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ON STOCHASTIC COMPARISONS FOR LOAD-SHARING SERIES AND PARALLEL SYSTEMS

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We study the allocation strategies for redundant components in the load-sharing series/parallel systems. We show that under the specified assumptions, the allocation of a redundant component to the stochastically weakest (strongest) component of a series (parallel) system is the best strategy to achieve its maximal reliability. The results have been studied under cumulative exposure model and for a general scenario as well. They have a clear intuitive meaning; however, the corresponding additional assumptions are not obvious, which can be seen from the proofs of our theorems.

Keywords: accelerated lifetime model, cumulative exposure model, hazard rate function, reversed hazard rate function, stochastic orders, virtual age

1. INTRODUCTION

One of the standard methods to enhance reliability of a system is to use redundancy in its structure (Barlow and Proschan [1]). The problem of optimal allocation of redundant components was addressed in numerous publications (cf. Boland, El-Neweihi, and Proschan [2], Brito, Zequeira, and Valdés [3], Misra and Misra [16], Misra et al. [17,18], Romera, Valdés, and Zequeira [20], Valdéz and Zequeira [25,26], Hazra and Nanda [7,8], Cha, Mi, and Yun [4], Li et al. [12,13], Yun and Cha [29,30] to name a few). The specific case of redundancy, that is, load sharing, had attracted much less attention (see, e.g., Kapur and Lamberson [9], Keccecioglu [10], Scheuer [21], Shechner [24], Lin, Chen, and Wang et al. [14], Yinghui and Jing [28], Wang et al. [27], Shao and Lamberson [23], Liu [15] and references therein).

The load-sharing systems can be often encountered in practice, as a tool for decreasing electrical or mechanical stresses and therefore, increasing the corresponding reliability characteristics. The popular examples are: generators used in a power plant, CPU in a multiprocessor computer system, cables in a suspension bridge, valves, or pumps used in a hydraulic system, bolts used to hold a mechanical system, etc. However, most of the relevant papers in the literature deal only with the specific case when the components of the load-sharing systems are described by the exponentially distributed lifetimes. There are very few references where the systems with components having arbitrary lifetime distributions have been considered (see, e.g., Liu [15], and Yun and Cha [29]). This is mostly because, for a general case, one must be able to define the initial age of a component in one regime (e.g., full load) after switching from the other regime (e.g., partial load).

In this paper, we consider the load-sharing series and parallel systems with components having arbitrary lifetime distributions when one of the components of the system shares load with another component. We want to obtain the best allocation strategy that maximizes reliability of our system, that is, which components of a system should be chosen for a load sharing with a given redundant component. For addressing this question, we have to: (1) Describe the mechanism of age correspondence for a component that is functioning in one regime and then is switched over to the other regime. (2) Develop the corresponding stochastic ordering technique for comparing different variants of allocation. This will be done in our paper, which requires the comprehensive analysis of the model and the detailed stochastic comparisons.

Similar and more general settings for systems *without load sharing* were extensively studied in the literature (see, e.g., Boland et al. [2], Brito et al. [3], Misra and Misra [16], Misra et al. [17,18], Romera et al. [20], Valdéz and Zequeira [25,26]). However, to the best of our knowledge, there are no relevant results in the literature for the load-sharing systems with arbitrary distributions of component's lifetimes. The results obtained in this paper have a clear intuitive meaning similar to the case of "ordinary" redundancy (without load sharing). On the other hand, due to the load sharing and the subsequent recalculation of age, a number of additional issues come into play that should be properly formulated and adequately described mathematically.

After this general discussion of the topic, let us start with the relevant notation. For any continuous random variable X, denote by $F_X(\cdot)$ the cumulative distribution function, the probability density function by $f_X(\cdot)$ (whenever exists), the survival function by $\bar{F}_X(\cdot)$, the hazard (failure) rate function by $r_X(\cdot)$, and the reversed hazard rate function by $\tilde{r}_X(\cdot)$.

Consider a parallel system formed by two components, namely, A and B. Assume that both A and B are initially sharing load, and after the failure of one component, the other one switches over to the full load condition. Without loss of generality, let the total load for a system be 1 and the components A and B share α and $(1-\alpha)$ of it, respectively. Denote by X and Y the random variables representing the lifetimes of A and B under a full load and by X^* and Y^* those for the load sharing, respectively. As the lifetime of a component in a partial load condition should be larger than that in the full load condition, we can use stochastic reasoning employed in accelerated life modeling (ALM) (see, e.g., Nelson [19] and Finkelstein [6]). Thus, in accordance with the linear version of the ALM, we can write for both components the following relationships:

$$F_{X^*}(t) = F_X(g(\alpha)t), \quad \text{for all } t \geq 0,$$

and

$$F_{Y^*}(t) = F_Y(h(1-\alpha)t), \quad \text{for all } t \geq 0,$$

where both $g(\cdot)$ and $h(\cdot)$ satisfy: (i) $0 \le g(\alpha) \le 1$ and $0 \le h(1-\alpha) \le 1$, for all $\alpha \in [0,1]$, and (ii) both $g(\cdot)$ and $h(\cdot)$ are strictly increasing functions. The simplest specific case of these functions is when $g(\alpha) = \alpha$ and $h(1-\alpha) = 1-\alpha$; however, our assumptions allow for a more practically important setting when dependence on the load-sharing factor α is more general.

Suppose now that the component A was operating under a partial load in $[0,u)$ and was switched to the full load at time $t = u$ due to the failure of the component B. Thus, we must be able to define the initial age of this component under the full load that corresponds to the time spent under the partial load before the switching. We will call this equivalent age the *virtual age* (see, e.g., Kijima [11] and Finkelstein [5]) and denote it by $\omega(u)$. The virtual age satisfies the following natural conditions: (i) $0 \le \omega(u) \le u$ for all $u \ge 0$, and (ii) $\omega(\cdot)$ is an increasing function. Let us call $\{\omega(\cdot), g(\cdot)\}$ the *set of model functions* for A. For the "reversed" scenario, when the component A fails before the component B, denote the virtual age of the component B by $\gamma(u)$ and the corresponding set of model functions by $\{\gamma(\cdot), h(\cdot)\}$. Denote also the lifetime of the described load-sharing system by $X \oplus Y$. Then the corresponding survival function is given by (cf. Yun and Cha [29])

$$\begin{aligned}\bar{F}_{X\oplus Y}(t) &= \bar{F}_X(g(\alpha)t)\bar{F}_Y(h(1-\alpha)t)\\ &\quad + \int_0^t \frac{\bar{F}_X(t-u+\omega(u))}{\bar{F}_X(\omega(u))}\bar{F}_X(g(\alpha)u)h(1-\alpha)f_Y(h(1-\alpha)u)du\\ &\quad + \int_0^t \frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u)g(\alpha)f_X(g(\alpha)u)du. \qquad \textbf{(1.1)}\end{aligned}$$

Indeed, the first term in the r.h.s corresponds to the case when both components did not fail in $[0,t)$, whereas the second and the third terms correspond to the cases when one of the components had failed and the other was functioning under the full load after that.

Consider now a series (resp. parallel) system formed by n-independent components with lifetimes $X_1, X_2, \ldots, X_n$. Let a redundant component for load sharing with the lifetime Y be available for allocation to one of the components of the system. Then, the natural question is: how to allocate Y in the system in order to maximize its reliability in a suitable stochastic sense? Define, for $i = 1, 2, \ldots, n$,

$$U_i = \min\{X_1, \ldots, X_{i-1}, X_i \oplus Y, X_{i+1}, \ldots, X_n\},$$

and

$$V_i = \max\{X_1, \ldots, X_{i-1}, X_i \oplus Y, X_{i+1}, \ldots, X_n\},$$

where the operation $X_i \oplus Y$ means that Y is sharing load with X_i. Thus, U_i (resp. V_i) represents the lifetime of a series (resp. parallel) system where the component Y is sharing load with the ith component X_i. Note also that this problem obviously does not exist for parallel systems without load sharing as all $n+1$ components are functioning independently, whereas in the case of load sharing, the corresponding dependence takes place.

In order to achieve an optimal (maximal) reliability of our system, certain measures for comparison of reliability characteristics should be employed. It is well known that stochastic ordering is a very useful tool for comparing lifetimes. Many different types of stochastic orders have been developed and studied in the literature. For example, usual stochastic order compares two reliability functions, hazard rate order compares two failure rate functions, whereas the reversed hazard rate order compares two reversed hazard rate functions (see Shaked and Shanthikumar [22] for encyclopaedic information on stochastic orders). For the sake of completeness, we give the following definitions of stochastic orders that will be used in our paper.

Definition 1.1: *Let* X *and* Y *be two continuous non-negative random variables with respective supports* (l_X, u_X) *and* (l_Y, u_Y)*, where* u_X *and* u_Y *may be positive infinite, and* l_X *and* l_Y *may be zero. Then,* X *is said to be smaller than* Y *in*

1. *Hazard rate (hr) order, denoted as* $X \le_{hr} Y$*, if*

$$\frac{\bar{F}_Y(x)}{\bar{F}_X(x)} \quad \text{is increasing in } x \in (0, \max(u_X, u_Y)),$$

which can equivalently be written as

$$r_X(x) \ge r_Y(x), \quad \text{where defined;}$$

2. *Reversed hazard (rh) rate order, denoted as* $X \le_{rh} Y$*, if*

$$\frac{F_Y(x)}{F_X(x)} \quad \text{is increasing in } x \in (\min(l_X, l_Y), \infty),$$

which can equivalently be written as

$$\tilde{r}_X(x) \le \tilde{r}_Y(x), \quad \text{where defined;}$$

3. *Usual stochastic (st) order, denoted as* $X \le_{\rm st} Y$*, if*

$$\bar{F}_X(x) \le \bar{F}_Y(x) \quad \text{for all } t \in (0, \infty).$$

The following diagram shows the chain of implications among the stochastic orders as discussed above.

$$\begin{array}{ccc} X \le_{\rm hr} Y & & \\ & \searrow & \\ & & X \le_{\rm st} Y. \\ & \nearrow & \\ X \le_{\rm rh} Y & & \end{array}$$

Thus, the hazard rate order and the reversed hazard rate order are stronger than usual stochastic order. Throughout the paper, increasing and decreasing, as usual, means non-increasing and non-decreasing, respectively. The random variables considered in this paper are all non-negative. For convenience of notation, we write $Z = \min\{X_3, X_4, \ldots, X_n\}$ and $W = \max\{X_3, X_4, \ldots, X_n\}$.

The rest of the paper is organized as follows. In Section 2, we consider allocation strategies for load-sharing series (resp. parallel) systems. We compare different variants of these systems with respect to the usual stochastic order and under the assumption of the cumulative exposure model. We generalize these results in Section 3. In Section 4, we discuss some simulation results. Finally, the concluding remarks are given in Section 5.

2. STOCHASTIC COMPARISONS UNDER THE CUMULATIVE EXPOSURE MODEL

As was stated in the Introduction, when a component of a load-sharing system is switched from the partial load to a full load, its initial age in the new regime should be defined. The cumulative exposure model or its equivalents (see, e.g., Nelson [19] and Finkelstein [6]) that is widely used in accelerated life testing, is a popular and efficient way of dealing with this problem. Consider the load-sharing system as discussed in (1.1). In accordance with the reasoning similar to the cumulative exposure model, for example, for the component X, we have the following relation for the initial age (to be called "virtual age") after the switching:

$$F_{X^*}(t) = F_X(g(\alpha)t) = F_X(\omega(t)),$$

which immediately gives us

$$\omega(t) = g(\alpha)t$$

for all $t \geq 0$. Thus, the virtual age of a component after the switching to the full load is smaller than the age before switching and, in accordance with our assumptions, is given by the linear function. Note that this function also defines the scale transformation in the argument of the corresponding distribution function under partial load (ALM). In a similar way, we can also define the virtual age $\gamma(t)$ for the component Y.

In what follows in this section, we will formulate and analyze several practically important stochastic comparisons of interest for the case of the linear virtual age described above. In the next section, we will not rely on this assumption and, similar to (1.1), will consider the case of general, not necessarily linear virtual age. For the sake of presentation, we will omit now all the proofs that are just specific cases of general results of the next section, for which the detailed proofs will be given. Thus, we believe that the contents of the current section can have a practical importance, whereas the results of Section 3 are more theoretical.

Suppose that we have two different components, and one redundant component that can be used in a load-sharing scenario with any of the two components. Then, the following theorem holds (see the proof of Theorem 3.1).

THEOREM 2.1: *Let $\{\omega_1(\cdot), g_1(\cdot)\}$, $\{\omega_2(u), g_2(\cdot)\}$ and $\{\gamma(\cdot), h(\cdot)\}$ be the sets of model functions for X_1, X_2 and Y, respectively. Suppose that the following conditions hold.*

(i) $X_1 \geq_{st} X_2$.

(ii) $\omega_1(u) = g_1(\alpha)u \leq g_2(\alpha)u = \omega_2(u)$ *and* $\gamma(u) = h(1-\alpha)u$, *for all* $0 \leq \alpha \leq 1$ *and* $u \geq 0$.

Then, $X_1 \oplus Y \geq_{st} X_2 \oplus Y$.

The second assumption is, in fact, two-fold. On the one hand, we are assuming the specific form of the model functions that correspond to the cumulative exposure model. On the other hand, it establishes the required inequality. For convenience, let us say that one component (system) is stronger (weaker) than the other one if its lifetime is larger (smaller) in the sense of the usual stochastic ordering. If a different ordering is used, then we will add the corresponding description where necessary.

Thus, this theorem states that if the redundant component for load sharing is allocated to the stronger component, then the system with load sharing will be also stronger. This result can be, of course, intuitively anticipated as the same holds for ordinary, not load-sharing systems with an active (hot) or standby (cold) redundant component. However, an important feature of our result is that we need additionally an ordering between virtual ages.

Otherwise, the proposed ordering does not necessarily hold as the following counterexample shows.

Counterexample 2.1: Let X_1 and X_2 be two independent random variables representing the lifetimes of two components with failure rates 1 and 1.2, respectively. Further, let Y be another random variable representing the lifetime of a redundant component with the failure rate 2. Assume that X_1, X_2, and Y are independent. Let $\omega_1(u) = g_1(\alpha)u = 0.5u$, $\omega_2(u) = g_2(\alpha)u = 0.25u$, and $\gamma(u) = h(1-\alpha)u = 0.5u$, for all $u \geq 0$. Then, $X_1 \geq_{\rm st} X_2$ but $\omega_1(u) \not\leq \omega_2(u)$. Denote: $e_0(t) = \bar{F}_{X_1 \oplus Y}(t) - \bar{F}_{X_2 \oplus Y}(t)$. Then, for all $t \geq 0$,

$$e_0(t) = \int_0^t e^{-u} \left[e^{-(t-u+0.5u)} - e^{-1.2(t-u+0.25u)} \right] du + \int_0^t e^{-2(t-u+0.5u)} \left[e^{-0.5u} - e^{-0.3u} \right] du.$$

It can be numerically calculated that $e_0(1.2) = -0.18$ and $e_0(4.5) = 0.002$. This shows that $e_0(t)$ is not always non-negative, and hence $X_1 \oplus Y \not\geq_{\rm st} X_2 \oplus Y$.

In the following theorem we show that if we allocate the load-sharing redundancy to the weakest component of a series system, then the resulting system becomes optimal in the sense of the usual stochastic order (see the proof of Theorem 3.2).

THEOREM 2.2: *Fix $n = 2$. Let $\{\omega_1(\cdot), g_1(\cdot)\}$, $\{\omega_2(u), g_2(\cdot)\}$ and $\{\gamma(\cdot), h(\cdot)\}$ be the sets of model functions for X_1, X_2 and Y, respectively. Suppose that the following conditions hold.*

(i) $X_1 \leq_{hr} X_2$.

(ii) $\omega_1(u) = g_1(\alpha)u \leq g_2(\alpha)u = \omega_2(u)$ *and* $\gamma(u) = h(1-\alpha)u$, *for all* $0 \leq \alpha \leq 1$ *and* $u \geq 0$.

Then, $U_1 \geq_{st} U_2$.

The intuitive meaning of Theorem 2.2 is also quite clear: to allocate the redundant component to the weakest component of the system. However, distinct from the previous theorem, the weakest component is defined in the sense of the hazard rate ordering. This is an important observation that could not be foreseen without a proper proof. The following counterexample shows that the condition $X_1 \leq_{\rm hr} X_2$ given in Theorem 2.2 can not be replaced by $X_1 \leq_{\rm st} X_2$.

Counterexample 2.2: Let X_1, X_2, and Y be independent random variables with cumulative distribution functions given by

$$F_{X_1}(t) = \begin{cases} \dfrac{t}{3}, & \text{for } 0 \leq t \leq 3, \\ 1, & \text{for } t \geq 3, \end{cases}$$

$$F_{X_2}(t) = \begin{cases} \dfrac{t^2}{3}, & \text{for } 0 \leq t \leq 1, \\ \dfrac{t^2+3}{12}, & \text{for } 1 \leq t \leq 3, \\ 1, & \text{for } t \geq 3, \end{cases}$$

and

$$F_Y(t) = 1 - e^{-3t}, \quad \text{for } t > 0,$$

respectively. Then, it is easy to verify that $X_1 \leq_{\text{st}} X_2$ but $X_1 \nleq_{\text{hr}} X_2$. Further, let $\omega_1(u) = g_1(\alpha)u = 0.01u$, $\omega_2(u) = g_2(\alpha)u = 0.1u$, and $\gamma(u) = h(1-\alpha)u = 0.9u$. Note that condition (ii) given in Theorem 2.2 is satisfied. Denote: $e_1(t) = \bar{F}_{U_1}(t) - \bar{F}_{U_2}(t)$. Then, for all $0 \leq t \leq 1$,

$$\begin{aligned} e_1(t) &= e^{-3t}(t/3 - t^2/3) \\ &\quad + \int_0^t 0.3\left[\left(1-\frac{t^2}{3}\right)\left(1-\frac{0.01u}{3}\right) - \left(1-\frac{t}{3}\right)\left(1-\frac{0.01u^2}{3}\right)\right] e^{-3(t-u+0.9u)}du \\ &\quad + \int_0^t 2.7\left[\left(1-\frac{t^2}{3}\right)\left(1-\frac{t-u+0.01u}{3}\right) - \left(1-\frac{t}{3}\right)\left(1-\frac{(t-u+0.1u)^2}{3}\right)\right] e^{-2.7u}du. \end{aligned}$$

It can be numerically calculated that $e_1(0.1) = 0.027$ and $e_1(1) = -0.025$. This shows that $e_1(t)$ is not always non-negative. Thus, $U_1 \ngeq_{\text{st}} U_2$.

The following counterexample shows that the result given in Theorem 2.2 does not hold without the condition (ii).

Counterexample 2.3: Let X_1, X_2, and Y be independent random variables with survival functions given by $\bar{F}_{X_1}(t) = e^{-2.1t^2}$, $t > 0$, and $\bar{F}_{X_2}(t) = e^{-2t^2}$, $t > 0$, and $\bar{F}_Y(t) = e^{-3t^2}$, $t > 0$, respectively. Assume that X_1, X_2 and Y are independent. Let $\omega_1(u) = g_1(\alpha)u = 0.5u$, $\omega_2(u) = g_2(\alpha)u = 0.25u$ and $\gamma(u) = h(1-\alpha)u = 0.5u$, for all $u \geq 0$. Then, $X_1 \leq_{\text{hr}} X_2$ but $\omega_1(u) \nleq \omega_2(u)$. Denote: $\upsilon(t) = \bar{F}_{U_1}(t) - \bar{F}_{U_2}(t)$. Then, for all $t \geq 0$,

$$\begin{aligned} \upsilon(t) &= e^{-0.75t^2}\left[e^{-2.52t^2} - e^{-2.22t^2}\right] \\ &\quad + \int_0^t e^{-3(t-0.5u)^2}\left[1.05ue^{-(2t^2+0.52u^2)} - 0.25ue^{-(2.1t^2+0.12u^2)}\right] du \\ &\quad + \int_0^t 1.5ue^{-0.75u^2}\left[e^{-2.1(t-0.5u)^2-2t^2} - e^{-2(t-0.75u)^2-2.1t^2}\right] du. \end{aligned}$$

It can be numerically calculated that $\upsilon(0.3) = 0.004$ and $\upsilon(1) = -0.012$. This indicates that $\upsilon(t)$ is not always non-negative, and hence $U_1 \ngeq_{\text{st}} U_2$.

We are turning now to analysis of parallel load-sharing systems. We show that the optimal strategy (in the sense of the usual stochastic order) is when the redundant component is allocated for the load sharing to the strongest component of the system (see the proof of Theorem 3.4). Note that this can be considered as a new problem, as in the case of the ordinary redundancy (not load sharing), obviously, it does not matter to which component to allocate the redundant component.

Theorem 2.3: *Let $\{\omega_1(\cdot), g_1(\cdot)\}$, $\{\omega_2(u), g_2(\cdot)\}$, and $\{\gamma(\cdot), h(\cdot)\}$ be the sets of model functions for X_1, X_2, and Y, respectively. Suppose that the following conditions hold.*

(i) $X_1 \geq_{rh} X_2$.

(ii) $\omega_1(u) = g_1(\alpha)u \leq g_2(\alpha)u = \omega_2(u)$ *and* $\gamma(u) = h(1-\alpha)u$, *for all* $0 \leq \alpha \leq 1$ *and* $u \geq 0$.

Then, $V_1 \geq_{st} V_2$.

Thus, in order to obtain a more reliable system (in the sense of the usual stochastic order), the components should be ordered in the sense of the stronger reversed hazard rate order, which is an interesting observation. The following important counterexample shows that the condition $X_1 \geq_{\text{rh}} X_2$ given in Theorem 2.3 cannot be replaced by $X_1 \geq_{\text{st}} X_2$.

Counterexample 2.4: Fix $n = 2$. Let X_1, X_2, and Y be independent random variables with cumulative distribution functions given by

$$F_{X_1}(t) = \begin{cases} \dfrac{t^2}{3}, & \text{for } 0 \leq t \leq 1, \\ \dfrac{t^2+3}{12}, & \text{for } 1 \leq t \leq 3, \\ 1, & \text{for } t \geq 3, \end{cases}$$

$$F_{X_2}(t) = \begin{cases} \dfrac{t}{3}, & \text{for } 0 \leq t \leq 3, \\ 1, & \text{for } t \geq 3, \end{cases}$$

and

$$F_Y(t) = 1 - e^{-3t}, \quad \text{for } t > 0,$$

respectively. Then, it is easy to verify that $X_1 \geq_{\text{st}} X_2$ but $X_1 \ngeq_{\text{rh}} X_2$. Further, let $\omega_1(u) = g_1(\alpha)u = 0.01u$, $\omega_2(u) = g_2(\alpha)u = 0.1u$ and $\gamma(u) = h(1-\alpha)u = 0.9u$. Note that condition (ii) given in Theorem 2.3 is satisfied. Denote $e_2(t) = F_{V_2}(t) - F_{V_1}(t)$. Then, for all $1 \leq t \leq 3$,

$$\begin{aligned} e_2(t) = &\int_0^t 0.3 \left[\left(\frac{t^2+3}{12}\right)\left(\frac{0.01u}{3}\right) - \left(\frac{t}{3}\right)\left(\frac{0.0001u^2+3}{12}\right)\right] e^{-3(t-u+0.9u)} du \\ &+ \int_0^t 2.7 \left[\left(\frac{t^2+3}{12}\right)\left(\frac{t-u+0.1u}{3}\right) - \left(\frac{t}{3}\right)\left(\frac{(t-u+0.01u)^2+3}{12}\right)\right] e^{-2.7u} du. \end{aligned}$$

It can be numerically calculated that $e_2(1) = -0.018$ and $e_2(2.6) = 0.028$. This indicates that $e_2(t)$ is not always non-negative. Thus, $V_1 \ngeq_{\text{st}} V_2$.

The following counterexample shows that the result given in Theorem 2.3 does not hold without condition (ii).

Counterexample 2.5: Fix $n = 2$. Let X_1, X_2, and Y be independent random variables with failure rates 0.3, 0.45, and 1, respectively. Assume that X_1, X_2, and Y are independent. Let $\omega_1(u) = g_1(\alpha)u = 0.2u$, $\omega_2(u) = g_2(\alpha)u = 0.04u$, and $\gamma(u) = h(1-\alpha)u = 0.8u$, for all

$u \geq 0$. Then, $X_1 \geq_{\rm rh} X_2$ but $\omega_1(u) \nleq \omega_2(u)$. Denote: $\eta(t) = F_{V_2}(t) - F_{V_1}(t)$. Then, for all $t \geq 0$,

$$\eta(t) = \int_0^t \left[\left(1 - e^{0.3t}\right)\left(1 - e^{0.45(t-0.96u)}\right) - \left(1 - e^{0.45t}\right)\left(1 - e^{0.3(t-0.8u)}\right)\right] 0.8e^{-0.8u} du$$
$$+ \int_0^t \left[\left(1 - e^{0.3t}\right)\left(1 - e^{0.018u}\right) - \left(1 - e^{0.45t}\right)\left(1 - e^{0.06u}\right)\right] 0.2e^{-(t-0.2u)} du.$$

It can be numerically calculated that $\eta(2.5) = -0.01$ and $\eta(8.5) = 0.01$. This indicates that $\eta(t)$ is not always non-negative, and hence $V_1 \ngeq_{\rm st} V_2$.

3. GENERAL SCENARIO

In the previous section, the assumption of the cumulative exposure model was used to obtain the corresponding virtual ages after switching. As was already mentioned, we will not assume in this section that the cumulative exposure model holds and will consider general forms of the virtual age functions. Thus, the theorems of this section are generalizations of the corresponding theorems of the previous section, where the more practical results were presented. Our presentation of the following results are more formal. It should be noted that our theorems here employ a number of additional assumptions, however, their probabilistic meaning is quite clear and can be easily interpreted.

The following theorem is a generalization of Theorem 2.1

THEOREM 3.1: *Let $\{\omega_1(\cdot), g_1(\cdot)\}$, $\{\omega_2(u), g_2(\cdot)\}$, and $\{\gamma(\cdot), h(\cdot)\}$ be the sets of model functions for X_1, X_2, and Y, respectively. Suppose that the following conditions hold.*

(i) $g_1(\alpha) \leq g_2(\alpha)$, *for all* $0 \leq \alpha \leq 1$, *and* $\omega_1(u) \leq \omega_2(u)$, *for all* $u \geq 0$.
(ii) $u - \gamma(u)$ *and* $\gamma(u) - h(1-\alpha)u$ *are increasing in* $u \geq 0$, *for all* $0 \leq \alpha \leq 1$.
(iii) $X_1 \geq_{hr} X_2$, *and* X_1 *or* X_2 *has log-concave survival function.*
(iv) Y *has log-concave survival function.*

Then, $X_1 \oplus Y \geq_{st} X_2 \oplus Y$.

PROOF: We only prove the result when X_1 has log-concave survival function. The result follows similarly for the other case. Note that

$$\bar{F}_{X_1 \oplus Y}(t) - \bar{F}_{X_2 \oplus Y}(t) = l_1(t) + l_2(t),$$

where

$$l_1(t) = \int_0^t h(1-\alpha) f_Y(h(1-\alpha)u) \left[\frac{\bar{F}_{X_1}(t-u+\omega_1(u))}{\bar{F}_{X_1}(\omega_1(u))} \bar{F}_{X_1}(g_1(\alpha)u) - \frac{\bar{F}_{X_2}(t-u+\omega_2(u))}{\bar{F}_{X_2}(\omega_2(u))} \bar{F}_{X_2}(g_2(\alpha)u)\right] du,$$

and

$$\begin{aligned} l_2(t) &= \bar{F}_Y(h(1-\alpha)t)\left[\bar{F}_{X_1}(g_1(\alpha)t) - \bar{F}_{X_2}(g_2(\alpha)t)\right] \\ &\quad - \int_0^t \bar{F}_Y(h(1-\alpha)u)\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))} d\left[\bar{F}_{X_1}(g_1(\alpha)u) - \bar{F}_{X_2}(g_2(\alpha)u)\right] \\ &= \int_0^t \left[\bar{F}_{X_1}(g_1(\alpha)u) - \bar{F}_{X_2}(g_2(\alpha)u)\right] d\left[\bar{F}_Y(h(1-\alpha)u)\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\right]. \end{aligned}$$

To prove the result it suffices to show that both $l_1(t)$ and $l_2(t)$ are non-negative. Note that

$$\begin{aligned} l_1(t) &\geq \int_0^t h(1-\alpha) f_Y(h(1-\alpha)u)\bar{F}_{X_2}(g_2(\alpha)u)\left[\frac{\bar{F}_{X_1}(t-u+\omega_1(u))}{\bar{F}_{X_1}(\omega_1(u))} - \frac{\bar{F}_{X_2}(t-u+\omega_2(u))}{\bar{F}_{X_2}(\omega_2(u))}\right] du \\ &\geq \int_0^t h(1-\alpha) f_Y(h(1-\alpha)u)\bar{F}_{X_2}(g_2(\alpha)u)\left[\frac{\bar{F}_{X_1}(t-u+\omega_2(u))}{\bar{F}_{X_1}(\omega_2(u))} - \frac{\bar{F}_{X_2}(t-u+\omega_2(u))}{\bar{F}_{X_2}(\omega_2(u))}\right] du \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the fact that $X_1 \geq_{\mathrm{hr}} X_2$ and $g_1(\alpha) \leq g_2(\alpha)$. The second inequality holds because X_1 has log-concave survival function, and $\omega_1(u) \leq \omega_2(u)$, whereas the third inequality follows from $X_1 \geq_{\mathrm{hr}} X_2$. Further, for all $u \geq 0$, we have

$$\begin{aligned} \frac{d}{du}\left(\frac{\bar{F}_Y(h(1-\alpha)u)}{\bar{F}_Y(\gamma(u))}\right) &= \frac{\bar{F}_Y(h(1-\alpha)u)}{\bar{F}_Y(\gamma(u))}\left[\gamma'(u)\frac{f_Y(\gamma(u))}{\bar{F}_Y(\gamma(u))} - h(1-\alpha)\frac{f_Y(h(1-\alpha)u)}{\bar{F}_Y(h(1-\alpha)u)}\right] \\ &\geq h(1-\alpha)\frac{\bar{F}_Y(h(1-\alpha)u)}{\bar{F}_Y(\gamma(u))}\left[\frac{f_Y(\gamma(u))}{\bar{F}_Y(\gamma(u))} - \frac{f_Y(h(1-\alpha)u)}{\bar{F}_Y(h(1-\alpha)u)}\right] \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the fact that both $\gamma(u)$ and $\gamma(u) - h(1-\alpha)u$ are increasing in $u \geq 0$. The second inequality holds because $\gamma(u) \geq h(1-\alpha)u$, for all $u \geq 0$, and Y has log-concave survival function. Thus,

$$\frac{\bar{F}_Y(h(1-\alpha)u)}{\bar{F}_Y(\gamma(u))} \quad \text{is increasing in } u \geq 0. \tag{3.1}$$

Since, $u - \gamma(u)$ is increasing in $u \geq 0$, we have that

$$\bar{F}_Y(t-u+\gamma(u)) \quad \text{is increasing in } u \geq 0. \tag{3.2}$$

Thus, from (3.1) and (3.2), we get that

$$\bar{F}_Y(h(1-\alpha)u)\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))} \quad \text{is increasing in } u \geq 0. \tag{3.3}$$

Again, $X_1 \geq_{\mathrm{hr}} X_2$ and $g_1(\alpha) \leq g_2(\alpha)$ imply that, for all $u \in [0, t]$,

$$\bar{F}_{X_1}(g_1(\alpha)u) - \bar{F}_{X_2}(g_2(\alpha)u) \geq 0. \tag{3.4}$$

Thus, on using (3.3) and (3.4), we have that $l_2(t) \geq 0$, and hence the result is proved. ■

Below we give an example of model functions which supports the above theorem.

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Example 3.1: Let $\gamma(u) = au$, $\omega_1(u) = bu$, and $\omega_2(u) = cu$, for all $0 \le a \le 1$ and $0 \le b \le c \le 1$. Further, let $g_1(\alpha) = \alpha^2$, $g_2(\alpha) = \alpha$, and $h(1-\alpha) = (1-\alpha)$ with $a \ge 1-\alpha$. Note that all the conditions of the model functions given in Theorem 3.1 are satisfied.

Remark 3.1: It can be easily verified that $u - \gamma(u)$ can never be decreasing in $u \ge 0$. It might happen that $u - \gamma(u)$ is non-monotone in some cases, but we could not find the corresponding example. Thus, we were not able so far to relax the condition that $u - \gamma(u)$ should be increasing in $u \ge 0$.

Remark 3.2: It is to be noted that the condition (i) given in Theorem 3.1 cannot be removed (see Counterexample 2.1).

The following counterexample shows that the condition "$\gamma(u) - h(1-\alpha)u$ is increasing in u" given in Theorem 3.1 cannot be relaxed.

Counterexample 3.1: Let X_1 and X_2 be two independent random variables representing the lifetimes of two components with failure rate 20.5 and 21, respectively. Further, let Y be another random variable representing the lifetime of a redundant component with survival function given by $\bar{F}_Y(t) = \exp\{-0.2t^2\}$, $t > 0$. Let $g_1(\alpha) = 0.01$, $g_2(\alpha) = 0.1$, $h(1-\alpha) = 0.9$, $\gamma(u) = 0.01u$, $\omega_1(u) = 0.2u$, and $\omega_2(u) = 0.3u$, for all $u \ge 0$. Note that all the conditions given in Theorem 3.1 are satisfied except $\gamma(u) - h(1-\alpha)u$ is not increasing in u. Denote: $\kappa(t) = \bar{F}_{X_1 \oplus Y}(t) - \bar{F}_{X_2 \oplus Y}(t)$. Then, for all $t \ge 0$,

$$\begin{aligned}\kappa(t) &= e^{-0.162t^2}\left[e^{-0.205t} - e^{-2.1t}\right] \\ &\quad - \int_0^t e^{-0.162u^2 - 0.2(t-0.99u)^2 + 0.2(0.01u)^2}\left[2.1e^{-2.1u} - 0.205e^{-0.205u}\right] du \\ &\quad + \int_0^t 0.324ue^{-0.162u^2}\left[e^{-20.5(t-0.99u)} - e^{-21(t-0.9u)}\right] du.\end{aligned}$$

It can be numerically obtained that $\kappa(1.4) = -0.021$ and $\kappa(4) = 0.035$. This indicates that $\kappa(t)$ changes sign over t, and hence $X_1 \oplus Y \ngeq_{\text{st}} X_2 \oplus Y$.

In the following theorem, we show that allocation of the redundant component to the stochastically weakest component of the system is the best strategy (in the sense of the usual stochastic order) for obtaining the optimal series system (see Theorem 2.2).

THEOREM 3.2: *Let $\{\omega_1(\cdot), g_1(\cdot)\}$, $\{\omega_2(u), g_2(\cdot)\}$ and $\{\gamma(\cdot), h(\cdot)\}$ be the sets of model functions for X_1, X_2, and Y, respectively. Suppose that (i) and (ii), and any one among (iii), (iv), (v), (vi) hold.*

- *(i)* *$X_1 \le_{hr} X_2$ and $g_1(\alpha) \le g_2(\alpha)$, for all $0 \le \alpha \le 1$.*
- *(ii)* *Y has log-concave survival function, and $u - \gamma(u)$ and $\gamma(u) - h(1-\alpha)u$ are increasing in $u \ge 0$, for all $0 \le \alpha \le 1$.*
- *(iii)* *X_1 has log-concave survival function, and $\max\{g_1(\alpha)u, \omega_1(u)\} \le \omega_2(u)$, for all $u \ge 0$ and $0 \le \alpha \le 1$.*
- *(iv)* *X_1 has log-convex survival function, and $g_1(\alpha)u \le \omega_2(u) \le \omega_1(u)$, for all $u \ge 0$ and $0 \le \alpha \le 1$.*

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(*v*) X_2 *has log-concave survival function, and* $g_1(\alpha)u \le \omega_1(u) \le \omega_2(u)$, *for all* $u \ge 0$ *and* $0 \le \alpha \le 1$.

(*vi*) X_2 *has log-convex survival function, and* $\max\{g_1(\alpha)u, \omega_2(u)\} \le \omega_1(u)$, *for all* $u \ge 0$ *and* $0 \le \alpha \le 1$.

Then, $U_1 \ge_{st} U_2$.

Proof: We only prove the result under conditions (*i*), (*ii*), and (*iii*). The result follows similarly for the other cases. Note that

$$\bar{F}_{U_1}(t) = \bar{F}_{X_1 \oplus Y}(t)\bar{F}_{X_2}(t)\bar{F}_Z(t),$$

and

$$\bar{F}_{U_2}(t) = \bar{F}_{X_2 \oplus Y}(t)\bar{F}_{X_1}(t)\bar{F}_Z(t).$$

Writing $\Im(t) = \bar{F}_{U_1}(t) - \bar{F}_{U_2}(t)$, we have

$$\Im(t) = k_1(t) + k_2(t),$$

where

$$\begin{aligned}
k_1(t) &= \bar{F}_Z(t)\bar{F}_Y(h(1-\alpha)t)\left[\bar{F}_{X_2}(t)\bar{F}_{X_1}(g_1(\alpha)t) - \bar{F}_{X_1}(t)\bar{F}_{X_2}(g_2(\alpha)t)\right] \\
&\quad + \int_0^t \bar{F}_Z(t)\bar{F}_Y(h(1-\alpha)u)\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\left[\bar{F}_{X_2}(t)g_1(\alpha)f_{X_1}(g_1(\alpha)u)\right. \\
&\quad \left. - \bar{F}_{X_1}(t)g_2(\alpha)f_{X_2}(g_2(\alpha)u)\right] du \\
&= \bar{F}_Z(t)\bar{F}_Y(t)\left[\bar{F}_{X_2}(t) - \bar{F}_{X_1}(t)\right] + \int_0^t \bar{F}_Z(t)\left[\bar{F}_{X_2}(t)\bar{F}_{X_1}(g_1(\alpha)u)\right. \\
&\quad \left. - \bar{F}_{X_1}(t)\bar{F}_{X_2}(g_2(\alpha)u)\right] d\left[\bar{F}_Y(h(1-\alpha)u)\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\right],
\end{aligned}$$

and

$$\begin{aligned}
k_2(t) = \int_0^t \bar{F}_Z(t)h(1-\alpha)f_Y(h(1-\alpha)u)&\left[\frac{\bar{F}_{X_1}(t-u+\omega_1(u))}{\bar{F}_{X_1}(\omega_1(u))}\bar{F}_{X_1}(g_1(\alpha)u)\bar{F}_{X_2}(t)\right. \\
&\left. - \frac{\bar{F}_{X_2}(t-u+\omega_2(u))}{\bar{F}_{X_2}(\omega_2(u))}\bar{F}_{X_2}(g_2(\alpha)u)\bar{F}_{X_1}(t)\right] du.
\end{aligned}$$

To prove the result, it is sufficient to show that both $k_1(t)$ and $k_2(t)$ are non-negative. From (i), we have

$$\bar{F}_{X_2}(t)\bar{F}_{X_1}(g_1(\alpha)t) - \bar{F}_{X_1}(t)\bar{F}_{X_2}(g_2(\alpha)t) \ge 0, \tag{3.5}$$

and

$$\bar{F}_{X_2}(t) - \bar{F}_{X_1}(t) \ge 0. \tag{3.6}$$

Further, (ii) implies that (see the proof of Theorem 3.1)

$$\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u \text{ is increasing in } u \ge 0. \tag{3.7}$$

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Thus, on using (3.5)–(3.7), we have that $k_1(t) \geq 0$. Again, (iii) implies that

$$\frac{\bar{F}_{X_1}(t-u+\omega_1(u))}{\bar{F}_{X_1}(\omega_1(u))} \geq \frac{\bar{F}_{X_1}(t-u+\omega_2(u))}{\bar{F}_{X_1}(\omega_2(u))}, \tag{3.8}$$

which can equivalently be written as

$$\begin{aligned}\frac{\bar{F}_{X_1}(t-u+\omega_1(u))}{\bar{F}_{X_1}(\omega_1(u))}\bar{F}_{X_1}(g_1(\alpha)u)\bar{F}_{X_2}(t) &\geq \frac{\bar{F}_{X_1}(t-u+\omega_2(u))}{\bar{F}_{X_1}(\omega_2(u))}\bar{F}_{X_1}(g_1(\alpha)u)\bar{F}_{X_2}(t)\\ &\geq \frac{\bar{F}_{X_2}(t-u+\omega_2(u))}{\bar{F}_{X_1}(\omega_2(u))}\bar{F}_{X_1}(g_1(\alpha)u)\bar{F}_{X_1}(t)\\ &\geq \frac{\bar{F}_{X_2}(t-u+\omega_2(u))}{\bar{F}_{X_2}(\omega_2(u))}\bar{F}_{X_2}(g_2(\alpha)u)\bar{F}_{X_1}(t),\end{aligned}$$

where the second and third inequalities follow from (i) and (iii). Thus, on using the above inequality, we get $k_2(t) \geq 0$. Hence, the result is proved. ■

An example of model functions, which satisfies Theorem 3.2, is discussed below.

Example 3.2: Let $\gamma(u) = a\ln(1+u)$, $\omega_1(u) = bu$, and $\omega_2(u) = cu$, for all $0 \leq a \leq 1$ and $0 \leq b \leq c \leq 1$. Further, let $g_1(\alpha) = \alpha^2$, $g_2(\alpha) = \alpha$, and $h(1-\alpha)u = (1-\alpha)\ln(1+u)$ with $a \geq 1-\alpha$. Note that all conditions for the model functions given in (i)–(iii) of Theorem 3.2 are satisfied.

Remark 3.3: It is to be noted that the conditions "$g_1(\alpha) \leq g_2(\alpha)$", "$\max\{g_1(\alpha)u, \omega_1(u)\} \leq \omega_2(u)$" and "$g_1(\alpha)u \leq \omega_1(u) \leq \omega_2(u)$" given in Theorem 3.2 cannot be relaxed (see Counterexample 2.3).

The following counterexample shows that the result given in Theorem 3.2 does not hold without the condition "$\gamma(u) - h(1-\alpha)u$ is increasing in u".

Counterexample 3.2: Fix $n = 2$. Let X_1 and X_2 be two independent random variables representing the lifetimes of two components with failure rates 19 and 18.99, respectively. Further, let Y be another random variable representing the lifetime of a redundant component with survival function given by $\bar{F}_Y(t) = \exp\{-t^2\}$, $t > 0$. Let $g_1(\alpha) = 0.01$, $g_2(\alpha) = 0.1$, $h(1-\alpha) = 0.9$, $\gamma(u) = 0.01u$, $\omega_1(u) = 0.2u$, and $\omega_2(u) = 0.3u$, for all $u \geq 0$. Note that all the conditions given in (i), (ii), and (iii) of Theorem 3.2 are satisfied except $\gamma(u) - h(1-\alpha)u$ is not increasing in u. Denote: $\kappa_0(t) = \bar{F}_{U_1}(t) - \bar{F}_{U_2}(t)$. Then, for all $t \geq 0$,

$$\begin{aligned}\kappa_0(t) = e^{-0.81t^2}&\left[e^{-19.18t} - e^{-20.899t}\right]\\ &- \int_0^t e^{-0.81u^2-(t-0.99u)^2+(0.01u)^2}\left[1.899e^{-(19t+1.899u)} - 0.19e^{-(18.99t+0.19u)}\right]du\\ &+ \int_0^t 1.62ue^{-(0.81u^2+37.99t)}\left[e^{-18.81u} - e^{-17.09u)}\right]du.\end{aligned}$$

It can be verified that $\kappa_0(t)$ is not always non-negative, and hence $U_1 \ngeq_{\text{st}} U_2$.

The following theorem shows that a similar result as in Theorem 3.2 holds under some weaker conditions, whenever both X_1 and X_2 have the same set of model functions.

THEOREM 3.3: *Let both* X_1 *and* X_2 *have the same set of model functions given by* $\{\omega(\cdot), g(\cdot)\}$, *and* Y *have the model function given by* $\{\gamma(\cdot), h(\cdot)\}$. *Assume that* $\omega(u) \geq g(\alpha)u$, *for all* $0 \leq \alpha \leq 1$ *and* $u \geq 0$. *If* $X_1 \leq_{hr} X_2$ *then* $U_1 \geq_{st} U_2$.

PROOF: From $\zeta(t) = \bar{F}_{U_1}(t) - \bar{F}_{U_2}(t)$, we have

$$\zeta(t) = k_3(t) + k_4(t), \tag{3.9}$$

where

$$\begin{aligned} k_3(t) = {} & \bar{F}_Z(t)\bar{F}_Y(h(1-\alpha)t)\left[\bar{F}_{X_2}(t)\bar{F}_{X_1}(g(\alpha)t) - \bar{F}_{X_1}(t)\bar{F}_{X_2}(g(\alpha)t)\right] \\ & + \int_0^t \bar{F}_Z(t)\bar{F}_Y(h(1-\alpha)u)\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))} g(\alpha)\left[\bar{F}_{X_2}(t) f_{X_1}(g(\alpha)u) \right. \\ & \left. - \bar{F}_{X_1}(t) f_{X_2}(g(\alpha)u)\right] du, \end{aligned}$$

and

$$\begin{aligned} k_4(t) = {} & \int_0^t \bar{F}_Z(t) h(1-\alpha) f_Y(h(1-\alpha)u)\left[\frac{\bar{F}_{X_1}(t-u+\omega(u))}{\bar{F}_{X_1}(\omega(u))}\bar{F}_{X_1}(g(\alpha)u)\bar{F}_{X_2}(t) \right. \\ & \left. - \frac{\bar{F}_{X_2}(t-u+\omega(u))}{\bar{F}_{X_2}(\omega(u))}\bar{F}_{X_2}(g(\alpha)u)\bar{F}_{X_1}(t)\right] du. \end{aligned}$$

Since, $X_1 \leq_{\mathrm{hr}} X_2$, we have, for all $0 \leq u \leq t < \infty$,

$$\bar{F}_{X_2}(t)\bar{F}_{X_1}(g(\alpha)t) - \bar{F}_{X_1}(t)\bar{F}_{X_2}(g(\alpha)t) \geq 0, \tag{3.10}$$

$$\bar{F}_{X_2}(t)\bar{F}_{X_1}(g(\alpha)u) - \bar{F}_{X_1}(t)\bar{F}_{X_2}(g(\alpha)u) \geq 0, \tag{3.11}$$

and

$$r_{X_1}(g(\alpha)u) \geq r_{X_2}(g(\alpha)u). \tag{3.12}$$

On using (3.11) and (3.12), we have

$$\bar{F}_{X_2}(t) f_{X_1}(g(\alpha)u) - \bar{F}_{X_1}(t) f_{X_2}(g(\alpha)u) \geq 0. \tag{3.13}$$

Thus, on using (3.10) and (3.13), we get that $k_3(t) \geq 0$. Further, $X_1 \leq_{\mathrm{hr}} X_2$ and $\omega(u) \geq g(\alpha)u$ imply that

$$\bar{F}_{X_1}(t-u+\omega(u))\bar{F}_{X_2}(t) - \bar{F}_{X_2}(t-u+\omega(u))\bar{F}_{X_1}(t) \geq 0, \tag{3.14}$$

and

$$\frac{\bar{F}_{X_1}(g(\alpha)u)}{\bar{F}_{X_1}(\omega(u))} \geq \frac{\bar{F}_{X_2}(g(\alpha)u)}{\bar{F}_{X_2}(\omega(u))}. \tag{3.15}$$

Thus, on using (3.14) and (3.15), we have $k_4(t) \geq 0$. Hence, the result is proved. ■

In the following theorem, we show that the best strategy to obtain the optimal parallel system is to allocate the redundancy to the stochastically strongest component of the system (see Theorem 2.4.)

THEOREM 3.4: *Let* $\{\omega_1(\cdot), g_1(\cdot)\}$, $\{\omega_2(u), g_2(\cdot)\}$ *and* $\{\gamma(\cdot), h(\cdot)\}$ *be the sets of model functions for* X_1, X_2 *and* Y, *respectively. Assume that* $X_1 \geq_{rh} X_2$. *Suppose that the following conditions hold.*

(i) $\omega_1(u) = g_1(\alpha)u \leq g_2(\alpha)u = \omega_2(u)$ *for all* $u \geq 0$ *and* $0 \leq \alpha \leq 1$.

(ii) Y *has log-concave survival function, and* $u - \gamma(u)$ *and* $\gamma(u) - h(1-\alpha)u$ *are increasing in* $u \geq 0$, *for all* $0 \leq \alpha \leq 1$.

Then, $V_1 \geq_{st} V_2$.

PROOF: Note that

$$\begin{aligned}\bar{F}_{X_1 \oplus Y}(t) &= \bar{F}_{X_1}(g_1(\alpha)t)\bar{F}_Y(h(1-\alpha)t) + \int_0^t \bar{F}_{X_1}(t-u+\omega_1(u))dF_Y(h(1-\alpha)u) \\ &\quad + \int_0^t \frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u)dF_{X_1}(g_1(\alpha)u) \\ &= \bar{F}_Y(h(1-\alpha)t) + \int_0^t \bar{F}_{X_1}(t-u+\omega_1(u))dF_Y(h(1-\alpha)u) \\ &\quad - \int_0^t F_{X_1}(g_1(\alpha)u)\, d\left[\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u)\right],\end{aligned}$$

which gives

$$\begin{aligned}F_{X_1 \oplus Y}(t) &= F_Y(h(1-\alpha)t) - \int_0^t \bar{F}_{X_1}(t-u+\omega_1(u))dF_Y(h(1-\alpha)u) \\ &\quad + \int_0^t F_{X_1}(g_1(\alpha)u)\, d\left[\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u)\right] \\ &= \int_0^t F_{X_1}(t-u+\omega_1(u))dF_Y(h(1-\alpha)u) \\ &\quad + \int_0^t F_{X_1}(g_1(\alpha)u)\, d\left[\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u)\right].\end{aligned}$$

From $\Im_2(t) = F_{V_2}(t) - F_{V_1}(t)$, we have

$$\Im_2(t) = s_1(t) + s_2(t),$$

where

$$s_1(t) = \int_0^t F_W(t)\left[F_{X_1}(t)F_{X_2}(t-u+\omega_2(u)) - F_{X_2}(t)F_{X_1}(t-u+\omega_1(u)\right] dF_Y(h(1-\alpha)u),$$

and

$$\begin{aligned}s_2(t) = \int_0^t F_W(t)\left[F_{X_1}(t)F_{X_2}(g_2(\alpha)u) - F_{X_2}(t)F_{X_1}(g_1(\alpha)u)\right] d \\ \times \left[\frac{\bar{F}_Y(t-u+\gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u)\right].\end{aligned}$$

Since, $X_1 \geq_{\text{rh}} X_2$ and $\omega_1(u) = g_1(\alpha)u \leq g_2(\alpha)u = \omega_2(u)$ we have, for all $u \in [0, t]$,

$$F_{X_1}(t)F_{X_2}(t - u + \omega_2(u)) - F_{X_2}(t)F_{X_1}(t - u + \omega_1(u) \geq 0, \tag{3.16}$$

and

$$F_{X_1}(t)F_{X_2}(g_2(\alpha)u) - F_{X_2}(t)F_{X_1}(g_1(\alpha)u) \geq 0. \tag{3.17}$$

Again, condition (ii) implies that (see proof of Theorem 3.1)

$$\frac{\bar{F}_Y(t - u + \gamma(u))}{\bar{F}_Y(\gamma(u))}\bar{F}_Y(h(1-\alpha)u \text{ is increasing in } u \geq 0. \tag{3.18}$$

Thus, on using (3.16)–(3.18) we get that $s_1(t) \geq 0$ and $s_2(t) \geq 0$, and hence the result follows. ■

Below we give an example of model functions that supports the above theorem.

Example 3.3: Let $\gamma(u) = au/(1+u)$, and $\omega_1(u) = g_1(\alpha)u = \alpha^2 u$, $\omega_2(u) = g_2(\alpha)u = \alpha u$, and $h(1-\alpha)u = (1-\alpha)u/(1+u)$ with $a \geq 1 - \alpha$. Note that all conditions of the model functions given in Theorem 3.4 are satisfied.

Remark 3.4: Condition (i) given in Theorem 3.4 cannot be relaxed (see Counterexample 2.5).

The following counterexample shows that the condition "$\gamma(u) - h(1-\alpha)u$ is increasing in u" given in Theorem 3.4 cannot be relaxed.

Counterexample 3.3: Fix $n = 2$. Let X_1 and X_2 be two independent random variables representing the lifetimes of two components with failure rates 5.45 and 5.55, respectively. Further, let Y be another random variable representing the lifetime of a redundant component with survival function given by $\bar{F}_Y(t) = \exp\{-t^2\}$, $t > 0$. Let $\omega_1(u) = g_1(\alpha)u = 0.01u$, $\omega_2(u) = g_2(\alpha)u = 0.1u$, $h(1-\alpha) = 0.9$, and $\gamma(u) = 0.01u$, for all $u \geq 0$. Note that all the conditions given in Theorem 3.4 are satisfied except $\gamma(u) - h(1-\alpha)u$ is not increased in u. Denote: $\xi(t) = F_{V_2}(t) - F_{V_1}(t)$. Then, for all $t \geq 0$,

$$\begin{aligned}
\xi(t) = &\int_0^t \left[\left(1 - e^{-5.45t}\right)\left(1 - e^{-5.55t(t-0.9u)}\right) - \left(1 - e^{-5.55t}\right)\right. \\
&\times \left.\left(1 - e^{-5.45t(t-0.99u)}\right)\right] 1.62ue^{-0.81u^2}du \\
&+ \int_0^t \left[\left(1 - e^{-5.45t}\right)\left(1 - e^{-0.555u}\right) - \left(1 - e^{-5.55t}\right)\left(1 - e^{-0.054u}\right)\right] \\
&\times 2(0.99t - 1.79u)e^{-\left[(t-0.99u)^2 - (0.01u)^2 + 0.81u^2\right]}du.
\end{aligned}$$

It can be numerically obtained that $\xi(0.8) = 0.009$ and $\xi(2) = -0.01$. This indicates that $\xi(t)$ is not always non-negative, and hence $V_1 \ngeq_{\text{st}} V_2$.

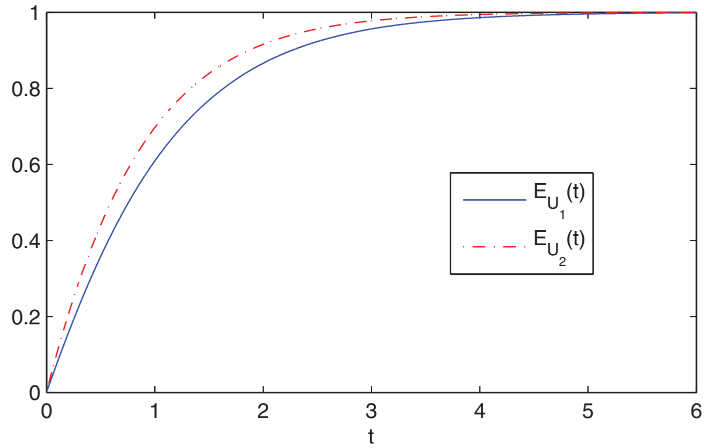

FIGURE 1. Plot of $E_{U_1}(t)$ and $E_{U_2}(t)$ against t.

4. SIMULATION

We illustrate the result given in Theorem 2.2 via the simulation example below. Other theorems of our paper can be illustrated in a similar way. It is possible to perform the comparison using the corresponding survival curves, as all our theorems eventually compare the systems lifetimes in the sense of the usual stochastic ordering.

Consider a system with $n = 3$ components in series. Let X_1, X_2, X_3, and Y be independent exponentially distributed random variables having respective failure rates $4, 2, 6$, and 5. We take $\omega_1(u) = g_1(\alpha)u = 0.04u$, $\omega_2(u) = g_2(\alpha)u = 0.2u$ and $\gamma(u) = h(1-\alpha)u = 0.8u$. Then $X_1 \oplus Y$ and $X_2 \oplus Y$ have the survival functions given by $\bar{F}_{X_1 \oplus Y}(t) = -23.81e^{-4.16t} + 25e^{-4t} - 0.19e^{-5t}, \;\; t > 0,$ and $\bar{F}_{X_2 \oplus Y}(t) = -0.833e^{-4.08t} + 1.92e^{-2t} - 0.087e^{-5t}, \;\; t > 0,$ respectively. Note that all the conditions given in Theorem 2.2 are satisfied by these random variables. Now we draw samples for X_1, X_2, X_3, $X_1 \oplus Y$, and $X_2 \oplus Y$, each of size $800,000$, from the respective distributions. We call them x_{1i}, x_{2i}, x_{3i}, $x_{1i} \oplus y_i$, and $x_{2i} \oplus y_i$, for $i = 1, 2, \ldots, \overline{800,000}$, respectively. Now we calculate, for $i = 1, 2, \ldots, \overline{800,000}$, $u_{1i} = \min\{x_{1i} \oplus y_i, x_{2i}, x_{3i}\}$ and $u_{2i} = \min\{x_{1i}, x_{2i} \oplus y_i, x_{3i}\}$, the realization of U_1 and U_2, respectively. Next we sort the combined sample (u_{1i}, u_{2i}), $i = 1, 2, \ldots, \overline{800,000}$, in an increasing order of magnitude, and their empirical distribution functions $E_{U_1}(\cdot)$ and $E_{U_2}(\cdot)$ are plotted in Figure 1. This shows that $E_{U_2}(t)$ dominates $E_{U_1}(t)$ for all t. Thus, we may conclude, based on the data set, that $U_1 \geq_{\text{st}} U_2$.

5. CONCLUDING REMARKS

In this paper, we have considered general load-sharing series and parallel systems. We have shown that, for a load-sharing series (resp. parallel) system, the best strategy for achieving maximal reliability is to allocate the redundant component to the weakest (resp. strongest) original component of the system. We have studied the proposed results under cumulative exposure model as well as in a general scenario. The accelerated lifetime model and the virtual age concept are used in order to calculate the reliability function of a general load-sharing system. To the best of our knowledge, there are no results in the literature that deal with allocation strategies for general load-sharing systems and, therefore, our paper might be considered as the first step in this direction. We have considered only series and parallel

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systems. The study of more general systems (e.g., k-out-of-n system and coherent system) can constitute a topic for the future research.

We conclude our discussion by mentioning the fact that the straightforward corollaries corresponding to Theorems 2.2, 2.3, 3.3, and 3.4 could be formulated similar to the one given below for Theorem 3.2.

Corollary 5.1: *Let $\{\omega_i(\cdot), g_i(\cdot)\}$ be the set of model functions for X_i, $i = 1, 2, \ldots, n$, and $\{\gamma(\cdot), h(\cdot)\}$ be that for Y. Suppose that (i) and (ii), and (iii) or (iv) hold.*

(i) $X_1 \leq_{hr} X_2 \leq_{hr} \cdots \leq_{hr} X_n$ *and* $g_1(\alpha) \leq g_2(\alpha) \leq \cdots \leq g_n(\alpha)$, *for all* $0 \leq \alpha \leq 1$.

(ii) Y *has log-concave survival function, and* $u - \gamma(u)$ *and* $\gamma(u) - h(1-\alpha)u$ *are increasing in* $u \geq 0$, *for all* $0 \leq \alpha \leq 1$.

(iii) *Let* n *be an even integer. Further,* $X_1, X_3, \ldots, X_{n-1}$ *or* $X_2, X_4, \ldots, X_n$ *have log-concave (resp. log-convex) survival functions, and* $g_n(\alpha)u \leq \omega_1(u) \leq \omega_2(u) \leq \cdots \leq \omega_n(u)$ *(resp.* $g_n(\alpha)u \leq \omega_n(u) \leq \omega_{n-1}(u) \leq \cdots \leq \omega_1(u)$*), for all* $u \geq 0$ *and* $0 \leq \alpha \leq 1$.

(iv) *Let* n *be an odd integer. Further,* $X_1, X_3, \ldots, X_n$ *or* $X_2, X_4, \ldots, X_{n-1}$ *have log-concave (resp. log-convex) survival functions, and* $g_n(\alpha)u \leq \omega_1(u) \leq \omega_2(u) \leq \cdots \leq \omega_n(u)$ *(resp.* $g_n(\alpha)u \leq \omega_n(u) \leq \omega_{n-1}(u) \leq \cdots \leq \omega_1(u)$*), for all* $u \geq 0$ *and* $0 \leq \alpha \leq 1$.

Then, $U_1 \geq_{st} U_2 \geq_{\mathrm{st}} \cdots \geq_{\mathrm{st}} U_n$.

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References

1. Barlow, R.E. & Proschan, F. (1965). *Mathematical theory of reliability and life testing*. New York: Wiley.
2. Boland, P.J., El-Neweihi, E., & Proschan, F. (1992). Stochastic order for redundancy allocations in series and parallel systems. *Advances in Applied Probability* 24: 161–171.
3. Brito, G., Zequeira, R.I. & Valdés, J.E. (2011). On the hazard rate and reversed hazard rate orderings in two-component series systems with active redundancies. *Statistics and Probability Letters* 81: 201–206.
4. Cha, J.H., Mi, J., & Yun, W.Y. (2008). Modelling a general standby system and evaluation of its performance. *Applied Stochastic Models in Business and Industry* 24: 159–169.
5. Finkelstein, M. (2007). On statistical and information-based virtual age of degrading system. *Reliability Engineering and System Safety* 92: 676–681.
6. Finkelstein, M. (2008). *Failure rate modelling for reliability and risk*. London: Springer–Verlag.
7. Hazra, N.K. & Nanda, A.K. (2014). Component redundancy versus system redundancy in different stochastic orders. *IEEE Transactions on Reliability* 63: 567–582.
8. Hazra, N.K. & Nanda, A.K. (2014). General standby component allocation in series and parallel systems. arXiv:1401.0132 [stat.AP].
9. Kapur, K.C. & Lamberson, L.R. (1977). *Reliability in engineering design*. New York: Wiley.
10. Keccecioglu, D. (1991). *Reliability engineering handbook*. New Jersey: Prentice–Hall.
11. Kijima, M. (1989). Some results for repairable systems with general repair. *Journal of Applied Probability* 26: 89–102.
12. Li, X., Wu, Y., & Zhang, Z. (2013). On allocation of general standby redundancy to series and parallel systems. *Communications in Statistics – Theory and Methods* 42: 4056–4069.

13. Li, X., Zhang, Z., & Wu, Y. (2009). Some new results involving general standby systems. *Applied Stochastic Models in Business and Industry* 25: 632–642.
14. Lin, H.H., Chen, K.H., & Wang, R.T. (1993). A multivariate exponential shared-load model. *IEEE Transactions on Reliability* 42: 165–171.
15. Liu, H. (1998). Reliability of a load-sharing k-out-of-n:G system: Non iid components with arbitrary distributions. *IEEE Transactions on Reliability* 47: 279–284.
16. Misra, A.K. & Misra, N. (2011). A note on active redundancy allocations in k-out-of-n systems. *Statistics and Probability Letters* 81: 1518–1523.
17. Misra, N., Misra, A.K., & Dhariyal, I.D. (2011). Active redundancy allocations in series systems. *Probability in the Engineering and Informational Sciences* 25: 219–235.
18. Misra, N., Misra, A.K., & Dhariyal, I.D. (2011). Standby redundancy allocations in series and parallel systems. *Journal of Applied Probability* 48: 43–55.
19. Nelson, W. (1990). *Accelerated testing: statistical methods, test plans, and data analysis*. New York: Wiley.
20. Romera, R., Valdés, J.E., & Zequeira, R.I. (2004). Active redundancy allocation in systems. *IEEE Transactions on Reliability* 53: 313–318.
21. Scheuer, E.M. (1988). Reliability of a m-out-of-n system when component failure induces higher failure rates in survivors. *IEEE Transactions on Reliability* 37: 73–74.
22. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer.
23. Shao, J. & Lamberson, L.R. (1991). Modeling a shared-load k-out-of-n:G system. *IEEE Transactions on Reliability* 40: 205–209.
24. Shechner, Z. (1984). A load-sharing model: the linear break down rule. *Naval Research Logistics* 31: 137–144.
25. Valdés, J.E. & Zequeira, R.I. (2003). On the optimal allocation of an active redundancy in a two component series system. *Statistics and Probability Letters* 63: 325–332.
26. Valdés, J.E. & Zequeira, R.I. (2006). On the optimal allocation of two active redundancies in a two component series system. *Operations Research Letters* 34: 49–52.
27. Wang, K.S., Huang, J.J., Tsai, Y.T., & Hsu, F.S. (2000). Study of loading policies for unequal strength shared-load system. *Reliability Engineering and System Safety* 67: 119–128.
28. Yinghui, T. & Jing, Z. (2008). New model for load-sharing k-out-of-n:G system with different components. *Journal of Systems Engineering and Electronics* 19: 748–751.
29. Yun, W.Y. & Cha, J.H. (2009). A stochastic model for a general load-sharing system under overload condition. *Applied Stochastic Models in Business and Industry* 26: 624–638.
30. Yun, W.Y. & Cha, J.H. (2010). Optimal design of a general warm standby system. *Reliability Engineering and System Safety* 95: 880–886.