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Lack of balance in continuously stratified rotating flows

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Periodic linear waves in a vertically sheared flow are considered in a continuously stratified layer of rotating fluid between homogeneous layers along a sloping bottom. This generalized Phillips' configuration has cyclonic horizontal shear and supports the Rossby modes related to the thickness variations of the homogeneous layers and inertia–gravity waves (IGW). While long Rossby modes with streamwise wavenumber $\kappa < f/V$ (f is the Coriolis parameter, V is the maximum velocity) can be approximated by a neutral balanced solution, short waves with $\kappa > f/V$ are found to have an inertial critical level and unbalanced gravity-wave-like structure beyond this level. Such ageostrophic unstable normal modes are shown explicitly to couple short Rossby waves with Doppler-shifted gravity waves. They exist even for small Froude number, although the growth rate of ageostrophic unstable modes is exponentially small in Froude number as in the Eady model. This lack of balance in continuously stratified flows agrees with the ultraviolet problem for Ripa's sufficient conditions of stability in a multi-layer model when the number of layers tends to infinity.

1. Introduction

Dynamics of the oceans and the planetary atmospheres are constrained by rapid rotation and strong density stratification. Their effects can be characterized by the Rossby number $Ro = |\zeta|/f$ ($\zeta = \partial_x V - \partial_y U$ is the vertical vorticity, related to the velocity shear along the horizontal coordinates (x, y), and f is the Coriolis parameter), and the Froude number $F = \Lambda/N$ ($\Lambda = \max(|\partial_z U|, |\partial_z V|)$ is the horizontal vorticity, related to the velocity shear along the vertical coordinate z, and N is the buoyancy frequency).† When both of them are small, slow motion related to the potential vorticity advection, and fast motion related to inertia–gravity waves (IGW) are in practice decoupled (Zeitlin, Reznik & Ben Jelloul 2003). Thus, the most important synoptic variability can be successfully described using so-called balanced models (close to hydrostatic and gradient balance) filtering out fast IGW (e.g. see Sutyrin 2004 and references therein).

Ultimate limitations on balance dynamics are often related to instabilities arising in the vicinity of the breakdown of certain ellipticity conditions necessary for the solvability of balance relations. These unbalanced instabilities may have an important role in providing dynamical routes to dissipation at small scales (McWilliams 2003). Unbalanced instabilities have been found by solving the eigenvalue problems for

† The Froude number is sometimes called the shear number; it is related to the Richardson number, $Ri = F^{-2}$. Owing to thermal-wind balance, $F = sN/f$, where s is the slope of isopycnal surfaces.

a variety of basic flows with either horizontal or vertical shear; their growth rate is typically exponentially small in the Rossby number, so they cannot be captured asymptotically using power-series expansions in Ro (see Vanneste & Yavneh 2007 and references therein).

A new approach for analysing unbalanced instabilities in multi-layer flows along sloping topography has been developed recently by assuming the variations of the layer thicknesses are small (Sutyrin 2007, hereinafter S07). The dispersion curves for the Rossby modes due to the layer thickness variations and the Poincaré modes of IGW were investigated to identify different types of instabilities. They occur if there is a pair of wave components which have almost the same Doppler-shifted frequency related to crossover of the branches when the Froude number is increased. Simple analytic criteria were derived for ageostrophic instabilities related to a resonance between the IGW and the Rossby modes. These criteria correspond exactly to violation of Ripa's sufficient conditions for the flow stability in the two-active-layer model.

When the number of layers increases, in the continuously stratified limit, Ripa (1991) concluded that there are no conditions on the basic flow that could ensure that the wave energy or momentum of an arbitrary perturbation be positive definite. This 'ultraviolet' problem did not allow the general formal stability of continuously stratified flow to be proved, as short enough disturbances can violate sufficient conditions. Calculation of the linear normal modes of a continuously stratified vertically sheared basic flow in the case of the horizontally homogeneous Eady problem ($V = \Lambda z$) has demonstrated that unbalanced instability is related to spatial coupling between the Rossby and gravity modes in the vicinity of inertial critical layers (Nakamura 1988; Plougonven, Muraki & Snyder 2005; Molemaker, McWilliams & Yavneh 2005).

Note that, due to thermal-wind balance, in the Eady problem the buoyancy $B = N^2 z + f\Lambda x$, i.e. the velocity varies along isopycnal surfaces: $V = \Lambda(B - f\Lambda x)/N^2$. Being spatially uniform, the Ertel potential vorticity depends on the Froude number

$$\Pi = fN^2 - \partial_z V \partial_x B = fN^2(1 - F^2).$$

In the Eady configuration, the normal modes are described by the third-order equation for the vertical velocity, which is not easy to analyse. Although horizontally uniform, such baroclinic shear flow has anticyclonic (negative) isopycnal vorticity. Thus, Π becomes negative when $F > 1$, while the isopycnal balance equations break down for $F > 1/\sqrt{2}$ due to loss of convexity of the streamfunction for anticyclonically sheared flow (Molemaker *et al.* 2005).

To consider a stratified flow in the limit of an infinite number of layers, we modify the Eady problem by adding a cyclonic (positive) horizontal shear to the basic velocity profile: $V(x, z) = \Lambda z + Ax$ and choosing $A = f\Lambda^2/N^2$ to compensate the velocity change along isopycnal surfaces, so that $V = \Lambda B/N^2$. In this case $\Pi \equiv fN^2$ does not depend on the Froude number, while $Ro = F^2$. This basic flow can be prescribed between two homogeneous layers, representing a generalized version of the Phillips' model. In the stratified layer this basic flow has cyclonic horizontal shear corresponding to positive potential vorticity, so that isopycnal balance equations do not break down for any Froude number, in contrast to the Eady model.

How could the balance dynamics be violated in this case? Assuming, again, that variations of the thickness of homogeneous layers are small, we can find explicit solutions for the unstable normal modes with inertial critical layers and demonstrate how lack of balance is related to Ripa's 'ultraviolet' problem. The remainder of this

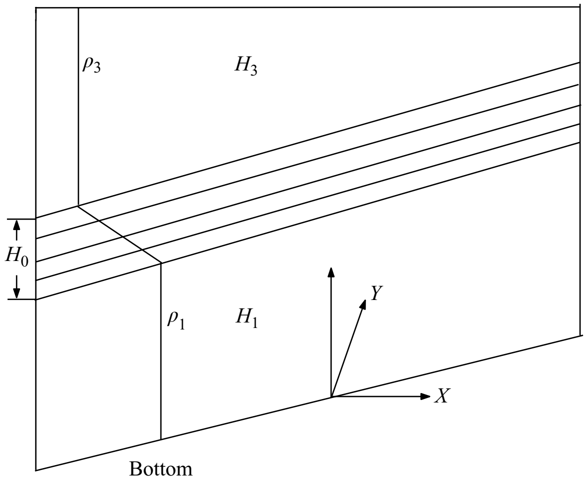

FIGURE 1. The baroclinic flow configuration along a sloping bottom in a channel.

paper is organized as follows. In § 2, basic equations for a continuously stratified isopycnal model are introduced. In § 3 they are linearized for a given basic state. In § 4, the solution for small variations of the layer thickness is derived. In § 5 the dispersion relation for the geostrophic baroclinic Rossby modes is obtained. In § 6, the ageostrophic instability is analysed due to a resonance between the Rossby mode and IGW. In § 7, the results are summarized and discussed.

2. Isopycnal model

Let us consider a continuously stratified layer of thickness H_0 between two homogeneous layers with densities ρ_j, depths H_j, pressure field P_j, and velocity vector $\boldsymbol{V}_j = (U_j, V_j)$, where $j = 1$, $j = 2$ and $j = 3$ represent variables in the lower, stratified middle and upper layer, respectively (see figure 1). The right-hand coordinate system corresponds to the depth topography $D(X)$, with the X-axis directed onshore, and the Y-axis parallel to the isobaths; t is time. In the middle layer all variables depend also on the normalized isopycnal coordinate $\eta = (\rho_2 - \rho_3)/(\rho_1 - \rho_3)$.

First, we non-dimensionalize the variables as follows:

$$(\hat{X}, \hat{Y}) = (X, Y) f/N H_0, \qquad \hat{t} = tf, \tag{2.1}$$

$$(\hat{U}_j, \hat{V}_j) = (U_j, V_j)/N H_0, \qquad \hat{H}_j = H_j/H_0, \qquad \hat{P}_j = P_j/\rho_1 N^2 H_0^2. \tag{2.2}$$

In the Boussinesq, hydrostatic, and f-plane approximations the non-dimensional momentum and continuity equations are

$$\partial_t \boldsymbol{V}_j + (\boldsymbol{V}_j \cdot \nabla)\boldsymbol{V}_j + \boldsymbol{k} \times \boldsymbol{V}_j = -\nabla P_j, \tag{2.3}$$

$$\partial_t H_j + \nabla \cdot (H_j \boldsymbol{V}_j) = 0, \tag{2.4}$$

where ∇ is the horizontal gradient operator, $\boldsymbol{k}$ is the vertical unit vector, and the hats for the non-dimensional variables are dropped. Here we assume $\rho_1 - \rho_3 \ll \rho_1$. In the stratified middle layer $H_2 = \partial_\eta \varXi$, where $\varXi$ is the depth of isopycnal surfaces normalized by H_0, so that the conditions at the boundaries of the middle layer are

$$\varXi(X, Y, 0, t) = H_3, \qquad P_2(X, Y, 0, t) = P_3, \tag{2.5}$$

$$\varXi(X, Y, 1, t) + H_1 + D(X) = \text{const}, \qquad P_2(X, Y, 1, t) = P_1, \tag{2.6}$$

and inside the middle layer the hydrostatic approximation gives $\nabla\partial_\eta P_2 = -\nabla\Xi$, so that

$$\nabla(P_1 - P_3) = -\nabla H_3 + \int_0^1 (\eta - 1)\nabla H_2 \,\mathrm{d}\eta = \nabla(H_1 + D(X)) + \int_0^1 \eta\nabla H_2 \,\mathrm{d}\eta, \qquad (2.7)$$

taking into account that $H_3 + \int_0^1 H_2 \,\mathrm{d}\eta + H_1 + D(X) = \text{const}$.

3. The basic state and linearization

We consider the basic state with uniform flows in the upper and lower layers:

$$U_j = 0, \qquad V_j = \frac{\mathrm{d}P_j}{\mathrm{d}X}, \qquad H_j = \overline{H}_j + S_j X, \qquad \Xi = H_3 + \eta H_0, \qquad (3.1)$$

$$V_1 = V_3 + F, \qquad V_2 = V_3 + \eta F, \qquad S_1 = F - S, \qquad S_2 = 0, \qquad S_3 = -F = \frac{\mathrm{d}\Xi}{\mathrm{d}X}, \qquad (3.2)$$

where S is the topographic slope normalized by f/N. We assume $V_1 > V_3$ and choose coordinates moving with the upper layer velocity V_3.

The linear stability of this flow is addressed by adding infinitesimal disturbances of the form $(\mathrm{i}u_j(X, \eta), v_j(X, \eta), p_j(X, \eta)) \exp(\mathrm{i}kY - \mathrm{i}\omega t)$ and linearizing. Here k is the streamwise wavenumber and ω is the disturbance frequency (a positive imaginary part implying instability). The linearized equations (2.3)–(2.6) can be written in the form

$$\sigma_j u_j - v_j + p_j' = 0, \qquad u_j - \sigma_j v_j + k p_j = 0, \qquad (3.3)$$

$$-\sigma_1 \xi(X, 1) + (H_1 u_1)' + k H_1 v_1 = 0, \qquad (3.4)$$

$$\sigma_2 \partial_\eta \xi + u_2' + k v_2 = 0, \qquad \xi = -\partial_\eta p_2 \qquad (3.5)$$

$$\sigma_3 \xi(X, 0) + (H_3 u_3)' + k H_3 v_3 = 0, \qquad (3.6)$$

where intrinsic frequencies $\sigma_1 = \omega - kF$, $\sigma_2 = \omega - \eta kF$, $\sigma_3 = \omega$, and a prime denotes $\mathrm{d}/\mathrm{d}X$. The system (3.3)–(3.6) is an eigenvalue problem to be solved with the boundary conditions $u_j = 0$ at $X = \pm L$ to obtain a set of normal modes which are orthogonal to each other in an appropriate norm. From (3.3) we can express (u_j, v_j) via p_j as

$$f_j u_j = \sigma_j p_j' - k p_j, \qquad f_j v_j = p_j' - \sigma_j k p_j, \qquad f_j = 1 - \sigma_j^2. \qquad (3.7)$$

Then from (3.4)–(3.6) we obtain the system for $p = p_2(X, \eta)$ and $\xi(X, \eta)$:

$$-f_1 \xi + (\overline{H}_1 + S_1 X)(k^2 p - p'') - S_1 p' = -\frac{k S_1 p}{\omega - kF} \quad \text{for} \quad \eta = 1, \qquad (3.8)$$

$$f_2 \partial_\eta \xi + k^2 p - p'' = 0, \qquad \xi = -\partial_\eta p, \qquad (3.9)$$

$$f_3 \xi + (\overline{H}_3 + S_3 X)(k^2 p - p'') - S_3 p' = -\frac{k S_3 p}{\omega} \quad \text{for} \quad \eta = 0, \qquad (3.10)$$

with the boundary conditions $(\omega - \eta kF)p' = kp$ at $X = \pm L$. The system (3.8)–(3.10) describes an infinite set of IGW and two topographic Rossby modes depending on the Froude number, F, the layer thicknesses $\overline{H}_1$, $\overline{H}_3$, the channel width, L, and the topographic slope S. Generally, it is a non-separable problem where eigenvalues $\omega(k)$ and eigenfunctions can be found numerically by discretizing the X- and η-intervals.

4. The solution for small thickness variations

In order to analyse different modes and their resonances explicitly, we further assume the changes in thickness of homogeneous layers to be small $|S_j|L \ll \overline{H}_j$. Then we seek a solution in the form of periodic Poincaré modes $\xi = W\exp(\mathrm{i}\alpha X)$ + c.c. (where $W(\eta)$ is a complex function, α is the cross-stream wavenumber, and c.c. means complex conjugate), neglecting small terms proportional to S_j on the left-hand sides of (3.8) and (3.10). (Here we exclude from consideration the boundary-trapped Kelvin modes of IGW analysed by Sakai 1989). From (3.8)–(3.10) for W we obtain

$$W = -\left(\overline{H}_1 + \frac{kS_1}{(\omega - kF)K^2}\right)\frac{\mathrm{d}W}{\mathrm{d}\eta} \qquad \text{for} \quad \eta = 1, \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_2\frac{\mathrm{d}W}{\mathrm{d}\eta}\right) = K^2W, \qquad K^2 = k^2 + \alpha^2. \tag{4.2}$$

$$W = \left(\overline{H}_3 + \frac{kS_3}{\omega K^2}\right)\frac{\mathrm{d}W}{\mathrm{d}\eta} \qquad \text{for} \quad \eta = 0. \tag{4.3}$$

In compare to the third-order equation for vertical velocity in the Eady problem, the second-order equation (4.2) is easily reduced to the hypergeometric equation

$$Z(1-Z)\frac{\mathrm{d}^2W}{\mathrm{d}Z^2} + (c - (1+a+b)Z)\frac{\mathrm{d}W}{\mathrm{d}Z} = abW, \qquad 2Z = 1 - \omega + \eta kF, \tag{4.4}$$

where $c = 1$, $a + b = 1$, and $ab = K^2/F^2k^2$. The two linear independent solutions for this case are given by (15.5.16)–(15.5.17) in Abramowitz & Stegan (1964). Using these solutions together with (4.1) and (4.3) allows eigenvalues $\omega(k, \alpha)$ for normal modes, to be found.

5. The baroclinic Rossby modes

Owing to small variations of the layer thicknesses, the Rossby wave branches can be found using the quasi-geostrophic approximation assuming $f_2 \approx 1$ to filter out the IGW when $kF \ll 1$. Then the solution to (4.2) has the form

$$W = \cosh(K\eta) + \chi\sinh(K\eta), \qquad \chi = \frac{CK}{CK^2\overline{H}_3 - F}, \tag{5.1}$$

where the complex phase speed $C = \omega/\kappa$ is defined by a quadratic equation similar to (38) in S07

$$\frac{C(C-F)}{S_1F}(Q_1Q_3 + \mu Q_1 + \mu Q_3 + K^2) + \frac{C}{F}(\mu + Q_3) - \frac{C-F}{S_1}(\mu + Q_1) = 1. \tag{5.2}$$

Here S_3 is replaced by $-F$, $Q_j \equiv K^2\overline{H}_j$ and $\mu = K\coth(K) \to 1$ when $K^2 \to 0$ for fixed Q_j that corresponds to a two-layer limit. When the lower layer thickness increases onshore, $S_1 > 0$, the growing mode of geostrophic baroclinic instability may exist for some range of wavenumbers due to a resonance between the Rossby modes related to the thickness gradient in the lower and upper layers. In particular, this regime for continuously stratified flow above a thin homogeneous layer over a topographic slope was studied by Poulin & Swaters (1999). When $S_1 < 0$, these geostrophic Rossby modes are neutral for all wavenumbers.

However, for large vertical shear when $F \gg |S_1|$, the Rossby modes are always decoupled because their Doppler-shifted phase speeds differ too much. The phase speed C_1 of the first mode (related to the thickness gradient in the lower layer) and C_2 for the second mode (related to the thickness gradient in the upper layer) can be approximated from (5.2) as:

$$C_1 - F \approx -\frac{S_1(Q_3 + \mu - 1)}{Q_3(Q_1 + \mu) + Q_1(\mu - 1) + K^2 - \mu}, \tag{5.3}$$

$$C_2 \approx \frac{(Q_1 + \mu)}{Q_3(Q_1 + \mu) + \mu Q_1 + K^2}. \tag{5.4}$$

A similar conclusion was obtained for the two-layer model (see figure 2 in S07).

6. Ageostrophic instability and lack of balance

Owing to small variations of the layer thicknesses in our setup, even for finite Froude number the intrinsic speed of both Rossby modes remains small: $|C_1 - F|$ is small because $|S_1| \ll 1$, while C_2 is small because of the thick upper layer $H_3 \gg 1$. Further, we consider only resonance between the first Rossby mode and IGW modes, denoting $C_1 = F + S_1\lambda$, where the renormalized phase speed λ is expressed from the condition (4.1) as

$$\lambda = -\frac{W'}{Q_1 W' + K^2 W}, \qquad W' \equiv \frac{\mathrm{d}W}{\mathrm{d}\eta} \qquad \text{at} \quad \eta = 1. \tag{6.1}$$

In a similar way, the resonance between IGW and the second Rossby mode can be considered by renormalizing C_2 by $1/\overline{H}_3 \ll 1$.

Using an asymptotic expansion in S_1 and $1/\overline{H}_3$, at the leading order we set $f_2 = 1 - k^2F^2(1-\eta)^2$, and simplify the condition (4.3) $W' = 0$ at $\eta = 0$. For long Rossby waves when $k < 1/F$, we see that $f_2 > 0$, and this solution decays monotonically with height above the lower layer similar to the QG solution (5.1), so it can be considered balanced.

For shorter waves when $k > 1/F$, the coefficient f_2 changes sign at the inertial critical isopycnal surface $\eta_c = 1 - 1/kF$. At this critical surface the solution for the isopycnal displacement $W(\eta)$ has a logarifmic singularity given by a hypergeometric solution which is similar to one found for the vertical velocity at the inertial critical level in the Eady problem (Plougonven *et al.* 2005). And similar to the Eady problem, above this critical surface the solution for our generalized Phillips' problem behaves as an unbalanced gravity wave (figure 2). Matching the solution at the critical isopycnal surface gives the complex value of λ: its real part gives the phase speed, while the imaginary part becomes non-zero for short waves $k > 1/F$ (figure 3). This solution describes an unstable coupled Rossby–IGW mode with the growth rate $kS_1\lambda_i$ which is proportional to S_1, unlike to $\sqrt{S_1}$ found for ageostrophic Rossby–Kelvin instability in the two active layer model (S07). The normalized growth rate $\sigma_m = \max(k\lambda_i)$ reaches maximum at $k_m = 1/F + \Delta k_m$ as shown in figure 3. When F is small, $k_m > 1/F$ is large, so that K is also large and the isopycnal displacement decays exponentially with η as seen from (5.1), thus the IGW amplitude at the inertial critical surface becomes exponentially small in F, similar to what was found for the Eady problem.

7. Discussion and conclusions

A generalized Phillips' model over sloping topography with small variations of the layer thickness allows the existence of coupling between IGW and Rossby waves to be demonstated, i.e. lack of balance, for short enough periodic perturbations ($k > 1/F$ or

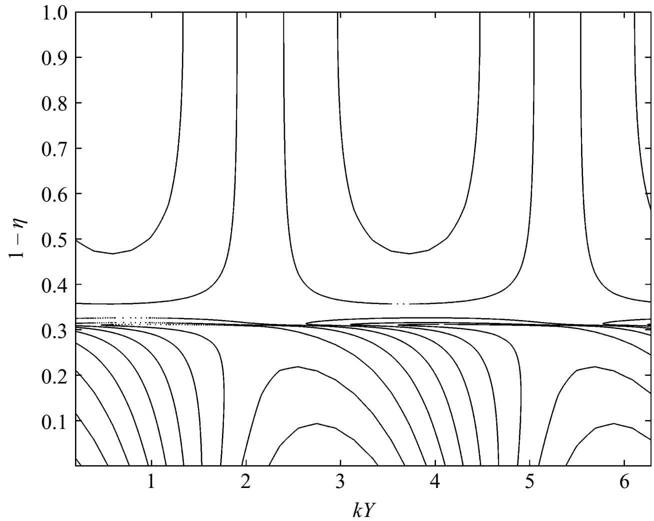

FIGURE 2. Spatial structure of the solution for isopycnal displacement, ξ, in the stratified layer for the fastest growing mode with $k=3.2$, $F=1$, $\alpha=0$, $\overline{H}_1=1$, and $S_1=0.1$. Sharp changes occur in the vicinity of the critical surface $1-\eta_c=1/kF=0.31$.

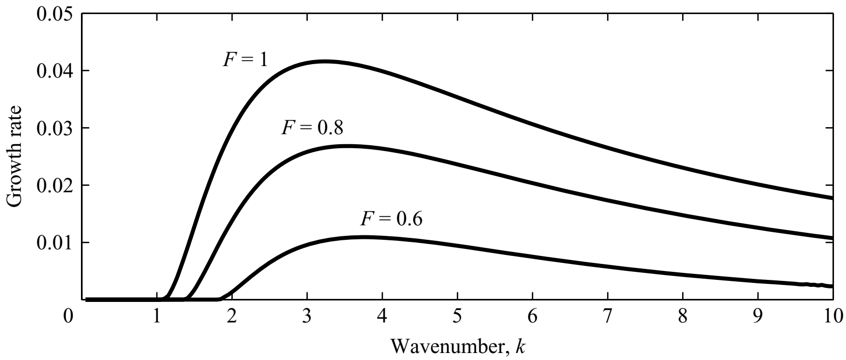

FIGURE 3. The normalized growth rate $k\lambda_i$ of the unstable modes for $\alpha=0$, $\overline{H}_1=1$, $S_1=0.1$ and different values of F.

$\kappa > f/V$) in the dimensional form ($\kappa = kf/NH_0$ and $V = FNH_0$). Thus, continuously stratified vertically sheared flows are unconditionally unstable in agreement with Ripa's conclusion that there are no conditions on the basic flow that could ensure that the wave energy or momentum of an arbitrary perturbation be positive definite. Being proportional to the gradient of the homogeneous layer thickness, the growth rate of ageostrophic unstable modes is exponentially small in Froude number, as in the Eady model. This is explained by the exponential decrease of the vertical scale of penetration of the Rossby mode into a stratified layer when the wavenumber $k > 1/F$ increases.

In practice, the exponential dependence of the growth rate on the Froude number means that these ageostrophic instabilities are exceedingly weak when F is small, i.e. for gentle isopycnal slope, $s \ll f/N$, but can become important suddenly as the isopycnal slope steepens towards f/N in the same fashion as noticed for the Rossby number approaching unity in horizontally sheared flows (Vanneste & Yavneh 2007).

Therefore, the growth of submesoscale ageostrophic perturbations and departures from balance in geophysical flows are extremely sensitive to the Froude and Rossby numbers (cf. Yamasaki & Peltier 2001; Heifetz & Farrel 2007).

A direct counterpart of this ageostrophic instability in two-layer models is the Rossby–Kelvin instability (e.g. Sakai 1989; S07). It appears only when the Froude number exceed some critical value, i.e. the two-layer models do not capture the exponential tail because of poor vertical structure. This illustrates some limitations of the layered models for the study of the interactions between balanced motion and fast waves in the continuously stratified atmosphere and ocean. Further investigations are needed to clarify finite-amplitude forms of ageostrophic instabilities. In particular, are they capable of reducing the upper limit of the Froude number $F=2$ ($Ri=1/4$) imposed by the classical (non-rotating) condition for the onset of Kelvin–Helmholtz instability?

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