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Large deviation theory for stochastic difference equations

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The probability density for the solution y_n of a stochastic difference equation is considered. Following Knessl *et al.* [1], it is shown to satisfy a master equation, which is solved asymptotically for large values of the index n. The method is illustrated by deriving the large deviation results for a sum of independent identically distributed random variables and for the joint density of two dependent sums. Then it is applied to a difference approximation to the Helmholtz equation in a random medium. A large deviation result is obtained for the probability density of the decay rate of a solution of this equation. Both the exponent and the pre-exponential factor are determined.

1 Introduction

A stochastic difference equation is a difference equation involving random coefficients or random functions. One goal of the theory of such equations is to determine the probability distribution of a solution that satisfies specified initial and/or boundary conditions. One may seek the mean value of the solution, the probability distribution for values near the mean, or the probability distribution for values far from the mean. In the case of sums of random variables, these quantities are given by the law of large numbers, the central limit theorem, and large deviation theory, respectively.

Our goal is to obtain large deviation results for solutions of stochastic difference equations. We shall use the method of Knessl *et al.* [1]. This involves deriving a master equation for the desired probability density, and then solving it asymptotically for large values of the index n in the difference equation. In §2 we illustrate the method by deriving the large deviation results for the sum y_n of n independent identically distributed random variables. We show that y_n satisfies a difference equation, and we use it to obtain the master equation for the density of y_n. Then we solve that equation asymptotically for large n. The result agrees with those of Cramer [2] for the exponent and Bahadur & Ranga Rao [3] for the pre-exponential factor, obtained by the method of characteristic functions, as we show in Appendix A.

In §3 we apply the method to the determination of the joint probability density of two dependent sums of random variables. We then specialize to the case of Gaussian

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summands, and obtain an explicit result. It agrees with the asymptotic expansion of the exact density, which we can find in this case.

In §4 we consider the Helmholtz equation in a random medium, and we discretize it to obtain a stochastic difference equation. We apply the method of the preceding sections to obtain the large deviation results for the probability density of the decay rate of a solution. In §5 we compare this result with the corresponding result of the central limit type, obtained by Kuske [4] using Hashminskii's [5] theorem for differential equations.

The usual analysis of large deviations deals with the exponent of the probability density for general situations. In the special case of linear stochastic difference equations, the present method determines not only the exponent but also the pre-exponential factor. The method should be applicable also to other physical phenomena governed by such equations, as described in Knessl *et al.* [1].

2 Sums of independent random variables

Let y_n be the sum of n independent identically distributed random variables x_i with density $p(x)$,

$$y_n = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{2.1}$$

We seek the probability density of y_n, which we denote by $q(y,n)$. To find q we use (2.1) to derive the following recurrence equation for y_n:

$$y_{n+1} = \frac{n}{n+1}y_n + \frac{x_{n+1}}{n+1}. \tag{2.2}$$

By using (2.2) we can express $q(y,n+1)$ in terms of $q(y,n)$ in the form

$$q(y,n+1) = \frac{n+1}{n}\int q\left(\frac{n+1}{n}y - \frac{x}{n}, n\right) p(x)dx. \tag{2.3}$$

Knessl *et al.* [1] showed how to solve master equations, such as (2.3), asymptotically for large values of n to obtain $q(y,n)$ for both large and small deviations. To obtain their result for (2.3), we use a modified version of the approach of Knessl *et al.* [1] in which we introduce a small positive parameter ϵ. (For more details, see Knessl *et al.* [1].) Then we extend it to more difficult equations.

We begin by introducing ϵ and defining $t = \epsilon n$. Then for large n we can treat t as of order unity. Next we write $q(y,n)$ in the form

$$q(y,n) = \epsilon^{\alpha}\hat{K}(y,t,\epsilon)e^{-t\psi(y)/\epsilon}. \tag{2.4}$$

Such a representation is possible for any q and, with it, (2.3) becomes

$$\hat{K}(y,t+\epsilon,\epsilon)e^{-(t+\epsilon)\psi(y)/\epsilon} = \frac{t+\epsilon}{t}\int \hat{K}(y+\frac{\epsilon(y-x)}{t}, t, \epsilon)e^{-(t/\epsilon)\psi[y+\epsilon(y-x)/t]}p(x)dx. \tag{2.5}$$

Now we make the assumption that $\hat{K}(y,t,\epsilon) = K(y,t) + O(\epsilon)$. Then we expand both sides of (2.5) in powers of ϵ and equate the coefficients of ϵ^0 and of ϵ. This leads to the two equations

$$e^{-\psi(y)} = \int e^{-(y-x)\psi'(y)}p(x)dx, \tag{2.6}$$

$$K_t e^{-\psi(y)} = \int e^{-(y-x)\psi'(y)}\left[\frac{y-x}{t}K_y + \frac{K}{t} - (y-x)^2\frac{\psi''(y)}{2t}K\right]p(x)dx. \tag{2.7}$$

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Equation (2.6) is a first-order differential equation of Colairant form for $\psi(y)$ and one of its solutions is

$$\psi(y) = \sup_z[zy - \ln M(z)]. \tag{2.8}$$

Here $M(z)$ is the moment generating function of x, defined by

$$M(z) \equiv \int e^{zx} p(x)dx. \tag{2.9}$$

Now (2.7) is an equation for K which can be rewritten as

$$tK_t = G_0 K - \frac{\psi''(y)}{2} G_2 K + G_1 K_y. \tag{2.10}$$

The three coefficients G_0, G_1 and G_2 are defined by

$$G_n = e^{\psi(y)} \int (y-x)^n e^{-(y-x)\psi'(y)} p(x)dx \quad \text{for} \quad n = 0, 1, 2. \tag{2.11}$$

Upon twice differentiating (2.6) with respect to y we find that the G_n satisfy the relation

$$G_2 = \frac{\psi'''(y)G_1 + \psi''(y)G_0}{(\psi''(y))^2}. \tag{2.12}$$

To solve (2.10) we assume that $K(y,t) = \sqrt{t}f(y)$. Then by using this in (2.10), together with (2.12) for G_2 and (2.11) which yields $G_0 = 1$, we obtain

$$G_1 \frac{f'(y)}{f(y)} = G_1 \frac{\psi'''(y)}{2\psi''(y)}. \tag{2.13}$$

The solution of this ordinary differential equation is $f(y) = c|\psi''(y)|^{1/2}$ where c is an arbitrary constant. Thus $K = ct^{1/2}|\psi''(y)|^{1/2} = c\epsilon^{1/2}|n\psi''(y)|^{1/2}$, and then (2.4) yields

$$q(y,n) \sim \epsilon^{\alpha+1/2} c\sqrt{n|\psi''(y)|}e^{-n\psi(y)}. \tag{2.14}$$

Since q is independent of ϵ, (2.14) shows that $\alpha = -1/2$. The constant c must be chosen to make the integral of q equal to one. Asymptotic evaluation of this integral by Laplace's method shows that this is the case if $1/c = \sqrt{2\pi}$. Thus, finally,

$$q(y,n) \sim \sqrt{\frac{n|\psi''(y)|}{2\pi}} e^{-n\psi(y)}. \tag{2.15}$$

In Appendix A we derive the asymptotic form of q by the usual method of characteristic functions, which was used by Cramer [2] to determine the exponent and by Bahadur & Ranga Rao [3] to obtain the pre-exponential factor. We then show that (2.15) agrees with their results. The form (2.15) follows from equations (7.8) and (7.13) of Knessl *et al.* [1] by setting their parameter $b = 1$.

3 Joint density of two sums

Next we shall show how the preceding method can be used to determine the joint density of two sums of random variables y_n and x_n. Again y_n is defined by (2.1) and x_n is given by

$$x_n = \frac{1}{n} \sum_{i=1}^{n} \xi_i. \tag{3.1}$$

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Then y_n is a mean of sample means, in which each sample is the previous sample plus one additional observation. Thus in each y_n the earlier samples are given more weight than later ones. The ξ_i are i.i.d. random variables with common density $\rho(\xi)$. One could combine (2.1) and (3.1) into an equation for a one-dimensional Markov chain with a more complicated coefficient of the noise, and then use the method of Knessl *et al.* [1]. Instead, we seek $q(y, x, n)$, the probability density that $y_n = y$ and $x_n = x$. This serves to illustrate the method for determining the joint probability density, which is also used in the next section.

For clarity, we write the recursion equation satisfied by $q(y_n, x_n, n)$, keeping the subscripts on the arguments,

$$q(y_{n+1}, x_{n+1}, n+1) = \int\int\int \Pr(y_{n+1}, x_{n+1}, n+1 | y_n, x_n, \xi_{n+1}) \times q(y_n, x_n, n)\rho(\xi_{n+1}) dy_n dx_n d\xi_{n+1}. \quad (3.2)$$

From (2.1) we get (2.2), and from (3.1) we have

$$x_{n+1} = \frac{n}{n+1}x_n + \frac{\xi_{n+1}}{n+1}. \quad (3.3)$$

Thus

$$\Pr(y_{n+1}, x_{n+1} | y_n, x_n, \xi_{n+1}) = \delta\left(y_{n+1} - \frac{n}{n+1}y_n - \frac{x_{n+1}}{n+1}\right) \times\delta\left(x_{n+1} - \frac{n}{n+1}x_n - \frac{\xi_{n+1}}{n+1}\right). \quad (3.4)$$

We use (3.4) in (3.2) and drop the subscripts $n+1$ from the arguments of q to obtain the master equation for $q(y, x, n)$:

$$q(y, x, n+1) = \left(\frac{n+1}{n}\right)^2 \int q(y + \frac{y-x}{n}, x + \frac{x-\xi}{n}, n)\rho(\xi)d\xi. \quad (3.5)$$

To solve (3.5) for n large, we write

$$q(y, x, n) = (K(y, x, n) + K_1(y, x, n) + \ldots)e^{-n\psi(y,x)}. \quad (3.6)$$

We assume that the leading term in the pre-exponential factor is K, and that the other terms are smaller for large values of n. We substitute (3.6) into (3.5) and expand various terms in powers of n^{-1} to get

$$K(y, x, n+1)e^{-(n+1)\psi(y,x)} = \left(\frac{n+1}{n}\right)^2 \int e^{-n(\psi(y,x)+n^{-1}\psi_y(y-x)+n^{-1}\psi_x(x-\xi))}$$
$$\times\left[1 - \frac{1}{n}\psi_{yx}(y-x)(x-\xi) - \frac{1}{2n}\psi_{yy}(y-x)^2 - \frac{1}{2n}\psi_{xx}(x-\xi)^2 + \ldots\right]$$
$$\times\left[K(y, x, n) + \frac{1}{n}K_y(y-x) + \frac{1}{n}K_x(x-\xi)\right] + K_1 + \ldots]\rho(\xi)d\xi. \quad (3.7)$$

Now we equate coefficients of n^0 and n^{-1}. This yields equations for ψ and K:

$$e^{-\psi} = e^{-(y-x)\psi_y - x\psi_x} \int e^{\xi\psi_x}\rho(\xi)d\xi, \quad (3.8)$$

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$$K_n(y,x,n)e^{-\psi} = \frac{1}{n}e^{-\psi_y(y-x)}\int e^{-\psi_x(x-\xi)}\Big\{\big[2K(y,x,n)+K_y(y,x,n)(y-x)$$
$$+K_x(y,x,n)(x-\xi)\big] - K\Big[\psi_{yx}(y-x)(x-\xi)$$
$$+\psi_{yy}\frac{(y-x)^2}{2}+\psi_{xx}\frac{(x-\xi)^2}{2}\Big]\Big\}\rho(\xi)d\xi. \quad (3.9)$$

Here subscripts indicate differentiation with respect to the subscript variable. The equation (3.9) is obtained by using (3.8) and neglecting the terms involving K_1 and those with coefficients n^{-2}, which are assumed to be of higher order in n^{-1}. In (3.9) the term K_n is included with the $O(n^{-1})$ terms. It is shown below that K is proportional to a power of n, so that this ordering is consistent.

Equation (3.8) is a nonlinear first-order partial differential equation for $\psi(y,x)$ and (3.9) is a similar linear equation for $K(y,x,n)$. Therefore both of them can be solved by the method of characteristics. But first we can simplify (3.9) by differentiating (3.8) with respect to y and x and using the resulting expressions to eliminate the integrals in (3.9). We get

$$nK_n = K_y(y-x) - K_x\frac{\psi_{yy}}{\psi_{xy}}(y-x) + \frac{\psi_{yy}}{2}K(y-x)^2 + 2K$$
$$-K\frac{\psi_{xx}}{2\psi_{xy}^2}\left[-(y-x)\psi_{yyy} - \psi_{yy} + \psi_{xyy}\frac{\psi_{yy}(y-x)}{\psi_{xy}} - (\psi_{yy})^2(y-x)^2\right]. \quad (3.10)$$

Let $h(x,y) = \psi_{yy}/\psi_{xy}$ and let g be the coefficient of K in (3.10). Then (3.10) becomes

$$nK_n = (y-x)K_y - (y-x)h(x,y)K_x + g(x,y)K. \quad (3.11)$$

The form of (3.11), or a consideration of its characteristics, indicates that it has solutions of the form

$$K(x,y,n) = C(x,y)n^{\alpha(y,x)}. \quad (3.12)$$

Substitution of (3.12) into (3.11) yields equations for α and C:

$$\alpha_y = h(x,y)\alpha_x, \quad (3.13)$$

$$(y-x)(C_y - h(x,y)C_x) = -g(x,y) + \alpha(x,y). \quad (3.14)$$

By using (3.12) in (3.6) we can write q in the form

$$q(y,x,n) \sim C(x,y)n^{\alpha(x,y)}e^{-n\psi(x,y)}. \quad (3.15)$$

Here ψ is a solution of (3.8), α is a solution of (3.13), and C is a solution of (3.14).

In general one must solve (3.8) using its characteristic curves and appropriate initial data on them. To demonstrate the type of result obtained from (3.8) and (3.9), we shall solve these equations for the specific case of ξ a Gaussian random variable with zero mean and variance unity. Evaluating the integrals in (3.8) and then taking logarithms yields

$$\psi = \psi_y(y-x) + x\psi_x - \frac{(\psi_x)^2}{2}. \quad (3.16)$$

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The appropriate solution of (3.16) is

$$\psi(y,x) = \frac{1}{2}y^2 - xy + x^2. \tag{3.17}$$

Then, substituting (3.17) into (3.10) yields (3.11) with $h(x,y) = -1$ and $g(x,y) = 1-(y-x)^2/2$. The equation in (3.13) yields $\alpha = \alpha(y-x)$, and the equation (3.14) is satisfied by $\alpha = g(x,y)$ and C =constant. We use these values of α and C, with (3.17) for ψ, in (3.15). Then we choose C to normalize q, which yields $C = 1/2\pi$, and we can write (3.15) as

$$q(y,x,n) \sim \frac{n^{1-(y-x)^2/2}}{2\pi} e^{-n((y-x)^2/2+x^2/2)}. \tag{3.18}$$

For this example one can compute q exactly by using properties of sums of Gaussian random variables. Both y_n and x_n are sums of Gaussian random variables, whose joint density is given by

$$q(y,x,n) = \frac{1}{2\pi(\det V)^{1/2}} \exp\left(-\frac{1}{2}(x,y)V^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right), \quad V = \frac{1}{n}\begin{pmatrix} 1 & 1 \\ 1 & 2-\frac{H_n}{n} \end{pmatrix}. \tag{3.19}$$

$$H_n = \sum_{j=1}^{n} \frac{1}{j} \tag{3.20}$$

For $n \gg 1$, the result (3.18) gives the correct asymptotic form of (3.19).

4 Density of decay rate

Let $u(x)$ satisfy the dimensionless one dimensional Helmholtz equation in the interval $0 < x < 1$, with the random refractive index $(1+\epsilon\xi(x))^{1/2}$:

$$\frac{d^2u}{dx^2} + k^2(1+\epsilon\xi(x))u = 0, \quad 0 < x < 1, \tag{4.1}$$

with $u(0) = 1$, and $u(1)$ bounded. Following Rytov *et al.* [6], we assume that the moments of ξ are

$$\langle \xi(x) \rangle = 0$$
$$\left\langle |\int_0^1 \xi(x)dx|^2 \right\rangle = 1. \tag{4.2}$$

We discretize by introducing mesh points $x_n = n/N$ and writing $u_n = u(x_n)$ and $\xi_n = \xi(x_n)$. Then, upon using the centred second difference to approximate d^2u/dx^2, we get the difference equation

$$u_{n+1} - 2u_n + u_{n-1} = -\frac{k^2}{N^2}(1+\epsilon\xi_n)u_n, \quad n = 0, \ldots, N. \tag{4.3}$$

From (4.2) the moments of ξ_n are

$$\langle \xi_n \rangle = 0$$
$$\langle \xi_n^2 \rangle = N. \tag{4.4}$$

Now we divide (4.3) by u_n and define $X_{n+1} = u_{n+1}/u_n$. Then (4.3) becomes

$$X_{n+1} = 2 - \frac{k^2}{N^2}(1+\epsilon\xi_n) - \frac{1}{X_n}, \quad n = 1, \ldots, N-1. \tag{4.5}$$

In terms of X_n we have, with $u_0 = 1$

$$u_n = \frac{u_n}{u_{n-1}} \frac{u_{n-1}}{u_{n-2}} \dots \frac{u_1}{u_0} u_0 = X_n X_{n-1} \dots X_1. \tag{4.6}$$

Thus we can write

$$|u_n| = \exp\left(\sum_{i=1}^{n} \ln|X_i|\right) = e^{n\gamma_n}. \tag{4.7}$$

Here we have introduced γ_n, defined by

$$\gamma_n = \frac{1}{n}\sum_{i=1}^{n} \ln|X_i|. \tag{4.8}$$

We shall calculate the density of γ_n. To accomplish this we consider the random variable Y_n which has mean zero:

$$Y_n \equiv \gamma_n - \gamma = \frac{1}{n}\sum_{i=1}^{n} (\ln|X_i| - \gamma). \tag{4.9}$$

Here $\gamma = E[\ln|X_i|]$ is the Lyapunov exponent describing the decay rate of u_n in the limit as $n \to \infty$. It is independent of i, and it has been calculated using the stationary probability density of X_i obtained in Kuske [4]. From the definitions above we see that γ_n tends to γ as n tends to infinity. We shall seek the joint probability density $q(y, x, n)$ of Y_n and X_n, from which we can obtain the density of $n^{-1} \ln|u_n|$ by using (4.7).

By proceeding as in §§2 and 3 we find that the master equation for $q(y, x, n)$ is

$$q(y, x, n+1) = \frac{n+1}{n} \int q\left(\frac{n+1}{n} y - \frac{\ln|x| - \gamma}{n}, \frac{1}{2 - \frac{k^2}{N^2} - x - \epsilon\frac{k^2}{N^2}\xi}, n\right) \times \frac{\rho(\xi)}{(2 - \frac{k^2}{N^2} - x - \epsilon\frac{k^2}{N^2}\xi)^2} d\xi. \tag{4.10}$$

In Kuske [4] it was shown that for $N \gg 1$ the values of X_n cluster around the value 1. Therefore we introduce the scaled variable

$$z = N(x - 1). \tag{4.11}$$

It also simplifies the analysis considerably to introduce another variable

$$\eta = y - \ln(|1 + z/N| + \gamma \tag{4.12}$$

and to consider the case $n = N$. That is, we consider the decay rate of an outgoing wave at the end of a slab with (non-dimensionalized) length 1. Accordingly, we write the density as $H(\eta, z, N) = q(y, x, N)$. Using the definitions (4.11) and (4.12) in (4.10) yields

$$H(\eta, z, N+1) = \frac{N+1}{N} \int H\left(\eta + \frac{\eta}{N}, \frac{z + k^2/N + \epsilon k^2\xi/N}{1 - k^2/N^2 - z/N - \epsilon k^2\xi/N^2}, N\right) \cdot \times \frac{\rho(\xi)}{(1 - k^2/N^2 - z/N - \epsilon k^2\xi/N^2)^2} d\xi. \tag{4.13}$$

We write H in the form

$$H(\eta, z, N) = [K(\eta, z, N) + K_1(\eta, z, N)/N + \dots]e^{-N^2\psi(\eta)}. \tag{4.14}$$

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Substituting a general form for the exponent, such as $N^{\mu}\psi(\eta, z)$, into (4.13) yields a more complicated equation for ψ. From this we have concluded that the correct form is given by (4.14). It is not surprising that we must write N^2 in the exponent, rather than N as in the previous sections, since $\mathrm{Var}(\log|X_i|) = O(N^2)$, as discussed below. Then we substitute (4.14) into (4.13) and expand for $1/N$ small. This corresponds to expanding H in the integrand about $H(\eta, z, N)$. This yields, to leading order, the following equation for ψ:

$$e^{-(N+1)^2\psi(\eta)} = e^{-N^2(\psi(\eta)+\psi'(\eta)\eta/N+\frac{\psi''(\eta)\eta^2}{2N^2})}. \tag{4.15}$$

Here the ξ integration has been performed, yielding a factor of 1 since the coefficient of $\rho(\xi)$ is, to leading order, independent of ξ. Terms of order N^{-1} have been included in the equation below for K. We equate the exponents in (4.15) and obtain a second-order differential equation for $\psi(\eta)$. The solution regular at $\eta = 0$ is

$$\psi = \mathscr{C}\eta^2. \tag{4.16}$$

The terms of order $1/N$ in (4.13) yield the equation for K.

$$NK_N = K - \eta^3\frac{\psi_{\eta\eta\eta}}{6}K + 2zK + \eta K_\eta + (z^2 + k^2)K_z + \frac{\epsilon^2 k^4}{2}K_{zz}. \tag{4.17}$$

The term K_{zz} comes in at this order since $\langle\xi^2\rangle = N$. From (4.16) it follows that $\psi_{\eta\eta\eta} = 0$. Therefore we seek a solution of (4.17) proportional to N with $K_\eta = 0$. Then K can be written as

$$K = ND(z). \tag{4.18}$$

The density q must vanish as $|x| \to \infty$, and therefore $D(z) \to 0$ as $|z| \to \infty$. The function $D(z)$ with $D(z) \to 0$ as $|z| \to \infty$ is found to be

$$D(z) = K_0 e^{-2(z^3/3+k^2 z)/(\epsilon^2 k^4)} \int_{-\infty}^{z} e^{2(z'^3/3+k^2 z')/(\epsilon^2 k^4)} dz'. \tag{4.19}$$

For $\epsilon^2 k^4 \ll 1$ one can write the solution of (4.17) as

$$D(z) \sim K_0\left(\frac{1}{z^2+k^2} + \epsilon^2 k^4 \frac{z}{2(z^2+k^2)^3}\right) \quad \text{for } \epsilon^2 k^4 \ll 1. \tag{4.20}$$

In (4.19) K_0 is a constant to be determined by normalization. Substituting (4.16) and (4.18) into (4.14) yields the leading-order form for $H(\eta, z, N)$. By replacing z and η by their definitions we get

$$q(y, x, N) \sim ND[N(x-1)]e^{-\mathscr{C}N^2(y-\ln|x|+\gamma)^2}. \tag{4.21}$$

To determine the constant $\mathscr{C}$ in (4.16) and (4.21) we consider the second moment

$$\int_{-\infty}^{\infty} y^2 q(y, x, N)dy \sim \frac{1}{2\mathscr{C}N^2} + E[(\ln|X_i| - \gamma)^2] = E[Y_N^2]. \tag{4.22}$$

The quantity $E[NY_N^2]$ has been calculated in Kuske [4] by using the definition (4.9), with the result

$$E[Y_N^2] \sim \frac{2\epsilon^2 k^2}{N^2}, \quad \epsilon \ll 1. \tag{4.23}$$

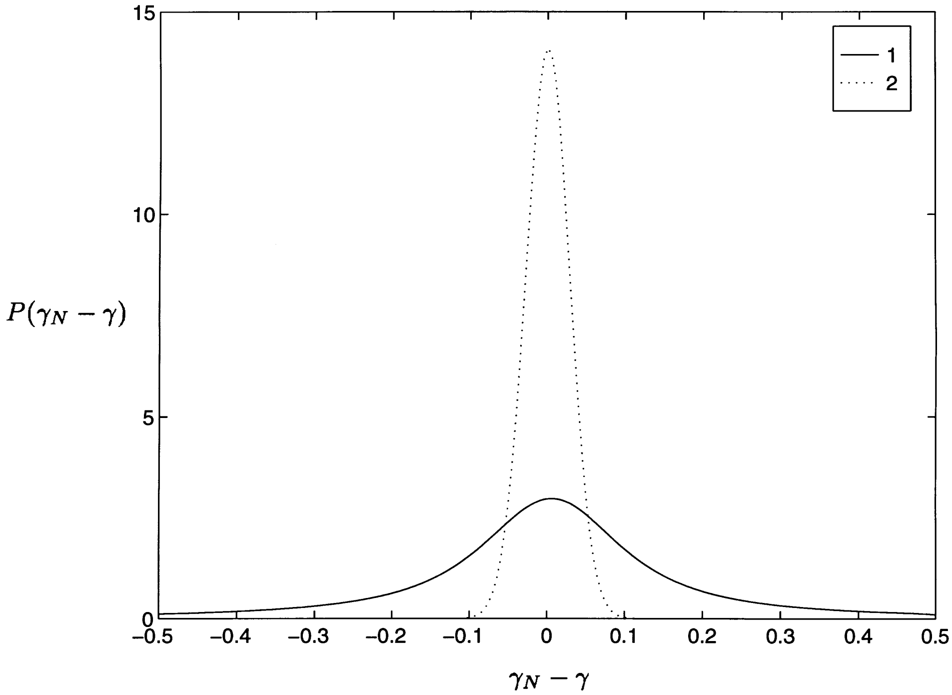

FIGURE 1. The density $p(y, N)$ of $Y_N = \gamma_N - \gamma$ for $N = 10$ as obtained using (1) the large deviation result (4.25) of this paper, and (2) the central limit type results of Kuske [4].

For $\epsilon^2 \ll 1$ the quantity $E[(\ln|X_i| - \gamma)^2]$ can be calculated analytically using (4.20), which yields

$$E[(\ln|X_i| - \gamma)^2] \sim \frac{\epsilon^4 k^4}{8N^2} + O(\epsilon^2 k^4 N^{-3}), \quad \epsilon \ll 1. \tag{4.24}$$

Upon using (4.23) and (4.24) in (4.22) we find that for $\epsilon \ll 1$ (4.22) yields $\mathscr{C} \sim 4\epsilon^{-2}k^{-2}$.

By integrating (4.21) over all values of x, we obtain the density $p(y, N)$ for $Y_N = \gamma_N - \gamma$:

$$p(y, N) \sim NK_0 \int_{-\infty}^{\infty} D[N(x-1)] e^{-\mathscr{C}N^2(y - \ln|x| + \gamma)^2} dx. \tag{4.25}$$

In Fig. 1 we compare this result with the density for $\gamma_N - \gamma$ obtained in Kuske [4] using Hasminskii's theorem [5], which is a theorem of central limit type for dependent random variables. We note that the width of the peak of the density is significantly larger for the present large deviations result and $N = 10$ and $N = 20$. This is not unexpected, since the central limit theorem is not a good approximation in the tails. As $N \to \infty$ the large deviations result approaches the central limit theorem result. In Fig. 2 these two results are compared with a numerical simulation of the density of $\gamma_N - \gamma$, where the random variables X_i in (4.9) are obtained from (4.5). The large deviation result (4.21) captures the behavior of the density of $\gamma_N - \gamma$ in the tails, and thus is a better approximation than the central limit theorem result.

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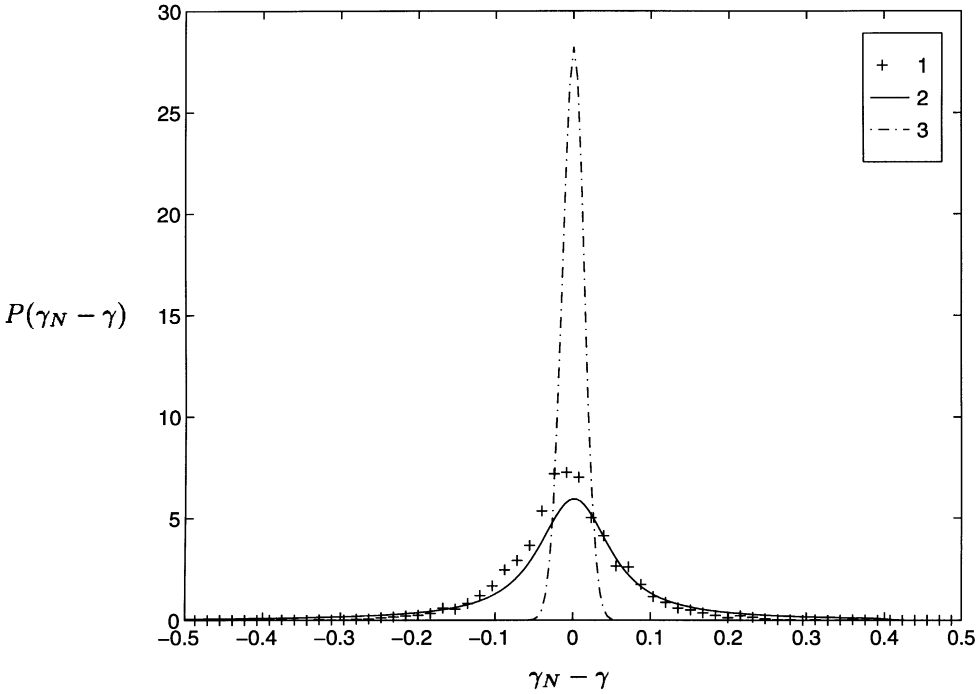

FIGURE 2. The density $p(y,N)$ of $Y_N = \gamma_N - \gamma$ for $N = 20$, compared with a numerical simulation of (4.5). (1) is the simulation result, (2) is the result of (4.25), and (3) is the result using the central limit type theorem. Note that (2) is close to (1) even though N is only 20.

5 Comparisons and conclusions

Now we shall compare the procedure and the results of the previous sections with certain related results. First we consider the result (2.15) for the probability density $q(y,n)$ of a sum y_n of i.i.d. random variables x_i. This result, like that in Knessl *et al.* [1], is based on the assumption that $p(x) > 0$ for all x. We contrast the different behaviours of the result (A 4), the density of a sum of n i.i.d. random variables x_i, for discrete and continuous random variables for which this assumption does not hold. For a continuous random variable whose density is nonzero on a finite interval $[a,b]$ the density of the sum y_n is given in (2.15) or (A 4). Near the boundary $y = b$ this density has the behaviour as in (B 4)

$$q(y,n) \sim \frac{\sqrt{n}}{\epsilon\sqrt{2\pi}} e^{n((b-y)/\epsilon + \ln p(b) + \ln \epsilon)} = \frac{e^n \epsilon^n p^n(b)\sqrt{n}}{\epsilon\sqrt{2\pi}}, \tag{5.1}$$

where $b - y \equiv \epsilon$. The behaviour near the boundary $y = a$ can be obtained in a similar manner.

For a sum of discrete Bernoulli random variables the result (2.15) becomes singular at the endpoints $y = \pm 1$ of the support of q. This can be seen by evaluating (2.15), which yields

$$q(y,n) \sim \sqrt{\frac{n}{2\pi}} \frac{(y+1)^{-(y+1)n/2}(1-y)^{(y-1)n/2}}{\sqrt{1-y^2}}. \tag{5.2}$$

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Instead of using (2.15) or (5.2) near $y = \pm 1$, we introduce a boundary layer expansion (B.9) near $y = 1$ and a similar expansion near $y = -1$. Then we can form a uniform composite expansion, in the manner described in Appendix B.

If one integrates out the dependence on y of the density $q(y, x, N)$ in (4.21), the marginal density for x which remains is (4.18) with D given by (4.19) or (4.20). This should be the density for Schrödinger's equation on a lattice with weak disorder. This equation was studied by Kuske *et al.* [8] using an equation similar to (4.5) above. For certain values of the parameters corresponding to those used in the discretized wave equation (4.3), the density for x was found to be exactly that given in (4.19) and (4.20), appropriately normalized.

The average Lyapunov exponent γ for the random oscillator problem, of which (4.1) is a special case, was considered in Arnold *et al.* [9]. For $u = re^{i\phi}$, an asymptotic expansion for the joint probability density of ϕ and the noise ξ was used to determine γ. The density for r was not needed to calculate γ. However, a system of stochastic differential equations was obtained for r and ϕ, from which one could write down the generator for the process. A consideration of this generator could lead to a density for r. The density of $\ln r$ could be compared to the density for γ_n obtained in the previous section. The main difference is that the approach of Arnold *et al.* [9] does not use discretized equations, while in the previous section we consider the continuous limit of the discretized Helmholtz equation. The discretization has the advantage that one can avoid an analysis of the (possibly) complicated generator for r and ϕ.

The method of §§2 and 3 can be applied to the sum y_n of random variables having Markovian dependence with the conditional probability density $p(x_n|x_{n-1}, \ldots, x_{n-k})$. It leads to an eigenvalue problem for an integral equation involving a k-dimensional integral. For the case $k = 1$, the joint probability density $p(y_n, x_n, n)$ of the sum y_n and the random variable x_n was considered by Lerman & Schuss [10]. The joint probability density is written as

$$p(y_n, x_n, n) = K(x_n, y_n, n)e^{-n\psi(y_n)} \tag{5.3}$$

Following a procedure similar to that of Knessl *et al.* [1] and of the previous sections of this paper, the following equation is obtained:

$$K(x_n, y_n, n) = e^{\psi(y)-y\psi'(y)} \int e^{x_{n-1}\psi'(y)} K(x_{n-1}, y_n, n) p(x_n|x_{n-1}) dx_{n-1} \tag{5.4}$$

Here $e^{\psi(y)-y\psi'(y)}$ is the eigenvalue of (5.4) and $K(x_n, y_n, n)$ is the eigenfunction. The solution to this equation has been discussed in Lerman & Schuss [10].

While (5.4) looks very similar to (3.8)–(3.9) and (4.13) with $H = Ke^{-N^2\psi}$, there are significant differences. In §§3 and 4, we considered sums of random variables which are related by the stochastic difference equations (3.3) and (4.5), driven by the iid random variables ξ_n. In these cases both the exponent ψ and the prefactor K in the joint probability densities were functions of x and y, in contrast to (5.3). Since the x_i are related by a stochastic difference equation, we could obtain partial differential equations for ψ and K. In general, this is not possible when only the conditional probability density $p(x_n|x_{n-1})$ is known.

Appendix A Usual derivation of asymptotic form of $q(y,n)$

The usual method of obtaining the probability density $q(y,n)$ is to introduce the characteristic function $S(k)$ of x_i

$$S(k) = \int_{-\infty}^{\infty} e^{ikx} p(x) dx. \tag{A 1}$$

Then the probability density function $q(y,n) = \Pr(y_n = y)$ is determined from the characteristic function $S^n(k)$ of y_n:

$$\begin{aligned} q(y,n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-inky} S^n(k) dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-inky+n\ln S(k)} dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{n\Phi(k)} dk. \end{aligned} \tag{A 2}$$

Here $\Phi(k) = -iky + \ln S(k)$. We evaluate this integral asymptotically for large n by the saddle point method. The equation for the saddle point is

$$iy = \frac{S'(k)}{S(k)}. \tag{A 3}$$

This equation has a pure imaginary root ik_0 with the sign of k_0 opposite to that of y. The steepest descent path through this point, defined by $\mathrm{Im}\Phi(k) = \mathrm{Im}\Phi(ik_0)$, is normal to the imaginary k axis at the saddle point $k = ik_0$. Consequently, the asymptotic behaviour of the integral in (A 2) is

$$q(y,n) \sim \frac{\sqrt{n}}{\sqrt{2\pi|\Phi''(ik_0)|}} e^{n(k_0 y + \ln S(ik_0))}. \tag{A 4}$$

To compare our result (2.15) with (A 4), we note from (2.8) that

$$\psi(y) = z_0 y - \ln M(z_0), \tag{A 5}$$

where z_0 satisfies $M'(z_0)/M(z_0) = y$. Comparison of this equation with (A 3) shows that $z_0 = -k_0$ since $M(z) = S(-iz)$. Thus the exponent in (2.15) is equal to that in (A 4). To show that the pre-exponential factors are equal, we differentiate twice the equation (A 5) and use the equation satisfied by z_0 to obtain

$$\psi''(y) = \frac{dz_0}{dy}. \tag{A 6}$$

Upon taking the y derivative of the equation for z_0 we get

$$1 = \frac{d}{dz_0}\left(\frac{M'(z_0)}{M(z_0)}\right)\frac{dz_0}{dy}. \tag{A 7}$$

From the last two equations we see that

$$(\psi''(y))^{-1} = \frac{d}{dz_0}\left(\frac{M'(z_0)}{M(z_0)}\right). \tag{A 8}$$

From the definition $\Phi = -iky + \ln S(k)$ and the relation between M and S, it follows that the right side of (A.8) is just $\Phi''(ik_0)$, so (A.8) yields $[\psi''(y)]^{-1} = \Phi''(ik_0)$. Therefore, the pre-exponential factors in (2.15) and (A 4) are equal, which shows that both methods yield the same asymptotic form for $q(y,n)$.

Appendix B Behaviour of q near endpoints of its support

Suppose that x_i is a continuous random variable whose density is positive over a finite interval (a, b) and zero outside it:

$$\begin{aligned} p(x) > 0, &\quad a \leqslant x \leqslant b, \\ p(x) = 0, &\quad \text{otherwise.} \end{aligned} \tag{B 1}$$

Then $q(y, n) = 0$ outside (a, b). We shall now determine the asymptotic behaviour of $q(y, n)$ within the interval but near the boundaries $y = b$ and $y = a$. We consider first the case of $b - y \equiv \epsilon \ll 1$. Since $k_0 \to -\infty$ as $y \to b$, we must evaluate $S(k)$ and $S'(k)$ in (A 3) for $|k_0| \gg 1$. Integration by parts in (A 1) yields

$$S(k) = e^{ikb}\left[\frac{p(b)}{ik} - \frac{p(a)}{ik}e^{ika-ikb} - e^{-ikb}\frac{1}{ik}\int_a^b e^{ikx}p'(x)dx\right] . \tag{B 2}$$

We use (B 2) and a similar expression for $S'(k)$ in the equation (A.3) for k_0. Since $|k_0|$ is large for $\epsilon \ll 1$, we neglect the terms with $e^{ika-ikb}\frac{p(a)}{ik}$ and $\frac{1}{(ik)^2}$, and we obtain

$$b - \epsilon \sim b - \frac{1}{|k_0|} . \tag{B 3}$$

This implies that $k_0 \sim -1/\epsilon$. Evaluating Φ and Φ'' at $k_0 = -1/\epsilon$, and using them in (A.4) yields

$$q(y, n) \sim \frac{\sqrt{n}}{\epsilon\sqrt{2\pi}}e^{n((b-y)/\epsilon + \ln p(b) + ln\epsilon)} = \frac{e^n\epsilon^n p^n(b)\sqrt{n}}{\epsilon\sqrt{2\pi}} . \tag{B 4}$$

Similarly, for y near a we obtain (B 4) with $p(b)$ replaced by $p(a)$ and $\epsilon = y - a$.

Next we consider the case of x_i a discrete random variable. For definiteness we take

$$p(x) = \frac{1}{2}[\delta(x-1) + \delta(x+1)] . \tag{B 5}$$

In this case $S(k) = \cos k$, and (A 3) becomes

$$y = -\tanh k_0 \tag{B 6}$$

for $k = ik_0$. Then (A 4) yields

$$q(y, n) \sim \sqrt{\frac{n}{2\pi\Phi''(ik_0)}}e^{n(yk_0 - \ln\sqrt{1-y^2})} = \sqrt{\frac{n}{2\pi}}\frac{(y+1)^{-(y+1)n/2}(1-y)^{(y-1)n/2}}{\sqrt{1-y^2}} . \tag{B 7}$$

Since this expression becomes infinite as $y \to \pm 1$, it is not valid near these points. This difficulty does not arise for x_i a continuous random variable, because $|\Phi(ik_0)|$ and $|1/\Phi''(ik_0)|$ both approach infinity as y approaches the boundary of its domain, in such a way that the result for $q(y, n)$ remains finite. In the discrete case, $\Phi(ik_0)$ is bounded while $1/\Phi''(ik_0)$ is infinite for $y = \pm 1$. Thus there are boundary layers at $y = \pm 1$.

To give a uniform expansion of $q(y, n)$, we consider the exact result for the probability of j 'successes' in n 'trials' based on (B.5):

$$p(j, n) = \binom{n}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)^{n-j} . \tag{B 8}$$

If success is '1' and failure '-1', then $y = (2j - n)/n$. Since the density $q(y, n)$ is symmetric about $y = 0$ we consider the boundary layer near $y = 1$ only. For y near 1, that is, for j near n, with $n \gg 1$, (B 8) behaves as

$$p(j,n) \sim \sqrt{\frac{n}{j}} \frac{n^n e^{-n+j}}{j^j (n-j)!} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{n-j} . \tag{B 9}$$

Now $q(y,n)$ given by (B.7) should match with $p(j,n)$ given by (B.9) in the intermediate region where $y \to 1$ with $j \gg 1$, but $j < n$, so that $n \gg n - j \gg 1$. Writing $1 - y = \epsilon = 2(n-j)/n$ in (B.7) and expanding yields

$$q(y,n) \sim \sqrt{\frac{n}{2\pi}} \frac{(2-\epsilon)^{-(2-\epsilon)n/2} \epsilon^{-\epsilon n/2}}{\sqrt{(2-\epsilon)\epsilon}} . \tag{B 10}$$

We see that (B.10) agrees with (B.9) in the intermediate region so (B.7) matches with (B 8). Therefore, the composite expansion for $q(y,n)$ on $y \in [0,1]$ is given by (B 7) + (B 9) − (B 10). The density $q(y,n)$ on $y \in [-1,0]$ is obtained by the symmetry of $q(y,n)$.

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