Combinatorics, Probability and Computing (2020), **29**, pp. 137–152
doi:10.1017/S0963548319000270

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ARTICLE

The Infinite limit of random permutations avoiding patterns of length three

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(Received 4 July 2018; revised 3 July 2019; first published online 14 October 2019)

Abstract

For $\tau \in S_3$, let μ_n^τ denote the uniformly random probability measure on the set of τ-avoiding permutations in S_n. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ with an appropriate metric and denote by $S(\mathbb{N}, \mathbb{N}^*)$ the compact metric space consisting of functions $\sigma = \{\sigma_i\}_{i=1}^\infty$ from $\mathbb{N}$ to $\mathbb{N}^*$ which are injections when restricted to $\sigma^{-1}(\mathbb{N})$; that is, if $\sigma_i = \sigma_j$, $i \neq j$, then $\sigma_i = \infty$. Extending permutations $\sigma \in S_n$ by defining $\sigma_j = j$, for $j > n$, we have $S_n \subset S(\mathbb{N}, \mathbb{N}^*)$. For each $\tau \in S_3$, we study the limiting behaviour of the measures $\{\mu_n^\tau\}_{n=1}^\infty$ on $S(\mathbb{N}, \mathbb{N}^*)$. We obtain partial results for the permutation $\tau = 321$ and complete results for the other five permutations $\tau \in S_3$.

2010 MSC Codes: Primary 60C05; Secondary 60B10, 05A05

1. Introduction and statement of results

We recall the definition of pattern avoidance for permutations. Let S_n denote the set of permutations of $[n] := \{1, \ldots, n\}$. If $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ and $\tau = \tau_1 \cdots \tau_m \in S_m$, where $2 \leqslant m < n$, then we say that σ contains τ as a pattern if there exists a subsequence $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ such that for all $1 \leqslant j, k \leqslant m$, the inequality $\sigma_{i_j} < \sigma_{i_k}$ holds if and only if the inequality $\tau_j < \tau_k$ holds. If σ does not contain τ, then we say that σ *avoids* τ. We consider here permutations on S_n that avoid a pattern $\tau \in S_3$. Let $S_n(\tau)$ denote the set of permutations in S_n that avoid τ. It is well known that $|S_n(\tau)| = C_n$, for all six permutations $\tau \in S_3$, where

$$C_n = \frac{\binom{2n}{n}}{n+1}$$

is the nth Catalan number [2]. Let μ_n^τ denote the uniformly random probability measure on $S_n(\tau)$. In this paper we investigate the limiting behaviour of the probability measures μ_n^τ as $n \to \infty$. In the limit we will obtain a probability measure not on the set of permutations of $\mathbb{N} := \{1, 2, \ldots\}$, but on a more general structure that we now describe.

Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ with the metric $d_{N^*}(i, j) = \sum_{k=i}^{j-1} 2^{-k}$, for $1 \leqslant i < j \leqslant \infty$. Denote by $S(\mathbb{N}, \mathbb{N}^*)$ the set of functions $\sigma = \{\sigma_i\}_{i=1}^\infty$ from $\mathbb{N}$ to $\mathbb{N}^*$ *which are injections when restricted to* $\sigma^{-1}(\mathbb{N})$; that is, if $\sigma_i = \sigma_j$, $i \neq j$, then $\sigma_i = \infty$. Let $S(\mathbb{N}, \mathbb{N}) \subset S(\mathbb{N}, \mathbb{N}^*)$ denote the subset of injections from $\mathbb{N}$ to $\mathbb{N}$, let $S_{\text{sur}}(\mathbb{N}, \mathbb{N}^*) \subset S(\mathbb{N}, \mathbb{N}^*)$ denote the subset of surjections from $\mathbb{N}$ to $\mathbb{N}^*$, and let $S_\infty \subset S(\mathbb{N}, \mathbb{N})$ denote the set of bijections from $\mathbb{N}$ to $\mathbb{N}$, that is, the set of permutations of $\mathbb{N}$.

The space $S(\mathbb{N}, \mathbb{N}^*)$ can be identified with the countably infinite product $\mathbb{N}^* \times \mathbb{N}^* \cdots$. Since $\mathbb{N}^*$ is a compact metric space, it follows that $S(\mathbb{N}, \mathbb{N}^*)$ is also a compact metric space with the metric

$$D(\sigma, \tau) := \sum_{i=1}^{\infty} \frac{d_{\mathbb{N}^*}(\sigma_i, \tau_i)}{2^i}.$$

For any $n \in \mathbb{N}$, we identify the set S_n of permutations of $[n]$ with the subset $\{\sigma \in S_\infty : \sigma_j = j, j > n\}$. Consequently, if μ_n is a probability measure on S_n, for each $n \in \mathbb{N}$, then $\{\mu_n\}_{n=1}^{\infty}$ may be considered as a sequence of probability measures on the compact metric space $S(\mathbb{N}, \mathbb{N}^*)$. Thus, any such sequence has a subsequence converging weakly to a probability measure on $S(\mathbb{N}, \mathbb{N}^*)$.

If one uses the above framework to study the limit of the uniform probability measure on S_n, then it is easy to show that the sequence of measures converges weakly to the degenerate distribution $\delta_{\infty^{(\infty)}}$ on the point $\infty^{(\infty)} \in S(\mathbb{N}, \mathbb{N}^*)$, where $\infty^{(\infty)}$ denotes the function $\sigma \in S(\mathbb{N}, \mathbb{N}^*)$ satisfying $\sigma_n = \infty$, for all $n \in \mathbb{N}$. On the other hand, consider the Mallows distribution on S_n with parameter $q > 0$. This is the probability measure that gives to any permutation $\sigma \in S_n$ a probability proportional to $q^{\mathrm{inv}(\sigma)}$, where $\mathrm{inv}(\sigma)$ denotes the number of inversions in the permutation σ; that is,

$$\mathrm{inv}(\sigma) = |\{(i, j) : 1 \leqslant i < j \leqslant n \text{ and } \sigma_i > \sigma_j\}|.$$

When $q = 1$, the Mallows measure is just the uniform measure. When $q \in (0, 1)$, the Mallows measure favours permutations with few inversions, and when $q > 1$, it favours permutations with many inversions. When $q > 1$, the sequence of Mallows distributions converges weakly to $\delta_{\infty^{(\infty)}}$, but when $q \in (0, 1)$, these distributions converge weakly to a non-trivial distribution on $S(\mathbb{N}, \mathbb{N}^*)$ which is in fact supported on the set of permutations S_∞. The form of this limiting distribution is regenerative. See [4, 5] for the limiting behaviour of the Mallows distribution, and see [10] and references therein for more on the general theory of regenerative infinite permutations.

Since the limit of the Mallows distribution with $q \in (0, 1)$ is a distribution on S_∞, the more general framework of $S(\mathbb{N}, \mathbb{N}^*)$ is not needed there. However, this more general framework is necessary for our study of the limiting behaviour of the measures $\{\mu_n^\tau\}_{n=1}^{\infty}$, for $\tau \in S_3$. It will turn out that the limiting distribution is trivial in two out of the six cases, while in three out of the other four cases, the limiting distribution has a regenerative structure. In order to describe this regenerative structure, we will need to consider permutations of subsets $I \subset \mathbb{N}$ not as functions with a domain, but rather just as images. We will call such an object a *permutation image* of I. Thus, for example, if $I = \{3, 4, 9\}$, then there are six permutation images of I, which we denote by (3 4 9), (3 9 4), (4 3 9), (4 9 3), (9 3 4), (9 4 3). We will denote a generic permutation image of I by σ_I^{im}. We also define $\infty^{(j)}$ to be the j-fold image of ∞:

$$\infty^{(j)} = \underbrace{(\infty\ \infty \cdots \infty)}_{j \text{ times}}, \quad j \in \mathbb{N}.$$

We will use these permutation images to build functions in $S(\mathbb{N}, \mathbb{N}^*)$. For example, if $I_1 = \{3, 4, 9\}$ and $I_2 = \{20, 22, 24, 26, 28, 30 \cdots\}$, and if the permutation images $\sigma_{I_i}^{\mathrm{im}}$, $i = 1, 2$, are given by $\sigma_{I_1}^{\mathrm{im}} =$ (9 3 4) and $\sigma_{I_2}^{\mathrm{im}} =$ (22 20 26 24 30 28 $\cdots$), then $\sigma := \sigma_{I_1}^{\mathrm{im}} * \sigma_{I_2}^{\mathrm{im}}$ denotes the function in $S(\mathbb{N}, \mathbb{N})$ given by $\sigma_1 = 9, \sigma_2 = 3, \sigma_3 = 4, \sigma_4 = 22, \sigma_5 = 20, \sigma_6 = 26, \ldots$, while $\sigma = \infty^{(2)} * \sigma_{I_1}^{\mathrm{im}} * \infty^{(1)} * \sigma_{I_2}^{\mathrm{im}}$ denotes the function in $S(\mathbb{N}, \mathbb{N}^*)$ given by $\sigma_1 = \infty, \sigma_2 = \infty, \sigma_3 = 9, \sigma_4 = 3, \sigma_5 = 4, \sigma_6 = \infty, \sigma_7 = 22, \sigma_8 = 20, \sigma_9 = 26, \ldots$.

The mathematical description of our results in the propositions and theorems that follow looks a bit complicated, so we deem it worthwhile to begin with a verbal synopsis of the results. In what follows, a permutation image of a *block* means a permutation image of a set of consecutive numbers.

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1. $\tau = 123$: Weak convergence to the trivial distribution $\delta_{\infty^{(\infty)}}$.

2. $\tau = 132$: Weak convergence to the trivial distribution $\delta_{\infty^{(\infty)}}$.

3. $\tau = 312$: Weak convergence to a limiting distribution which is supported on $S(\mathbb{N}, \mathbb{N}) - S_\infty$, and whose structure is a concatenation that alternates uniformly random 312-avoiding permutations images of random finite blocks of infinite expected length with permutation images of random singletons, each random singleton being the largest value smaller than the values in the preceding finite block permutation image. The random finite blocks are obtained in a regenerative fashion, and their lengths are i.i.d.

4. $\tau = 231$: Weak convergence to a limiting distribution which is supported on $S_{\text{sur}}(\mathbb{N}, \mathbb{N}^*)$, and whose structure is a concatenation which alternates uniformly random 231-avoiding permutations images of random finite contiguous blocks of infinite expected length with permutation images of the singleton $\infty^{(1)}$. The contiguous random finite blocks are obtained in a regenerative fashion, and their lengths are i.i.d.

5. $\tau = 213$: Weak convergence to a limiting distribution which is supported on $S(\mathbb{N}, \mathbb{N}^*) - S(\mathbb{N}, \mathbb{N}) - S_{\text{sur}}(\mathbb{N}, \mathbb{N}^*)$, and whose structure is a concatenation which alternates permutation images of blocks of ∞ of random finite length with permutation images of singletons whose values increase along the concatenation. The values of the singletons are obtained in a regenerative fashion, and the lengths of the blocks of ∞ are i.i.d. and have infinite expectation.

6. $\tau = 321$: Here we only have partial results. The limit of any weakly convergent subsequence is a concatenation of a Geom $(1/2)$ number of uniformly random block irreducible (for the definition, see the paragraph preceding Lemma 1.5) 321-avoiding permutations of finite contiguous blocks, whose union is $[N]$, for some random N. The blocks, whose lengths have infinite expectation, are obtained in a regenerative fashion. If in fact, the limit is in S_∞, then the continuation Z of the concatenation is supported on block irreducible 321-avoiding permutations of the infinite set $\{N+1, \ldots\}$. Thus, the regenerative structure only maintains itself for a finite length.

Remark. Note that the supports of the limiting distributions in cases (3), (4) and (5) are all disjoint from one another.

We now state our results in full.

Proposition 1.1.
(i) *Let* $\tau = 123$. *Then* $\lim_{n\to\infty} \mu_n^\tau = \delta_{\infty^{(\infty)}}$.
(ii) *Let* $\tau = 132$. *Then* $\lim_{n\to\infty} \mu_n^\tau = \delta_{\infty^{(\infty)}}$.

To present the rest of the results, we need to introduce some more definitions. The distribution of the random variable X defined below will play a fundamental role in our results.

$$P(X = n) = \frac{C_n}{2 \cdot 4^n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where C_n is the nth Catalan number.

Remark. As is well known [9],

$$\frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n \quad \text{for } |x| < \frac{1}{4}.$$

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Since $C_n \sim \pi^{-1/2}\, 4^n n^{-3/2}$, if follows that the series converges for $x = 1/4$, and

$$\sum_{n=0}^{\infty} C_n \left(\frac{1}{4}\right)^n = 2.$$

Thus, (1.1) does indeed define a distribution. It also follows that $EX^p < \infty$ for $p \in (0, 1/2)$ but not for $p = 1/2$.

Let Y denote a random variable with distribution Geom (1/2):

$$P(Y = n) = \left(\frac{1}{2}\right)^n, \quad n \in \mathbb{N}. \tag{1.2}$$

Define

$$\begin{aligned} &T_0^X = T_0^Y = 0,\\ &T_n^X = \sum_{j=1}^n X_j, \quad T_n^Y = \sum_{j=1}^n Y_j, \quad n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ are mutually independent i.i.d. sequences with X_1 distributed according to (1.1) and Y_1 distributed according to (1.2).

We define pattern avoidance for permutation images in the obvious way; for example the permutation image (5 3 9 1) is 123-avoiding, but is not 321-avoiding (because of the terms 5 3 1). For fixed $\tau \in S_3$ and for all finite blocks $I \subset \mathbb{N}$, define the random permutation images Π_I^τ of I as follows:

$$\begin{aligned} &\Pi_I^\tau \text{ is uniformly distributed over } \tau\text{-avoiding permutation images of the}\\ &\text{finite block } I \subset \mathbb{N} \text{ and } \{P_I^\tau : I \subset \mathbb{N}, |I| < \infty\} \text{ are independent.} \end{aligned} \tag{1.4}$$

Note on notation. Henceforth we will frequently use the following notation for blocks: $[a, b] := \{a, \ldots, b\} \subset \mathbb{N}$, for $a, b \in \mathbb{N}$ with $a \leqslant b$.

Theorem 1.2. *Let $\tau = 312$. Let $\{X_n\}_{n=1}^\infty, \{Y_n\}_{n=1}^\infty$ and $\{\Pi_I^{312} : I \subset \mathbb{N}\}$ be mutually independent random variables with $\{\Pi_I^{312} : I \subset \mathbb{N}\}$ as in (1.4) and with $\{X_n\}_{n=1}^\infty$, $\{Y_n\}_{n=1}^\infty$, T_n^X, T_n^Y as in the paragraph containing (1.3). Then $\lim_{n\to\infty} \mu_n^\tau$ is the distribution of the $(S(\mathbb{N}, \mathbb{N}) - S_\infty)$-valued random variable*

$$\begin{aligned} &*_{n=1}^{\infty}\, \Pi^{312}_{[T_n^Y + T_{n-1}^X + 1, T_n^Y + T_n^X]} * (T_n^Y + T_{n-1}^X)\\ &\quad := \Pi^{312}_{[T_1^Y + 1, T_1^Y + T_1^X]} * (T_1^Y) * \Pi^{312}_{[T_2^Y + T_1^X + 1, T_2^Y + T_2^X]} * (T_2^Y + T_1^X) * \cdots . \end{aligned}$$

Theorem 1.3. *Let $\tau = 231$. Let $\{X_n\}_{n=1}^\infty$ and $\{\Pi_I^{231} : I \subset \mathbb{N}\}$ be mutually independent random variables with $\{\Pi_I^{231} : I \subset \mathbb{N}\}$ as in (1.4) and with $\{X_n\}_{n=1}^\infty$ and T_n^X as in the paragraph containing (1.3). Then $\lim_{n\to\infty} \mu_n^\tau$ is the distribution of the $S_{sur}(\mathbb{N}, \mathbb{N}^*)$-valued random variable*

$$*_{n=1}^{\infty}\, \Pi^{231}_{[T_{n-1}^X + 1, T_n^X]} * \infty^{(1)} := \Pi^{231}_{[1, T_1^X]} * \infty^{(1)} * \Pi^{231}_{[T_1^X + 1, T_2^X]} * \infty^{(1)} * \cdots .$$

For the next result, we will need some additional notation. Define

$$\begin{aligned} &T_0^{\hat{X}} = T_0^{Y^{(1)}} = T_0^{Y^{(2)}} = 0,\\ &T_n^{\hat{X}} = \sum_{j=1}^n \hat{X}_j, \quad T_n^{Y^{(i)}} = \sum_{j=1}^n Y_j^{(i)}, \quad n \in \mathbb{N},\ i = 1, 2, \end{aligned} \tag{1.5}$$

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where $\{\hat{X}_n\}_{n=1}^{\infty}, \{Y_n^{(1)}\}_{n=1}^{\infty}$ and $\{Y_n^{(2)}\}_{n=1}^{\infty}$ are mutually independent i.i.d. sequences with $\hat{X}_1 \overset{\text{dist}}{=} X+1$, where X is as in (1.1), and $Y_1^{(i)} \overset{\text{dist}}{=} Y$, $i=1,2$, where Y is as in (1.2).

Let

$$\chi_{0,1} \overset{\text{dist}}{=} \text{Ber}\left(\frac{1}{2}\right): P(\chi_{0,1}=0)=P(\chi_{0,1}=1)=\frac{1}{2}. \tag{1.6}$$

For $J=\{J_n\}_{n=1}^{\infty}$, where $J_n \in \mathbb{N}$, and $I=(i_1,i_2,\ldots) \subset \mathbb{N}$ an increasing sequence, define

$$\infty^{(J)} * I := *_{n=1}^{\infty} \infty^{(J_n)} * (i_n) = \infty^{(J_1)} * (i_1) * \infty^{(J_2)} * (i_2) * \cdots . \tag{1.7}$$

Theorem 1.4. *Let $\tau=213$. Let $\{X_n\}_{n=1}^{\infty}$, $\{\hat{X}_n\}_{n=1}^{\infty}$, $\{Y_n^{(i)}\}_{n=1}^{\infty}$, $i=1,2$, and $\chi_{0,1}$ be mutually independent random variables with $\{X_n\}_{n=1}^{\infty}$ as in the paragraph containing (1.3), with $\{\hat{X}_n\}_{n=1}^{\infty}, \{Y_n^{(i)}\}_{n=1}^{\infty}, T_n^{\hat{X}}, T_n^{Y^{(i)}}$, $i=1,2$, as in the paragraph containing (1.5), and with $\chi_{0,1}$ as in (1.6). Then $\lim_{n\to\infty} \mu_n^{\tau}$ is the distribution of the $S(\mathbb{N},\mathbb{N}^*)-S(\mathbb{N},\mathbb{N})-S_{sur}(\mathbb{N},\mathbb{N}^*)$-valued random variable*

$$\chi_{0,1}\cdot(\infty^{(J)} * I^{(1)}) + (1-\chi_{0,1})\cdot(\infty^{(J)} * I^{(0)}),$$

where $\infty^{(J)} * I^{(i)}$ is as in (1.7), with

$$J=\{X_n\}_{n=1}^{\infty}$$

and

$$I^{(1)} = \cup_{n=0}^{\infty}\left[T_n^{Y^{(1)}} + T_{T_n^{Y^{(2)}}}^{\hat{X}} + 1,\, T_{n+1}^{Y^{(1)}} + T_{T_n^{Y^{(2)}}}^{\hat{X}}\right],$$

$$I^{(0)} = \cup_{n=0}^{\infty}\left[T_n^{Y^{(1)}} + T_{T_{n+1}^{Y^{(2)}}}^{\hat{X}} + 1,\, T_{n+1}^{Y^{(1)}} + T_{T_{n+1}^{Y^{(2)}}}^{\hat{X}}\right],$$

with $I^{(i)}$, $i=0,1$, understood as the increasing sequence of its elements.

For the final pattern, $\tau=321$, we need some more notation and another concept. Let $I\subset\mathbb{N}$ be a (possibly infinite) block of integers, and let $S_{(I)}$ denote the set of permutations of the block I. (In this notation, $S_n = S_{([n])}$.) Let $\sigma \in S_{(I)}$ and write I generically as $I=\{j+i: 0\leqslant i<n^*\}$, where $n^*\in\mathbb{N}^*$. If there does not exist a k satisfying $0\leqslant k<n^*$ and such that σ maps $\{j,\ldots,j+k\}$ to itself, then we call σ a *block irreducible* permutation in $S_{(I)}$. Let $S_{(I)}(321)$ denote the set of 321-avoiding permutations in $S_{(I)}$, and let $S_{(I)}^{\text{b-irr}}(321)$ denote the set of block irreducible permutations in $S_{(I)}(321)$. We will prove the following lemma.

Lemma 1.5. *Let $I=\{m,\ldots,m+j-1\}$, for some $m,j\in\mathbb{N}$. Then*

$$|S_{(I)}^{b\text{-}irr}(321)| = C_{j-1}, \quad j\geqslant 1. \tag{1.8}$$

Remark. Of course, $|S_{(I)}(321)|=C_j$, for I as in the lemma.

Let $\mathcal{I}$ denote the class of all finite blocks $I\subset\mathbb{N}$. Define the random permutations $\{\Pi_{(I)}^{321;\,\text{b-irr}}\}_{I\in\mathcal{I}}$ as follows:

$$\begin{aligned}&\Pi_{(I)}^{321;\,\text{b-irr}} \text{ is uniformly distributed over the set } S_{(I)}^{\text{b-irr}}(321) \text{ of 321-avoiding}\\ &\text{block irreducible permutations of } I\in\mathcal{I} \text{ and } \{\Pi_{(I)}^{321;\,\text{b-irr}}\}_{I\in\mathcal{I}} \text{ are independent.}\end{aligned} \tag{1.9}$$

Proposition 1.6. *Let $\tau=321$. Let $\{\hat{X}_n\}_{n=1}^{\infty}$, Y and $\{\Pi_{(I)}^{321;\,b\text{-}irr}: I\subset\mathcal{I}\}$ be mutually independent random variables with $\{\Pi_{(I)}^{321;\,b\text{-}irr}: I\subset\mathcal{I}\}$ as in (1.9), with $\{\hat{X}_n\}_{n=1}^{\infty}$ and $T_n^{\hat{X}}$ as in the paragraph*

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containing (1.5) *and with* Y *as in* (1.2). *Then the distribution of any weakly converging subsequence of* $\{\mu_n^{321}\}_{n=1}^{\infty}$ *is the distribution of an* $S(\mathbb{N},\mathbb{N}^*)$*-valued random variable of the form*

$$\left(*_{n=0}^{Y-2}\Pi^{321;\,b\text{-}irr}_{([T_n^{\hat{X}}+1,T_{n+1}^{\hat{X}}])}\right)*Z,$$

for some appropriate Z. *If the limiting distribution is in fact supported on* S_∞, *then the random variable* Z, *conditioned on* $Y=y$ *and* $T_{y-1}^{\hat{X}}=M$, *is almost surely a 321-avoiding block irreducible permutation of the infinite set* $\{M+1,M+2,\ldots\}$.

Note that in Theorems 1.2–1.4 the length of each segment in the regenerative structure is distributed as $X+1$, and the length of the first n segments is given by T_n^X+n. Thus, it is of interest to determine the growth rate of T_n^X.

Proposition 1.7.

$$\lim_{n\to\infty}\frac{T_n^X}{n^2}\overset{\text{dist}}{=}W, \tag{1.10}$$

where W *is the one-sided stable distribution with stability parameter* $1/2$ *and characteristic function*

$$\phi(t)=Ee^{-itW}=\exp\left(-\frac{\sqrt{2}}{2}\,|t|^{1/2}(1+i\,\text{sgn}(t))\right).$$

Remark. We note that Proposition 1.7 is a particular case of the general theory of the convergence of scaled sums of i.i.d. random variables with infinite expectation to limiting stable distributions [3].

Since our results pertain to weak convergence of infinite sequences of random variables, they relate to the asymptotic distribution of arbitrarily long, finite initial segments of the permutation. For other limiting results for permutations, of a completely different nature, see [7] and [1], which concern the asymptotic density of the number of occurrences of a certain pattern within certain classes of permutations, and [8] and [6], which consider the limiting 'shape' of permutations avoiding a particular pattern of length three.

In Section 2 we will state and prove several preliminary facts that will be used in the proofs of the main results, and we will prove Lemma 1.5. The five sections that follow Section 2 give the proofs respectively of Proposition 1.1, Theorems 1.2–1.4 and Proposition 1.6. In the final section we prove Proposition 1.7.

Important note regarding the proofs. The same basic idea is used in the proofs of Theorems 1.2–1.4 (via Lemma 2.1 in Section 2). A variant of that idea is used for the proof of Proposition 1.6 (via Lemma 1.5). However, to write down a complete and entirely rigorous proof is extremely tedious and may well obscure the relative simplicity of the ideas behind the proofs. Thus, for the proof of Theorem 1.2, we begin with a rather verbal explanation of the proof, and then prove completely rigorously the first few steps of the proof. From this, it will be clear that one can precede similarly to obtain the entire proof. After that, for the proofs of Theorems 1.3 and 1.4 and Proposition 1.6, we will only give the rather verbal explanation, the rigorous proof following very similarly to that of Theorem 1.2. On the other hand, the proof of Proposition 1.1 is short and direct.

2. Some preliminary results

We begin with the proof of Lemma 1.5, which appeared in the introductory section.

Proof of Lemma 1.5. It suffices to prove the lemma for $S_j^{\text{b-irr}}(321) := S_{([j])}^{\text{b-irr}}(321)$. For $1 \leqslant j \leqslant n < \infty$, let $S_n^{\text{b-irr};j}(321)$ denote the set of permutations in $S_n(321)$ which map $[j]$ to $[j]$ but do not map $[k]$ to $[k]$ for $1 \leqslant k < j$. (In this notation $S_n^{\text{b-irr};n}(321) = S_n^{\text{b-irr}}(321)$.) We have

$$|S_n(321)| = C_n = \sum_{j=1}^{n} |S_n^{\text{b-irr};j}(321)|, \quad 1 \leqslant n < \infty. \tag{2.1}$$

It is well known that a permutation in S_n belongs to $S_n(321)$ if and only if it is composed of two increasing subsequences [2]. Thus, $\sigma \in S_n^{\text{b-irr};j}(321)$ if and only if $\sigma = \tau * \nu$, where $\tau \in S_j^{\text{b-irr}}(321)$ and $\nu \in S_{([j+1,n])}(321)$, that is, ν is a 321-avoiding permutation of $[j+1, n]$. Of course, the number of 321-avoiding permutations of $[j+1, n]$ is C_{n-j}. Thus, $|S_n^{\text{b-irr};j}(321)| = |S_j^{\text{b-irr}}(321)|C_{n-j}$. Substituting this in (2.1) gives

$$C_n = \sum_{j=1}^{n} |S_j^{\text{b-irr}}(321)|C_{n-j}, \quad n \geqslant 1. \tag{2.2}$$

On the other hand, the fundamental recurrence relation for Catalan numbers [9] gives

$$C_n = \sum_{j=1}^{n} C_{j-1}C_{n-j}, \quad n \geqslant 1. \tag{2.3}$$

Equating (2.2) and (2.3) successively for $n = 1, 2, \ldots$ shows that $|S_j^{\text{b-irr}}(321)| = C_{j-1}$, for all $j \geqslant 1$. □

Remark. From the proof of the lemma, we obtain the following fact, which will be used later:

$$|S_n^{\text{b-irr};j}(321)| = C_{j-1}C_{n-j}. \tag{2.4}$$

The following lemma states a well-known fact about permutations avoiding certain patterns of length three. For completeness, we provide the short proof.

Lemma 2.1. *For $1 \leqslant j \leqslant n$,*

(i) $\mu_n^{312}(\sigma_1^{-1} = j) = \mu_n^{213}(\sigma_1^{-1} = j) = \dfrac{C_{j-1}C_{n-j}}{C_n}$,

(ii) $\mu_n^{231}(\sigma_n^{-1} = j) = \mu_n^{132}(\sigma_n^{-1} = j) = \dfrac{C_{j-1}C_{n-j}}{C_n}$.

Proof. A 312-avoiding permutation $\sigma \in S_n$ has the property that all of the numbers in the positions to the left of the position occupied by 1 are smaller than all of the numbers in the positions to the right of the position occupied by 1. That is, if $\sigma_1^{-1} = j_1$, then $\{2, \ldots, j_1\}$ appear in the first $j_1 - 1$ positions of σ and $\{j_1 + 1, \ldots, n\}$ appear in the last $n - j_1$ positions of σ. In fact then, it follows that a permutation $\sigma \in S_n$ satisfying $\sigma_1^{-1} = j$ will be 312-avoiding if and only if $(\sigma_1, \ldots, \sigma_{j-1})$ is a 312-avoiding permutation image of $\{2, \ldots, j_1\}$ and $(\sigma_{j+1}, \ldots, \sigma_n)$ is a 312-avoiding permutation image of $\{j+1, \ldots, n\}$. The proof of the lemma for the case μ_n^{312} now follows from the fact that there are C_{j-1} 321-avoiding permutation images of $\{2, \ldots, j\}$ and C_{n-j} 312-avoiding permutation images of $\{j+1, \ldots, n\}$.

The proof for μ_n^{213} follows similarly, using the fact that a 213-avoiding permutation $\sigma \in S_n$ has the property that all of the numbers in the positions to the left of the position occupied by 1 are

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larger than all of the numbers in the positions to the right of the position occupied by 1. The proof for μ_n^{231} (μ_n^{132}) follows similarly from the fact that a 231-avoiding (132-avoiding) permutation $\sigma \in S_n$ has the property that all of the numbers in the positions to the left of the position occupied by n are smaller (larger) than all of the numbers in the positions to the right of the position occupied by n. □

Lemma 2.2. *For $n \in \mathbb{N}$, let ν_n be the probability measure on $\mathbb{N}^*$ satisfying*

$$\nu_n(j) = \frac{C_{j-1}C_{n-j}}{C_n}, \quad j \in [n],$$
$$\nu_n(j) = 0, \quad j \in \mathbb{N}^* - [n].$$

Then $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to the probability measure ν on $\mathbb{N}^*$ satisfying

$$\nu(j) = C_{j-1}\left(\frac{1}{4}\right)^j, \quad j \in \mathbb{N},$$
$$\nu(\infty) = \frac{1}{2}.$$

Remark. Note that $X + 1$ has the distribution of $\nu(\cdot\,|\mathbb{N})$, where X is as in (1.1).

Proof. A direct calculation shows that for each fixed j,

$$\lim_{n\to\infty} \frac{C_{n-j}}{C_n} = \left(\frac{1}{4}\right)^j.$$

Thus,

$$\lim_{n\to\infty} \nu_n(j) = C_{j-1}\left(\frac{1}{4}\right)^j \quad \text{for } j \geqslant 1.$$

As noted in the remark following (1.1),

$$\sum_{n=0}^{\infty} C_n\left(\frac{1}{4}\right)^n = 2.$$

Thus,

$$\sum_{j=1}^{\infty} C_{j-1}\left(\frac{1}{4}\right)^j = \frac{1}{2}.$$

This proves the lemma. □

3. Proof of Proposition 1.1

Proof of (i). For fixed $j, M \in \mathbb{N}$, we give an upper bound on $\mu_n^{123}(\sigma_j = M)$, for $n \geqslant j \vee M$. To construct a permutation $\sigma \in S_n(123)$ satisfying $\sigma_j = M$, there are certainly no more than $(n-1)\cdots(n-j+1)$ ways to choose the values of $\{\sigma_1, \ldots, \sigma_{j-1}\}$. Having chosen $\{\sigma_1, \ldots, \sigma_{j-1}\}$, there are at least $n - M - j + 1$ values larger than M among the numbers $\{\sigma_{j+1}, \ldots, \sigma_n\}$. Since $\sigma_j = M$, all the values larger than M among $\{\sigma_{j+1}, \ldots, \sigma_n\}$ must appear in decreasing order. Thus, at least $n - M - j + 1$ of the values among $\{\sigma_{j+1}, \ldots, \sigma_n\}$ must appear in decreasing order. So with regard to $n - M - j + 1$ such values, the only choice we have is which $n - M - j + 1$ spaces out of $n - j$

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spaces to use for them. Therefore, we conclude that

$$\mu_n^{123}(\sigma_j = M) \leqslant \frac{1}{C_n}(n-1)\cdots(n-j+1)\binom{n-j}{n-M-j+1}(M-1)! \leqslant \frac{n^{j+M-2}}{C_n}.$$

Thus, for any $j, L \in \mathbb{N}$,

$$\lim_{n\to\infty} \mu_n^{123}(\sigma_j \leqslant L) \leqslant \lim_{n\to\infty} L\frac{n^{j+L-2}}{C_n} = 0.$$

From this it follows that the distribution of any weak limit of $\{\mu_n^{123}\}_{n=1}^{\infty}$ must be supported on the singleton $\infty^{(\infty)}$.

Proof of (ii). For fixed $j, M \in \mathbb{N}$, we give an upper bound on $\mu_n^{132}(\sigma_j = M)$, for $n \geqslant j \vee M$. To construct a permutation $\sigma \in S_n(132)$ satisfying $\sigma_j = M$, there are certainly no more than $(n-1)\cdots(n-j+1)$ ways to choose the values of $\{\sigma_1, \ldots, \sigma_{j-1}\}$. Having chosen $\{\sigma_1, \ldots, \sigma_{j-1}\}$, there are at least $n-M-j+1$ values larger than M among the numbers $\{\sigma_{j+1}, \ldots, \sigma_n\}$. Since $\sigma_j = M$, all the values larger than M among $\{\sigma_{j+1}, \ldots, \sigma_n\}$ must appear in increasing order. Thus, at least $n-M-j+1$ of the values among $\{\sigma_{j+1}, \ldots, \sigma_n\}$ must appear in increasing order. So with regard to $n-M-j+1$ such values, the only choice we have is which $n-M-j+1$ spaces out of $n-j$ spaces to use for them. Therefore, we conclude that

$$\mu_n^{132}(\sigma_j = M) \leqslant \frac{1}{C_n}(n-1)\cdots(n-j+1)\binom{n-j}{n-M-j+1}(M-1)! \leqslant \frac{n^{j+M-2}}{C_n}.$$

The proof is now completed as it was in part (i). □

4. Proof of Theorem 1.2

We will need the following additional notation. For a permutation image $\sigma_I^{\text{im}} = (i_1\ i_2 \cdots i_l)$ of a block $I = \{j+1, \ldots, j+l\}$, let $\sigma_I^{\text{im}} - j$ denote the permutation $\tau \in S_l$ given by $\tau_k = i_k - j$, $k \in [l]$. Also, for any $I \subset \mathbb{N}$, let Σ_I^{im} denote the collection of all permutation images of I.

By Lemma 2.1,

$$\mu_n^{312}(\sigma_1^{-1} = j_1) = \frac{C_{j_1-1}C_{n-j_1}}{C_n}, \quad j_1 \in [1, n]. \tag{4.1}$$

From the proof of (4.1) in Lemma 2.1, it follows that

$$\mu_n^{312}(\sigma_{I_1}^{\text{im}} * (1) * \sigma_{I_2}^{\text{im}}) | \sigma_1^{-1} = j_1) = \mu_{j_1-1}^{312}(\sigma_{I_1}^{\text{im}} - 1)\mu_{n-j_1}^{312}(\sigma_{I_2}^{\text{im}} - j_1), \tag{4.2}$$

where $1 \leqslant j_1 \leqslant n$, $\sigma_{I_1}^{\text{im}}$ is a permutation image of $I_1 = [2, j_1]$ and $\sigma_{I_2}^{\text{im}}$ is a permutation image of $I_2 = [j_1+1, n]$.

As noted at the end of the first section, we first give a rather verbal explanation of the proof. From (4.1) and Lemma 2.2 with the remark following it, along with (1.1) and (1.3), it follows that as $n \to \infty$, σ_1^{-1} will be carried off to ∞ with probability 1/2, and will converge to the distribution $X_1 + 1$ with probability 1/2. Consider the former case. Let $\sigma_1^{-1} = j_1$ be very large. Akin to the proof of Lemma 2.1, since the first $j_1 - 1$ places constitute a 312-avoiding permutation of $[2, j_1]$, it follows that from among these numbers, all the numbers in the positions to the left of σ_2^{-1} are smaller than all the numbers in positions to the right of σ_2^{-1}. Thus, the same reasoning as in (4.1) gives

$$\mu_n^{312}(\sigma_2^{-1} = j_2 | \sigma_1^{-1} = j_1) = \frac{C_{j_2-1}C_{j_1-j_2-1}}{C_{j_1-1}}, \quad j_2 \in [1, j_1 - 1].$$

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Thus, as $j_1 \to \infty$, it follows that σ_2^{-1} will be carried off to ∞ with probability 1/2 and will converge to the distribution $X_1 + 1$ with probability 1/2. Continuing like this, eventually, we will arrive as some $m \in \mathbb{N}$ such that $\sigma_1^{-1}, \ldots, \sigma_{m-1}^{-1}$ were all carried off to ∞, but σ_m^{-1} converges to the distribution $X_1 + 1$. Note that the probability of this occurring at any specific m is $(1/2)^m$; that is, this occurrence time has the distribution of Y_1, as in the paragraph containing (1.3). Thus, what we see so far is that the numbers $1, \ldots, Y_1 - 1$ have escaped to ∞, the number Y_1 is in position $X_1 + 1$, and by (4.2), the first X_1 positions are occupied by a permutation image σ_I^{im} of $I = [Y_1 + 1, Y_1 + X_1]$ and this permutation image has the uniform distribution on 312-avoiding permutation images of $[Y_1 + 1, Y_1 + X_1]$. Stating this in the notation of (1.3) and (1.4), we have that the first $T_1^X + 1$ positions look like

$$\Pi^{312}_{[T_1^Y+1, T_1^Y+T_1^X]} * (T_1^Y).$$

This is just as in the statement of the theorem. Now everything after position $T_1^X + 1$ is iterated, with the smallest number still available there being $T_1^Y + T_1^X + 1$. By the same reasoning, the first of these numbers that does not run off to ∞ will be $T_2^Y + T_1^X$, its position will be $T_2^X + 2$ and in positions $[T_1^X + 2, T_2^X + 1]$ will appear a uniformly random 312-avoiding permutation image of $[T_2^Y + T_1^X + 1, T_2^Y + T_2^X]$, that is,

$$\Pi^{312}_{[T_2^Y+T_1^X+1, T_2^Y+T_2^X]}.$$

We now have the initial part of the limiting random variable being

$$\Pi^{312}_{[T_1^Y+1, T_1^Y+T_1^X]} * (T_1^Y) * \Pi^{312}_{[T_2^Y+T_1^X+1, T_2^Y+T_2^X]} * (T_2^Y + T_1^X),$$

as in the theorem.

We now turn to the rigorous proof. Using Lemma 2.2 and the remark following it, along with (4.1) and (4.2), it follows that

$$\begin{aligned}\lim_{n\to\infty} \mu_n^{312}(\sigma_{I_1}^{\text{im}} * (1) * \Sigma^{\text{im}}_{[j_1+1,n]})) &= \frac{1}{2} P(X_1 = j_1 - 1)\mu^{312}_{j_1-1}(\sigma_{I_1}^{\text{im}} - 1)\\ &= P(T_1^X = j_1 - 1, T_1^Y = 1)\mu^{312}_{j_1-1}(\sigma_{I_1}^{\text{im}} - 1), \qquad (4.3)\end{aligned}$$

for $j_1 \in \mathbb{N}$ and $\sigma_{I_1}^{\text{im}}$ a permutation image of $I_1 = [2, j_1]$, where T_1^X and T_1^Y are as in (1.3).

Repeating the procedure that yielded (4.1) and (4.2), we have

$$\begin{aligned}\mu_n^{312}(\sigma^{-1}_{j_1+1} = j_2 | \sigma_1^{-1} = j_1) &= \frac{C_{j_2-j_1-1}C_{n-j_2}}{C_{n-j_1}}, \quad j_2 \in [j_1 + 1, n],\\ \mu_n^{312}(\sigma_2^{-1} = j_2 | \sigma_1^{-1} = j_1) &= \frac{C_{j_2-1}C_{j_1-j_2-1}}{C_{j_1-1}}, \quad j_2 \in [1, j_1 - 1],\end{aligned} \qquad (4.4)$$

and then we have

$$\begin{aligned}&\mu_n^{312}(\sigma_{I_1}^{\text{im}} * (1) * \sigma_{I_2}^{\text{im}} * (j_1 + 1) * \sigma_{I_3}^{\text{im}}) | \sigma_1^{-1} = j_1, \sigma^{-1}_{j_1+1} = j_2)\\ &= \mu^{312}_{j_1-1}(\sigma_{I_1}^{\text{im}} - 1)\mu^{312}_{j_2-j_1-1}(\sigma_{I_2}^{\text{im}} - j_1)\mu^{312}_{n-j_2}(\sigma_{I_3}^{\text{im}} - j_2), \quad 1 \leqslant j_1 < j_2 \leqslant n, \qquad (4.5)\end{aligned}$$

where $\sigma_{I_1}^{\text{im}}$ is a permutation image of $I_1 = [2, j_1]$, $\sigma_{I_2}^{\text{im}}$ is a permutation image of $I_2 = [j_1 + 2, j_2]$ and $\sigma_{I_3}^{\text{im}}$ is a permutation image of $I_3 = [j_2 + 1, n]$, and we have

$$\begin{aligned}&\mu_n^{312}(\sigma_{I_1}^{\text{im}} * (2) * \sigma_{I_2}^{\text{im}} * (1) * \sigma_{I_3}^{\text{im}}) | \sigma_1^{-1} = j_1, \sigma_2^{-1} = j_2)\\ &= \mu^{312}_{j_2-1}(\sigma_{I_1}^{\text{im}} - 2)\mu^{312}_{j_1-1-j_2}(\sigma_{I_2}^{\text{im}} - j_2 - 1)\mu^{312}_{n-j_1}(\sigma_{I_3}^{\text{im}} - j_1), \quad 1 \leqslant j_2 < j_1 \leqslant n, \qquad (4.6)\end{aligned}$$

where $\sigma_{I_1}^{\text{im}}$ is a permutation image of $I_1 = [3, j_2 + 1]$, $\sigma_{I_2}^{\text{im}}$ is a permutation image of $I_2 = [j_2 + 2, j_1]$ and $\sigma_{I_3}^{\text{im}}$ is a permutation image of $I_3 = [j_1 + 1, n]$.

Using Lemma 2.2 along with (4.1), (4.6) and the second equation in (4.4), we have

$$\begin{aligned}\lim_{M\to\infty}\lim_{n\to\infty}&\mu_n^{312}(\sigma_{I_1}^{\text{im}}*(2)*\Sigma^{\text{im}}_{[j_2+2,n]},\ \sigma_1^{-1}\geqslant M)\\ &=\frac{1}{4}P(X_1=j_2-1)\mu_{j_2-1}^{312}(\sigma_{I_1}^{\text{im}}-2)\\ &=P(T_1^X=j_2-1,\,T_1^Y=2)\mu_{j_2-1}^{312}(\sigma_{I_1}^{\text{im}}-2),\end{aligned}\tag{4.7}$$

for $j_2\in\mathbb{N}$ and $\sigma_{I_1}^{\text{im}}$ a permutation image of $I_1=[3,j_2+1]$.

Repeating the procedure yet again, we have

$$\mu_n^{312}(\sigma_{j_2+1}^{-1}=j_3|\sigma_1^{-1}=j_1,\sigma_{j_1+1}^{-1}=j_2)=\frac{C_{j_3-j_2-1}C_{n-j_3}}{C_{n-j_3}},\quad j_3\in[\,j_2+1,\dots,n],\ j_1<j_2,\tag{4.8}$$

and then applying this to (4.6) we have

$$\begin{aligned}&\mu_n^{312}(\sigma_{I_1}^{\text{im}}*(3)*\sigma_{I_2}^{\text{im}}*(2)*\sigma_{I_3}^{\text{im}}*(1)*\sigma_{I_4}^{\text{im}})|\sigma_1^{-1}=j_1,\sigma_2^{-1}=j_2,\sigma_3^{-1}=j_3)\\ &\quad=\mu_{j_3-1}^{312}(\sigma_{I_1}^{\text{im}}-3)\mu_{j_2-1-j_3}^{312}(\sigma_{I_2}^{\text{im}}-j_3-2)\mu_{j_1-1-j_2}^{312}(\sigma_{I_3}^{\text{im}}-j_2-1)\mu_{n-j_1}^{312}(\sigma_{I_4}^{\text{im}}-j_1),\\ &\qquad 1\leqslant j_3<j_2<j_1\leqslant n,\end{aligned}\tag{4.9}$$

where $\sigma_{I_1}^{\text{im}}$ is a permutation image of $I_1=[4,j_3+2]$, $\sigma_{I_2}^{\text{im}}$ is a permutation image of $I_2=[\,j_3+3,j_2+1]$, $\sigma_{I_3}^{\text{im}}$ is a permutation image of $I_3=[\,j_2+2,j_1]$ and $\sigma_{I_4}^{\text{im}}$ is a permutation image of $I_4=[\,j_1+1,n]$. Using Lemma 2.2 along with (4.1), the second equation in (4.4), (4.8) and (4.9), we have

$$\begin{aligned}\lim_{M\to\infty}\lim_{n\to\infty}&\mu_n^{312}(\sigma_{I_1}^{\text{im}}*(3)*\Sigma^{\text{im}}_{[j_3+3,n]},\ \sigma_1^{-1}\geqslant M,\ \sigma_2^{-1}\geqslant M)\\ &=\frac{1}{8}P(X_1=j_3-1)\mu_{j_3-1}^{312}(\sigma_{I_1}^{\text{im}}-3)=P(T_1^X=j_3-1,\,T_1^Y=3)\mu_{j_3-1}^{312}(\sigma_{I_1}^{\text{im}}-3),\end{aligned}\tag{4.10}$$

for $j_3\in\mathbb{N}$ and $\sigma_{I_1}^{\text{im}}$ a permutation image of $I_1=[4,j_3+2]$.

It is clear from (4.3), (4.7) and (4.10) that if we continue in this vein we obtain

$$\begin{aligned}\lim_{M\to\infty}\lim_{n\to\infty}&\mu_n^{312}(\sigma_{I_1}^{\text{im}}*(i)*\Sigma^{\text{im}}_{[j_i+i,n]},\ \sigma_1^{-1}\geqslant M,\dots\ \sigma_{i-1}^{-1}\geqslant M)\\ &=\left(\frac{1}{2}\right)^iP(X_1=j_i-1)\mu_{j_i-1}^{312}(\sigma_{I_1}^{\text{im}}-i)=P(T_1^X=j_i-1,\,T_1^Y=i)\mu_{j_i-1}^{312}(\sigma_{I_1}^{\text{im}}-i),\end{aligned}\tag{4.11}$$

for $j_i\in\mathbb{N}$ and $\sigma_{I_1}^{\text{im}}$ a permutation image of $I_1=[i+1,j_i+i-1]$. This shows that a random variable whose distribution is that of a weakly convergent subsequence of $\{\mu_n^{312}\}_{n=1}^{\infty}$ must be of the form $\Pi^{312}_{[T_1^Y+1,T_1^Y+T_1^X]}*(T_1^Y)*Z$, for some random Z distributed on $S(\mathbb{N}-[1,T_1^X+1],\mathbb{N}^*)$.

We now need to continue and peel off the next component from Z. We just show the following step. Using Lemma 2.2 along with (4.1), (4.5) and the first equation in (4.4), we have

$$\begin{aligned}\lim_{n\to\infty}&\mu_n^{312}(\sigma_{I_1}^{\text{im}}*(1)*\sigma_{I_2}^{\text{im}}*(1+j_1)*\Sigma_{[j_2+1,n]}))\\ &=\frac{1}{4}P(X_1=j_1-1)P(X_2=j_2-j_1-1)\mu_{j_1-1}^{312}(\sigma_{I_1}^{\text{im}}-1)\mu_{j_2-j_1-1}^{312}(\sigma_{I_2}^{\text{im}}-j_1)\qquad(4.12)\\ &=P(X_1=j_1-1,X_2=j_2-j_1-1,\,Y_1=1,\,Y_2=1)\mu_{j_1-1}^{312}(\sigma_{I_1}^{\text{im}}-1)\mu_{j_2-j_1-1}^{312}(\sigma_{I_2}^{\text{im}}-j_1),\end{aligned}$$

where $\sigma^{\text{im}}_{I_1}$ is a permutation image of $I_1=[2,j_1]$ and $\sigma^{\text{im}}_{I_2}$ is a permutation image of $I_2=[j_1+2,j_2]$. Continuing in this vein will give us for all $(k_1,k_2)\in\mathbb{N}\times\mathbb{N}$,

$$\begin{aligned}&\lim_{n\to\infty}\mu^{312}_n(\sigma^{\text{im}}_{I_1}*(k_1)*\sigma^{\text{im}}_{I_2}*(k_1+j_1-1+k_2)*\Sigma^{\text{im}}_{[k_1+k_2+j_2-1,n]})) \qquad (4.13)\\&=\left(\frac{1}{2}\right)^{k_1+k_2}P(X_1=j_1-1)P(X_2=j_2-j_1-1)\mu^{312}_{j_1-1}(\sigma^{\text{im}}_{I_1}-1)\mu^{312}_{j_2-j_1-1}(\sigma^{\text{im}}_{I_2}-j_1)\\&=P(X_1=j_1-1,X_2=j_2-j_1-1,Y_1=k_1,Y_2=k_2)\mu^{312}_{j_1-1}(\sigma^{\text{im}}_{I_1}-1)\mu^{312}_{j_2-j_1-1}(\sigma^{\text{im}}_{I_2}-j_1),\end{aligned}$$

where $\sigma^{\text{im}}_{I_1}$ is a permutation image of $I_1=[k_1+1,k_1+j_1-1]$ and σ_{I_2} is a permutation image of $I_2=[k_1+j_1+k_2,k_1+k_2+j_2-2]$. This shows that a random variable whose distribution is that of a weakly convergent subsequence of $\{\mu^{312}_n\}_{n=1}^\infty$ must be of the form

$$\Pi^{312}_{[T^Y_1+1,T^Y_1+T^X_1]}*(T^Y_1)*\Pi^{312}_{[T^Y_2+T^X_1+1,T^Y_2+T^X_2]}*(T^Y_2+T^X_1)*Z,$$

for some random Z distributed on $S(\mathbb{N}-[1,T^X_2+2],\mathbb{N}^*)$. The proof is completed by iterating on this regenerative structure. □

5. Proof of Theorem 1.3

As noted at the end of the introductory section, we will give a rather verbal explanation of the proof, the completely rigorous proof following via the same considerations and methods used in the proof of Theorem 1.2. By Lemma 2.1,

$$\mu^{231}_n(\sigma^{-1}_n=j_1)=\frac{C_{j_1-1}C_{n-j_1}}{C_n},\quad j_1\in[1,n]. \tag{5.1}$$

From the proof of (5.1) in Lemma 2.1 it follows that

$$\mu^{231}_n(\sigma^{\text{im}}_{I_1}*(n)*\sigma^{\text{im}}_{I_2})|\sigma^{-1}_n=j_1)=\mu^{231}_{j_1-1}(\sigma^{\text{im}}_{I_1})\mu^{231}_{n-j_1}(\sigma^{\text{im}}_{I_2}-j_1+1), \tag{5.2}$$

where $1\leqslant j_1\leqslant n$, $\sigma^{\text{im}}_{I_1}$ is a permutation image of $I_1=[1,j_1-1]$ and $\sigma^{\text{im}}_{I_2}$ is a permutation image of $I_2=[j_1,n-1]$.

From (5.1) and Lemma 2.2 with the remark following it, along with (1.1) and (1.3), it follows that as $n\to\infty$, σ^{-1}_n will be carried off to ∞ with probability 1/2, and will converge to the distribution X_1+1 with probability 1/2. Consider the latter case. Then as $n\to\infty$, the position $\sigma^{-1}_n=j_1$ will converge in distribution to X_1+1, and by (5.2) the first X_1 positions will constitute a uniformly random 231-avoiding permutation of $[1,X_1]$. Thus, the initial segment of any weakly convergent subsequence of $\{\mu^{231}_n\}_{n=1}^\infty$ looks like $\Pi^{231}_{[1,X_1]}*(\infty)=\Pi^{231}_{[1,T^X_1]}*(\infty)$.

Now consider the former case. Let $\sigma^{-1}_n=j_1$ be very large. The first j_1-1 positions are occupied by a uniformly random 231-avoiding permutation of $[1,j_1-1]$. Then, in particular, the position of j_1-1 will satisfy

$$\mu^{231}_n(\sigma^{-1}_{j_1-1}=j_2|\sigma^{-1}_n=j_1)=\frac{C_{j_2-1}C_{j_1-j_2-1}}{C_{j_1-1}}\quad\text{for } j_2\in[1,j_1-1].$$

Since j_1 is going to ∞ (as $n\to\infty$), $\sigma^{-1}_{j_1-1}$ will be carried off to ∞ with probability 1/2 and will converge in distribution to X_1+1 with probability 1/2. Consider the latter case. Then just as in the latter case in the previous paragraph, the initial segment of any weakly convergent subsequence of $\{\mu^{231}_n\}_{n=1}^\infty$ will look like $\Pi^{231}_{[1,T^X_1]}*(\infty)$. On the other hand, in the former case, we iterate the process we have just described. So far we have assumed that the former case has prevailed twice. Eventually, after say i times in a row of the former case prevailing, the latter case will finally prevail, and then as above it will follow that the initial segment of any weakly convergent subsequence of

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$\{\mu_n^{231}\}_{n=1}^{\infty}$ looks like $\Pi^{231}_{[1,T_1^X]} * (\infty)$. This process now regenerates on the rest of the domain, that is, on $[T_1^X + 2, \infty)$, giving as the next piece, $\Pi^{231}_{[T_1^X+1,T_2^X]} * (\infty)$, and so on. □

6. Proof of Theorem 1.4

As noted at the end of the introductory section, we will give a rather verbal explanation of the proof, the completely rigorous proof following via the same considerations and methods used in the proof of Theorem 1.2. By Lemma 2.1,

$$\mu_n^{213}(\sigma_1^{-1} = j_1) = \frac{C_{j_1-1}C_{n-j_1}}{C_n}, \quad j_1 \in [1, n]. \tag{6.1}$$

From the proof of (6.1) in Lemma 2.1 it follows that

$$\mu_n^{213}(\sigma_{I_1}^{\text{im}} * (1) * \sigma_{I_2}^{\text{im}}) | \sigma_1^{-1} = j_1) = \mu_{j_1-1}^{213}(\sigma_{I_1}^{\text{im}} - n + j_1 - 1)\mu_{n-j_1}^{213}(\sigma_{I_2}^{\text{im}} - 1), \tag{6.2}$$

where $1 \leqslant j_1 \leqslant n$, $\sigma_{I_1}^{\text{im}}$ is a permutation image of $I_1 = [n - j_1 + 2, n]$ and $\sigma_{I_2}^{\text{im}}$ is a permutation image of $I_2 = [2, n - j_1 + 1]$. From (6.1) and Lemma 2.2 with the remark following it, it follows that as $n \to \infty$, with probability 1/2, $n - \sigma_1^{-1}$ will converge in distribution to $\hat{X}_1 - 1$, and with probability 1/2, σ_1^{-1} will converge in distribution to $X_1 + 1$.

Consider the latter case. Then as $n \to \infty$, the position $\sigma_1^{-1} = j_1$ will converge in distribution to $X_1 + 1$, and by (6.2) the distribution of the permutation image $\sigma_{I_1}^{\text{im}}$ of $I_1 = [n - j_1 + 2, n]$ will converge to the degenerate distribution $\delta_{\infty^{(X_1)}}$. Thus, in this case, the initial segment of any weakly convergent subsequence of $\{\mu_n^{213}\}_{n=1}^{\infty}$ looks like $\infty^{(X_1)} * (1)$.

Now consider the former case. By (6.2), conditioned on $\sigma_1^{-1} = j_1$, the final $n - j_1$ positions in the permutation are a random 213-avoiding permutation image of $[2, n - j_1 + 1]$. Thus, since $n - \sigma_1^{-1} = n - j_1$ is converging in distribution to $\hat{X}_1 - 1$, and consequently $\sigma_1^{-1} = j_1$ is converging in distribution to ∞, it follows that the values $[1, \hat{X}_1]$ get swept away to ∞. Thus the support of any weakly convergent subsequence of $\{\mu_n^{213}\}_{n=1}^{\infty}$ will be on functions in $S(\mathbb{N}, \mathbb{N}^* - [1, \hat{X}_1])$.

Iterating the above scenarios, we see that with probability 1/2, the latter case will prevail during the first $Y_1^{(1)}$ iterations, then the former case will prevail for the next $Y_1^{(2)}$ iterations, then the latter case for the next $Y_2^{(1)}$ iterations, then the former for the next $Y_2^{(2)}$ iterations, *etc.*, while also with probability 1/2, the former case will prevail for the first $Y_1^{(2)}$ iterations, then the latter for the next $Y_1^{(1)}$ iterations, *etc.* These two possibilities, each with probability 1/2, are represented in the statement of the theorem by the random variable $\chi_{0,1}$, with $\chi_{0,1} = 1$ if the first of these two possibilities occurs. Let us say that the first of these two possibilities occurs, the second possibility being handled similarly. Then the latter case prevails on the first $Y_1^{(1)}$ iterations. This results in the initial segment of any weakly convergent subsequence of $\{\mu^{213}\}_{n=1}^{\infty}$ looking like

$$\infty^{(X_1)} * (1) * \infty^{(X_2)} * (2) * \cdots \infty^{(X_{Y_1^{(1)}})} * (Y_1^{(1)}).$$

After this, the former case prevails for $Y_1^{(2)}$ iterations. This causes the values

$$[Y_1^{(1)} + 1, Y_1^{(1)} + T^{\hat{X}}_{Y_1^{(2)}}]$$

to get swept out to ∞. After this, the latter case prevails again for $Y_2^{(1)}$ iterations. This results in the next segment of any weakly convergent subsequence looking like

$$\infty^{(X_{Y_1^{(1)}+1})} * (Y_1^{(1)} + T^{\hat{X}}_{Y_1^{(2)}} + 1) * \cdots \infty^{(X_{Y_1^{(1)}+Y_2^{(1)}})} * (Y_1^{(1)} + T^{\hat{X}}_{Y_1^{(2)}} + Y_2^{(1)}),$$

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or equivalently, like

$$\infty^{(X_{Y_1^{(1)}+1})} * (Y_1^{(1)} + T^{\hat{X}}_{Y_1^{(2)}} + 1) * \cdots \infty^{(X_{T_2^{Y(1)}})} * (T_2^{Y^{(1)}} + T^{\hat{X}}_{Y_1^{(2)}}).$$

In the notation of the theorem, we thus see in these two segments the beginning of $\infty^{(J)} * I^{(1)}$, revealed for $I^{(1)}$ up to

$$\cup_{n=0}^{1}[T_n^{Y^{(1)}} + T^{\hat{X}}_{T_n^{Y(2)}} + 1, T_{n+1}^{Y^{(1)}} + T^{\hat{X}}_{T_n^{Y(2)}}] = [1, Y_1^{(1)}] \cup [Y^{(1)} + T^{\hat{X}}_{Y_1^{(2)}} + 1, T_2^{Y^{(1)}} + T^{\hat{X}}_{Y_1^{(2)}}].$$

The above procedure now regenerates again and so on. □

7. Proof of Proposition 1.6

As noted at the end of the introductory section, we will give a rather verbal explanation of the proof, the completely rigorous proof following via the same considerations and methods used in the proof of Theorem 1.2. Recall the definition of $S_n^{\text{b-irr};j}(321)$ from the proof of Lemma 1.5. For $\sigma \in S_n(321)$, let $\mathcal{J}_n(\sigma) = \min\{j \geqslant 1 : \sigma \in S_n^{\text{b-irr};j}(321)\}$. Then by (2.4), we have

$$\mu_n^{321}(\sigma \in \mathcal{J}_n^{-1}(j)) = \frac{C_{j-1}C_{n-j}}{C_n}, \quad 1 \leqslant j \leqslant n, \ n \geqslant 1. \tag{7.1}$$

Also, by the considerations in the proof of Lemma 1.5, we have

$$\begin{aligned} &\mu_n^{321}(\sigma = \tau * \nu^{\text{im}} | \sigma \in \mathcal{J}_n^{-1}(j)) = \mu_j^{321;\text{b-irr}}(\tau)\mu_{n-j}^{321}(\nu^{\text{im}} - j), \text{ for } \tau \in S_j^{\text{b-irr}}(321) \\ &\text{and } \nu^{\text{im}} \text{ a 321-avoiding permutation image of } [j+1, n], \end{aligned} \tag{7.2}$$

where $\mu_j^{321;\,\text{b-irr}}$ denotes the uniformly probability measure on $S_j^{\text{b-irr}}(321)$.

From (7.1) and Lemma 2.2, it follows that

$$\lim_{n\to\infty} \mu_n^{321}(\sigma \in \mathcal{J}_n^{-1}(j)) = \frac{1}{2}P(\hat{X}_1 = j), \quad j = 1, 2, \ldots,$$
$$\lim_{M\to\infty} \lim_{n\to\infty} \mu_n^{321}(\sigma \in \mathcal{J}_n^{-1}([M, \infty)) = \frac{1}{2}.$$

Using this with (7.2) shows that with probability 1/2, the distribution of any weakly convergent subsequence of $\{\mu_n^{321}\}_{n=1}^{\infty}$ will begin with a segment whose distribution is that of $\Pi^{321;\,\text{b-irr}}_{[1,\hat{X}]}$, and alternatively, with probability 1/2, if a weakly convergent subsequence converges to a limiting distribution on S_∞, then that limiting distribution is supported on permutations with no irreducible block. Using regeneration and iterating the above procedure proves the proposition. □

8. Proof of Proposition 1.7

We have $T_n^X = \sum_{j=1}^{n} X_j$, where $\{X_n\}_{n=1}^{\infty}$ are i.i.d. with distribution given in (1.1). To prove the proposition, it suffices to show that

$$\lim_{n\to\infty} E\exp\left(-it\frac{T_n^X}{n^2}\right)$$

is equal to the characteristic function appearing in the statement of the proposition. We have

$$E\exp\left(-it\frac{T_n^X}{n^2}\right) = \left(E\exp\left(-i\frac{t}{n^2}X_1\right)\right)^n, \tag{8.1}$$

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and

$$E\exp(-isX_1) = \frac{1}{2}\sum_{m=0}^{\infty} e^{-ism}\frac{C_m}{4^m}. \tag{8.2}$$

By the remark after (1.1), it follows that

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{m=0}^{\infty} C_m z^m$$

defines an analytic function for $|z| < 1/4$, and that the equality continues to hold for $|z| = 1/4$, where

$$\sqrt{w} = |w|^{1/2}\exp\left(\frac{1}{2}i\mathrm{Arg}(w)\right),$$

for $\mathrm{Re}(w) > 0$ and $\mathrm{Arg}(w) \in (-\pi/2, \pi/2)$. Thus, from (8.2) with $s = t/n^2$, we have

$$E\exp\left(-i\frac{t}{n^2}X_1\right) = \frac{1-\sqrt{1-\exp(-i(t/n^2))}}{\exp(-i(t/n^2))}. \tag{8.3}$$

Writing

$$1-\exp\left(-i\frac{t}{n^2}\right) = 1-\cos\frac{t}{n^2} + i\sin\frac{t}{n^2},$$

we see that

$$1-\exp\left(-i\frac{t}{n^2}\right) = \frac{t^2}{n^4} + i\frac{t}{n^2} + O\left(\frac{1}{n^6}\right), \quad \text{as } n\to\infty.$$

Consequently,

$$\sqrt{1-\exp\left(-i\frac{t}{n^2}\right)} = (1+o(1))\frac{|t|^{1/2}}{n}\exp\left(i\,\mathrm{sgn}(t)\left(\frac{\pi}{4}+o(1)\right)\right), \quad \text{as } n\to\infty. \tag{8.4}$$

From (8.1), (8.3) and (8.4), we have

$$\begin{aligned}
&\lim_{n\to\infty} E\exp\left(-it\frac{T_n^X}{n^2}\right)\\
&= \lim_{n\to\infty}\exp\left(i\frac{t}{n}\right)\left(1-\frac{1}{n}(1+o(1))|t|^{1/2}\exp\left(i\,\mathrm{sgn}(t)\left(\frac{\pi}{4}+o(1)\right)\right)\right)^n\\
&= \exp\left(-|t|^{1/2}\exp\left(i\,\mathrm{sgn}(t)\frac{\pi}{4}\right)\right) = \exp\left(-\frac{\sqrt{2}}{2}|t|^{1/2}(1+i\,\mathrm{sgn}(t))\right).
\end{aligned} \tag{8.5}$$

□

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Cite this article: Pinsky RG (2020). The infinite limit of random permutations avoiding patterns of length three. *Combinatorics, Probability and Computing* **29**, 137–152. https://doi.org/10.1017/S0963548319000270