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STABILITY OF AN EXPONENTIAL-MONOMIAL FUNCTIONAL EQUATION

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Abstract

Let N be a fixed positive integer and $f : \mathbb{R} \to \mathbb{C}$. As a generalisation of the superstability of the exponential functional equation we consider the functional inequalities

$$\left|f(\sqrt[N]{x^N + y^N}) - f(x)f(y)\right| \le \phi(x),$$
$$\left|f(\sqrt[N]{x^N + y^N}) - f(x)f(y)\right| \le \psi(x, y)$$

for all $x, y \in \mathbb{R}$, where $\phi : \mathbb{R} \to \mathbb{R}^+$ is an arbitrary function and $\psi : \mathbb{R}^2 \to \mathbb{R}^+$ satisfies a certain condition.

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1. Introduction

Throughout, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of real numbers, nonnegative real numbers and complex numbers respectively and $\delta \ge 0$. A function $E : \mathbb{R} \to \mathbb{C}$ is called an *exponential function* if $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

The Ulam problem for functional equations goes back to 1940.

Problem 1.1 (Ulam [11]). Suppose that f is a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot, \cdot)$ such that

$$d(f(xy), f(x)f(y)) \le \delta \quad \text{for all } x, y \in G_1.$$

Does there exist a group homomorphism h and $\theta_\delta > 0$ such that

$$d(f(x), h(x)) \le \theta_\delta \quad \text{for all } x \in G_1?$$

This problem was solved affirmatively by Hyers under the assumption that G_2 is a Banach space (see Hyers [7], Hyers *et al.* [8]). In the case of functions $f : \mathbb{R} \to \mathbb{R}$, it is known that if f satisfies

$$|f(x + y) - f(x)f(y)| \le \delta \tag{1.1}$$

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for all $x, y \in \mathbb{R}$, then f is either a bounded function satisfying $|f(x)| \le \frac{1}{2}(1 + \sqrt{1 + 4\delta})$ for all $x \in \mathbb{R}$ or an exponential function (see Baker [1], Baker *et al.* [2]). Székelyhidi [10] generalised this result to the case when the difference in (1.1) is bounded for each fixed y (or, equivalently, for each fixed x).

During the thirty-first International Symposium on Functional Equations, Th. M. Rassias posed a problem concerning the behaviour of solutions of the functional inequality

$$|f(x + y) - f(x)f(y)| \le \theta(\|x\|^p + \|y\|^p) \tag{1.2}$$

for all $x, y \in \mathbb{R}$ and for some $\theta > 0, p > 0$ (see [9, page 211]). In response, Găvruţă investigated the stability of (1.2) in [6] (see also [9, Theorem 9.6]). A refined version of the result can be found in [5].

We consider the Hyers–Ulam stability of the N-radical functional equation

$$f(\sqrt[N]{x^N + y^N}) = f(x)f(y) \tag{1.3}$$

for all $x, y \in \mathbb{R}$, that is, we consider the functional inequalities

$$\left|f(\sqrt[N]{x^N + y^N}) - f(x)f(y)\right| \le \phi(x), \tag{1.4}$$

$$\left|f(\sqrt[N]{x^N + y^N}) - f(x)f(y)\right| \le \psi(x, y) \tag{1.5}$$

for all $x, y \in \mathbb{R}$, where $\phi : \mathbb{R} \to \mathbb{R}^+$ is an arbitrary function and $\psi : \mathbb{R}^N \to \mathbb{R}^+$ is a symmetric even function in each variable such that there exist positive constants a_1, a_2 with

$$\psi(x, y) \le a_1(\psi(x, x) + \psi(y, y)), \tag{1.6}$$

$$\psi(\sqrt[N]{x^N + y^N}, z) \le a_2(\psi(x, z) + \psi(y, z)) \tag{1.7}$$

for all $x, y, z \in \mathbb{R}$. In Section 2, we consider the functional inequality (1.4) and, in Section 3, we consider the functional inequality (1.5).

Remark 1.2. It is easy to see that if ψ satisfies (1.6) and (1.7), then there exist positive constants c_1, c_2 such that

$$\psi(\sqrt[N]{2}x,\ \sqrt[N]{2}x) \le c_1\psi(x, x), \tag{1.8}$$

$$\psi(\sqrt[N]{2x^N + y^N}, z) \le c_2\psi(x, x) + \beta(y, z) \tag{1.9}$$

for all $x, y, z \in \mathbb{R}$, where $\beta : \mathbb{R}^2 \to \mathbb{R}^+$ is an appropriately chosen function.

2. Stability with perturbations of one variable

In this section, we consider the functional equation (1.3) and the functional inequality (1.4). We first exhibit in Lemma 2.1 the general solutions of the functional equation (1.3). We exclude the trivial case when $f(x) = 0$ for all $x \in \mathbb{R}$.

A slightly different description of the solutions to (1.3) is given in [3, Corollary 2.2].

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A mapping $f : \mathbb{R} \to \mathbb{C}$ is called *a general monomial of degree N* if it satisfies the functional equation

$$\Delta_y^N f(x) - N!f(y) = 0 \tag{2.1}$$

for all $x, y \in \mathbb{R}$, where the difference operator Δ_y is defined by $\Delta_y f(x) = f(x+y) - f(x)$ for all $x, y \in \mathbb{R}$ and Δ_y^n is defined by $\Delta_y^{n+1} f = \Delta_y(\Delta_y^n f)$ for $n = 1, 2, \ldots$. Using iteration, we can see that

$$\Delta_y^N f(x) = \sum_{k=0}^{N} \binom{N}{k} (-1)^k f(x + (N-k)y)$$

for all $x, y \in \mathbb{R}$.

Lemma 2.1. *All nontrivial solutions of the functional equation (1.3) are of the form*

$$f(x) = e^{p_N(x)} \quad \text{for all } x \in \mathbb{R}, \quad \text{or} \quad f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases} \tag{2.2}$$

where $p_N : \mathbb{R} \to \mathbb{R}$ is a monomial function of degree N.

Proof. Replacing y by $-y$ in (1.3) shows that f is an even function if N is even. Replacing y by x in (1.3) gives $f(x)^2 = f(\sqrt[N]{2x^N})$ for all $x \in \mathbb{R}$ and it follows that $f(x) \geq 0$ for all $x \in R$. Putting $x = y = 0$ in (1.3) gives $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, putting $y = 0$ in (1.3) gives $f(\sqrt[N]{x^N}) = f(x)f(0) = 0$ for all $x \in \mathbb{R}$, which implies that $f(x) = 0$ for all $x \in \mathbb{R}$. Thus, we have $f(0) = 1$.

First, we assume that $f(a) = 0$ for some $a \in \mathbb{R}$. Putting $y = a$ in (1.3) gives $f(\sqrt[N]{x^N + a^N}) = f(x)f(a) = 0$ for all $x \in \mathbb{R}$, which implies that $f(x) = 0$ for all $x \geq |a|$. Putting $x = y = |a|/\sqrt[N]{2}$ in (1.3) gives $f(|a|/\sqrt[N]{2})^2 = f(\sqrt[N]{|a|^N}) = 0$. By induction, $f(|a|/\sqrt[N]{2}^k) = 0$ for all positive integers k. Let $c > 0$ be given. Since we can choose a positive integer k so that $|a|/\sqrt[N]{2}^k \leq c$, we have $f(c) = 0$. Thus, we have $f(x) = 0$ for all $x \neq 0$, which gives the second case of (2.2).

Now we assume that $f(x) > 0$ for all $x \neq 0$. Set $g(x) = \ln f(x)$ for all $x \in \mathbb{R}$, so that

$$g(\sqrt[N]{x^N + y^N}) = g(x) + g(y) \tag{2.3}$$

for all $x, y \in \mathbb{R}$. Putting $y = x = 0$ in (2.3) shows that $g(0) = 0$ and so putting $y = -x$ in (2.3) shows that g is even if N is even and odd if N is odd. By iteration,

$$g\Big(\sqrt[N]{x_0^N + x_1^N + \cdots + x_{m-1}^N}\Big) = g(x_0) + g(x_1) + \cdots + g(x_{m-1}) \tag{2.4}$$

for all $x_0, x_1, \ldots, x_{m-1} \in \mathbb{R}$. Putting $x_0 = x_1 = \cdots = x_{m-1} = x$ in (2.4) gives

$$g(\sqrt[N]{m}x) = g(\sqrt[N]{mx^N}) = mg(x) \tag{2.5}$$

for all $x \in \mathbb{R}$ and all positive integers m. We first consider the case when N is odd. Since $g(x) = x^N$ satisfies the functional equation (2.1),

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} (x + (N-k)y)^N = N!y^N \tag{2.6}$$

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for all $x, y \in \mathbb{R}$. In (2.4), set $m = N+1$ and $x_k = (-1)^k \binom{N}{k}^{1/N}(x + (N-k)y)$ for $k = 0, 1, \ldots, N$. By (2.5) and (2.6),

$$\begin{aligned} N!g(y) = g(\sqrt[N]{N!y^N}) &= g\Big(\Big(\sum_{k=0}^{N}(-1)^k\binom{N}{k}(x+(N-k)y)^N\Big)^{1/N}\Big) \\ &= \sum_{k=0}^{N} g\Big((-1)^k\sqrt[N]{\binom{N}{k}}(x+(N-k)y)\Big) \\ &= \sum_{k=0}^{N}(-1)^k\binom{N}{k}g(x+(N-k)y)) = \Delta_y^N g(x) \end{aligned} \tag{2.7}$$

for all $x, y \in \mathbb{R}$. Now we consider the case when N is even. From (2.4),

$$\begin{aligned} &g(x_1) + g(x_3) + \cdots + g(x_{N-1}) + g\Big(\sqrt[N]{x_0^N + x_2^N + \cdots + x_N^N - x_1^N - x_3^N - \cdots - x_{N-1}^N}\Big) \\ &= g\Big(\sqrt[N]{x_0^N + x_2^N + \cdots + x_N^N}\Big) \\ &= g(x_0) + g(x_2) + \cdots + g(x_N) \end{aligned}$$

for all $x_0, x_1, \ldots, x_N \in \mathbb{R}$ with $x_0^N + x_2^N + \cdots + x_n^N - x_1^N - x_3^N - \cdots - x_{N-1}^N \geq 0$, which implies that

$$g\Big(\Big(\sum_{k=0}^{N}(-1)^k x_k^N\Big)^{1/N}\Big) = \sum_{k=0}^{N}(-1)^k g(x_k) \tag{2.8}$$

for all $x_0, x_1, \ldots, x_N \in \mathbb{R}$ with $\sum_{k=0}^{N}(-1)^k x_k^N \geq 0$. Putting $x_k = \binom{N}{k}^{1/N}(x + (N-k)y)$ for $k = 0, 1, 2, \ldots, N$ in (2.8), we again get (2.7). Thus, g is a monomial function of degree N and $f(x) = e^{p_N(x)}$, which gives the first case of (2.2). This completes the proof. □

Theorem 2.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality*

$$|f(x)f(y) - f(\sqrt[N]{x^N + y^N})| \leq \phi(x) \tag{2.9}$$

for all $x, y \in \mathbb{R}$. Then either f is a bounded function satisfying

$$|f(x)| \leq \tfrac{1}{2}(1 + \sqrt{1 + 4\phi(x)}) \tag{2.10}$$

for all $x \in \mathbb{R}$ or f satisfies the functional equation (1.3).

Proof. First, we assume that f is bounded. Using the triangle inequality with (2.9) and letting $M := \sup_{x\in\mathbb{R}} |f(x)|$,

$$|f(x)f(y)| \leq |f(\sqrt[N]{x^N + y^N})| + \phi(x) \leq M + \phi(x) \tag{2.11}$$

for all $x, y \in \mathbb{R}$. Taking the supremum of the left-hand side of (2.11) with respect to y,

$$|f(x)|M \leq M + \phi(x)$$

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for all $x \in \mathbb{R}$, which implies that

$$M(|f(x)| - 1) \leq \phi(x)$$

for all $x \in \mathbb{R}$. The inequality (2.10) holds for all $x \in \mathbb{R}$ such that $|f(x)| \leq 1$. If $|f(x)| > 1$, then

$$|f(x)|(|f(x)| - 1) \leq \phi(x) \tag{2.12}$$

for all $x \in \mathbb{R}$. Fixing x and solving the quadratic inequality (2.12) gives (2.10).

Now we assume that f is unbounded. Choosing a sequence $y_n \in \mathbb{R}, n = 1, 2, 3, \ldots,$ such that $|f(y_n)| \to \infty$ as $n \to \infty$, putting $y = y_n, n = 1, 2, 3, \ldots,$ in (2.9), dividing the result by $|f(y_n)|$ and letting $n \to \infty$ gives

$$f(x) = \lim_{n\to\infty} \frac{f(\sqrt[N]{x^N + y_n^N})}{f(y_n)} \tag{2.13}$$

for all $y_n, x \in \mathbb{R}$. Multiplying both sides of (2.13) by $f(y)$ and using (2.9) and (2.13),

$$f(y)f(x) = \lim_{n\to\infty} \frac{f(y)f(\sqrt[N]{x^N + y_n^N})}{f(y_n)} = \lim_{n\to\infty} \frac{f(\sqrt[N]{y^N + x^N + y_n^N}) + R(y_n, x, y)}{f(y_n)} \tag{2.14}$$

for all $y_n, x, y \in \mathbb{R}$, where $R(y_n, x, y) = f(y)f(\sqrt[N]{x^N + y_n^N}) - f(\sqrt[N]{y^N + x^N + y_n^N})$. From (2.9),

$$|R(y_n, x, y)| \leq \left|\phi(\sqrt[N]{x^N + y^N})\right| \tag{2.15}$$

for all $y_n, x, y \in \mathbb{R}$. Dividing (2.15) by $|f(y_n)|$ gives

$$\frac{R(y_n, x, y)}{f(y_n)} \to 0 \quad \text{as } n \to \infty.$$

Thus, from (2.13) and (2.14),

$$f(y)f(x) = \lim_{n\to\infty} \frac{f(\sqrt[N]{(\sqrt[N]{x^N + y^N})^N + y_n^N})}{f(y_n)} = f(\sqrt[N]{x^N + y^N})$$

for all $x, y \in \mathbb{R}$. The proof is complete. □

Remark 2.3. An analogous result to Theorem 2.2 can be derived from the much more involved [4, Theorem 2]. The estimation resulting from [4, (18)] is better than (2.11) when the parameter $\delta(t)$ satisfies $\delta(t) \leq M^2 - M$.

3. Stability with perturbations of all variables

In this section, we consider the functional inequality (1.5).

Let $\mathbb{R}^* = \{x \in \mathbb{R} : \psi(x, x) \neq 0\}$. From (1.8), $\sup_{x\in\mathbb{R}^*} \psi(\sqrt[N]{2}x, \sqrt[N]{2}x)/\psi(x, x) < \infty$. From now on, we set $\lambda = \max\{1, \sup_{x\in\mathbb{R}^*} \psi(\sqrt[N]{2}x, \sqrt[N]{2}x)/\psi(x, x)\}$.

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THEOREM 3.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality*

$$|f(\sqrt[N]{x^N + y^N}) - f(x)f(y)| \le \psi(x, y) \tag{3.1}$$

for all $x, y \in \mathbb{R}$. Then either f satisfies

$$|f(x)| \le \tfrac{1}{2}(\sqrt{\lambda} + \sqrt{\lambda + 4\psi(x, x)}) \tag{3.2}$$

for all $x \in \mathbb{R}$ or f satisfies the functional equation (1.3).

PROOF. Let $L > 0$ be a positive real number and let $\Phi_L(x) = \max\{1, L\psi(x, x)\}$. Then

$$\sup_{x\in\mathbb{R}} \frac{\Phi_L(\sqrt[N]{2}x)}{\Phi_L(x)} \le \lambda \tag{3.3}$$

for all $L > 0$. Also, it is easy to see that

$$\min\{1, L\}\Phi_1(x) \le \Phi_L(x) \le \max\{1, L\}\Phi_1(x) \tag{3.4}$$

for all $x \in \mathbb{R}$ and $L > 0$. From (3.4), either

$$\sup_{x\in\mathbb{R}} \frac{|f(x)|}{\sqrt{\Phi_L(x)}} := M_L < \infty \quad \text{for all } L > 0 \tag{3.5}$$

or

$$\sup_{x\in\mathbb{R}} \frac{|f(x)|}{\sqrt{\Phi_L(x)}} = \infty \quad \text{for all } L > 0. \tag{3.6}$$

First, we assume that (3.5) holds. Replacing y by x in (3.1) and using the triangle inequality in the result,

$$|f(x)|^2 \le |f(\sqrt[N]{2x^N})| + \psi(x, x) \le |f(\sqrt[N]{2x^N})| + \frac{1}{L}\Phi_L(x) \tag{3.7}$$

for all $x \in \mathbb{R}$ and $L > 0$. Dividing (3.7) by $\Phi_L(x)$ and using (3.3) and (3.5),

$$\begin{aligned}\Big(\frac{|f(x)|}{\sqrt{\Phi_L(x)}}\Big)^2 &\le \frac{\big|f(\sqrt[N]{2x^N})\big|}{\Phi_L(x)} + \frac{1}{L} \le M_L\frac{\sqrt{\Phi_L(\sqrt[N]{2x^N})}}{\Phi_L(x)} + \frac{1}{L}\\ &\le M_L\sqrt{\frac{\Phi_L(\sqrt[N]{2}x)}{\Phi_L(x)}} + \frac{1}{L} \le M_L\sqrt{\lambda} + \frac{1}{L}\end{aligned} \tag{3.8}$$

for all $x \in \mathbb{R}$ and $L > 0$. Taking the supremum of the left-hand side of (3.8),

$$M_L^2 - \sqrt{\lambda}M_L - \frac{1}{L} \le 0. \tag{3.9}$$

Solving the quadratic inequality (3.9),

$$M_L \le \frac{1}{2}\Big(\sqrt{\lambda} + \sqrt{\lambda + \frac{4}{L}}\Big). \tag{3.10}$$

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From (3.5) and (3.10),

$$|f(x)| \le \frac{1}{2}\Big(\sqrt{\lambda} + \sqrt{\lambda + \frac{4}{L}}\Big)\sqrt{\max\{1, L\psi(x,x)\}} \tag{3.11}$$

for all $x \in \mathbb{R}$ and $L > 0$. Fix an $x_0 \in \mathbb{R}$. If $\psi(x_0, x_0) > 0$, then applying (3.11) with $L := 1/\psi(x_0, x_0)$ gives

$$|f(x)| \le \frac{1}{2}(\sqrt{\lambda} + \sqrt{\lambda + 4\psi(x_0, x_0)})\sqrt{\max\Big\{1, \frac{\psi(x,x)}{\psi(x_0,x_0)}\Big\}} \tag{3.12}$$

for all $x \in \mathbb{R}$. Putting $x = x_0$ in (3.12),

$$|f(x_0)| \le \tfrac{1}{2}(\sqrt{\lambda} + \sqrt{\lambda + 4\psi(x_0, x_0)}). \tag{3.13}$$

On the other hand, if $\psi(x_0, x_0) = 0$, then, from (3.11),

$$|f(x_0)| \le \frac{1}{2}\Big(\sqrt{\lambda} + \sqrt{\lambda + \frac{4}{L}}\Big) \tag{3.14}$$

for all $L > 0$. Letting $L \to \infty$ in (3.14),

$$|f(x_0)| \le \sqrt{\lambda} = \tfrac{1}{2}(\sqrt{\lambda} + \sqrt{\lambda + 4\psi(x_0, x_0)}). \tag{3.15}$$

Thus, from (3.13) and (3.15) we reach the alternative (3.2) in the theorem.

Secondly, we assume that (3.6) holds. Then we can choose a sequence $x_n \in \mathbb{R}$ for $n = 1, 2, \ldots$ such that

$$\frac{\psi(x_n, x_n)}{|f(x_n)|^2} + \frac{1}{|f(x_n)|^2} \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Replacing (x, y) by $(\sqrt[N]{x^N + y^N}, z)$ in (3.1),

$$|f(\sqrt[N]{x^N + y^N})f(z) - f(\sqrt[N]{x^N + y^N + z^N})| \le \psi(\sqrt[N]{x^N + y^N}, z) \tag{3.17}$$

for all $x, y, z \in \mathbb{R}$. Multiplying both sides of (3.1) by $|f(z)|$,

$$|f(x)f(y)f(z) - f(\sqrt[N]{x^N + y^N})f(z)| \le \psi(x,y)|f(z)| \tag{3.18}$$

for all $x, y, z \in \mathbb{R}$. Using the triangle inequality with (3.17) and (3.18),

$$|f(x)f(y)f(z) - f(\sqrt[N]{x^N + y^N + z^N})| \le \psi(\sqrt[N]{x^N + y^N}, z) + \psi(x,y)|f(z)| \tag{3.19}$$

for all $x, y, z \in \mathbb{R}$.

Replacing both x and y by x_n in (3.19), dividing the result by $|f(x_n)|^2$ and using (1.9),

$$\begin{aligned}\Big|\frac{f(\sqrt[N]{2x_n^N + z^N})}{f(x_n)^2} - f(z)\Big| &\le \frac{\psi(\sqrt[N]{2}x_n, z) + \psi(x_n, x_n)|f(z)|}{|f(x_n)|^2} \\ &\le \frac{(c_2 + |f(z)|)\psi(x_n, x_n) + \beta(0, z)}{|f(x_n)|^2}\end{aligned} \tag{3.20}$$

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for all $x_n, z \in \mathbb{R}$ and c_2 a positive constant. Letting $n \to \infty$ in (3.20) and using (3.16),

$$f(z) = \lim_{n\to\infty} \frac{f(\sqrt[N]{2x_n^N + z^N})}{f(x_n)^2} \tag{3.21}$$

for all $x_n, z \in \mathbb{R}$. Multiplying both sides of (3.21) by $f(w)$ and using (3.1),

$$f(z)f(w) = \lim_{n\to\infty} \frac{f(\sqrt[N]{2x_n^N + z^N})f(w)}{f(x_n)^2} = \lim_{n\to\infty} \frac{f(\sqrt[N]{2x_n^N + z^N + w^N}) + R(x_n, z, w)}{f(x_n)^2} \tag{3.22}$$

for all $x_n, z, w \in \mathbb{R}$, where $R(x_n, z, w) = f(\sqrt[N]{2x_n^N + z^N})f(w) - f(\sqrt[N]{2x_n^N + z^N + w^N})$. Using (1.9),

$$|R(x_n, z, w)| \le \psi(\sqrt[N]{2x_n^N + z^N}, w) \le c_2\psi(x_n, x_n) + \beta(z, w) \tag{3.23}$$

for all $x_n, z, w \in \mathbb{R}$. Using (3.16) in (3.23),

$$\frac{R(x_n, z, w)}{f(x_n)^2} \to 0 \quad \text{as } n \to \infty.$$

Thus, from (3.21) and (3.22),

$$f(z)f(w) = \lim_{n\to\infty} \frac{f(\sqrt[N]{2x_n^N + z^N + w^N})}{f(x_n)^2} = f(\sqrt[N]{x^N + w^N})$$

for all $z, w \in \mathbb{R}$. The proof is complete. □

Remark 3.2. As a matter of fact, fixing $x \in \mathbb{R}$ and taking the infimum of the right-hand side of (3.11) with respect to $L > 0$ we get the inequality (3.2).

Remark 3.3. Let $p_j, q_j, a_j, j = 1, 2, \ldots, m$, be sequences of nonnegative real numbers. Then

$$\psi(x, y) = \sum_{j=1}^{m} a_j |x|^{p_j} |y|^{q_j}$$

satisfies (1.6) and (1.7) and, if $p = \max\{p_j + q_j : j = 1, 2, \ldots, m\}$, then $\lambda = \sqrt[N]{2^p}$.

As a direct consequence of Theorem 3.1, we obtain the Hyers–Ulam–Rassias stability of the Gaussian functional equation, which is the case $N = 2$ of the following corollary.

Corollary 3.4. *Let $p, q, r, \theta_1, \theta_2$ be given nonnegative real numbers. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality*

$$\left|f(x)f(y) - f(\sqrt[N]{x^N + y^N})\right| \le \theta_1 |x|^p |y|^q + \theta_2(|x|^r + |y|^r)$$

for all $x, y \in \mathbb{R}$. Then either f satisfies

$$|f(x)| \le \tfrac{1}{2}\Big(\sqrt[2N]{2^\mu} + \sqrt{\sqrt[N]{2^\mu} + 4\theta_1 |x|^{p+q} + 8\theta_2 |x|^r}\Big)$$

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for all $x \in \mathbb{R}$, *where*

$$\mu = \begin{cases} \max\{p+q, r\} & \text{if } \theta_1\theta_2 \neq 0, \\ p+q & \text{if } \theta_1 \neq 0, \theta_2 = 0, \\ r & \text{if } \theta_1 = 0, \theta_2 \neq 0, \end{cases}$$

or f *satisfies the functional equation (1.3).*

REMARK 3.5. Corollary 3.4 reduces to Hyers–Ulam–Rassias stability if $\theta_1 = 0$, to Ulam–Găvruţă–Rassias stability if $\theta_2 = 0$ and to Ulam–Rassias stability if $\theta_1 = \theta_2$.

References

[1] J. A. Baker, 'The stability of the cosine functional equation', *Proc. Amer. Math. Soc.* **80** (1980), 411–416.

[2] J. A. Baker, J. Lawrence and F. Zorzitto, 'The stability of the equation $f(x+y) = f(x)f(y)$', *Proc. Amer. Math. Soc.* **74** (1979), 242–246.

[3] J. Brzdęk, 'Remarks on solutions to the functional equations of the radical type', *Adv. Theor. Nonlinear Anal. Appl.* **1**(2) (2017), 125–135.

[4] J. Brzdęk, A. Najdecki and B. Xu, 'Two general theorems on superstability of functional equations', *Aequationes Math.* **89** (2015), 771–783.

[5] J. Chung, 'General stability of the exponential and Lobačevskiĭ functional equations', *Bull. Aust. Math. Soc.* **94**(2) (2016), 278–285.

[6] P. Găvruţă, 'An answer to a question of Th. M. Rassias and J. Tabor on mixed stability of mappings', *Bul. St. Univ. 'Politehnica' Timisoara Ser. Mat. Fiz.* **42**(56) (1997), 1–6.

[7] D. H. Hyers, 'On the stability of the linear functional equation', *Proc. Natl. Acad. Sci. USA* **27** (1941), 222–224.

[8] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables* (Birkhauser, Basel, 1998).

[9] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis* (Springer, New York, 2011).

[10] L. Székelyhidi, 'On a theorem of Baker, Lawrence and Zorzitto', *Proc. Amer. Math. Soc.* **84** (1982), 95–96.

[11] S. M. Ulam, *Problems in Modern Mathematics* (Wiley, New York, 1940), Ch. 6.

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