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SOME INEQUALITIES FOR THE GENERALIZED GRÖTZSCH FUNCTION

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Abstract For $a \in (0, \frac{1}{2}]$ and $r \in (0,1)$, let $\mu_a(r)$ be the so-called generalized Grötzsch function which appears in Ramanujan's generalized modular equations. In this paper, several sharp inequalities for $\mu_a(r)$ are obtained and a conjecture on $\mu_a(r)$, which was presented by Qiu and Vuorinen in 1999, is proved.

Keywords: generalized elliptic integrals; generalized Grötzsch function; monotonicity; inequalities

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1. Introduction

Throughout this paper, we let $r' = \sqrt{1-r^2}$ for $r \in (0,1)$. For $x > 0$, let

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\, \mathrm{d}t \quad \text{and} \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{1.1}$$

be the classical Euler *gamma function* and *psi function*, respectively. For real numbers a, b and c with $c \neq 0, -1, -2, \dots$, the *Gaussian hypergeometric function* is defined by [**1**]

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) \equiv \sum_{n=0}^\infty \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \quad \text{for } |x| < 1. \tag{1.2}$$

Here $(a,0) = 1$ for $a \neq 0$, and (a,n) is the *shifted factorial function*

$$(a,n) \equiv a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)} \tag{1.3}$$

for $n \in \boldsymbol{N} \equiv \{k : k \text{ is a positive integer}\}$. For $r \in (0,1)$ and $a \in (0,1)$, the *generalized elliptic integrals*(cf. [**5**, § 5.5]) are defined as

$$\left.\begin{aligned} \mathcal{K}_a &= \mathcal{K}_a(r) \equiv \tfrac{1}{2}\pi F(a, 1-a; 1; r^2), \\ \mathcal{K}'_a &= \mathcal{K}'_a(r) \equiv \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) &= \tfrac{1}{2}\pi, \qquad \mathcal{K}_a(1) = \infty, \end{aligned}\right\} \tag{1.4}$$

and

$$\left.\begin{aligned}\mathcal{E}_a &= \mathcal{E}_a(r) \equiv \tfrac{1}{2}\pi F(a-1, 1-a; 1; r^2),\\ \mathcal{E}'_a &= \mathcal{E}'_a(r) \equiv \mathcal{E}_a(r'),\\ \mathcal{E}_a(0) &= \tfrac{1}{2}\pi, \qquad \mathcal{E}_a(1) = \frac{\sin(\pi a)}{2(1-a)}.\end{aligned}\right\} \tag{1.5}$$

In the particular case $a = \frac{1}{2}$, the functions $\mathcal{K}_a$ and $\mathcal{E}_a$ reduce to $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively, which are the well-known complete elliptic integrals of the first and second kind, respectively (cf. [**6**]). By symmetry of a and b in (1.2), it is obvious that $\mathcal{K}_a = \mathcal{K}_{1-a}$ for $a \in (0, \frac{1}{2}]$. Hence, we may assume that $a \in (0, \frac{1}{2}]$ in the following. For $a \in (0, \frac{1}{2}]$ and $r \in (0, 1)$, define the *generalized Grötzsch function*

$$\mu_a(r) \equiv \frac{\pi}{2\sin(\pi a)} \frac{\mathcal{K}'_a(r)}{\mathcal{K}_a(r)}, \qquad \mu(r) = \mu_{1/2}(r). \tag{1.6}$$

The function $\mu_a(r)$ plays a very important role in some fields of mathematics. For instance, it is indispensable in geometric function theory, quasiconformal theory and the theory of Ramanujan's modular equations (see [**2**–**4**, **7**]). In the general case $a \in (0, \frac{1}{2})$, however, the known properties of $\mu_a(r)$ are fewer than those of $\mu(r)$, which is the modulus of the Grötzsch ring domain in the plane. One of the tasks for the study of the properties of $\mu_a(r)$ is to extend the known results for $\mu(r)$ to the function $\mu_a(r)$. On the other hand, the comparison of $\mu_a(r)$ and $\mu(r)$ will enable us to use the known bounds of $\mu(r)$ to give estimates for $\mu_a(r)$. In this paper we shall extend some well-known results for $\mu(r)$ to $\mu_a(r)$, and give bounds of $\mu_a(r)$ in terms of $\mu(r)$ as well as in terms of elementary functions. We shall also prove a conjecture concerning $\mu_a(r)$ [**9**], and show some properties of $\mathcal{K}_a$.

2. Main results

We now state our main results. Our first result answers the question of whether the well-known identities [**2**, (5.2) and (5.4)]

$$2\mu\left(\frac{2\sqrt{r}}{1+r}\right) \equiv \mu(r), \qquad \mu\left(\frac{1-r}{1+r}\right) \equiv 2\mu(r') \tag{2.1}$$

can be extended to $\mu_a(r)$.

Theorem 2.1.

(i) *For each $r \in (0, 1)$, the function*

$$f(a, r) \equiv 2\mu_a\left(\frac{2\sqrt{r}}{1+r}\right) - \mu_a(r)$$

is strictly decreasing in a from $(0, \frac{1}{2}]$ onto $[0, \infty)$. In particular, for all $r \in (0, 1)$ and $a \in (0, \frac{1}{2}]$,

$$2\mu_a\left(\frac{2\sqrt{r}}{1+r}\right) \geqslant \mu_a(r), \tag{2.2}$$

with equality if and only if $a = \frac{1}{2}$.

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(ii) *For each* $r \in (0,1)$, *the function*

$$g(a,r) \equiv \mu_a\left(\frac{1-r}{1+r}\right) - 2\mu_a(r')$$

is strictly increasing in a *from* $(0,\frac{1}{2}]$ *onto* $(-\infty,0]$. *In particular, for all* $r \in (0,1)$ *and* $a \in (0,\frac{1}{2}]$,

$$\mu_a\left(\frac{1-r}{1+r}\right) \leqslant 2\mu_a(r'), \tag{2.3}$$

with equality if and only if $a = \frac{1}{2}$.

The next theorem gives comparisons of $\mu_a(r)$ and $\mu(r)$.

Theorem 2.2.

(i) *For each* $r \in (0,1)$, *the function* $h(a,r) \equiv a^2\mu_a(r)$ *is strictly increasing in* a *from* $(0,\frac{1}{2}]$ *onto* $(0,\frac{1}{4}\mu(r)]$.

(ii) *For each* $a \in (0,\frac{1}{2}]$, *the function* $H(r) \equiv \mu_a(r)/\mu(r)$ *is strictly increasing from* $(0,1)$ *onto* $(1, 1/\sin^2(\pi a))$. *In particular,*

$$\mu(r) \leqslant \mu_a(r) \leqslant \frac{1}{\sin^2(\pi a)}\mu(r), \tag{2.4}$$

with equality in each instance if and only if $a = \frac{1}{2}$.

Applying Theorem 2.2, one can easily derive monotonicity properties of certain functions defined in terms of $\mu_a(r)$ and some elementary functions. As an example, we give the following theorem.

Theorem 2.3.

(i) *The function*

$$f_1(r) \equiv \frac{\mu_a(r)}{\sqrt{r'}\log(4/r)}$$

is strictly increasing from $(0,1)$ *onto* $(1,\infty)$.

(ii) *The function*

$$f_2(r) \equiv \frac{\mu_a(r)}{\operatorname{arth}\sqrt[4]{r'}}$$

is strictly increasing from $(0,1)$ *onto* $(1,\infty)$.

(iii) *The function* $f_3(r) \equiv \mu_a(r)\operatorname{arth}\sqrt[4]{r}$ *is strictly increasing from* $(0,1)$ *onto*

$$\left(0, \frac{\pi^2}{4\sin^2(\pi a)}\right).$$

Remark 2.4. In [**11**, Theorem 1.14], the monotonicity properties of $f(a,r)$ and $g(a,r)$ as functions of $r \in (0,1)$, have been obtained, and [**11**, Theorem 1.22] says that the function $\mu_a(r)$ is strictly decreasing from $(0,\frac{1}{2}]$ onto $[\mu(r),\infty)$ with respect to a for given r.

In [**9**], it was conjectured that the function $f_2(r) \equiv \mu_a(r)/\operatorname{arth}\sqrt[4]{r'}$ is increasing from $(0,1)$ onto $(1,\infty)$. Theorem 2.3 (ii) gives the proof of this conjecture.

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3. Preliminaries

In this section, we establish the following technical lemma, which shows some properties of $\mathcal{K}_a$ and is needed in the proofs of the main theorems.

Lemma 3.1.

(i) *For* $x \geqslant 0$ *and* $a \in (0, \frac{1}{2})$, *let* $b = 1-a$. *Then the function* $f_1(x) \equiv \Psi(x+a)-\Psi(x+b)$ *is strictly increasing and concave from* $[0, \infty)$ *onto* $[\Psi(a) - \Psi(b), 0)$.

(ii) *For each* n, *the function* $f_2(a) \equiv (a, n)(1-a, n)$ *is strictly increasing on* $(0, \frac{1}{2}]$, *while the function* $g(a) \equiv f_2(a)/a$ *is strictly decreasing on* $(0, \frac{1}{2}]$.

(iii) *For* $a \in (0, \frac{1}{2}]$, *the function*

$$f_3(r) \equiv \frac{\partial}{\partial a} \log \mathcal{K}_a(r)$$

is strictly increasing on $(0, 1)$.

(iv) *The function* $f_4(a) \equiv \mathcal{K}_a/a$ *is strictly decreasing from* $(0, \frac{1}{2}]$ *onto* $[2\mathcal{K}(r), \infty)$.

(v) *The function*

$$f_5(r) \equiv \frac{\mathcal{K}(r)}{\mathcal{K}_a(r)}$$

is strictly increasing from $(0, 1)$ *onto* $(1, 1/\sin(\pi a))$.

Proof. (i) It is well-known that Ψ' and Ψ'' are strictly decreasing and increasing, respectively, on $(0, \infty)$. Since $a < b$, $f_1'(x) = \Psi'(x + a) - \Psi'(x + b) > 0$, and $f_1''(x) = \Psi''(x + a) - \Psi''(x + b) < 0$. Thus, the result for f_1 follows.

(ii) It follows from (1.3) that

$$f_2(a) = (a, n)(1 - a, n) = \frac{\Gamma(n + a)}{\Gamma(a)} \frac{\Gamma(n + 1 - a)}{\Gamma(1 - a)}.$$

By logarithmic differentiation, we have

$$f_2'(a) = f_2(a)[f_1(n) - f_1(0)],$$

which is a product of two positive functions by part (i). Hence, the monotonicity of f_2 follows.

By the reflection property of the gamma function [**12**],

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)},$$

we get

$$g(a) = \frac{\sin(\pi a)}{\pi a}[\Gamma(n + a)\Gamma(n + 1 - a)]. \tag{3.1}$$

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Since

$$\frac{\mathrm{d}}{\mathrm{d}a}\log[\Gamma(n+a)\Gamma(n+1-a)] = f_1(n) < 0,$$

$\Gamma(n+a)\Gamma(n+1-a)$ is strictly decreasing in a. Clearly, $\sin(\pi a)/(\pi a)$ is strictly decreasing in a on $(0, \frac{1}{2}]$. Hence, the right-hand side of (3.1) is a product of two positive and strictly decreasing functions, so that g is strictly decreasing in a.

(iii) By (1.2), we have

$$f_3(r) = \frac{\sum_{n=0}^{\infty} A_n r^{2n}}{\sum_{n=0}^{\infty} B_n r^{2n}},$$

where $A_n = f_2(a)[f_1(n) - f_1(0)]/(n!)^2$ and $B_n = f_2(a)/(n!)^2$. Since $A_n/B_n = f_1(n) - f_1(0)$ is strictly increasing in n by part (i), f_3 is strictly increasing in r by [**8**, Lemma 2.1].

(iv) By (1.2),

$$f_4(a) = \frac{\pi}{2}\left[\sum_{n=0}^{\infty} \frac{g(a)}{(n!)^2} r^{2n}\right],$$

and hence the monotonicity of f_4 follows from part (ii). The limiting values are clear.

(v) Write

$$C_n = \frac{(\frac{1}{2}, n)^2}{(a,n)(1-a,n)}.$$

Then

$$\begin{aligned} C_{n+1} - C_n &= \frac{(\frac{1}{2}, n)^2}{(a,n)(1-a,n)}\left[\frac{(\frac{1}{2}+n)^2}{(a+n)(1-a+n)} - 1\right] \\ &= \frac{(\frac{1}{2}, n)^2}{(a,n)(1-a,n)} \frac{\frac{1}{4} - a(1-a)}{(a+n)(1-a+n)} \\ &> 0 \end{aligned}$$

and hence C_n is strictly increasing in n. By (1.2),

$$\frac{\mathcal{K}(r)}{\mathcal{K}_a(r)} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}, n)^2}{(n!)^2} r^{2n} \left(\sum_{n=0}^{\infty} \frac{(a,n)(1-a,n)}{(n!)^2} r^{2n}\right)^{-1},$$

which is strictly increasing by [**8**, Lemma 2.1]. $f_5(0^+) = 1$ is clear. By l'Hôpital's rule,

$$f_5(1^-) = \lim_{r\to 1^-} \frac{\mathcal{E} - r'^2\mathcal{K}}{2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)} = \frac{1}{\sin(\pi a)}.$$

This completes the proof of the lemma. □

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4. Proofs of main theorems

Proof of Theorem 2.1. (i) Let $x = 2\sqrt{r}/(1+r)$. Then $x' = (1-r)/(1+r)$, $\mathrm{d}x/\mathrm{d}r = x'x/2r$.

Differentiation gives

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}r}\left[2\mu_a\left(\frac{2\sqrt{r}}{1+r}\right) - \mu_a(r)\right]\\ &= -2\frac{1}{xx'^2F(a,1-a;1;x^2)^2}\frac{xx'}{2r} + \frac{1}{rr'^2F(a,1-a;1;r^2)^2}\\ &= \frac{1}{rr'^2}\left[\frac{1}{F(a,1-a;1;r^2)^2} - (1+r)^2\frac{1}{F(a,1-a;1;x^2)^2}\right].\end{aligned}$$

By the *Landen inequalities* [**10**, Theorem 1.2] and Lemma 3.1 (iii), we have

$$\begin{aligned}\frac{\partial(\partial f/\partial r)}{\partial a} &= \frac{1}{rr'^2}\left[-2\frac{F_a'(a,1-a;1;r^2)}{F(a,1-a;1;r^2)^3} + 2(1+r)^2\frac{F_a'(a,1-a;1;x^2)}{F(a,1-a;1;x^2)^3}\right]\\ &\geqslant \frac{2}{rr'^2}\left[\frac{F_a'(a,1-a;1;x^2)}{F(a,1-a;1;x^2)F(a,1-a;1;r^2)^2} - \frac{F_a'(a,1-a;1;r^2)}{F(a,1-a;1;r^2)^3}\right]\\ &= \frac{2}{rr'^2F(a,1-a;1;r^2)^2}[f_3(x) - f_3(r)]\\ &> 0,\end{aligned}$$

where

$$F_a'(a,1-a;1;r^2) = \frac{\partial}{\partial a}F(a,1-a;1;r^2).$$

It follows that $\partial f/\partial r$ is strictly increasing in $a \in (0,\frac{1}{2})$. Hence, for $0 < a < b \leqslant \frac{1}{2}$, we have

$$\frac{\partial f(a,r)}{\partial r} < \frac{\partial f(b,r)}{\partial r}.$$

By integration,

$$\int_r^1 \frac{\partial f(a,r)}{\partial r}\,\mathrm{d}r < \int_r^1 \frac{\partial f(b,r)}{\partial r}\,\mathrm{d}r;$$

hence,

$$f(a,r) > f(b,r).$$

This yields the monotonicity of $f(a,r)$ in a.

For $r \in (0,1)$, $f(\frac{1}{2},r) = 0$ by (2.1). Write $A(a,n) = (a,n)(1-a,n)/(n!)^2$ and $B(a,n) = \Gamma(n+a)\Gamma(n+1-a)/(n!)^2$. By (1.2) and (1.6),

$$\begin{aligned}f(a,r) &= 2\frac{\pi}{2\sin(\pi a)}\frac{\sum_{n=0}^{\infty}A(a,n)x'^{2n}}{\sum_{n=0}^{\infty}A(a,n)x^{2n}} - \frac{\pi}{2\sin(\pi a)}\frac{\sum_{n=0}^{\infty}A(a,n)r'^{2n}}{\sum_{n=0}^{\infty}A(a,n)r^{2n}}\\ &= \frac{\pi}{\sin(\pi a)}\left\{\frac{\sum_{n=0}^{\infty}\sin(\pi a)\pi^{-1}B(a,n)x'^{2n}}{\sum_{n=0}^{\infty}\sin(\pi a)\pi^{-1}B(a,n)x^{2n}} - \frac{1}{2}\frac{\sum_{n=0}^{\infty}\sin(\pi a)\pi^{-1}B(a,n)r'^{2n}}{\sum_{n=0}^{\infty}\sin(\pi a)\pi^{-1}B(a,n)r^{2n}}\right\}\end{aligned}$$

$$= \frac{\pi}{\sin(\pi a)}\left\{\frac{1+\sin(\pi a)\pi^{-1}\sum_{n=1}^{\infty}B(a,n)x'^{2n}}{1+\sin(\pi a)\pi^{-1}\sum_{n=1}^{\infty}B(a,n)x^{2n}} - \frac{1}{2}\frac{1+\sin(\pi a)\pi^{-1}\sum_{n=1}^{\infty}B(a,n)r'^{2n}}{1+\sin(\pi a)\pi^{-1}\sum_{n=1}^{\infty}B(a,n)r^{2n}}\right\}$$

$$\sim \frac{\pi}{2\sin(\pi a)} \quad (a\to 0),$$

and hence $f(0^+,r)=\infty$. The inequality (2.2) and its equality case are clear.

(ii) Let $t=(1-r)/(1+r)$. Then $r'=2\sqrt{t}/(1+t)$, and $g(a,r)=-f(a,t)$. Hence, the assertion about g follows from part (i). □

Proof of Theorem 2.2. (i) By [**3**, Theorem 4.1(5)], we have

$$h_r(a,r)\equiv\frac{\mathrm{d}h}{\mathrm{d}r}=-\frac{\pi^2}{4}\frac{a^2}{rr'^2\mathcal{K}_a^2},$$

which is strictly decreasing in a by Lemma 3.1 (iv). Therefore, for $0<a<b\leqslant\frac{1}{2}$,

$$\int_r^1 h_r(a,r)\,\mathrm{d}r>\int_r^1 h_r(b,r)\,\mathrm{d}r.$$

This gives

$$a^2\mu_a(r)<b^2\mu_b(r),$$

and hence $h(a,r)$ is strictly increasing in a. Clearly, $h(\frac{1}{2},r)=\frac{1}{4}\mu(r)$. Let $A(a,n)$ and $B(a,n)$ be as in the proof of Theorem 2.1 (i). Then

$$\begin{aligned}h(0^+,r)&=\lim_{a\to0^+}a^2\frac{\pi}{2\sin(\pi a)}\frac{\sum_{n=0}^{\infty}A(a,n)r'^{2n}}{\sum_{n=0}^{\infty}A(a,n)r^{2n}}\\&=\lim_{a\to0^+}a\frac{\pi a}{\sin(\pi a)}\frac{\sum_{n=0}^{\infty}\sin(\pi a)\pi^{-1}B(a,n)r'^{2n}}{\sum_{n=0}^{\infty}\sin(\pi a)\pi^{-1}B(a,n)r^{2n}}\\&=\lim_{a\to0^+}a\frac{\pi a}{\sin(\pi a)}\frac{1+\sin(\pi a)\pi^{-1}\sum_{n=1}^{\infty}B(a,n)r'^{2n}}{1+\sin(\pi a)\pi^{-1}\sum_{n=1}^{\infty}B(a,n)r^{2n}}\\&=0.\end{aligned}$$

(ii) Write $H_1(r)=\mu_a(r)$ and $H_2(r)=\mu(r)$. Then $H_1(1^-)=H_2(1^-)=0$ and, by [**3**, Theorem 4.1(5)],

$$\frac{H_1'(r)}{H_2'(r)}=\frac{-\frac{1}{4}\pi^2(rr'^2\mathcal{K}_a^2)^{-1}}{-\frac{1}{4}\pi^2(rr'^2\mathcal{K}^2)^{-1}}=\left(\frac{\mathcal{K}}{\mathcal{K}_a}\right)^2. \tag{4.1}$$

Hence, the monotonicity of H follows from Lemma 3.1 (v) and [**3**, Lemma 5.1].

The limiting values follow from l'Hôpital's rule, (4.1) and Lemma 3.1 (v). The second inequality in (2.4) is clear, while the first inequality in (2.4) holds by [**11**, Theorem 1.22]. □

Proof of Theorem 2.3. Parts (i) and (ii) follow from Theorem 2.2 (ii) and the corresponding results for $\mu(r)$ (see [**2**, Theorem 5.13 (5), (6)]). Part (iii) follows from part (ii) and the identity

$$\mu_a(r)\mu_a(r')=\frac{\pi^2}{4\sin^2(\pi a)}.$$

□

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