J. Austral. Math. Soc. Ser. B 36(1994), 101–106

THE TAYLOR SERIES FOR BANDLIMITED SIGNALS

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(Received 28 April 1992; revised 27 November 1992)

Abstract

We study a special type of infinite product, called an infinite product of Cardano type, and we obtain its Taylor series. We prove that Hadamard's factorization of bandlimited signals is given by an infinite product of Cardano type, and apply our results to obtain the Taylor series for bandlimited signals.

1. Introduction

Let Q be a polynomial of degree q, normalized by $Q(0) = 1$ and with roots $[w_n]_{n=1,\dots,q}$. Cardano's formulae allow us to represent Q in the form

$$Q(z) = \prod_{n=1}^{q}\left(1 - \frac{z}{w_n}\right) = \sum_{k=0}^{q}(-1)^k D_k z^k,$$

where $D_0 = 1$ and $D_k = \sum_{1\le i_1<\dots<i_k\le q} w_{i_1}^{-1}\dots w_{i_k}^{-1}$ for $1 \le k \le q$.

The main purpose of this paper is to extend this result to bandlimited signals. Let f be a bandlimited signal which has been normalized by $f(0) = 1$. We investigate when we can write

$$f(z) = \prod_{n\ge 1}\left(1 - \frac{z}{w_n}\right) = \sum_{k\ge 0}(-1)^k D_k z^k, \tag{1}$$

where $\Lambda = \{w_n : n \ge 1\}$ is the set of zeros of the function f, and where $D_0 = 1$ and for $k \ge 1$:

$$D_k = \lim_n \sum_{1\le i_1<\dots<i_k\le n} w_{i_1}^{-1}\dots w_{i_k}^{-1}.$$

Notice that Λ is a countable set because in our conditions f is analytic in the entire plane [3].

Several questions arise when we consider this problem :

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(i) The convergence in $\mathbb{C}$ of the infinite product $\prod_{n\geq 1}(1 - z/w_n)$.
(ii) The existence of D_k and convergence of the series $\sum_{k\geq 0}(-1)^k D_k z^k$.
(iii) Conditions on a bandlimited signal f so that (1) can be satisfied.

The paper is organized as follows. In Section 1, (i) and (ii) are studied. We show that the second equality in (1) is valid if rank $\Lambda = m$ and $B_1, \dots, B_m$ exist, where $B_j = \sum_{n\geq 1} w_n^{-j}$. In Section 2 we discuss the question (iii). Then, applying a result of Tichmarsh [4], we prove that a bandlimited signal f, normalized by $f(0) = 1$, which has a symmetric spectrum with respect to the origin, can be written as

$$f(z) = \prod_{n\geq 1}\left(1 - \frac{z}{w_n}\right) \quad \text{where } |w_1| \leq |w_2| \leq \dots \leq |w_n| \dots .$$

Then, using the results obtained in Section 1, we prove that f satisfies (1).

2. Infinite products of Cardano type

In what follows, $\Lambda = [w_n]_{n\geq 1}$ is an unbounded set of nonzero complex numbers, with no limit points and rank m; that is, m is the smallest integer such that

$$\sum_{n\geq 1} |w_n|^{-(m+1)} < +\infty.$$

DEFINITION. $\prod_{n\geq 1}(1 - z/w_n)$ is said to be an *infinite product of Cardano type* if it coincides with $\sum_{k\geq 0}(-1)^k D_k z^k$ and defines an entire function, where

$$D_k = \lim_n D_k^{(n)}, \qquad \text{and} \qquad D_k^{(n)} = \sum_{1\leq i_1\leq \dots \leq i_k \leq n} w_{i_n}^{-1} \dots w_{i_k}^{-1}. \tag{2}$$

We shall write $\Lambda \in CT$ when the associated product is of Cardano type.

This definition leads us firstly to study the existence of the D_k. We introduce the following notation: if $n \geq k$

$$E_{i_1,\dots,i_k}^{(n)} = \sum_{t_i\neq t_j,\ i\neq j} w_{t_1}^{-i_1} \dots w_{t_k}^{-i_k}, \quad 1 \leq t_j \leq n, \quad i_j \in \mathbb{N}; \qquad B_k^{(n)} = \sum_{t=1}^{n} w_t^{-k}.$$

When the limits exist, we write:

$$E_{i_1,\dots,i_k} = \lim_n E_{i_1,\dots,i_k}^{(n)} \tag{3}$$

and

$$B_k = \lim_n B_k^{(n)}. \tag{4}$$

To simplify, we shall denote $E_{1,1,1,1,1}^{(n)} = E_{1,.(5..,1}^{(n)}$, for example.

THEOREM 1. *D_k exists for every k if and only if B_k exists for every k. Moreover*

$$D_{k+1} = \frac{(-1)^k}{k+1}\left[B_{k+1} + \sum_{t=1}^{k}(-1)^t D_t B_{k+1-t}\right]. \tag{5}$$

PROOF. First we prove that if $n \geq k$ we have

$$E^{(n)}_{1,.(k..,1}B^{(n)}_j = kE^{(n)}_{j+1,1,.(k-1..,1} + E^{(n)}_{j,1,.(k..,1}, \quad \text{for } j,k \in \mathbb{N}. \tag{6}$$

$$E^{(n)}_{j,1,.(k..,1} = (-1)^k k!\left[\sum_{t=0}^{k}(-1)^t D^{(n)}_t B^{(n)}_{k+j-t}\right], \quad \text{for } j,k \in \mathbb{N}. \tag{7}$$

$$E^{(n)}_{j,1,.(k-j..,1} \doteq (-1)^{j-1}(k-j)!\left[kD^{(n)}_k + \sum_{t=1}^{j-1}(-1)^t D^{(n)}_{k-t}B^{(n)}_t\right],$$
$$\text{for } 2 \leq j \leq k-1 \text{ and } k \geq 3. \tag{8}$$

Indeed, when $n > k$ we obtain

$$\begin{aligned}
E^{(n)}_{1,.(k..,1}B^{(n)}_j &= \sum_{i=1}^{k}\left[\sum_{t_p \neq t_q,\ p\neq q} w^{-1}_{t_1}\dots w^{-(j+1)}_{t_i}\dots w^{-1}_{t_k}\right] + \sum_{t_p\neq t_q,\ p\neq q} w^{-1}_{t_1}\dots w^{-1}_{t_k} w^{-j}_{t_{k+1}} \\
&= \sum_{i=1}^{k} E^{(n)}_{1,.(i-1..,1,j+1,1,.(k-i..,1} + E^{(n)}_{1,.(k..,1,j} \\
&= kE^{(n)}_{j+1,1,.(k-1..,1} + E^{(n)}_{j,1,.(k-1..,1}
\end{aligned}$$

and it proves (6).

On the other hand, for $k = 1$, (7) is deduced from (6) and from the equalities: $E^{(n)}_1 = B^{(n)}_1 = D^{(n)}_1$ and $E^{(n)}_k = B^{(n)}_k$. Now, from (6) we have

$$\begin{aligned}
E^{(n)}_{j,1,.(k..,1} &= E^{(n)}_{1,.(k..,1}B^{(n)}_j - kE^{(n)}_{j+1,1,.(k-1..,1} \\
&= E^{(n)}_{1,.(k..,1}B^{(n)}_j - k\left[E^{(n)}_{1,.(k-1..,1}B^{(n)}_{j+1} - (k-1)E^{(n)}_{j+2,1,.(k-2..,1}\right].
\end{aligned}$$

Then by using induction, (7) is proved by noting that $E^{(n)}_{1,.(k..,1} = k!D^{(n)}_k$.

From (6), considering $k-1$ instead of k, we obtain

$$E^{(n)}_{1,.(k-1..,1}B^{(n)}_1 = (k-1)E^{(n)}_{2,1,.(k-2..,1} + E^{(n)}_{1,.(k..,1}.$$

Then it follows that

$$E^{(n)}_{2,1,.(k-2..,1} = (-1)(k-2)!\left[kD^{(n)}_k - D^{(n)}_{k-1}B^{(n)}_1\right].$$

Therefore (8) is verified for $j = 2$. Now, (8) turns out to be immediate using induction on j.

As a consequence of this we have:

(i) If $D_1, \dots, D_k$ and $B_j, \dots, B_{j+k}$ exist for any $j \geq 1$, then $E_{j,1,.(k..,1}$ exists for any $j \geq 1$.
(ii) If $D_2, \dots, D_k$ and $B_1, \dots, B_k$ exist with $k \geq 3$, then $E_{j,1,.(k-j..,1}$ exists for $2 \leq j \leq k-1$.

Next we are going to prove the Theorem. Assume that D_m exists for every m and that $B_2, \dots, B_k$ exist. Then from (6) $D_1^{(n)} B_k^{(n)} = B_{k+1}^{(n)} + E_{k,1}^{(n)}$. Since $D_1, \dots, D_{k+1}$ and $B_1, \dots, B_{k-1}$ exist, it follows from (ii) that $E_{j,1,.(k+1-j..,1}$ exists for $2 \leq j \leq k$, and so $E_{k,1}$ must exist. Now, taking limits in the last equality we obtain B_{k+1}.

Conversely, the existence of B_k for any $k \in \mathbb{N}$ implies that D_1 exists. Using induction, (i) and the equality

$$(k+1)!D_{k+1}^{(n)} = k!D_k^{(n)} B_1^{(n)} - kE_{2,1,.(k-1..,1}^{(n)}$$

obtained from (6), the existence of D_{k+1} is proved by taking limits in this last expression. Finally, (5) is easily obtained by putting $j = 1$ and taking limits in (7).

THEOREM 2. *$\Lambda \in CT$ if and only if B_k exists for $1 \leq k \leq m$.*

PROOF. Since rank $\Lambda = m$, it is known [2], that the canonical product associated with Λ converges uniformly on compact subsets (c.u.c.) of the plane, that is

$$P_m(z) = \prod_{n\geq 1} \left(1 - \frac{z}{w_n}\right) \exp\left[\sum_{j=1}^{m} \frac{z^j}{j w_n^j}\right] \text{ c.u.c..}$$

If $B_1, B_2, \dots, B_m$ exist, it is not difficult to prove that

$$\prod_{n\geq 1} \exp\left[-\sum_{j=1}^{m} \frac{z^j}{j w_n^j}\right] = \exp\left[-\sum_{j=1}^{m} B_j \frac{z^j}{j}\right],$$

and therefore

$$P(z) = \prod_{n\geq 1} \left(1 - \frac{z}{w_n}\right) \text{ c.u.c..}$$

Then

$$\frac{P'(z)}{P(z)} = \sum_{n\geq 1} \frac{1}{z - w_n} \text{ c.u.c. in } D(0, R) = \{z : |z| < 1\},$$

where $R < |w_n|$, for all n. If we let $g(z) = \sum_{n\geq 1} \frac{1}{z-w_n}$ then $g \in H(D(0, R))$ and $g^{(k)}(0) = (-1)k!B_{k+1}$ for $k = 0, 1, 2, \dots$. As

$$P^{(k+1)}(0) = \sum_{t=0}^{k} \binom{k}{t} P^{(t)}(0) g^{(k-t)}(0),$$

it follows using induction that

$$P^{(k+1)}(0) = (-1)k!\left[B_{k+1} + \sum_{t=0}^{k}(-1)^t D_t B_{k+1-t}\right].$$

Taking into account (5), we have that

$$\prod_{n\geq 1}\left(1 - \frac{z}{w_n}\right) = \sum_{k\geq 0}(-1)^k D_k z^k.$$

The converse is immediate.

Notice that in the case $m = 0$, $\prod_{n\geq 1}(1 - z/w_n)$ converges absolutely and c.u.c.. Moreover $\sum_{n\geq 1} |w_n|^{-k}$ exists for every k and consequently the last result is trivial.

3. Hadamard's factorization and the Taylor series for bandlimited signals

A function f on $\mathbb{R}$ is a bandlimited signal if its Fourier transform is zero outside a finite interval, i.e.

$$\hat{f}(t) = \int_{-\infty}^{+\infty} f(z)e^{-itz}\,dz = 0 \quad \text{for } |t| > \sigma,$$

and its energy,

$$E = \int_{-\infty}^{+\infty} |f(z)|^2\,dz,$$

is finite.

It is easy to prove [3] that if f is a bandlimited signal then f is analytic in the entire plane and of exponential type. We obtain the Taylor series for a bandlimited signal f.

THEOREM 3. *Let f be a bandlimited signal, normalized by $f(0) = 1$ and which has a symmetric spectrum with respect to the origin. Then*

$$f(z) = \sum_{k\geq 0}(-1)^k D_k z^k \quad \text{in } \mathbb{C}.$$

PROOF. Let $F = f_o\phi$, where $\phi(z) = -iz$, and let $z_1 = r_1 e^{i\theta_1}, \ldots, z_n = r_n e^{i\theta_n}, \ldots$ be the zeros of F, arranged so that r_n is a nondecreasing function of n. Then we can write

$$F(z) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} \hat{f}(t)e^{tz}\,dt,$$

where there is no number $\alpha > -\sigma$ such that $\int_{-\sigma}^{\alpha} |\hat{f}(t)|\,dt = 0$ and no number $\beta < \sigma$ such that $\int_{\beta}^{\sigma} |\hat{f}(t)|\,dt = 0$. It is known [4] that $F(z) = \prod_{n\geq 1}(1 - z/z_n)$ where the product is conditionally convergent. Besides,

$$\sum_{n\geq 1} \frac{\cos\theta_n}{r_n} = -\mathrm{Re}\left(\frac{F'(0)}{F(0)}\right) \qquad \text{and} \qquad \sum_{n\geq 1} \frac{\sin\theta_n}{r_n} = -\mathrm{Im}\left(\frac{F'(0)}{F(0)}\right).$$

From the above results, we have that $f(z) = \prod_{n\geq 1}(1 - z/w_n)$, where $w_n = -iz_n$. Then, $[w_n]_{n\geq 1}$ is a rearrangement of the set $Z_f = \{z \in C \mid f(z) = 0\}$.

To finish, we prove that $Z_f \in CT$. As

$$B_1 = \sum_{n\geq 1} \frac{1}{w_n} = \sum_{n\geq 1} \frac{\sin\theta_n}{r_n} + i \sum_{n\geq 1} \frac{\cos\theta_n}{r_n},$$

necessarily B_1 exists. On the other hand, as f is an entire function of exponential type, B_m exists for all $m \geq 2$ [1]. Therefore $Z_f \in CT$.

Acknowledgement

This research was supported by a grant of I. E. R.

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