Bull. Aust. Math. Soc. **88** (2013), 359–368
doi:10.1017/S0004972712001013

THE NATURAL PARTIAL ORDER ON THE SEMIGROUP OF ALL TRANSFORMATIONS OF A SET THAT REFLECT AN EQUIVALENCE RELATION

LEI SUN ✉ **and XIANGJUN XIN**

(Received 13 September 2012; accepted 18 October 2012; first published online 25 January 2013)

Abstract

Let $\mathcal{T}_X$ be the full transformation semigroup on a set X and E be a nontrivial equivalence relation on X. Denote

$$T_{\exists}(X) = \{f \in \mathcal{T}_X : \forall x, y \in X, (f(x), f(y)) \in E \Rightarrow (x, y) \in E\},$$

so that $T_{\exists}(X)$ is a subsemigroup of $\mathcal{T}_X$. In this paper, we endow $T_{\exists}(X)$ with the natural partial order and investigate when two elements are related, then find elements which are compatible. Also, we characterise the minimal and maximal elements.

2010 *Mathematics subject classification*: primary 20M20.

Keywords and phrases: transformation semigroup, natural partial order, compatibility, minimal (maximal) elements.

1. Introduction

In [4] Mitsch defined a partial order on an arbitrary semigroup S by

$$a \leq b \quad \text{if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

and this is called the *natural partial order* on S. Later Kowol and Mitsch in [2] studied various properties of this partial order on the full transformation semigroup $\mathcal{T}_X$ consisting of all total transformations of an arbitrary nonempty set X. Then Marques-Smith and Sullivan in [3] extended some of the previous work to the semigroup $\mathcal{P}_X$ of all partial transformations on X. Sullivan in [11] investigated the partial order on the linear transformation semigroup $P(V)$ for a vector space V. In [10] Singha *et al.* considered the partial order on the partial Baer–Levi semigroup, and so on (see [12, 13]).

This paper is supported by National Natural Science Foundation of China (No. 10971086).

Let E be an equivalence relation on the set X. The subsemigroup of $\mathcal{T}_X$ defined by

$$T_E(X) = \{f \in \mathcal{T}_X : \forall x, y \in X, (x, y) \in E \Rightarrow (f(x), f(y)) \in E\}$$

was mainly studied in [5–9] and the natural partial order on the semigroup $T_E(X)$ was investigated in [12]. Inspired by the semigroup $T_E(X)$, the authors in [1] considered the semigroup

$$T_\exists(X) = \{f \in \mathcal{T}_X : \forall x, y \in X,\ (f(x), f(y)) \in E \Rightarrow (x, y) \in E\}$$

which differs greatly from the semigroup $T_E(X)$. The transformation $f \in T_\exists(X)$ reflects the equivalence relation E. Clearly, $T_\exists(X)$ is also a subsemigroup of $\mathcal{T}_X$ and contains the identity transformation id_X on X. Moreover, if $E = X \times X$, then $T_\exists(X) = \mathcal{T}_X$. If $E = \triangle = \{(x, x) : x \in X\}$, then

$$T_\exists(X) = \{f \in \mathcal{T}_X : f \text{ is injective}\}.$$

So to this extent it is regarded as a generalisation of $\mathcal{T}_X$.

In this paper, we assume the set X is finite or infinite, the equivalence relation E is nontrivial (that is, $E \neq X \times X$ and $E \neq \triangle$) and X/E, which is the partition of X induced by E, is finite or infinite, and consider the semigroup $T_\exists(X)$ endowed with the natural partial order. Denote by fg the transformation obtained by performing first g and then f. Then the natural partial order can be written, for $f, g \in T_\exists(X)$, as

$$f \leq g \quad \text{if and only if } f = kg = gh \text{ and } f = kf \text{ for some } k, h \in T_\exists(X).$$

This paper is organised as follows. In Section 2 we give a characterisation of the natural partial order on the semigroup $T_\exists(X)$. In Section 3 we find the elements which are compatible with the natural partial order. And in Section 4 we characterise the minimal and maximal elements.

The following lemma describes an essential property of $T_\exists(X)$.

Lemma 1.1 [1]. *Let $f \in T_\exists(X)$. Then for each $A \in X/E$, $f(A) \subseteq \bigcup_{i \in I} B_i$ where I is some index set and $B_i \in X/E$.*

2. Characterisation

Let $\pi(f)$ be the partition of X induced by $f \in \mathcal{T}_X$, namely,

$$\pi(f) = \{f^{-1}(y) : y \in f(X)\}.$$

Denote

$$Z(f) = \{A \in X/E : A \cap f(X) = \emptyset\}.$$

Let $\mathcal{A}, \mathcal{B}$ be two collections of subsets of X. If for each $A \in \mathcal{A}$, there exists some $B \in \mathcal{B}$ such that $A \subseteq B$, then $\mathcal{A}$ is said to *refine* $\mathcal{B}$. For $A \subseteq X$, let

$$\overline{f(A)} = \{B \in X/E : B \cap f(A) \neq \emptyset\}.$$

The following theorem gives a characterisation of this partial order.

THEOREM 2.1. *Let $f, g \in T_\exists(X)$. Then $f \leq g$ if and only if the following statements hold.*

(1) *$\pi(g)$ refines $\pi(f)$ and $|Z(g)| \leq |Z(f)|$.*
(2) *If $(f(x), f(y)) \in E$ for some distinct $x, y \in X$, then $(g(x), g(y)) \in E$.*
(3) *If $g(x) \in f(X)$ for some $x \in X$, then $f(x) = g(x)$.*
(4) *For each $A \in X/E$, there exists a unique $B \in X/E$ such that $f(A) \subseteq g(B)$.*

PROOF. Suppose that $f \leq g$. Then there exist some $k, h \in T_\exists(X)$ such that

$$f = kg = gh \quad \text{and} \quad f = kf.$$

It follows from $f = kg$ that $\pi(g)$ refines $\pi(f)$. By $f(X) = kg(X)$, $f(X) \cap k(A) = \emptyset$ for each $A \in Z(g)$. Then there is some $B \in Z(f)$ such that $B \cap k(A) \neq \emptyset$. By $k \in T_\exists(X)$, $|Z(g)| \leq |Z(f)|$ and (1) holds. Let $(f(x), f(y)) \in E$ for some distinct $x, y \in X$, that is, $(kg(x), kg(y)) \in E$. Then, by $k \in T_\exists(X)$, $(g(x), g(y)) \in E$ and (2) holds. Now if $g(x) \in f(X)$ for some $x \in X$, then $g(x) = f(y)$ for some $y \in X$. So

$$f(x) = kg(x) = kf(y) = f(y) = g(x)$$

and (3) holds. For each $A \in X/E$, let $\overline{h(A)} = \{B_i : i \in I\}$ where $B_i \in X/E$ and I is some index set. Then, for each $M \in \overline{f(A)}$,

$$f(A) \cap M = gh(A) \cap M \subseteq g\Big(\bigcup_{i \in I} B_i\Big) \cap M.$$

By $g \in T_\exists(X)$, we know that g does not map the different E-classes to the same E-class. So there is a unique $i \in I$ such that $f(A) \cap M \subseteq g(B_i) \cap M$. Write $B = B_i$ and then $f(A) \cap M \subseteq g(B) \cap M$. Therefore, $f(A) \subseteq g(B)$ and (4) holds.

Conversely, suppose that conditions (1)–(4) hold. Then, by $|Z(g)| \leq |Z(f)|$, there is a map

$$\rho : \mathcal{M} = \Big\{\bigcup A : A \in Z(g)\Big\} \to \mathcal{N} = \Big\{\bigcup B : B \in Z(f)\Big\}$$

such that $(x, y) \notin E \Rightarrow (\rho(x), \rho(y)) \notin E$ for any $x, y \in \mathcal{M}$. We define k on each E-class A. There are two cases to consider.

Case 1. $A \cap g(X) = \emptyset$. For each $z \in A$, let $k(z) = \rho(z)$.

Case 2. $A \cap g(X) \neq \emptyset$. For each $z \in A \cap g(X)$, then $z = g(x)$ for some $x \in X$ and define $k(z) = f(x)$. Fix a point $z_A \in A \cap g(X)$ and let $k(z) = k(z_A)$ for each $z \in A - g(X)$. If some $x' \in X$ satisfies $z = g(x') = g(x)$, then $f(x') = f(x)$ since $\pi(g)$ refines $\pi(f)$. Thus k is well defined on A. Consequently, k is well defined on all of X. Moreover, $k(A) \subseteq f(X)$.

Now we verify that $k \in T_\exists(X)$. Let $x \in A_1$ and $y \in A_2$ for some distinct $A_1, A_2 \in X/E$. We discuss three cases.

Case 1. $A_1 \cap g(X) = \emptyset$ and $A_2 \cap g(X) = \emptyset$. Then $(k(x), k(y)) = (\rho(x), \rho(y)) \notin E$.

https://doi.org/10.1017/S0004972712001013 Published online by Cambridge University Press

Case 2. $A_1 \cap g(X) = \emptyset$ and $A_2 \cap g(X) \neq \emptyset$. We discuss two subcases.

Case 2.1. $y \in A_2 \cap g(X)$. Then $k(x) = \rho(x)$ and $k(y) \in f(X)$. So $(k(x), k(y)) \notin E$.

Case 2.2. $y \in A_2 - A_2 \cap g(X)$. In this case $k(y) = k(z_{A_2})$ where z_{A_2} is a fixed point in $A_2 \cap g(X)$. So $(k(x), k(y)) = (k(x), k(z_{A_2})) \notin E$ (by Case 2.1).

Case 3. $A_1 \cap g(X) \neq \emptyset$ and $A_2 \cap g(X) \neq \emptyset$. We discuss three subcases.

Case 3.1. $x \in A_1 \cap g(X)$ and $y \in A_2 \cap g(X)$. Then $x = g(x')$, $y = g(y')$ for some distinct $x', y' \in X$. We assert that $(k(x), k(y)) \notin E$. Indeed, if $(k(x), k(y)) \in E$, namely, $(f(x'), f(y')) \in E$, then, by (2), we have $(g(x'), g(y')) \in E$, that is, $(x, y) \in E$, a contradiction.

Case 3.2. $x \in A_1 - A_1 \cap g(X)$ and $y \in A_2 \cap g(X)$. Then we have $k(x) = k(z_{A_1})$ and $(k(z_{A_1}), k(y)) \notin E$ (by Case 3.1). So $(k(x), k(y)) \notin E$.

Case 3.3. $x \in A_1 - A_1 \cap g(X)$ and $y \in A_2 - A_2 \cap g(X)$. Then $k(x) = k(z_{A_1})$, $k(y) = k(z_{A_2})$ and $(k(z_{A_1}), k(z_{A_2})) \notin E$ (by Case 3.1). So $(k(x), k(y)) \notin E$.

In any case $k \in T_\exists(X)$. It is clear that $f = kg$. We show that $f = kf$. For each $x \in X$, by (4), there exists some $y \in X$ such that $f(x) = g(y)$ and it follows from (3) that $f(y) = g(y)$. So

$$f(x) = f(y) = kg(y) = kf(x)$$

which means that $f = kf$.

Finally, we define h on X. For each $A \in X/E$ and each $x \in A$, there exists a unique $B \in X/E$ such that $y \in B$ and $f(x) = g(y)$. Define $h(x) = y$ as required. By $f, g \in T_\exists(X)$ and the uniqueness of the E-class B associated with each E-class A, we have $h \in T_\exists(X)$. It is clear that $f = gh$. This completes the proof. □

Corollary 2.2. *Let $f, g \in T_\exists(X)$. Then the following statements hold.*

(1) *If $f \leq g$, then $f(X) \subseteq g(X)$.*
(2) *If $f \leq g$ and $f(X) = g(X)$, then $f = g$.*
(3) *If $f \leq g$ and $\pi(f) = \pi(g)$, then $f = g$.*

Proof. (1) This follows from Theorem 2.1(4).

(2) This follows from Theorem 2.1(3).

(3) By (1), $f(X) \subseteq g(X)$. If $f(X) \subset g(X)$ (where $f(X) \subset g(X)$ means that $f(X)$ is a proper subset of $g(X)$), then take $y \in g(X) - f(X)$ and let $g(x) = y$ for some $x \in X$. So $f(x) = g(x')$ for some $x' \in X$ $(x' \neq x)$. By Theorem 2.1(3), $f(x') = g(x')$ which implies that $f(x') = f(x)$. Since $\pi(f) = \pi(g)$, we have $g(x') = g(x)$. Observing that $g(x') = f(x)$, $g(x) = y$, we deduce that $f(x) = y$, a contradiction. Therefore, $f(X) = g(X)$. By (2), $f = g$. □

3. Compatibility

A transformation $h \in T_\exists(X)$ is said to be *strictly left compatible* with the partial order if $hf < hg$ for all $f < g$. *Strict right compatibility* is defined dually.

THEOREM 3.1. *Let $h \in T_{\exists}(X)$. Then h is strictly left compatible if and only if h is injective and $h(A) \subseteq B \in X/E$ for each $A \in X/E$.*

PROOF. Suppose that h is strictly left compatible. We claim that h is injective. Indeed, let $h(a) = h(b)$ for some distinct $a, b \in C \in X/E$. Assume that C is a disjoint union of nonempty sets C_1 and C_2 (namely, $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$) and $a \in C_1, b \in C_2$. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} a & \text{if } x \in C \\ x & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} a & \text{if } x \in C_1 \\ b & \text{if } x \in C_2 \\ x & \text{otherwise,} \end{cases}$$

respectively. Clearly, $f, g \in T_{\exists}(X)$ and $f \neq g$. It is straightforward to show $f < g$. Then $hf < hg$ and $hf(X) \subset hg(X)$. However, by the assumption $h(a) = h(b)$, $hf(C) = hg(C)$ and $hf(D) = hg(D)$ for any other E-class D which implies that $hf(X) = hg(X)$, a contradiction. It follows that h is injective.

To verify the remaining conclusion, assume without loss of generality that $\overline{h(A)} = \{B_1, B_2\}$ for some $A \in X/E$. Denote

$$A_1 = \{x \in A : h(x) \in B_1\} \quad \text{and} \quad A_2 = \{x \in A : h(x) \in B_2\}.$$

Then A is a disjoint union of nonempty sets A_1 and A_2. Take $x' \in A_1$ and define $f : X \to X$ by

$$f(x) = \begin{cases} x' & \text{if } x \in A \\ x & \text{otherwise.} \end{cases}$$

Clearly, $f \in T_{\exists}(X)$, $f \neq \mathrm{id}_X$ and $f < \mathrm{id}_X$. Thus $hf < h\,\mathrm{id}_X$. However, taking $y' \in A_2$, we have $(hf(x'), hf(y')) \in E$, $h\,\mathrm{id}_X(x') \in B_1$, $h\,\mathrm{id}_X(y') \in B_2$ which means that $(hf(x'), hf(y')) \in E$ does not imply $(h\,\mathrm{id}_X(x'), h\,\mathrm{id}_X(y')) \in E$, a contradiction.

Conversely, let $f, g \in T_{\exists}(X)$ and $f < g$. Clearly, $\pi(hg)$ refines $\pi(hf)$. Write

$$\overline{f(X)} = \{A_i : i \in I\} \quad \text{and} \quad \overline{g(X)} = \{B_j : j \in J\},$$

where I, J are some index sets. Since h maps any E-class to one E-class, let $h(A_i) \subseteq C_i$ and $h(B_j) \subseteq D_j$ for each $i \in I$, $j \in J$. Then $\overline{hf(X)} = \{C_i : i \in I\}$ and $\overline{hg(X)} = \{D_j : j \in J\}$. By $|Z(g)| \leq |Z(f)|$, we have $|\overline{f(X)}| \leq |\overline{g(X)}|$ and $|\overline{hf(X)}| \leq |\overline{hg(X)}|$. So $|Z(hg)| \leq |Z(hf)|$ and hf, hg satisfy Theorem 2.1(1). Let $(hf(x), hf(y)) \in E$ for some distinct $x, y \in X$. Then $(f(x), f(y)) \in E$. By $f < g$, we deduce $(g(x), g(y)) \in E$. Thus $(hg(x), hg(y)) \in E$ which implies that hf, hg satisfy Theorem 2.1(2). It is clear that hf, hg satisfy Theorem 2.1(3). For each $A \in X/E$ and $M \in \overline{hf(A)}$, we have $hf(A) \cap M \neq \emptyset$ and there is some $N \in \overline{f(A)}$ such that $h(f(A) \cap N) \cap M \neq \emptyset$. Thus, by $f < g$, $f(A) \cap N \subseteq g(B) \cap N$ for a unique $B \in X/E$. So it follows that

$$hf(A) \cap M = h(f(A) \cap N) \cap M \subseteq h(g(B) \cap N) \cap M = hg(B) \cap M,$$

and $hf(A) \subseteq hg(B)$. This means that hf, hg satisfy Theorem 2.1(4). Therefore, $hf < hg$. □

https://doi.org/10.1017/S0004972712001013 Published online by Cambridge University Press

Note that if X/E is finite, then $|\overline{h(A)}| = 1$ for each $h \in T_{\exists}(X)$ and $A \in X/E$. So Theorem 3.1 is simplified as follows.

COROLLARY 3.2. *Let X/E be finite and $h \in T_{\exists}(X)$. Then h is strictly left compatible if and only if h is injective.*

THEOREM 3.3. *Let $h \in T_{\exists}(X)$. Then h is strictly right compatible if and only if h is surjective.*

PROOF. Suppose that h is strictly right compatible. We assert that h is surjective. Indeed, for some $A \in X/E$, let $h(A) \cap B \subset B$ for some $B \in \overline{h(A)}$. Take $a \in B - h(A) \cap B$, $b \in h(A) \cap B$ and define $f, g : X \to X$ by

$$f(x) = \begin{cases} a & \text{if } x \in h(A) \cap B \\ x & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} b & \text{if } x \in h(A) \cap B \\ x & \text{otherwise,} \end{cases}$$

respectively. Then $f, g \in T_{\exists}(X)$ and $f \neq g$. To see that $f < g$, let $g(x) = g(y)$ for some distinct $x, y \in X$. Then $x, y \in h(A) \cap B$ and $f(x) = f(y)$ which means that $\pi(g)$ refines $\pi(f)$. Clearly, $|Z(g)| = |Z(f)| = 0$. So f, g satisfy Theorem 2.1(1). If $g(x) \in f(X) = X - h(A) \cap B$ for some $x \in X$, then, by the definition of g, $f(x) = g(x) = x$ which implies that f, g satisfy Theorem 2.1(3). Observing that

$$f(B) = f((h(A) \cap B) \cup (B - h(A) \cap B)) = \{a\} \cup (B - h(A) \cap B) = B - h(A) \cap B$$

and

$$g(B) = g((h(A) \cap B) \cup (B - h(A) \cap B)) = \{b\} \cup (B - h(A) \cap B),$$

that is, $f(B) \subset g(B)$, together with $f(C) = g(C)$ for any other E-class C, we have that f, g satisfy Theorem 2.1(2) and (4). Thus $f < g$ and $fh < gh$. However,

$$fh(A) \cap B = f(h(A) \cap B) \cap B = \{a\}$$

and

$$gh(A) \cap B = g(h(A) \cap B) \cap B = \{b\}, \quad gh(C) \cap B = h(C) \cap B = \emptyset$$

where $C \in X/E$ $(C \neq A)$, which implies that there is no E-class D such that $fh(A) \cap B \subseteq gh(D) \cap B$. So fh, gh do not satisfy Theorem 2.1(4), a contradiction.

Conversely, let $f, g \in T_{\exists}(X)$ and $f < g$. Clearly, fh, gh satisfy Theorem 2.1(1)–(3). For each $A \in X/E$ and $M \in \overline{fh(A)}$, $fh(A) \cap M \neq \emptyset$ and there is some $N \in \overline{h(A)}$ such that $f(h(A) \cap N) \cap M \neq \emptyset$. Since h is surjective, $h(A) \cap N = N$. Then $f(N) \cap M \subseteq g(C) \cap M$ for a unique $C \in X/E$. So there is a unique $B \in X/E$ such that $h(B) \cap C = C$. It follows that

$$\begin{aligned} fh(A) \cap M = f(h(A) \cap N) \cap M = f(N) \cap M \subseteq g(C) \cap M &= g(h(B) \cap C) \cap M \\ &= gh(B) \cap M, \end{aligned}$$

that is, $fh(A) \subseteq gh(B)$, which means that fh, gh satisfy Theorem 2.1(4). Therefore, $fh < gh$. □

https://doi.org/10.1017/S0004972712001013 Published online by Cambridge University Press

4. Minimal and maximal elements

We begin by determining the minimal elements of $T_\exists(X)$.

THEOREM 4.1. *Let $f \in T_\exists(X)$. Then f is minimal if and only if for each $A \in X/E$, $|f(A) \cap M| = 1$ for each $M \in \overline{f(A)}$.*

PROOF. The sufficiency is clear, so we only show the necessity. If $|f(A) \cap M| \geq 2$, denote $A' = \{x \in A : f(x) \in M\}$, then take $a \in f(A) \cap M$ and define

$$g(x) = \begin{cases} a & \text{if } x \in A' \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, $g \in T_\exists(X)$, $g \neq f$ and $g < f$, which leads to a contradiction. □

Before characterising the maximal elements of $T_\exists(X)$ we need some terminology. For a transformation $f \in T_\exists(X)$ and $A \in X/E$, we say that $f|_A$ is *defect-divided* if A is a disjoint union of nonempty sets A_1 and A_2 such that $f|_{A_1}$ is not injective, $f(A) \cap M = M$ for each $M \in \overline{f(A_1)}$ and $f|_{A_2}$ is injective, $f(A) \cap N \subset N$ for some $N \in \overline{f(A_2)}$. And we say that $f|_A$ is *surjection-divided* if $f|_A$ is not injective and $f(A) \cap M = M$ for each $M \in \overline{f(A)}$.

THEOREM 4.2. *Let $f \in T_\exists(X)$. Then f is maximal if and only if one of the following statements holds.*

(1) *f is injective or surjective.*
(2) *There is some E-class A such that $f|_A$ is defect-divided. For any other E-class B, either $f|_B$ is surjection-divided or $f|_B$ is injective.*
(3) *There are some distinct $A, B \in X/E$ such that $f|_A$ is surjection-divided and $f|_B$ is injective and $f(B) \cap N \subset N$ for some $N \in \overline{f(B)}$. For any other E-class C, $f|_C$ is injective and $f(C) \cap N' = N'$ for each $N' \in \overline{f(C)}$.*

PROOF. Let f be maximal. Suppose to the contrary that none of (1)–(3) holds. Assume that $f|_A$ is not injective for some $A \in X/E$. Then we claim that $f|_A$ is surjection-divided. Indeed, if $f(A) \cap M \subset M$ for some $M \in \overline{f(A)}$, let A be a disjoint union of nonempty sets A_1 and A_2 with the property that $f|_{A_1}$ is not injective and $f|_{A_2}$ is injective. Then $M \notin \overline{f(A_1)}$. Otherwise, let $f(x_1) = f(x_2) \in M'$ for some distinct $x_1, x_2 \in A_1$ and take $a \in M - f(A) \cap M$. Then define $g : X \to X$ by

$$g(x) = \begin{cases} a & \text{if } x = x_1 \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, $g \in T_\exists(X)$, $g \neq f$. It is straightforward to show that $f < g$. So f is not maximal, a contradiction. It follows that $M \in \overline{f(A_2)}$. This also means that $f(A) \cap N = N$ for each $N \in \overline{f(A_1)}$. Thus $f|_A$ is defect-divided, a contradiction. It follows that $f|_A$ is surjection-divided. On the other hand, since f is not surjective, let $f(B) \cap C \subset C$ for some $B, C \in X/E$ $(B \neq A)$. We assert that $f|_B$ is injective. Indeed, if $f|_B$ is not injective, then let B be a disjoint union of nonempty sets B_1 and B_2 with the property that $f|_{B_1}$

is not injective and $f|_{B_2}$ is injective. By the above approach, we deduce that $f|_B$ is defect-divided, a contradiction. Thus $f|_B$ is injective. Hence we find two E-classes A, B with the property that $f|_A$ is surjection-divided and $f|_B$ is injective, $f(B) \cap C \subset C$, a contradiction.

Conversely, let $f \leq g$. There are three cases to consider.

Case 1. f is injective or surjective. If f is injective, then $\pi(f) = \pi(g)$. By Corollary 2.2(3), $f = g$. So f is maximal. And if f is surjective, then $f(X) = g(X)$. By Corollary 2.2(2), $f = g$. So f is also maximal.

Case 2. f satisfies (2). Let A be a disjoint union of nonempty sets A_1 and A_2 such that $f|_{A_1}$ is not injective, $f(A) \cap M = M$ for each $M \in \overline{f(A_1)}$ and that $f|_{A_2}$ is injective, $f(A) \cap N \subset N$ for some $N \in \overline{f(A_2)}$. Since $f \leq g$, by Theorem 2.1(4), for each $M \in \overline{f(A_1)}$, there exists a unique $A' \in X/E$ such that

$$M = f(A) \cap M \subseteq g(A') \cap M \subseteq M$$

which implies that $f(A) \cap M = g(A') \cap M = M$. So if $g(x) \in M$ for some $x \in A'$, then $g(x) \in g(A') \cap M = f(A) \cap M$. According to Theorem 2.1(3), $f(x) = g(x)$ and $f(x) \in f(A) \cap M$ which implies that $A' = A$. This also means that $f(A_1) = g(A_1)$. Moreover, by Corollary 2.2(3), $f(A_2) = g(A_2)$. It follows that $f(A) = g(A)$. For any other E-class B, we also have $f(B) = g(B)$. Hence $f(X) = g(X)$ and $f = g$. Therefore, f is maximal.

Case 3. f satisfies (3). Then for each $M \in \overline{f(A)}$ there exists a unique $A' \in X/E$ such that

$$M = f(A) \cap M \subseteq g(A') \cap M \subseteq M.$$

Similarly to Case 2, we deduce that $A' = A$ and $f(A) = g(A)$. By Corollary 2.2(3) again, $f(B) = g(B)$ and $f(C) = g(C)$ as well. Thus $f(X) = g(X)$. So $f = g$ and f is maximal. □

To illustrate the maximal elements of Theorem 4.2(2) and (3), we present two examples.

Example 4.3. Let $X = \{1, 2, \ldots\}$ and $E = \bigcup_{i=1}^{\infty}(A_i \times A_i)$ where $A_1 = \{1, 2, 3, \ldots, 10\}$, $A_2 = \{11, 12\}, A_3 = \{13, 14, 15\}, A_4 = \{16, 17, 18, 19\}, A_5 = \{20, 21, 22, 23, 24\}, \ldots$. Let $f \in T_\exists(X)$ satisfy

$$f|_{A_1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 11 & 13 & 13 & 14 & 15 & 17 & 19 & 16 \end{pmatrix}$$

and $f|_{A_i}$ be injective, $f(A_i) \subset A_{i+3}$ $(i \geq 2)$. Clearly, $A_1 = \{1, 3, 4, 5\} \cup \{2, 6, 7, 8, 9, 10\}$, $f(1) = f(3) = 11 \in A_2$, $f(4) = f(5) = 13 \in A_3$, $f(A_1) \cap A_2 = A_2$, $f(A_1) \cap A_3 = A_3$, $f(A_1) \cap A_4 \subset A_4$. Then $f|_{A_1}$ is defect-divided. Moreover, $f|_{A_i}$ is injective $(i \geq 2)$. Then f is a maximal element of the kind belonging to Theorem 4.2(2).

EXAMPLE 4.4. Let $X = \{1, 2, \ldots, 18\}$ and $E = \bigcup_{i=1}^{4}(A_i \times A_i)$ where $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6, 7\}$, $A_3 = \{8, 9, 10, 11, 12\}$ and $A_4 = \{13, 14, 15, 16, 17, 18\}$. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 5 & 4 & 6 & 1 & 3 & 1 & 2 & 15 & 13 & 14 & 18 & 17 & 8 & 9 & 9 & 10 & 11 & 12 \end{pmatrix}.$$

Clearly, $f \in T_\exists(X)$ and $f|_{A_1}$ is injective, $f(A_1) \subset A_2$ and $f|_{A_2}$ is surjection-divided, $f|_{A_3}$ is injective, $f(A_3) \subset A_4$ and $f|_{A_4}$ is surjection-divided. Then f is a maximal element of the kind belonging to Theorem 4.2(3).

As a consequence of Theorem 4.2, we have the following conclusion.

COROLLARY 4.5. *Let $f \in T_\exists(X)$. Then the following statements hold.*

(1) *If X is finite and all E-classes have the same size, then f is maximal if and only if f is a permutation preserving E.*

(2) *If X/E is finite, then f is maximal if and only if f is either injective, or surjective, or there are some distinct $A, B \in X/E$ such that $f|_A$ is surjection-divided and $f|_B$ is injective and $f(B) \cap N \subset N$ for some $N \in \overline{f(B)}$, and for any other E-class C, $f|_C$ is injective and $f(C) \cap M = M$ for each $M \in \overline{f(C)}$.*

By the way, if X/E is infinite, then there may be a maximal element of the kind belonging to both Theorem 4.2(2) and (3). Even if X/E is finite and all E-classes have the same size, then there may be a maximal element of the kind belonging to Theorem 4.2(3).

EXAMPLE 4.6. Let $X = \{1, 2, \ldots\}$ and $E = \bigcup_{i=1}^{3}(A_i \times A_i)$, where $A_1 = \{1, 4, 7, \ldots\}$, $A_2 = \{2, 5, 8, \ldots\}$ and $A_3 = \{3, 6, 9, \ldots\}$. Choose

$$f(x) = \begin{cases} 3n + 3 & \text{if } x = 3n \\ 3n - 1 & \text{if } x = 3n + 2 \\ x & \text{otherwise,} \end{cases}$$

where n is a natural number. Clearly, $f \in T_\exists(X)$. Then $f|_{A_1}$ is injective, $f(A_1) = A_1$, $f|_{A_2}$ is surjection-divided ($f(2) = f(5) = 2$, $f(A_2) \cap A_2 = A_2$) and $f|_{A_3}$ is injective, $f(A_3) \subset A_3$ ($3 \notin f(A_3)$). So f is a maximal element of the kind belonging to Theorem 4.2(3).

Acknowledgement

We would like to thank the referee for valuable suggestions and comments which helped to improve the presentation of this paper.

References

[1] L. Z. Deng, J. W. Zeng and T. J. You, 'Green's relations and regularity for semigroups of transformations that preserve reverse direction equivalence', *Semigroup Forum* **83** (2011), 489–498.

[2] G. Kowol and H. Mitsch, 'Naturally ordered transformation semigroups', *Monatsh. Math.* **102** (1986), 115–138.
[3] M. P. O. Marques-Smith and R. P. Sullivan, 'Partial orders on transformation semigroups', *Monatsh. Math.* **140** (2003), 103–118.
[4] H. Mitsch, 'A natural partial order for semigroups', *Proc. Amer. Math. Soc.* **97** (1986), 384–388.
[5] H. S. Pei, 'Equivalences, α-semigroups and α-congruences', *Semigroup Forum* **49** (1994), 49–58.
[6] H. S. Pei, 'A regular α-semigroup inducing a certain lattice', *Semigroup Forum* **53** (1996), 98–113.
[7] H. S. Pei, 'Some α-semigroups inducing certain lattices', *Semigroup Forum* **57** (1998), 48–59.
[8] H. S. Pei, 'On the rank of the semigroup $T_E(X)$', *Semigroup Forum* **70** (2005), 107–117.
[9] H. S. Pei, 'Regularity and Green's relations for semigroups of transformations that preserve an equivalence', *Comm. Algebra* **33** (2005), 109–118.
[10] B. Singha, J. Sanwong and R. P. Sullivan, 'Partial order on partial Baer–Levi semigroups', *Bull. Aust. Math. Soc.* **81** (2010), 197–206.
[11] R. P. Sullivan, 'Partial orders on linear transformation semigroups', *Proc. Roy. Soc. Edinburgh Sect. A* **135** (2005), 413–437.
[12] L. Sun, H. S. Pei and Z. X. Cheng, 'Naturally ordered transformation semigroups preserving an equivalence', *Bull. Aust. Math. Soc.* **78** (2008), 117–128.
[13] L. Sun and L. M. Wang, 'Natural partial order in semigroups of transformations with invariant set', *Bull. Aust. Math. Soc.* **87** (2013), 94–107.

LEI SUN, School of Mathematics and Information Science,
Henan Polytechnic University, Henan, Jiaozuo 454003, PR China
e-mail: sunlei97@163.com

XIANGJUN XIN, Department of Mathematics and Information Science,
Zhengzhou University of Light Industry, Henan, Zhengzhou 450002, PR China
e-mail: xin_xiang_jun@126.com