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STARLIKENESS AND CONVEXITY OF CAUCHY TRANSFORMS ON REGULAR POLYGONS

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Abstract

For $n \geq 3$, let $Q_n \subset \mathbb{C}$ be an arbitrary regular n-sided polygon. We prove that the Cauchy transform F_{Q_n} of the normalised two-dimensional Lebesgue measure on Q_n is univalent and starlike but not convex in $\widehat{\mathbb{C}} \setminus Q_n$.

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1. Introduction

Let μ be a regular Borel measure with compact support $K \subset \mathbb{C}$. The Cauchy transform of μ is defined by

$$F(z) = F_\mu(z) = \int_K \frac{d\mu(w)}{z - w}. \tag{1.1}$$

If $z \notin K$, this integral is well defined; if $z \in K$, the integral is defined in the sense of the Cauchy principal value. It is well known that F_μ is continuous in the whole plane if K has positive area and μ is the two-dimensional Lebesgue measure [9, page 2]. Lund, Strichartz and Vinson [13] initiated an investigation of the Cauchy transform F_μ of a self-similar measure. They gave a condition on μ such that F_μ is Hölder continuous in $\mathbb{C}$ and proposed the Cantor set conjecture for the Cauchy transform on the Sierpiński gasket. This conjecture was verified by Dong and Lau [4].

The Sierpiński gasket is constructed by infinite iterations of a regular triangle. For the general case, the iterated function system

$$S_j(z) = e^{2j\pi i/n} + \rho(z - e^{2j\pi i/n}), \quad \rho \in (0, 1),\ j = 0, 1, \ldots, n - 1,$$

induces a self-similar measure $\mu_{n,\rho}$ and an attractor $K_{n,\rho}$. The chaotic behaviour of the Cauchy transform $F_{\mu_{n,\rho}}$ near $K_{n,\rho}$ was studied by Dong *et al.* in [2, 3, 5, 6, 12]. In particular, $K_{4,1/2}$ is the square with vertices $\{1, i, -1, -i\}$ and $\mu_{4,1/2}$ is the normalised Lebesgue measure on $K_{4,1/2}$. It is shown in [6] that the Cauchy transform $F_{\mu_{4,1/2}}$ is

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univalent outside the square $K_{4,1/2}$. Indeed, the authors proved in [7] that $F_{\mu_{4,1/2}}$ is starlike but not convex in $\widehat{\mathbb{C}} \setminus K_{4,1/2}$.

In this paper we generalise the results in [7] to regular n-sided polygons. Let P_n be the regular n-sided polygon with vertices $\{\epsilon_k = e^{(2k+1)\pi i/n} : k = 0, 1, \ldots, n-1\}$ and let $\mu_n = 2/(n\sin(2\pi/n))\mathcal{L}^2$ be the restriction of the normalised two-dimensional Lebesgue measure to P_n. By (1.1), we can write the Cauchy transform F_n of μ_n as

$$F_n(z) = \int_{P_n} \frac{d\mu_n(w)}{z-w}, \quad z \in \widehat{\mathbb{C}}. \tag{1.2}$$

The normalisation of μ_n ensures that F_n is normalised. Then F_n is continuous in the whole plane $\widehat{\mathbb{C}}$ [9, page 2]. In [7, 16], the authors proved that if $n = 3, 4$, then F_n is univalent and starlike but not convex in $\widehat{\mathbb{C}} \setminus P_n$. In these two papers the authors calculated the specific expressions for the real and imaginary parts of F_3 and F_4. However, it is almost impossible to write the specific expressions for F_n for $n \geq 5$. We use the symmetry of F_n to deal with the cases $n \geq 5$. Our method is different from [7] and [16] and also applicable to $n = 4$. Hence we have the following main theorem.

THEOREM 1.1. *If $n \geq 3$, then F_n is univalent and starlike but not convex in $\widehat{\mathbb{C}} \setminus P_n$.*

Let Q_n be an arbitrary regular n-sided polygon and let F_{Q_n} be the Cauchy transform of the normalised two-dimensional Lebesgue measure restricted to Q_n. Since any two regular n-sided polygons are similar, F_{Q_n} and F_n have the same univalence, starlikeness and convexity. Hence the following theorem follows immediately from Theorem 1.1.

THEOREM 1.2. *If $n \geq 3$, then F_{Q_n} is univalent and starlike but not convex in $\widehat{\mathbb{C}} \setminus Q_n$.*

2. Preliminaries

Let $\mathrm{Int}(P_n)$ be the interior of P_n and $c_n = 1/(n\sin(2\pi/n))$. By the Cauchy–Pompeiu formula [1, Theorem 2.1],

$$F_n(z) = 2\pi c_n \bar{z} + ic_n \int_{\partial P_n} \frac{\overline{w}\,dw}{w-z}, \quad z \in \mathrm{Int}(P_n). \tag{2.1}$$

The Sokhotski–Plemelj formula [11, Theorem 7.8] implies

$$F_n(z) = ic_n \int_{\partial P_n} \frac{\overline{w}\,dw}{w-z}, \quad z \in \widehat{\mathbb{C}} \setminus P_n. \tag{2.2}$$

Both equations (2.1) and (2.2) contain the expression

$$f_n(z) := \int_{\partial P_n} \frac{\overline{w}\,dw}{w-z}, \quad z \notin \partial P_n. \tag{2.3}$$

Our main tool in proving Theorem 1.1 is the second derivative of f_n.

PROPOSITION 2.1. *For $n \geq 3$, let f_n be as in (2.3). Then*

$$f_n''(z) = \frac{-2iz^{n-3}}{c_n(1+z^n)}, \quad z \notin \partial P_n.$$

PROOF. Put $\epsilon_{-1} = \epsilon_{n-1}$ and $\epsilon_n = \epsilon_0$. Then

$$f_n''(z) = \int_{\partial P_n} \frac{2\overline{w}\,dw}{(w-z)^3} = \sum_{k=0}^{n-1} \int_{\epsilon_k}^{\epsilon_{k+1}} \frac{2\overline{w}\,dw}{(w-z)^3}.$$

Let $w = \epsilon_k + t(\epsilon_{k+1} - \epsilon_k)$. The last integral above can be written as

$$\int_0^1 \frac{2(\overline{\epsilon_{k+1}} - \overline{\epsilon_k})\,dt}{(\epsilon_k + t(\epsilon_{k+1} - \epsilon_k) - z)^2} + \int_0^1 \frac{2\,(\overline{\epsilon_k}\epsilon_{k+1} - \epsilon_k\overline{\epsilon_{k+1}} + z(\overline{\epsilon_{k+1}} - \overline{\epsilon_k}))\,dt}{(\epsilon_k + t(\epsilon_{k+1} - \epsilon_k) - z)^3}.$$

These two integrals are not difficult to calculate and then sum from 0 to $n-1$ to give

$$f_n''(z) = 2i \sum_{k=0}^{n-1} \frac{\sin(2\pi/n)}{\epsilon_k^2(\epsilon_k - z)} = -i2n \sin\frac{2\pi}{n} \frac{z^{n-3}}{1+z^n}, \quad z \notin \partial P_n. \qquad \square$$

In the following, we study the Laurent expansion of F_n in $|z| > 1$. By (2.2), the Laurent series of F_n in $|z| > 1$ can be written as

$$F_n(z) = -ic_n \int_{\partial P_n} \frac{\overline{w}}{z} \sum_{k=0}^{\infty} \frac{w^k}{z^k} dw = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}, \quad |z| > 1,$$

where $a_k = -ic_n \int_{\partial P_n} \overline{w} w^k dw$. It follows from Proposition 2.1 that

$$F_n''(z) = \frac{2z^{n-3}}{z^n + 1} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)a_k}{z^{k+3}}, \quad |z| > 1.$$

Comparing the coefficients, we obtain the following result.

COROLLARY 2.2. *If $n \geq 3$, then*

$$F_n(z) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(nk+1)(nk+2)z^{nk+1}}, \quad |z| > 1, \tag{2.4}$$

$$F_n''(z) = \frac{2z^{n-3}}{1+z^n}, \quad z \in \mathbb{C} \setminus P_n.$$

We now write another expression for F_n. The polygon P_n is a union of n-similar triangles. If T_1 is the triangle with vertices $\{0, e^{-\pi i/n}, e^{\pi i/n}\}$, then

$$\int_{T_1} \frac{d\mathcal{L}^2(w)}{z-w} = i \cos\frac{\pi}{n}[(1 - ze^{-\pi i/n}) \log(1 - z^{-1}e^{\pi i/n}) - (1 - ze^{\pi i/n}) \log(1 - z^{-1}e^{-\pi i/n})].$$

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This implies

$$F_n(z) = 2c_n \sum_{k=0}^{n-1} e^{-2k\pi i/n} \int_{T_1} \frac{d\mathcal{L}^2(w)}{ze^{-2k\pi i/n} - w}$$
$$= \frac{2}{n} \sum_{k=0}^{n-1} \overline{\epsilon_k}^2 (z - \epsilon_k) \log(1 - z^{-1}\epsilon_k), \quad z \in \mathbb{C} \setminus P_n, \tag{2.5}$$

where the branches of $\log(1 - z^{-1}\epsilon_k)$ are chosen as follows. If $z \in (\cos(\pi/n), +\infty)$, then

$$\begin{cases} \arg(1 - z^{-1}\epsilon_k) = -\arg(1 - z^{-1}\epsilon_{n-1-k}) \in (-\alpha_{n,k}, 0), & 0 \le k \le [n/2] - 1, \\ \arg(1 - z^{-1}\epsilon_{[n/2]}) = 0 & \text{if } n \text{ is odd}, \end{cases} \tag{2.6}$$

where

$$\alpha_{n,k} = \arctan\Big(\frac{\sin((2k+1)\pi/n)}{\cos(\pi/n) - \cos((2k+1)\pi/n)}\Big).$$

REMARK 2.3. If $z \in \mathbb{C} \setminus P_n$, we can also calculate $F_n''(z)$ from (2.5).

Noting that P_n is invariant under the rotation $e^{2\pi i/n}z$, we have the so-called n-fold symmetry of F_n.

LEMMA 2.4 (n-fold symmetry). *If $n \ge 3$, then $F_n(z) = e^{-2\pi i/n} F_n(e^{-2\pi i/n} z)$ for all $z \in \mathbb{C}$.*

PROOF. By the definition of F_n,

$$F_n(e^{-2\pi i/n} z) = 2c_n e^{2\pi i/n} \int_{P_n} \frac{d\mathcal{L}^2(w)}{z - e^{2\pi i/n} w} = 2c_n e^{2\pi i/n} \int_{e^{2\pi i/n} P_n} \frac{d\mathcal{L}^2(w)}{z - w} = e^{2\pi i/n} F_n(z). \qquad \square$$

Since P_n is symmetric with respect to the x-axis, by definition (1.2) F_n also has the same property.

LEMMA 2.5. *If $n \ge 3$, then $F_n(\overline{z}) = \overline{F_n(z)}$ for $z \in \mathbb{C}$. In particular, $F_n(x) \in \mathbb{R}$ for $x \in \mathbb{R}$.*

A set $E \subset \mathbb{C}$ is said to be starlike with respect to a point w_0 if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. The set $E \subset \mathbb{C}$ is said to be convex if it is starlike with respect to each of its points; that is, if the linear segment joining any two points of E lies entirely in E (see Duren [8, page 40]). We say that an analytic function f is starlike (or convex) in a domain $D \subset \widehat{\mathbb{C}}$ if $f(D)$ is a starlike (or convex) domain (with respect to w_0). In this paper we always assume $w_0 = 0$.

We give a result on univalent and starlike functions in an unbounded domain.

LEMMA 2.6. *Let $\gamma \subset \mathbb{C}$ be a Jordan curve and D be the unbounded connected component of $\widehat{\mathbb{C}} \setminus \gamma$. Suppose that $f : D \to \mathbb{C}$ is analytic and has a continuous extension to $\overline{D}$. Then f is univalent and starlike in D if and only if $f(\partial D)$ is a Jordan curve and $\arg f(z)$ is decreasing when z moves anticlockwise on ∂D.*

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PROOF. Without loss of generality, we assume $f(\infty) = 0$. By the Riemann mapping theorem, we can choose a conformal map $\varphi : \mathbb{D} \to D$ such that $\varphi(0) = \infty$. Then the starlikeness of f in D is equivalent to the starlikeness of $f \circ \varphi$ in $\mathbb{D}$. The rest of the proof is similar to that of Theorem 2.10 in [8] (or Lemma 3.1 in [16]). □

3. Starlikeness and convexity

In this section, we study the starlikeness and convexity of F_n in $\widehat{\mathbb{C}} \setminus P_n$ for $n \geq 4$. By Lemma 2.6, we need to study $\arg F_n(z)$ on ∂P_n. According to the n-fold symmetry of F_n, we only consider F_n on one edge of P_n (see Figure 1).

If $n \geq 3$, we parameterise the side $\partial P_n \cap \{z : |\arg z| \leq \pi/n\}$ and its image by

$$z(t) = \cos\frac{\pi}{n} + it\sin\frac{\pi}{n} \quad \text{and} \quad F_n(z(t)) = u(t) + iv(t), \quad t \in [-1, 1]. \tag{3.1}$$

For convenience, we write $z(t)$ in exponential form

$$z(t) = \rho(\theta)e^{i\theta}, \quad \text{where } \rho(\theta) = \frac{\cos(\pi/n)}{\cos\theta}, \; \theta = \arctan\Big(t\tan\frac{\pi}{n}\Big) \in \Big[-\frac{\pi}{n}, \frac{\pi}{n}\Big]. \tag{3.2}$$

By Corollary 2.2 and (2.5), we obtain the smoothness of u and v.

PROPOSITION 3.1. *If $n \geq 3$, then u and v are C^∞ in $(-1, 1)$. Moreover,*

$$u''(t) + iv''(t) = -\frac{2(\sin(\pi/n))^2\rho^{n-3}(\theta)}{|1 + z^n(t)|^2}(R_n(\theta) + iI_n(\theta)), \quad t \in (-1, 1),$$

where $R_n(\theta) = \cos(n-3)\theta + \rho^n(\theta)\cos 3\theta$ and $I_n(\theta) = \sin(n-3)\theta - \rho^n(\theta)\sin 3\theta$.

PROOF. By the continuity of $F_n(z)$ and (2.5),

$$u(t) + iv(t) = \lim_{r\to 1^+} F_n(rz(t)) = \frac{2}{n}\sum_{k=0}^{n-1}\overline{\epsilon_k}^2(z(t) - \epsilon_k)\log\Big(1 - \frac{\epsilon_k}{z(t)}\Big), \quad t \in (-1, 1). \tag{3.3}$$

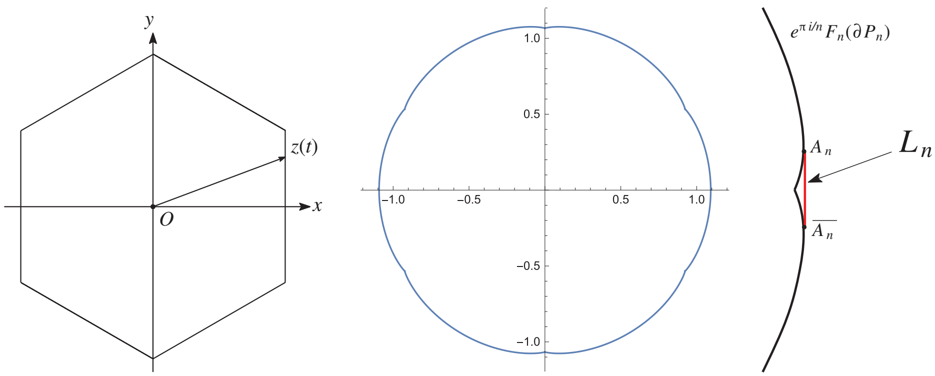

FIGURE 1. Left: the polygon P_6. Middle: the image $F_6(\widehat{\mathbb{C}} \setminus P_6)$. Right: $e^{\pi i/n}F_n(\widehat{\mathbb{C}} \setminus P_n)$ is not convex.

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Since $\log(1-\epsilon_k/z(t))$ is C^∞ in $(-1,1)$, u and v are C^∞ in $(-1,1)$. Taking the derivatives of both sides of the above equation, we obtain the second conclusion. □

In the following, we deal with $R_n(\theta)$ and $I_n(\theta)$.

LEMMA 3.2. (i) *If $n\geq 4$, then $\cos(n-3)\theta\cos^n\theta$ is decreasing in $(0,\pi/n)$.*
(ii) *If $a\geq b>0$, then $\sin ax/\sin bx$ is decreasing in $(0,\pi/a)$.*

PROOF. (i) The derivative of $\cos(n-3)\theta\cos^n\theta$ is

$$-\cos^{n-1}\theta\sin(n-3)\theta\sin\theta\,[(n-3)\cot\theta+n\cot(n-3)\theta]\,.$$

It is not hard to check $(n-3)\cot\theta+n\cot(n-3)\theta>(n-3)\cot(\pi/n)-n\cot(3\pi/n)>0$. Hence $\cos(n-3)\theta\cos^n\theta$ is decreasing in $(0,\pi/n)$.

(ii) The conclusion comes from a calculation of the derivative of $\sin ax/\sin bx$. □

PROPOSITION 3.3. *If $n\geq 4$, then $R_n(\theta)>0$ and $I_n(\theta)>0$ for $\theta\in[0,\pi/n)$.*

PROOF. By Lemma 3.2, $R_n(\theta)\cos^n\theta=\cos(n-3)\theta\cos^n\theta+(\cos(\pi/n))^n\cos 3\theta$ is decreasing. Hence $R_n(\theta)>0$.

Note that

$$\frac{I_n(\theta)}{\sin\theta}=\frac{\sin(n-3)\theta}{\sin\theta}-\frac{\cos^n(\pi/n)(4\cos^2\theta-1)}{\cos^n\theta}.$$

It is easy to check that $(4\cos^2\theta-1)/\cos^n\theta$ is increasing in $(0,\pi/n)$. This, together with Lemma 3.2, shows that $I_n(\theta)/\sin\theta$ is decreasing. Then $I_n(\theta)>0$, since $I_n(\pi/n)=0$. □

We now show the monotonicity of u and v.

PROPOSITION 3.4. *For $n\geq 4$, let u and v be given by (3.1). Then:*

(i) *u is strictly decreasing in $[0,1]$ and $u(t)=u(-t)>0$;*
(ii) *v is strictly decreasing in $[-1,1]$ and $v(t)=-v(-t)$.*

PROOF. By Lemma 2.5,

$$u(t)=u(-t)\quad\text{and}\quad v(t)=-v(-t),\quad t\in[-1,1].$$

So it is sufficient to consider $t\in(0,1)$. Let θ be defined as in (3.2). Then $\theta\in(0,\pi/n)$ for $t\in(0,1)$. Propositions 3.1 and 3.3 show that $u''(t)<0$ and $v''(t)<0$ in $(0,1)$.

(i) It follows from (2.5) that

$$F'(x)=\frac{2}{n}\sum_{k=0}^{n-1}\overline{\epsilon_k}^2\log\Big(1-\frac{\epsilon_k}{x}\Big),\quad x>\cos\frac{\pi}{n}.$$

By (3.3),

$$u'(0)+iv'(0)=\lim_{x\to\cos(\pi/n)} i\sin\frac{\pi}{n}F'(x).$$

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Since $F_n(x) \in \mathbb{R}$ and $F_n'(x) \in \mathbb{R}$, we see that $u'(0) = 0$ and hence $u(t)$ is strictly decreasing in $[0, 1]$. Therefore

$$u(t) > u(1) = \lim_{r \to 1^+} \operatorname{Re} F(re^{\pi i/n}) > 0,$$

where the last inequality comes from the Laurent series (2.4).

(ii) Corollary 2.2 implies

$$\lim_{x \to +\infty} x^2 F_n'(x) = -1 \quad \text{and} \quad F_n''(x) = \frac{2x^{n-3}}{1 + x^n} > 0, \quad x > \cos \frac{\pi}{n}.$$

We obtain $F_n'(x) < 0$, and hence $v'(t) < v'(0) < 0$. □

REMARK 3.5. If $n = 3$, it is easy to check that (i) holds, but (ii) does not hold. In fact, there exists $t_0 \in (0, 1)$ such that $v'(t) < 0$ in $(0, t_0)$ and $v'(t) > 0$ in $(t_0, 1)$.

To study the starlikeness of F_n, we now analyse $\arg F_n(z(t))$. Since $\arg F_n(z(t))$ is C^∞ in $(-1, 1)$,

$$\frac{d}{dt} \arg F_n(z(t)) = \operatorname{Im}\left(\frac{u'(t) + iv'(t)}{u(t) + iv(t)}\right) = \frac{u(t)v'(t) - u'(t)v(t)}{u^2(t) + v^2(t)}, \quad t \in (-1, 1). \tag{3.4}$$

By Proposition 3.4, we see that the numerator of the last term is negative. This is the most important step in proving the starlikeness of F_n.

THEOREM 3.6. *If $n \geq 4$, F_n is univalent and starlike in $\widehat{\mathbb{C}} \setminus P_n$.*

PROOF. It follows from (3.4) and Propositions 3.4 that

$$\frac{d}{dt} \arg F_n(z(t)) < 0, \quad t \in (-1, 1).$$

From the n-fold symmetry of F_n,

$$F_n(e^{2k\pi i/n} z(t)) = e^{-2k\pi i/n} F_n(z(t)), \quad k = 0, 1, \ldots, n-1, \ t \in [-1, 1].$$

Hence $\arg F(z)$ is decreasing when z moves anticlockwise on ∂P_n.

Since $F_n(z(0)) \in \mathbb{R}$, equation (2.6) implies that $\arg F_n(z(0)) = 0$. Consequently, by the continuity of $\arg F_n(z(t))$, the total variation of $\arg F_n(z(t))$ in $[-1, 1]$ is $-2\pi/n$, that is,

$$\Delta_{t \in [-1,1]} \arg F_n(z(t)) = \arg F_n(z(1)) - \arg F_n(z(-1)) = -\frac{2\pi}{n}.$$

Using the n-fold symmetry again, the total variation of $\arg F_n(z)$ on ∂P_n is -2π. By the argument principle, $F_n(\partial P_n)$ is a Jordan curve. The theorem follows from Lemma 2.6. □

Observing Figure 1, we can see that the domain $F_n(\widehat{\mathbb{C}} \setminus P_n)$ is not convex. Before proving this fact, we need a lemma.

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LEMMA 3.7. *Let $R_n(\theta)$ and $I_n(\theta)$ be given as in Proposition 3.1. If $n \ge 4$, then*

$$R_n(\theta)\cos\frac{\pi}{n} - I_n(\theta)\sin\frac{\pi}{n} > 0, \quad \theta \in \Big(-\frac{\pi}{n}, \frac{\pi}{n}\Big).$$

PROOF. It is easy to check $R_n(-\theta) = R_n(\theta)$ and $I_n(-\theta) = -I_n(\theta)$. By Proposition 3.3, we only need to consider $\theta \in [0, \pi/n)$. Note that

$$R_n(\theta)\cos\frac{\pi}{n} - I_n(\theta)\sin\frac{\pi}{n} = \cos\Big((n-3)\theta + \frac{\pi}{n}\Big) + \rho^n(\theta)\cos\Big(3\theta - \frac{\pi}{n}\Big). \tag{3.5}$$

It is obvious that $\cos(3\theta - \pi/n) > 0$ for $\theta \in [0, \pi/n)$ and $n \ge 4$. One can easily verify

$$\cos\Big((n-3)\theta + \frac{\pi}{n}\Big) > 0 \ \text{ with } \ \theta \in \Big[0, \frac{\pi}{n}\Big) \Longleftrightarrow 0 \le \theta < \frac{(n-2)\pi}{2n(n-3)}. \tag{3.6}$$

If $n = 4$, the lemma follows from (3.6). If $n \ge 5$, by (3.5) and (3.6), we only need to consider

$$\frac{(n-2)\pi}{2n(n-3)} < \theta < \frac{\pi}{n}. \tag{3.7}$$

Let

$$f(\theta) = \log\Big(\rho^n(\theta)\cos\Big(3\theta - \frac{\pi}{n}\Big)\Big) - \log\Big(-\cos\Big((n-3)\theta + \frac{\pi}{n}\Big)\Big).$$

Then $f'(\theta) = n\tan\theta - 3\tan(3\theta - \pi/n) + (n-3)\tan((n-3)\theta + \pi/n)$. The equation (3.7) implies

$$\frac{\pi}{2} < (n-3)\theta + \frac{\pi}{n} < \pi - 3\theta + \frac{\pi}{n}, \quad 0 < \theta < 3\theta - \frac{\pi}{n} < \frac{\pi}{2}.$$

This gives $f'(\theta) < 0$. Thus $f(\theta) > 0$, since $f(\pi/n) = 0$. The proof is complete. □

THEOREM 3.8. *If $n \ge 4$, F_n is not a convex function in $\widehat{\mathbb{C}} \setminus P_n$.*

PROOF. Since a rotation does not change the convexity, we prove that the domain $e^{\pi i/n}F_n(\widehat{\mathbb{C}} \setminus P_n)$ is not convex (see Figure 1). For $t \in [-1, 1]$, we introduce the notation $z(t) = \cos\pi/n + it\sin\pi/n$ and $e^{\pi i/n}F_n(z(t)) = x(t) + iy(t)$. Then

$$x(t) = u(t)\cos\frac{\pi}{n} - v(t)\sin\frac{\pi}{n}, \quad y(t) = u(t)\sin\frac{\pi}{n} + v(t)\cos\frac{\pi}{n}, \quad t \in (-1, 1).$$

It follows from Proposition 3.1 and Lemma 3.7 that

$$x''(t) = -\frac{2\rho^{n-3}(\theta)(\sin\pi/n)^2}{|1 + z^n(t)|^2}\Big(R_n(\theta)\cos\frac{\pi}{n} - I_n(\theta)\sin\frac{\pi}{n}\Big) < 0, \quad t \in (-1, 1).$$

Using (3.3), we decompose $x'(t)$ as $x'(t) = X_1(t) - X_2(t)$, where

$$X_1(t) = -\frac{2\sin\pi/n}{n}\sum_{k=1}^{n-1}\mathrm{Im}\Big(\epsilon_1\overline{\epsilon_k}^2\log\Big(1 - \frac{\epsilon_k}{z(t)}\Big)\Big),$$

$$X_2(t) = \frac{2\sin\pi/n}{n}\Big(\cos\frac{\pi}{n}\arg\Big(1 - \frac{\epsilon_0}{z(t)}\Big) - \sin\frac{\pi}{n}\log\Big|1 - \frac{\epsilon_0}{z(t)}\Big|\Big).$$

Since $\sum_{k=1}^{n-1}\overline{\epsilon_k}^2\log(1-\epsilon_k/z)$ is analytic in $\{z : |\arg z| < 2\pi/n\}$, it follows that $\sup_{-1<t<1}|X_1(t)| < \infty$. The choices of the branches of the logarithms (see (2.6)) imply

$$\left|\arg\left(1-\frac{\epsilon_0}{z(t)}\right)\right| < 2\pi, \quad t \in (-1, 1).$$

It is easy to check that $\lim_{t\to 1}\log|1-\epsilon_0/z(t)| = -\infty$ and so $\lim_{t\to 1} x'(t) = -\infty$. On the other hand, the proof of Proposition 3.4 gives $x'(0) > 0$. Hence there exists a unique $T_n \in (0, 1)$ such that $x'(T_n) = 0$. This means that T_n is the unique maximum point of $\operatorname{Re} e^{\pi i/n}F_n(z(t))$ for $t \in [-1, 1]$. Let $A_n = e^{\pi i/n}F_n(z(T_n))$ and $L_n = (A_n, \overline{A_n})$ denote the open line segment jointing A_n and $\overline{A_n}$ (see Figure 1). Then

$$L_n \cap (e^{\pi i/n}F_n(\widehat{\mathbb{C}} \setminus P_n)) = \emptyset,$$

which shows that $e^{\pi i/n}F_n(\widehat{\mathbb{C}} \setminus P_n)$ is not convex. □

Finally, we prove Theorem 1.2.

PROOF OF THEOREM 1.2. Let Q_n be a regular n-sided polygon, with centre a and circumradius R. Since Q_n is similar to P_n, there exists a constant $\theta \in [0, 2\pi)$ such that $Q_n = Re^{i\theta}P_n + a$. Let $|Q_n|$ denote the area of Q_n. It is easy to check that

$$F_{Q_n}(z) = \frac{1}{|Q_n|}\int_{Q_n}\frac{d\mathcal{L}^2(w)}{z-w} = R^{-1}e^{-i\theta}F_n(R^{-1}e^{-i\theta}(z-a)).$$

Thus F_{Q_n} and F_n have the same univalence. Obviously, rotation and scaling transformations do not change the starlikeness and convexity of a set. Hence F_{Q_n} and F_n have the same starlikeness and convexity. □

4. Open question

For an arbitrary regular n-sided polygon Q_n, we have proved that the Cauchy transform F_{Q_n} is univalent and starlike in $\widehat{\mathbb{C}} \setminus Q_n$. For many general polygons, we used Mathematica to draw images of the corresponding Cauchy transforms. From these images, we observed that:

(1) if a polygon P is convex, then the corresponding Cauchy transform F_P is univalent and starlike in $\widehat{\mathbb{C}} \setminus P$;
(2) there exists a nonconvex polygon P' such that the corresponding Cauchy transform $F_{P'}$ is not univalent in $\widehat{\mathbb{C}} \setminus P'$.

Note that any convex set can be approximated by convex polygons. Based on our observations, we make the following conjecture.

CONJECTURE 4.1. If $K \subset \mathbb{C}$ is a compact convex set, then the Cauchy transform of the two-dimensional Lebesgue measure restricted to K is univalent and starlike in $\widehat{\mathbb{C}} \setminus K$.

For general convex polygons, the corresponding Cauchy transforms do not have symmetry (see Lemmas 2.4 and 2.5) and Proposition 3.4 may not hold. Since our proofs strongly depend on Proposition 3.4, our method fails for general convex polygons.

The well-known criterion for univalence due to Nehari may shed some light on Conjecture 4.1. For f analytic in $\mathbb{D}$, the Schwarzian derivative of f is defined by

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \quad |z| < 1.$$

THEOREM 4.2 (Nehari's criterion, [14]). *Let f be an analytic function in $\mathbb{D}$. If the Schwarzian derivative of f satisfies*

$$|\{f, z\}| \leq \frac{2}{(1-|z|^2)^2} \quad \text{or} \quad |\{f, z\}| \leq \frac{\pi^2}{2}, \quad |z| < 1,$$

then f is univalent.

Nehari's criterion is a sufficient condition for univalence, but not necessary [10, 15]. We give two examples for which Conjecture 4.1 is true, but one of them satisfies the conditions of Nehari's criterion and the other does not.

EXAMPLE 4.3. The Cauchy transform of the two-dimensional Lebesgue measure restricted to the closed unit ball $\overline{\mathbb{D}}$ is

$$F_{\overline{\mathbb{D}}}(z) = \int_{\overline{\mathbb{D}}} \frac{d\mathcal{L}^2(w)}{z - w} = \frac{\pi}{z}, \quad |z| > 1.$$

Obviously, $F_{\overline{\mathbb{D}}}$ is univalent and starlike in $|z| > 1$, that is, Conjecture 4.1 is true for $\overline{\mathbb{D}}$. The Schwarzian derivative of $F_{\overline{\mathbb{D}}}(1/z)$ is 0, hence it satisfies the sufficient conditions in Nehari's criterion.

EXAMPLE 4.4. Let Δ be the regular triangle with vertices $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. The Cauchy transform of the normalised two-dimensional Lebesgue measure restricted to Δ is

$$F_\Delta(z) = \frac{4}{3\sqrt{3}} \int_\Delta \frac{d\mathcal{L}^2(w)}{z - w}, \quad z \in \mathbb{C}.$$

By Theorem 1.2 or [16, Theorem 1.1], F_Δ is univalent and starlike in $\widehat{\mathbb{C}} \setminus \Delta$.

For the triangle Δ, we can use the Schwarz–Christoffel formula and the Schwarz reflection principle to construct a conformal mapping $\varphi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus \Delta$ such that:

(i) $\varphi(0) = \infty$, $\varphi(-1) = -\frac{1}{2}$ and $\varphi(1) = 1$;

(ii) $\varphi((0, 1]) = [1, +\infty)$ and $\varphi([-1, 0)) = (-\infty, -\frac{1}{2}]$;

(iii) $\varphi(w) = 1 + c\int_0^{-(1-w^3)^2/4w^3} (\xi - 1)^{-1/2}\xi^{-1/6}\,d\xi$, where $0 < \arg w < \pi/3$ and the constant c is given by $c = e^{-\pi i/3}\Gamma(\frac{1}{3})(\sqrt{3\pi}\,\Gamma(\frac{5}{6}))^{-1}$.

One can verify directly that

$$\lim_{\substack{x\to 1^- \\ x\in(0,1)}} |(1-x^2)^2\{F_\Delta\circ\varphi, x\}| = \frac{32}{9} > 2.$$

Thus $F_\Delta \circ \varphi$ does not satisfy the sufficient conditions of Nehari's criterion.

REMARK 4.5. For a general convex set, it seems difficult to give an explicit formula for the Schwarzian derivative. However, for convex polygons, we may hope to invoke the argument principle which shows that the Cauchy transform in Conjecture 4.1 is univalent if and only if the boundary image of the Cauchy transform is a Jordan curve.

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References

[1] S. R. Bell, *The Cauchy Transform, Potential Theory and Conformal Mapping*, 2nd edn (Chapman and Hall/CRC, Boca Raton, FL, 2016).
[2] X. H. Dong and K. S. Lau, 'Cauchy transforms of self-similar measures: the Laurent coefficients', *J. Funct. Anal.* **202**(1) (2003), 67–97.
[3] X. H. Dong and K. S. Lau, 'An integral related to the Cauchy transform on the Sierpinski gasket', *Exp. Math.* **13**(4) (2004), 415–419.
[4] X. H. Dong and K. S. Lau, 'Cantor boundary behavior of analytic functions', in: *Recent Developments in Fractals and Related Fields*, eds. J. Barrel and S. Seuret (Birkhäuser, Boston, MA, 2010), 283–294.
[5] X. H. Dong, K. S. Lau and J. C. Liu, 'Cantor boundary behavior of analytic functions', *Adv. Math.* **232** (2013), 543–570.
[6] X. H. Dong, K. S. Lau and H. H. Wu, 'Cauchy transform of self-similar measures: Starlikeness and univalence', *Trans. Amer. Math. Soc.* **369**(7) (2017), 4817–4842.
[7] X. H. Dong, Y. Wang and P. F. Zhang, 'The starlikeness of Cauchy transform on square', *Comput. Methods Funct. Theory.* **19**(2) (2019), 341–351.
[8] P. Duren, *Univalent Functions* (Springer-Verlag, New York, 1983).
[9] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics, 297 (Springer-Verlag, Berlin, 1972).
[10] E. Hille, 'Remarks on a paper by Zeev Nehari', *Bull. Amer. Math. Soc.* **55** (1949), 552–553.
[11] R. Kress, *Linear Integral Equations*, 3rd edn (Springer, New York, 2014).
[12] J. C. Liu, X. H. Dong and S. M. Peng, 'A note on Cantor boundary behavior', *J. Math. Anal. Appl.* **408**(2) (2013), 795–801.
[13] J. P. Lund, R. Strichartz and J. Vinson, 'Cauchy transforms of self-similar measures', *Exp. Math.* **7**(3) (1998), 177–190.
[14] Z. Nehari, 'The Schwarzian derivative and schlicht functions', *Bull. Amer. Math. Soc.* **55** (1949), 545–551.
[15] Z. Nehari, 'Some criteria of univalence', *Proc. Amer. Math. Soc.* **5** (1954), 700–704.
[16] P. F. Zhang and X. H. Dong, 'Starlikeness and convexity of Cauchy transform on equilateral triangle', *Complex Var. Elliptic Equ.* **65**(9) (2020), 1590–1600.

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