocal systems of BVPs with asymptotically sublinear boundary conditions, *Appl. Analysis Disc. Math.* **6** (2012), 174–193.
10. C. S. Goodrich, On nonlocal BVPs with boundary conditions with asymptotically sublinear or superlinear growth, *Math. Nachr.* **285** (2012), 1404–1421.
11. C. S. Goodrich, On a first-order semipositone discrete fractional boundary value problem, *Arch. Math.* **99** (2012), 509–518.
12. C. S. Goodrich, On nonlinear boundary conditions satisfying certain asymptotic behavior, *Nonlin. Analysis* **76** (2013), 58–67.
13. C. S. Goodrich, On a nonlocal BVP with nonlinear boundary conditions, *Results Math.* **63** (2013), 1351–1364.
14. C. S. Goodrich, Positive solutions to differential inclusions with nonlocal, nonlinear boundary conditions, *Appl. Math. Computat.* **219** (2013), 11071–11081.
15. J. Graef and J. R. L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlin. Analysis* **71** (2009), 1542–1551.
16. G. Infante, Positive solutions of some nonlinear BVPs involving singularities and integral BCs, *Discrete Contin. Dynam. Syst.* S **1** (2008), 99–106.
17. G. Infante, Nonlocal boundary value problems with two nonlinear boundary conditions, *Commun. Appl. Analysis* **12** (2008), 279–288.
18. G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlin. Analysis* **71** (2009), 1301–1310.
19. G. Infante and P. Pietramala, Eigenvalues and non-negative solutions of a system with nonlocal BCs, *Nonlin. Stud.* **16** (2009), 187–196.
20. G. Infante and P. Pietramala, Perturbed Hammerstein integral inclusions with solutions that change sign, *Commentat. Math. Univ. Carolinae* **50** (2009), 591–605.
21. G. Infante and P. Pietramala, A third order boundary value problem subject to nonlinear boundary conditions, *Math. Bohem.* **135** (2010), 113–121.
22. G. Infante and P. Pietramala, Multiple non-negative solutions of systems with coupled nonlinear BCs, *Math. Meth. Appl. Sci.* (2013), DOI:10.1002/mma.2957.
23. G. Infante and J. R. L. Webb, Nonlinear non-local boundary-value problems and perturbed Hammerstein integral equations, *Proc. Edinb. Math. Soc.* **49** (2006), 637–656.
24. G. Infante, F. Minhós and P. Pietramala, Non-negative solutions of systems of ODEs with coupled boundary conditions, *Commun. Nonlin. Sci. Numer. Simulation* **17** (2012), 4952–4960.

25. T. Jankowski, Existence of positive solutions to third order differential equations with advanced arguments and nonlocal boundary conditions, *Nonlin. Analysis* **75** (2012), 913–923.
26. P. Kang and Z. Wei, Three positive solutions of singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential equations, *Nonlin. Analysis* **70** (2009), 444–451.
27. W. G. Kelley and A. C. Peterson, *The theory of differential equations: classical and qualitative* (Prentice Hall, Englewood Cliffs, NJ, 2004).
28. K. Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. Lond. Math. Soc. (2)* **63** (2001), 690–704.
29. P. Pietramala, A note on a beam equation with nonlinear boundary conditions, *Bound. Value Probl.* **2011** (2011), 376782.
30. J. R. L. Webb, Nonlocal conjugate type boundary value problems of higher order, *Nonlin. Analysis* **71** (2009), 1933–1940.
31. J. R. L. Webb, Solutions of nonlinear equations in cones and positive linear operators, *J. Lond. Math. Soc. (2)* **82** (2010), 420–436.
32. J. R. L. Webb, Remarks on a non-local boundary value problem, *Nonlin. Analysis* **72** (2010), 1075–1077.
33. J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. Lond. Math. Soc. (2)* **74** (2006), 673–693.
34. J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *Nonlin. Diff. Eqns Applic.* **15** (2008), 45–67.
35. J. R. L. Webb and G. Infante, Non-local boundary value problems of arbitrary order, *J. Lond. Math. Soc. (2)* **79** (2009), 238–258.
36. Z. Yang, Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlin. Analysis* **62** (2005), 1251–1265.
37. Z. Yang, Positive solutions of a second-order integral boundary value problem, *J. Math. Analysis Applic.* **321** (2006), 751–765.
38. Z. Yang, Existence and nonexistence results for positive solutions of an integral boundary value problem, *Nonlin. Analysis* **65** (2006), 1489–1511.
39. Z. Yang, Existence of nontrivial solutions for a nonlinear Sturm–Liouville problem with integral boundary conditions, *Nonlin. Analysis* **68** (2008), 216–225.
40. E. Zeidler, *Nonlinear functional analysis and its applications, I, Fixed-point theorems* (Springer, 1986).