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ON THE DEGREE DISTANCE OF SOME COMPOSITE GRAPHS

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Abstract

Let G be a connected simple graph. The degree distance of G is defined as $D'(G) = \sum_{u \in V(G)} d_G(u) D_G(u)$, where $D_G(u)$ is the sum of distances between the vertex u and all other vertices in G and $d_G(u)$ denotes the degree of vertex u in G. In contrast to many established results on extremal properties of degree distance, few results in the literature deal with the degree distance of composite graphs. Towards closing this gap, we study the degree distance of some composite graphs here. We present explicit formulas for $D'(G)$ of three composite graphs, namely, double graphs, extended double covers and edge copied graphs.

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1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, we let $d_G(v)$ be the degree of a vertex v in G and $d_G(u, v)$ be the distance between two vertices u and v in G. Other notation and terminology not defined here will conform to those in [2].

Let

$$D'(G) = \sum_{u \in V(G)} d_G(u) D_G(u) \quad \text{where } D_G(u) = \sum_{v \in V(G)} d_G(u, v).$$

This graph parameter $D'(G)$ is called the *degree distance* of G and was introduced by Dobrynin and Kochetova [7] and Gutman [9] as a weighted version of the *Wiener index*

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} D_G(u)$$

(see [6]). When G is a tree on n vertices, these two parameters are closely related; we have $D'(G) = 4W(G) - n(n-1)$ (see [9]).

In the literature, many results on the degree distance $D'(G)$ have been put forward in past decades and they mainly deal with extreme properties of $D'(G)$. Tomescu [15] showed that the star is the unique graph with minimum degree distance within the

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class of n-vertex connected graphs. Tomescu [16] deduced properties of graphs with minimum degree distance in the class of n-vertex connected graphs with $m \geq n-1$ edges. The same result was also obtained by Bucicovschi and Cioabă in [3]. Tomescu [17] determined among connected n-vertex graphs the three smallest graphs with respect to degree distance. Dankelmann *et al.* [5] gave asymptotically sharp upper bounds for the degree distance. Tomescu [18] characterized the unicyclic and bicyclic graphs with minimum degree distances. Du and Zhou [8] determined the maximum degree distance of n-vertex unicyclic graphs with given maximum degree, and the first seven maximum degree distances of n-vertex unicyclic graphs for $n \geq 6$. Ilić *et al.* [12] characterized those n-vertex unicyclic graphs with girth k and having minimum or maximum degree distance. Furthermore, they determined the unique graph having maximum degree distance among bicyclic graphs, and resolved a recent conjecture of Tomescu [19]. Also, Ilić *et al.* [11] calculated the degree distance of partial Hamming graphs. For other related results along this line, the reader is referred to [4, 10, 13].

It is well known that many graphs arise from simpler graphs via various graph operations. Hence, it is important to understand how certain invariants of such composite graphs are related to the corresponding invariants of the original graphs.

In contrast to many established results on degree distance, few results in the literature deal with the degree distance of composite graphs. Towards closing this gap, we study degree distance of some composite graphs in this paper. We present explicit formulas for $D'(G)$ of three composite graphs, namely, double graphs, extended double covers and edge copied graphs.

2. Main results

The *double graph* G^* of a given graph G is constructed by making two copies of G, including the initial edge set of each copy, and adding edges u_1v_2 and u_2v_1 for every edge uv of G.

For each vertex u in G, we call the corresponding vertices u_1 and u_2 in G^* the *clone vertices* of u.

Concerning the degree distance of double graphs, we have the following result.

THEOREM 2.1. *Let G be a nontrivial connected graph of order n and size m, and let G^* be its double graph. Then*

$$D'(G^*) = 8D'(G) + 16m.$$

PROOF. It is obvious that if G is connected, then G^* is also connected. For the sake of convenience, we label the vertices of G as $v_1, \ldots, v_n$. Suppose that for each $i = 1, \ldots, n$, x_i and y_i are the corresponding clone vertices in G^* of v_i. Now let v_i be a given vertex in G. From the definition of double graph, for any vertex v_j in G different from v_i, we have $d_{G^*}(x_i, x_j) = d_{G^*}(x_i, y_j) = d_{G^*}(y_i, x_j) = d_{G^*}(y_i, y_j) = d_G(v_i, v_j)$. Also, we have $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$ for $i = 1, \ldots, n$. Moreover, $d_{G^*}(x_i, y_i) = 2$ for $i = 1, \ldots, n$, since there exists at least one vertex, say x_k (or y_k), such that both x_i and y_i are adjacent to x_k (or y_k).

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We first consider the value of $D_{G^*}(x_i)$. We have

$$\begin{aligned} D_{G^*}(x_i) &= \sum_{1\le j\le n} d_{G^*}(x_i, x_j) + \sum_{1\le j\le n; j\ne i} d_{G^*}(x_i, y_j) + d_{G^*}(x_i, y_i) \\ &= \sum_{1\le j\le n} d_G(v_i, v_j) + \sum_{1\le j\le n} d_G(v_i, v_j) + 2 \\ &= 2D_G(v_i) + 2. \end{aligned}$$

By symmetry, we have $D_{G^*}(x_i) = D_{G^*}(y_i)$ for each $i = 1, \ldots, n$, so

$$\begin{aligned} D'(G^*) &= \sum_{u\in V(G^*)} d_{G^*}(u) D_{G^*}(u) = 2\sum_{i=1}^{n} d_{G^*}(x_i) D_{G^*}(x_i) \\ &= 2\sum_{i=1}^{n} 2d_G(v_i)(2D_G(v_i) + 2) = 8\sum_{i=1}^{n} d_G(v_i)(D_G(v_i) + 1) \\ &= 8\sum_{i=1}^{n} d_G(v_i) D_G(v_i) + 8\sum_{i=1}^{n} d_G(v_i) = 8D'(G) + 16m, \end{aligned}$$

as expected. □

The construction of the *extended double cover* was introduced by Alon [1] in 1986. For a simple graph G with vertex set $V = \{v_1, \ldots, v_n\}$, the extended double cover of G, denoted by $G^\star$, is the bipartite graph with bipartition $(X; Y)$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, in which x_i and y_j are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G. As before, we call x_i and y_i the clone vertices of v_i for each $i = 1, \ldots, n$.

Let x be a vertex in a connected graph G and $n_x(k)$ denote the number of vertices at distance k from x. Then

$$D_G(x) = \sum_{k=1}^{ec_G(x)} k n_x(k), \tag{2.1}$$

where $ec_G(x)$ is the eccentricity of vertex x and

$$\sum_{k=1}^{ec_G(x)} n_x(k) = n - 1. \tag{2.2}$$

The following theorem reveals a relation between $D'(G)$ and $D'(G^\star)$.

Theorem 2.2. *Let G be a nontrivial connected graph of order n and size m, and let $G^\star$ be its extended double cover. Then*

$$D'(G^\star) = 6D'(G) + 12W(G) + 4m(n-1) + 2n(n-1).$$

Proof. If G is connected, then clearly $G^\star$ is also connected. Assume that

$$v_{i1} v_{i2} \cdots v_{i(t+1)}, \quad 1 \le t \le n-1,$$

is a path of length t in G. For each $j = 1, \ldots, t+1$, we let x_{ij} and y_{ij} be the two clone vertices of v_{ij} in $G^\star$. From the definition of extended double over, one can easily

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deduce that

$$\max\{d_{G^\star}(x_{i1}, x_{i(t+1)}), d_{G^\star}(x_{i1}, y_{i(t+1)})\} = t + 1 = d_G(v_{i1}, v_{i(t+1)}) + 1,$$

no matter whether t is odd or even. Hence for each vertex v_i $(i = 1, \ldots, n)$ in G and its two clone vertices x_i and y_i in $G^\star$, $ec_{G^\star}(x_i) = ec_{G^\star}(y_i) = ec_G(v_i) + 1$. Also, it can be shown that there are exactly two vertices at distance 1, two vertices at distance 2,. . . , two vertices at distance t, and one vertex at distance $t + 1$ from x_{i1} among the set of vertices $\{x_{i2}, \ldots, x_{i(t+1)}; y_{i1}, \ldots, y_{i(t+1)}\}$. Let $n_{v_i}(t)$ denote the number of vertices in G at distance t from the vertex v_i. By the above analysis, there are exactly $2n_{v_i}(t)$ vertices in $G^\star$ at distance t from x_i (or y_i), and $n_{v_i}(t)$ vertices in $G^\star$ at distance $t + 1$ from x_i (or y_i). Obviously, for each v_i in G and the corresponding clone vertices x_i and y_i in $G^\star$, we have $d_{G^\star}(x_i) = d_{G^\star}(y_i) = d_G(v_i) + 1$.

We first compute $D_{G^\star}(x_i)$. In view of the equations (2.1) and (2.2),

$$\begin{aligned}
D_{G^\star}(x_i) = \sum_{w\in V(G^\star)} d_{G^\star}(x_i, w) &= \sum_{t=1}^{ec_G(v_i)} (2tn_{v_i}(t) + n_{v_i}(t)(t + 1)) \\
&= \sum_{t=1}^{ec_G(v_i)} (3tn_{v_i}(t) + n_{v_i}(t)) = 3 \sum_{t=1}^{ec_G(v_i)} tn_{v_i}(t) + \sum_{t=1}^{ec_G(v_i)} n_{v_i}(t) \\
&= 3D_G(v_i) + (n - 1).
\end{aligned}$$

By symmetry,

$$\begin{aligned}
D'(G^\star) &= 2 \sum_{i=1}^{n} d_{G^\star}(x_i)D_{G^\star}(x_i) \\
&= 2 \sum_{i=1}^{n}(d_G(v_i) + 1)(3D_G(v_i) + (n - 1)) \\
&= 6 \sum_{i=1}^{n} d_G(v_i)D_G(v_i) + 6 \sum_{i=1}^{n} D_G(v_i) + 2(n - 1) \sum_{i=1}^{n} d_G(v_i) + 2n(n - 1) \\
&= 6D'(G) + 12W(G) + 4m(n - 1) + 2n(n - 1),
\end{aligned}$$

as claimed. □

Yan and Yeh [20] constructed a composite graph by the following procedure. Suppose that G is a connected graph of order n and size m, and let $V(G) = \{v_1, \ldots, v_n\}$. For each edge v_iv_j, we now introduce a new vertex v_{ij} and connect it to both v_i and v_j. Denote the resulting graph by $T(G)$. It is easy to see that $T(G)$ has $m + n$ vertices and $3m$ edges. The graph $T(G)$ has remarkable properties. For example, Yan and Yeh obtained a remarkable result concerning the *Hosoya index* $Z(T(G))$ of $T(G)$, namely, $Z(T(G)) = (d_G(v_1) + 1)(d_G(v_2) + 1) \cdots (d_G(v_n) + 1)$.

Now, we construct a new composite graph $T^e(G)$ by the following procedure. Suppose that G is a connected graph of order n and size m. For each edge uv in G, we introduce a new edge u_1v_1 and add edges uu_1 and vv_1 as well. The resulting graph is called the *edge copied graph* of G and denoted by $T^e(G)$. It is easy to see that $T^e(G)$

has $2m + n$ vertices and $4m$ edges. Here we should note that each vertex u in G has been copied $d_G(u)$ times in $T^e(G)$.

Klavzăr and Gutman [14] defined the *modified Schultz index* $S^*(G)$ for a connected graph G by

$$S^*(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u)d_G(v)d_G(u, v).$$

This can be rewritten as

$$S^*(G) = \frac{1}{2} \sum_{i=1}^{n} d_G(v_i) \sum_{j=1}^{n} d_G(v_j)d_G(v_i, v_j). \tag{2.3}$$

We next present a formula for the degree distance of an edge copied graph.

Theorem 2.3. *Let G be a nontrivial connected graph of order n and size m, and let $T^e(G)$ be its edge copied graph. Then*

$$D'(T^e(G)) = 4D'(G) + 8S^*(G) + 24m^2 + 4mn - 8m.$$

Proof. Since G is connected, clearly $T^e(G)$ is also connected. For convenience, we label the vertices of G as $v_1, \ldots, v_n$. Also, we denote by A the set of newly added vertices of $T^e(G)$, so that $V(T^e(G)) = A \cup \{v_1, \ldots, v_n\}$. By the definition of edge copied graphs, for each v_i, we have $d_{T^e(G)}(v_i) = 2d_G(v_i)$ and for each x in A, we have $d_{T^e(G)}(x) = 2$.

It can be seen that for each vertex v_i and any vertex x in A, we have $d_{T^e(G)}(v_i, x) = d_G(v_i, v_k) + 1$, where v_k is a neighbour of x. Moreover, for any given v_i and each v_k, there are $d_G(v_k)$ vertices x in A such that $v_k x \in E(T^e(G))$ and $d_{T^e(G)}(v_i, x) = d_G(v_i, v_k) + 1$.

Also, we have $d_{T^e(G)}(v_i, v_j) = d_G(v_i, v_j)$.

We first compute $D_{T^e(G)}(v_i)$ for each vertex v_i in $T^e(G)$, $i = 1, \ldots, n$. By the above analysis, we obtain

$$\begin{aligned}\sum_{x\in A} d_{T^e(G)}(v_i, x) &= \sum_{k=1}^{n} d_G(v_k)(d_G(v_i, v_k) + 1) \\ &= \sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 2m.\end{aligned}$$

Hence

$$\begin{aligned}D_{T^e(G)}(v_i) &= \sum_{1\le j\le n} d_{T^e(G)}(v_i, v_j) + \sum_{x\in A} d_{T^e(G)}(v_i, x) \\ &= \sum_{1\le j\le n} d_G(v_i, v_j) + \sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 2m \\ &= D_G(v_i) + \sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 2m.\end{aligned}$$

Thus, for each v_i,

$$\begin{aligned} d_{T^e(G)}(v_i)D_{T^e(G)}(v_i) &= 2d_G(v_i)\Big(D_G(v_i) + \sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 2m\Big) \\ &= 2d_G(v_i)D_G(v_i) + 2d_G(v_i)\sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 4md_G(v_i). \end{aligned}$$

Let $x \in A$ and v_j be a neighbour of x. Let $y \in A \setminus \{x\}$ and v_l be a neighbour of y. Then

$$\begin{aligned} D_{T^e(G)}(x) &= \sum_{i=1}^{n} d_{T^e(G)}(x, v_i) + \sum_{y\in A\setminus\{x\}} d_{T^e(G)}(x, y) \\ &= \sum_{i=1}^{n}(d_G(v_i, v_j) + 1) + \sum_{l=1}^{n} d_G(v_l)(d_G(v_j, v_l) + 2) - 2 \\ &= D_G(v_j) + \sum_{l=1}^{n} d_G(v_l)d_G(v_j, v_l) + n + 4m - 2, \end{aligned}$$

where

$$\sum_{y\in A\setminus\{x\}} d_{T^e(G)}(x, y) = \sum_{l=1}^{n} d_G(v_l)(d_G(v_j, v_l) + 2) - 2$$

holds due to the fact that there are $d_G(v_l)$ vertices y adjacent to v_l in $T^e(G)$ when $l \neq j$ and the fact that when $l = j$ there are $d_G(v_j) - 1$ vertices y adjacent to v_j, not including the vertex x.

For each $x \in A$, we thus have

$$\begin{aligned} d_{T^e(G)}(x)D_{T^e(G)}(x) &= 2\Big(D_G(v_j) + \sum_{l=1}^{n} d_G(v_l)d_G(v_j, v_l) + n + 4m - 2\Big) \\ &= 2D_G(v_j) + 2\sum_{l=1}^{n} d_G(v_l)d_G(v_j, v_l) + 2n + 8m - 4 \end{aligned}$$

and then

$$\begin{aligned} D'(T^e(G)) &= \sum_{i=1}^{n} d_{T^e(G)}(v_i)D_{T^e(G)}(v_i) + \sum_{x\in A} d_{T^e(G)}(x)D_{T^e(G)}(x) \\ &= \sum_{i=1}^{n}\Big(2d_G(v_i)D_G(v_i) + 2d_G(v_i)\sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 4md_G(v_i)\Big) \\ &\quad + \sum_{x\in A}\Big(2D_G(v_j) + 2\sum_{l=1}^{n} d_G(v_l)d_G(v_j, v_l) + 2n + 8m - 4\Big) \\ &= \sum_{i=1}^{n}\Big(2d_G(v_i)D_G(v_i) + 2d_G(v_i)\sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 4md_G(v_i)\Big) \\ &\quad + \sum_{j=1}^{n} d_G(v_j)\Big(2D_G(v_j) + 2\sum_{l=1}^{n} d_G(v_l)d_G(v_j, v_l) + 2n + 8m - 4\Big) \end{aligned}$$

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$$= 2D'(G) + 2\sum_{i=1}^{n} d_G(v_i)\sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 8m^2 + 2D'(G)$$

$$+ 2\sum_{j=1}^{n} d_G(v_j)\sum_{l=1}^{n} d_G(v_l)d_G(v_j, v_l) + 4mn + 16m^2 - 8m$$

$$= 4D'(G) + 4\sum_{i=1}^{n} d_G(v_i)\sum_{k=1}^{n} d_G(v_k)d_G(v_i, v_k) + 24m^2 + 4mn - 8m$$

$$= 4D'(G) + 8S^*(G) + 24m^2 + 4mn - 8m,$$

in view of (2.3). This completes the proof. □

3. Concluding remarks

In this paper, we have given explicit formulas for the degree distance of three composite graphs, namely, double graphs, extended double covers and edge copied graphs. One may consider the degree distance of other composite graphs formed by graph operations such as union, join, Cartesian product, disjunction product, and so on.

For a graph G, its *kth iterated double graph* G^{k*} is defined by $G^{k*} = (G^{(k-1)*})^*$ for $k \geq 2$, with $G^{1*} = G^*$. We write $G^{0*} = G$ for consistency. The *kth iterated extended double cover* and *kth iterated edge copied graph* can be defined in an analogous way. One can use the formulas we obtained in Theorems 2.1, 2.2 and 2.3 to obtain the degree distances of these iterated composite graphs.

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