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Geometric Poisson brackets on Grassmannians and conformal spheres

M. Eastwood
Centre for Mathematics and Its Applications,
Mathematical Sciences Institute, Australian National University,
Canberra, ACT 0200, Australia (michael.eastwood@anu.edu.au)

G. Marí Beffa
Mathematics Department, University of Wisconsin, Madison,
Wisconsin 53706, USA (maribeff@math.wisc.edu)

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We relate the geometric Poisson brackets on the 2-Grassmannian in $\mathbb{R}^4$ and on the $(2, 2)$ Möbius sphere. We show that, when written in terms of local moving frames, the geometric Poisson bracket on the Möbius sphere does not restrict to the space of differential invariants of Schwarzian type. But when the concept of conformal natural frame is transported from the conformal sphere into the Grassmannian, and the Poisson bracket is written in terms of the Grassmannian natural frame, it restricts and results in either a decoupled system or a complexly coupled system of Korteweg–de Vries (KdV) equations, depending on the character of the invariants. We also show that the bi-Hamiltonian Grassmannian geometric brackets are equivalent to the non-commutative KdV bi-Hamiltonian structure. Both integrable systems and Hamiltonian structure can be brought back to the conformal sphere.

1. Introduction

Given a flat homogeneous space, one can define a Hamiltonian structure on the space of differential invariants (curvatures) of parametrized curves [14]. These structures are often linked to completely integrable partial differential equations and to their geometric realizations (invariant curve flows on the homogeneous space inducing the integrable system on its invariants). There has recently been a flurry of literature studying the existence of these integrable systems and their associated geometric flows (see, for example, [1, 3–5, 11, 17, 19, 20] and the references therein).

We shall say a differential invariant I is of Schwarzian type whenever

$$\phi^* I = ((\phi')^2 I) \circ \phi^{-1} + S(\phi) \circ \phi^{-1},$$

where $\phi^* I$ represents the pullback of I by a diffeomorphism ϕ and where $S(\phi)$ indicates the Schwarzian derivative of ϕ (the Schwarzian derivative itself behaves this way under reparametrization). In [15] Marí Beffa conjectured that the nature of the geometry (and its invariants) was linked to the type of integrable systems that could be realized. In particular, she conjectured that the existence of differential invariants

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of projective (or Schwarzian) type would result in geometric realizations of equations of Korteweg–de Vries (KdV) type (as it appeared in [13, 15], for example), while the existence of invariants of Riemannian type would result in geometric realizations of nonlinear Schrödinger (NLS), modified KdV (mKdV) and sine–Gordon equations (as in [1, 17, 20], for example). In particular, one can obtain a geometric realization of the KdV equation by flows in $\mathbb{RP}^1$, of generalized KdV equations by flows on $\mathbb{RP}^n$, of a system of complexly coupled KdV equations by conformal flows and of a decoupled system of KdV equations by a flow in the Lagrangian Grassmannian. Eastwood conjectured that this dichotomy might be related to the existence of preferred parametrizations, projective versus affine, as in [6]. Many of these geometric realizations are given by flows for which all non-Schwarzian invariants vanish or are constant (i.e. initial curves are restricted). That is, they are in fact completely integrable level sets of associated curve flows which are not completely integrable themselves. More interestingly, the existence of these integrable level sets is always linked to the reduction of a bi-Hamiltonian Poisson structure to the submanifold of vanishing non-Schwarzian invariants (as in [13, 15]).

Most of the above examples are particular cases of flat parabolic geometries, that is, homogeneous spaces of the form G/P, G semisimple and P a parabolic subgroup. (In fact, they are instances of parabolic geometries associated to $|1|$-gradings of the algebra.) One such case does not seem to behave in the way the other cases do, namely the spinor case $G = O(2n, 2n)$. Marí Beffa showed in [15] that, even though spinor curves do possess differential invariants of Schwarzian type, the geometric Poisson structure associated to flows of spinors does not reduce to the submanifold of vanishing non-Schwarzian invariants. Furthermore, the flow expected to possess a decoupled system of KdV equations as a level set does not preserve this submanifold, so that one cannot find a geometric realization of a system of decoupled KdV by flows of spinors.

In this paper we study the flat Grassmannian case of two-dimensional planes in $\mathbb{R}^4$. This homogeneous space can be identified with the homogeneous manifold $\mathrm{SL}(2+2, \mathbb{R})/P$, for a properly chosen parabolic subgroup P. The notation $\mathrm{SL}(2+2)$ refers to the action of $\mathrm{SL}(4)$ on the manifold, as shown in § 3. Since the manifold is flat, the Cartan connection of the manifold will be given by the Maurer–Cartan form (our results are local). As we shall see, the group $\mathrm{SL}(2 + 2, \mathbb{R})$ is a double cover if $\mathrm{O}(3, 3)$ and the cover induces an equivalency of parabolic geometries. Indeed, the oriented conformal sphere $\mathrm{SO}(3, 3)/P$ can also be viewed as the spin manifold $\mathrm{Spin}(3, 3)/P$, itself isomorphic to $\mathrm{SL}(2 + 2)/P$. At the infinitesimal level the isomorphism is given by an isomorphism of the Lie algebras and their associated gradations. Our original intention was to translate our knowledge of the conformal case into the Grassmannian. As it turned out, we also ended up learning more about the conformal case from the Grassmannian situation.

Moving frames and differential invariants for curves in Grassmannian manifolds $\mathrm{Gr}(p, q) = \mathrm{SL}(p + q)/P$, where P is a suitable chosen parabolic subgroup, are not well known in general. For the case $\mathrm{Gr}(nr, r)$, a special type of non-local invariants was found in [18]. These invariants correspond to a Laguerre–Forsyth canonical form for the Serret–Frenet equations, and we shall use them at the end of the paper. In § 3 we shall find a local moving frame along curves in $\mathrm{Gr}(2, 2)$ and we shall find the differential invariants they generate. We shall show that two of the four generating invariants are invariants of Schwarzian type. In theorem 3.8 we shall find explicitly the most general form of an invariant Grassmannian flow and we shall show that even those who have normalized coefficients do not preserve the submanifold of vanishing non-Schwarzian invariants; when these invariants vanish, the invariant flow blows up. This implies that the geometric Poisson bracket does not restrict to the space where the non-Schwarzian invariants vanish, much like the situation in the spinor case [15]. This seems to be somehow counterintuitive: integrable level sets do exist in the conformal sphere of signature $(n, 0)$ for which we can find a complexly coupled system of KdV equations.

We show that the Grasmannian problem lies in the choice of moving frame. For this we notice that a *local* choice (i.e. depending on the curves and its derivatives) of moving frame in the conformal sphere results in the same type of problem as the Grassmannian case had. On the other hand, if we choose a *natural moving frame*, a generalization of the non-local natural Euclidean frame, then the level set is preserved and both Hamiltonian structures can be reduced. In § 5 we define natural frames both for the conformal sphere of signature $(2, 2)$ and for Grassmannian curves. We then prove that we can find a Grassmannian geometric realization inducing a complexly coupled system of KdV equations on the Grassmannian curvatures of projective type. Furthermore, we show that there is also a geometric realization of a decoupled system of two KdV equations.

Finally, we show that, when written in terms of the moving frame generating the non-local invariants appearing in [18], the bi-Hamiltonian geometric structure *on the complete Grassmannian* is equal to the non-commutative KdV bi-Hamiltonian structure. We also show that the non-commutative KdV equation has a Grassmannian geometric realization. The non-commutative KdV equation and its bi-Hamiltonian structures were defined in [16]. Given the relation to the conformal sphere, these also produce conformal bi-Hamiltonian structures and a geometric realization for this system. The only conformal realizations that were previously known were those of the coupled KdV system. Using the isomorphism, we also prove that the Poisson brackets on the conformal $(2, 2)$ sphere are the non-commutative KdV structures.

2. Grassmannian-conformal parabolic equivalence

2.1. Description of the manifolds

Let us first realize $\mathrm{Gr}_2(\mathbb{R}^4)$ as the homogeneous space $\mathrm{SL}(4, \mathbb{R})/P_G$, where P_G is the parabolic subgroup of $\mathrm{SL}(4, \mathbb{R})$ defined by matrices of the form

$$\begin{pmatrix} A & \mathbf{0} \\ C & B \end{pmatrix},$$

where $A, B, C, \mathbf{0} \in M_{2\times 2}$ and where $\det A \det B = 1$. The subindex G in P_G indicates its association to the Grassmannian. Its Lie algebra $\mathfrak{p}_G$ is defined by similarly shaped matrices with vanishing trace. This quotient corresponds to a gradation of the algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, where $\mathfrak{p}_G = \mathfrak{g}_1 \oplus \mathfrak{g}_0$, and $\mathfrak{g}_{-1}$ is defined by the upper right block (the dual to $\mathfrak{g}_1$).

We now describe the conformal sphere with signature $(2,2)$. Let $J \in M_{6\times 6}$ be the matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

For the purposes of this paper, we shall define the group $\mathrm{SO}(3,3)$ as the identity component of the group $\mathrm{O}(3,3)$ realized as

$$\mathrm{O}(3,3) = \{A \in \mathrm{GL}(6,\mathbb{R}) \text{ such that } A^{\mathrm{T}} J A = J\}.$$

With this realization the Lie algebra will be given by matrices that are skew symmetric with respect to the secondary diagonal; that is, $X \in \mathfrak{so}(3,3)$ whenever $X^{\mathrm{T}} J + JX = 0$:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} & 0 & -x_{15} \\ x_{31} & x_{32} & x_{33} & 0 & -x_{24} & -x_{14} \\ x_{41} & x_{42} & 0 & -x_{33} & -x_{23} & -x_{13} \\ x_{51} & 0 & -x_{42} & -x_{32} & -x_{22} & -x_{12} \\ 0 & -x_{51} & -x_{41} & -x_{31} & -x_{21} & -x_{11} \end{pmatrix}. \tag{2.2}$$

Next, define P_C (the subindex C indicates its association with the conformal sphere) to be the parabolic subgroup of $\mathrm{SO}(3,3)$ given by the stabilizer of the line

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ * \end{pmatrix} \in \mathbb{R}^6$$

under the linear action of $\mathrm{SO}(3,3)$, i.e. the stabilizer of the base point in $\mathbb{RP}^5$ under the projective action. Its orbit is the non-singular quadric

$$Q \equiv \{[v] \in \mathbb{RP}_5 \text{ such that } v^{\mathrm{T}} J v = 0\} = \mathrm{SO}(3,3)/P_C.$$

The parabolic Lie algebra $\mathfrak{p}_C$ is given by those elements in $\mathfrak{so}(3,3)$, as in (2.2), for which $x_{1i} = 0$, $i = 2,3,4,5$. As before, the quotient is related to a gradation of the algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ with $\mathfrak{g}_{-1}$ dual to $\mathfrak{g}_1$ and $\mathfrak{p}_C = \mathfrak{g}_1 \oplus \mathfrak{g}_0$.

2.2. Isomorphism between the homogeneous manifolds

It is well known that there exists an isomorphism of homogeneous spaces

$$\mathrm{Gr}_2(\mathbb{R}^4) \cong Q.$$

The isomorphism is induced by a homomorphism at the Lie group level. Specifically, for $A \in \mathrm{SL}(4,\mathbb{R})$, define $\Phi(A) \in \mathrm{Hom}(\Lambda^2\mathbb{R}^4, \Lambda^2\mathbb{R}^4)$ by the usual induced action on simple vectors $v \wedge w$. That is,

$$\Phi(A)(v \wedge w) = Av \wedge Aw.$$

We can identify $\Lambda^2\mathbb{R}^4$ with $\mathbb{R}^6$ through the choice of basis

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_4 \wedge e_2, e_3 \wedge e_4\}.$$

Under this representation, $\Phi(A)$ is defined by an $\mathrm{SO}(3,3)$ matrix. Through straightforward calculations we see that if $A, D \in M_{2\times 2}$, $B, C \in M_{2\times 2}$ with $\det B \det C = 1$, then

$$\Phi\left(\begin{pmatrix} I & A \\ \mathbf{0} & I \end{pmatrix}\right) = \begin{pmatrix} 1 & a_{21} & a_{22} & -a_{11} & a_{12} & \det A \\ 0 & 1 & 0 & 0 & 0 & -a_{12} \\ 0 & 0 & 1 & 0 & 0 & a_{11} \\ 0 & 0 & 0 & 1 & 0 & -a_{22} \\ 0 & 0 & 0 & 0 & 1 & -a_{21} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

$$\Phi\left(\begin{pmatrix} I & \mathbf{0} \\ \mathcal{D} & I \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ d_{12} & 1 & 0 & 0 & 0 & 0 \\ d_{22} & 0 & 1 & 0 & 0 & 0 \\ -d_{11} & 0 & 0 & 1 & 0 & 0 \\ d_{21} & 0 & 0 & 0 & 1 & 0 \\ \det D & -d_{21} & d_{11} & -d_{22} & -d_{12} & 1 \end{pmatrix}, \tag{2.4}$$

$$\Phi\left(\begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix}\right) = \begin{pmatrix} \det B & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{11}c_{11} & b_{11}c_{12} & b_{12}c_{11} & -b_{12}c_{12} & 0 \\ 0 & b_{11}c_{21} & b_{11}c_{22} & b_{12}c_{21} & -b_{12}c_{22} & 0 \\ 0 & b_{21}c_{11} & b_{21}c_{12} & b_{22}c_{11} & -b_{22}c_{12} & 0 \\ 0 & -b_{21}c_{21} & -b_{21}c_{22} & -b_{22}c_{21} & b_{22}c_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \det C \end{pmatrix}. \tag{2.5}$$

The map Φ is a double cover of $\mathrm{SO}(3,3)$ by $\mathrm{SL}(4,\mathbb{R})$, which also maps P_G into P_C. Notice that Φ is a double cover on the parabolic subgroups, while it is one-to-one between the sections of $\mathrm{SL}(4,\mathbb{R})/P_G$ and $\mathrm{SO}(3,3)/P_C$ defined by (2.3). Therefore, the map induces the desired isomorphism between homogeneous spaces. Clearly, Φ induces a graded map at the Lie algebra level.

3. The local geometry of Grassmannian curves

Let $\mathrm{SL}(p+p) \subset \mathrm{GL}(p+p)$ be the simple linear group acting on $\mathrm{gl}(p)$ according to the action of $\mathrm{SL}(2p)$ on the homogeneous space $M = \mathrm{SL}(2p)/H$, where $H \subset \mathrm{SL}(2p)$ are matrices of the form

$$\begin{pmatrix} E & \mathbf{0} \\ C & D \end{pmatrix}$$

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with $E, C, D \in M_{p\times p}$. Assume we are in a neighbourhood of the identity and so we can locally factor an element $g \in \mathrm{SL}(2p)$ into the product

$$g = \begin{pmatrix} I & \mathbf{0} \\ Z & I \end{pmatrix} \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \begin{pmatrix} I & Y \\ \mathbf{0} & I \end{pmatrix}.$$

Then, a representation for the homogeneous manifold M is given by the section defined by matrices of the form

$$\begin{pmatrix} I & u \\ \mathbf{0} & I \end{pmatrix}.$$

Using this section we can write the $\mathrm{SL}(p+p)$ action on $\mathrm{gl}(p)$ as determined by the relation

$$g \begin{pmatrix} I & u \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} I & g\cdot u \\ \mathbf{0} & I \end{pmatrix} h,$$

where $h \in H$. This relation determines the action uniquely to be

$$g \cdot u = A(u+Y)(B + ZA(u+Y))^{-1}. \tag{3.1}$$

3.1. Group-based moving frames for Grassmannian generic curves

In this section we shall use the normalization method described in [7] to find a group-based moving frame along generic parametrized curves in the manifold $\mathrm{Gr}_2(\mathbb{R}^4)$. Let $J^{(k)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$ be the k-jet space of curves in $\mathrm{Gr}_2(\mathbb{R}^4)$, i.e. the set of equivalence classes of curves in $\mathrm{Gr}_2(\mathbb{R}^4)$ up to k-contact order. An m-order left (respectively, right) group-based moving frame is a map

$$\rho\colon J^{(m)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4)) \to \mathrm{SL}(2p)$$

equivariant with respect to the prolonged action of $\mathrm{SL}(2p)$ on $J^{(k)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$ and the left (respectively, right) action of $\mathrm{SL}(2p)$ on itself. Let $u_r = \mathrm{d}^r u/\mathrm{d}x^r$, where x is the parameter. In the case at hand, the prolonged action is defined by the relation

$$g \cdot u_r = (g \cdot u)_r,$$

where the right-hand side represents the formula given by differentiating r times the action $g \cdot u$, and writing it in terms of $x, u, u_1, \dots, u_r$, the coordinates in $J^{(k)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$, $r \leqslant k$.

In order to find a left-moving frame ρ along a generic curve $u(x)$, we shall normalize the prolonged action of $\mathrm{SL}(2p)$ on $J^{(m)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$, up to a certain order m. The choice of normalization (which defines a cross-section on the prolonged orbits of the group) is not unique and can be arbitrary as far as we retain full rank. If ρ is a left-moving (respectively, right-moving) frame, we call $K = \rho^{-1}\rho_x$ (respectively, $K = \rho_x\rho^{-1}$) its associated *Maurer–Cartan matrix*. A theorem by Hubert [9] states that if ρ is found via a normalization process, the entries of K and their derivatives functionally generate all other differential invariants of the curve. Different choices of normalization will give rise to different Maurer–Cartan matrices and different invariants. Our particular choices are made seeking both simplicity and a direct relation between the invariants (to this end the normalization constants are coordinated in both examples). Simplicity is important; a complicated Maurer–Cartan matrix will result in a difficult Hamiltonian study.

At each step we shall normalize fully, i.e. we shall normalize as many terms as permitted by the rank of the action. The terms that cannot be normalized will be differential invariants of the action. This process will determine an element g completely in terms of u and its derivatives. It is known [7] that $\rho^{-1} = g$ found through this process is a right-moving frame. A left-moving frame is given by its inverse ρ.

We proceed to determine the (right) frame for the case at hand.

STEP 1 (zeroth normalization equation). For first normalization constant we shall choose $i_0 = 0$. The first normalization equations will be equations of zero differential order:

$$g \cdot u = A(u + Y)(B + ZA(u + Y))^{-1} = i_0 = 0,$$

which are satisfied by the choice $Y = -u$. We have no zero-order differential invariants.

STEP 2 (first normalization equation). The next equations are the first-order normalization equations $g \cdot u_1 = i_1$. We shall make the normalization choice $i_1 = I$. After substituting the previous normalization choice ($u + Y = 0$), the equation becomes

$$\begin{aligned} g \cdot u_1 &= Au_1(B + ZA(u + Y))^{-1} \\ &\qquad - A(u + Y)(B + ZA(u + Y))^{-1}Z(B + ZA(u + Y))^{-1} \\ &= Au_1B^{-1} = i_1 = I. \end{aligned} \tag{3.2}$$

This equation is satisfied by the choice

$$A = Bu_1^{-1}.$$

Since $g \in \mathrm{SL}(2p)$, we also have $\det A \det B = 1$ and so

$$\det B = (\det u_1)^{1/2}.$$

As expected, we have no first-order differential invariants. Let us define

$$F = (B + ZA(u + Y)).$$

STEP 3 (second normalization equation). After differentiating again, the second-order normalizations are found by substituting previous normalization values (in this case $u + Y = 0$ and $A = Bu_1^{-1}$) and making the result equal to a constant i_2. In this case we choose $i_2 = 0$. That is

$$\begin{aligned} g \cdot u_2 &= Au_2F^{-1} - 2Au_1F^{-1}ZAu_1F^{-1} \\ &\qquad - A(u + Y)F^{-1}Z(Au_2F^{-1} - 2Au_1F^{-1}ZAu_1F^{-1}) \\ &= Bu_1^{-1}u_2B^{-1} - 2Z = i_2 = 0. \end{aligned} \tag{3.3}$$

This is solved with the choice

$$Z = \tfrac{1}{2}Bu_1^{-1}u_2B^{-1}$$

and we have no second-order invariants. At this point it remains only to determine B (although not its determinant).

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Step 4 (third normalization equation). The third-order normalization equations are, after some simplification, given by

$$g \cdot u_3 = B(u_1^{-1}u_3 - \tfrac{3}{2}u_1^{-1}u_2u_1^{-1}u_2)B^{-1} = i_3. \tag{3.4}$$

Definition 3.1. We define

$$S(u) = u_1^{-1}u_3 - \tfrac{3}{2}u_1^{-1}u_2u_1^{-1}u_2,$$

the *Schwarzian derivative of the Grassmannian curve* u.

The equation $BS(u)B^{-1} = i_3$ does not have full rank on B for any choice of i_3; we can at most reduce $S(u)$ to a certain normal form under conjugation. Let us assume now that $p = 2$. The action $U \to BUB^{-1}$ has two invariants, namely, the determinant and trace of U. That means that the rank of this action is 2 and we shall be able to use at most two third-order normalization equations. Therefore, there will be two third-order differential invariants given by the entries that cannot be further normalized. Notice that these are Grassmannian invariants of Schwarzian type.

The following result is a consequence of theorems that can be found, for example, in [8].

Proposition 3.2. *Two generating and (functionally) independent third-order differential invariants for a Grassmannian curve* u *in* M *are given by the determinant and the trace of its Schwarzian derivative. That is, any other third-order differential invariant of* u *must be a function of these two.*

There are many possible choices for i_3. Our choice will be to normalize i_3 depending on the nature of its eigenvalues (real or complex). Let us write

$$S(u) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In this case i_3 is given by

$$\begin{aligned} i_3 &= BS(u)B^{-1} \\ &= (\det u_1)^{-1/2} \begin{pmatrix} d(as_1 + bs_3) - c(as_2 + bs_4) & -b(as_1 + bs_3) + a(as_2 + bs_4) \\ d(cs_1 + ds_3) - c(cs_2 + ds_4) & -b(cs_1 + ds_3) + a(cs_2 + ds_4) \end{pmatrix}. \end{aligned}$$

Let us define $a/b = \alpha$ and $c/d = \beta$.

In the generic case of real eigenvalues, diagonalizing i_3 is equivalent to solving the same equation for both α and β, namely

$$P(\alpha) = \alpha^2 s_2 + \alpha(s_4 - s_1) - s_3 = 0, \qquad P(\beta) = \beta^2 s_2 + \beta(s_4 - s_1) - s_3 = 0. \tag{3.5}$$

Since $\det B \neq 0$, we need this equation to have two different solutions and we need α and β to be such solutions (the discriminant condition on this equation is the same as that for the characteristic equation for $S(u)$). Therefore,

$$i_3 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

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In the generic case of complex eigenvalues, the normal form will be

$$\begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix} \tag{3.6}$$

and the corresponding conditions to achieve it are

$$2\alpha\beta s_2 + (\beta+\alpha)(s_4 - s_1) - 2s_3 = 0, \qquad b^2 P(\alpha) = d^2 P(\beta), \tag{3.7}$$

where $P(\alpha)$ and $P(\beta)$ are given as in (3.5).

In both cases, the condition

$$(\alpha - \beta) bd = \det B = (\det u_1)^{1/2} \tag{3.8}$$

together with the two equations solve for a, c and d in terms of b in the real case, or for α, b and d in terms of β in the complex case. The real case is straightforward; the complex case might require a little more explanation. From (3.7) we obtain

$$\alpha = \frac{\beta(s_4 - s_1) - 2s_3}{s_1 - s_4 - 2\beta s_2} \tag{3.9}$$

and

$$b^4 = \frac{1}{4}\frac{(2\beta s_2 + s_4 - s_1)^2 \det u_1}{P(\alpha)P(\beta)}. \tag{3.10}$$

Using (3.9), one can see that

$$P(\alpha) = -\frac{P(\beta)}{(2\beta s_2 + s_4 - s_1)^2}\Delta \tag{3.11}$$

where $\Delta = (s_4 - s_1)^2 + 4s_3 s_2$ is the discriminant of the characteristic polynomial of $S(u)$, and hence negative in our case. Therefore,

$$b^4 = \frac{1}{4}\frac{-\Delta(2\beta s_2 + s_4 - s_1)^4 \det u_1}{P(\beta)^2}.$$

From (3.7) and (3.11), we also get

$$d^4 = -\frac{1}{4}\frac{\Delta^3 \det u_1}{P(\beta)^2}. \tag{3.12}$$

Thus, the only condition we need to be able to solve for b and d is $\det u_1 > 0$, which was required early on, in view of (3.8).

Step 5 (fourth normalization equation). Finally, if we write the fourth-order normalization equations and we use the previous normalization we shall also obtain an equation of the form

$$BR(u)B^{-1} = i_4, \tag{3.13}$$

for some fourth-order matrix $R(u)$ involving derivatives of u, but not B. We do not really need its explicit expression here, but we need to choose the last normalization equation carefully so as to simplify later calculations.

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Real case. Assume first that we are in the real case. If we denote $R(u)$ by

$$R(u) = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix},$$

then normalizing its $(1,2)$ entry, for example, will give us the remaining missing equation. Although one usually chooses constants to normalize, we can also choose expressions that depend on invariants, as we simply need a section of the prolonged orbit [7]. The resulting generation of invariants will remain unchanged, although the generators will be different. Assume that we have diagonalized the Schwarzian derivative and it is given by

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Generically, $k_1 \neq k_2$ and we can assume that $k_1 - k_2$ will not vanish near the base point. Therefore, we choose the last normalization equation to be

$$(\det u_1)^{-1/2} b^2 (\alpha^2 r_2 + \alpha(r_4 - r_1) - r_3) = k_2 - k_1. \tag{3.14}$$

(Notice that zero is a singular value for the normalization.) Recall that $k_2 - k_1 = \pm(\Delta)^{1/2}$, depending on the case. We can exchange k_1 and k_2 if this equation cannot be solved locally. (In fact, we can also choose any multiple of $k_2 - k_1$ instead of $k_2 - k_1$ and obtain similar results.) Thus, with a proper choice, this equation can be generically solved.

Complex case. Assume now that we are in the complex case. In order to find β we can choose several different normalizations. The one we shall choose here is to make the $(1,1)$ entry of the equation equal to the $(2,2)$ entry. The corresponding equation is

$$\alpha\beta r_2 + \tfrac{1}{2}(\alpha + \beta)(r_4 - r_1) + r_3 = 0.$$

After substituting the value of α and simplifying, we obtain a quadratic equation for β, which can be solved locally in generic cases. As before, $k_1 = \frac{1}{2}(s_1 + s_4)$ and $k_2 = \pm\frac{1}{2}(-\Delta)^{1/2}$, depending on the case. Different normalizations will be needed for those generic cases for which the equation cannot be solved. Later on we shall need to change this frame into a more convenient one.

This completes the calculation of the moving frame and we can now find its associated Maurer–Cartan matrix. Unlike in the classical case, the entries of the Maurer–Cartan matrix will produce a complete set of independent and generating differential invariants [9]. However, before doing this we shall write a final description of the generating invariants that can be obtained directly from this normalization.

THEOREM 3.3. *A generic curve in* $\mathrm{Gr}_2(\mathbb{R}^4)$ *has a system of four functionally independent and generating differential invariants, two of order* 3 *and one of each order* 4 *and* 5.

Proof. Standard arguments that can be found in [7] tell us that, in the case of real Schwarzian eigenvalues, the diagonal of i_4 in (3.13) equals the derivative of the

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diagonalization of $S(u)$. In the complex case, the same arguments show that if

$$i_3 = \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix},$$

then i_4 in (3.13) will be of the form

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} (k_1)_x & -(k_2)_x \\ (k_2)_x & (k_1)_x \end{pmatrix}.$$

In the real case, (3.13) adds one independent fourth-order invariant to our group of third-order ones, namely the $(2, 1)$ entry of (3.13) (which has not been normalized). In the complex case, our normalization implies $\alpha_1 = 0$. Hence, β_1 will be the additional functionally independent fourth-order invariant.

One more fifth-order invariant exists: one that is not a function of the previous three invariants and their derivatives. The description of this generator can be found by following the normalization process in [7]. If we write the fifth-order normalization equations as above, there is one entry that corresponds to a previously normalized entry in i_4, namely $(1, 2)$ in the real case and $(2, 1)$ in the complex one. That entry is an independent fifth-order differential invariant. □

The explicit expression is not relevant. Indeed, we shall make use of a different, but more convenient, choice.

3.2. Grassmannian Maurer–Cartan matrix associated to the left-moving frame

Definition 3.4. Let ρ be a left-moving frame. The matrix

$$K = \rho^{-1}\rho_x$$

is called the (left) Maurer–Cartan matrix associated to ρ (it is the horizontal component of the pullback by ρ of the Maurer–Cartan form of the group G). The entries of K are clearly differential invariants; theorem 4.2 in [9] and its corollary state that they generate all other differential invariants for the curve. Since the right-moving frame associated to a left one is its inverse, the right Maurer–Cartan matrix is the negative of the left one.

Some of the work in [7] provides what are normally called horizontal recurrence formulae. These formulae can be used to recurrently calculate the entries of K. The following proposition is a reformulation of these recurrence formulae for the case at hand.

Proposition 3.5. *Let K be the left Maurer–Cartan matrix associated to a moving frame ρ. Assume that $\rho \cdot u_s = i_s$, $s = 0, 1, 2, \dots$. Then*

$$K \cdot i_s = i_{s+1} - (i_s)_x, \tag{3.15}$$

$s = 0, 1, 2, \dots$, where the dot in $K \cdot i_s$ represents the sth infinitesimal prolonged action of the Lie algebra on the element of the jet space i_s.

We shall use this recurrence relation to find the left Maurer–Cartan matrix associated to the left-moving frame $\rho = g^{-1}$, where g was found in our previous section. Denote the Maurer–Cartan matrix by

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

If

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \in \mathfrak{g},$$

we can calculate the infinitesimal version of (3.1) to be

$$V \cdot u = V_{11}u - uV_{22} - uV_{21}u + V_{12}.$$

Given that $i_0 = 0$ and $i_1 = I$ (see (3.2)), for $s = 0$ (3.15) becomes

$$K \cdot \mathbf{0} = K_{12} = I.$$

The infinitesimal version of the action in (3.2) is given by

$$V \cdot u^{(1)} = V_{11}u_1 - u_1V_{22} - u_1V_{21}u - uV_{21}u_1.$$

Now, $i_2 = 0$, so for $s = 1$, (3.15) becomes

$$K \cdot i_1 = K_{11} - K_{22} = i_2 - (i_1)_x = 0, \quad K_{11} = K_{22}.$$

Again, the infinitesimal version of the action in (3.3) is given by

$$V \cdot u^{(2)} = V_{11}u_2 - u_2V_{22} - 2u_1V_{21}u_1 + R_2,$$

where R_2 are terms that vanish whenever $u = 0$. From here, when $s = 2$, (3.15) becomes

$$K \cdot i_2 = -2K_{21} = i_3, \quad K_{21} = -\tfrac{1}{2}i_3.$$

Finally, the infinitesimal version of the action in (3.4) is given by

$$V \cdot u^{(3)} = V_{11}u_3 - u_3V_{22} - 3(u_2V_{21}u_1 + u_1V_{21}u_2) + R_3,$$

where R_3 vanishes whenever $u = 0$. Therefore, for $s = 3$, (3.15) becomes

$$K_{11}i_3 - i_3K_{11} = i_4 - (i_3)_x. \tag{3.16}$$

Real case. In the case of real Schwarzian eigenvalues, both sides of (3.16) have a vanishing diagonal, and so the off-diagonal entries of K_{11} are determined by the entries of i_4. That is, if

$$K_{21} = -\tfrac{1}{2}i_3 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

then straightforward calculations show that the off-diagonal entries of K_{11} are given by

$$\begin{pmatrix} 0 & 1 \\ \kappa_3 & 0 \end{pmatrix},$$

where κ_3 is an independent fourth-order differential invariant determined by the one appearing in i_4 through (3.16). One can find κ_3 explicitly, but we do not really need its expression here and we shall change frames soon. Notice that the choice of $k_2 - k_1$ in the normalization of (3.14) implies that the $(1,2)$ entry of K_{11} is constant and equal to 1. Recall that zero was a singular value.

We do not need to go to the fourth order. Indeed, since $K_{11} = K_{22}$, we do know that the traces of both matrices vanish. Therefore, only one unknown entry of K remains, namely the one determining its trace. On the other hand, we do know that the entries of K generate all differential invariants, and there is one fifth-order invariant that is unaccounted for. Therefore, we can conclude that the missing entry, κ_4, is such a generator without actually calculating it explicitly. If we were to need its explicit formula, we could use the recurrence formulae further to find them. Thus, in the real case

$$K_{11} = \begin{pmatrix} \kappa_4 & 1 \\ \kappa_3 & -\kappa_4 \end{pmatrix}.$$

Complex case. In the case of complex Schwarzian eigenvalues

$$i_3 = \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}, \qquad K_{21} = -\tfrac{1}{2} i_3 = \begin{pmatrix} \kappa_1 & -\kappa_2 \\ \kappa_2 & \kappa_1 \end{pmatrix}.$$

As we pointed out before, i_4 can be split into

$$i_4 = \begin{pmatrix} \alpha_1 + (k_1)_x & -(k_2)_x \\ (k_2)_x & -\alpha_1 + (k_1)_x \end{pmatrix},$$

since, with our normalization, $\beta_1 = 0$. If $K_{11} = (k_{ij})$, then (3.16) becomes

$$\begin{pmatrix} k_2(k_{12} + k_{21}) & -2k_{11}k_2 \\ -2k_2k_{11} & -k_2(k_{12} + k_{21}) \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 \\ \beta_1 & 0 \end{pmatrix}$$

and so

$$K_{11} = \begin{pmatrix} \kappa_3 & -c \\ c & -\kappa_3 \end{pmatrix},$$

where c is still to be found and κ_3 is a fourth-order invariant obtained from α_1. As before, c needs to be the missing fifth-order generator κ_4, since it is the only undetermined entry. Finally,

$$K_{11} = \begin{pmatrix} \kappa_3 & -\kappa_4 \\ \kappa_4 & -\kappa_3 \end{pmatrix}.$$

We have finally proved the following theorem.

THEOREM 3.6. *Let ρ be the left-moving frame determined in the previous section. Its Maurer–Cartan matrix is given by*

$$K = \rho^{-1}\rho_x = \begin{pmatrix} \kappa_4 & 1 & 1 & 0 \\ \kappa_3 & -\kappa_4 & 0 & 1 \\ \kappa_1 & 0 & \kappa_4 & 1 \\ 0 & \kappa_2 & \kappa_3 & -\kappa_4 \end{pmatrix} \tag{3.17}$$

in the generic case of real eigenvalues, and by

$$K = \begin{pmatrix} \kappa_3 & -\kappa_4 & 1 & 0 \\ \kappa_4 & -\kappa_3 & 0 & 1 \\ \kappa_1 & -\kappa_2 & \kappa_3 & -\kappa_4 \\ \kappa_2 & \kappa_1 & \kappa_4 & -\kappa_3 \end{pmatrix} \tag{3.18}$$

in the generic case of complex eigenvalues. In both cases κ_1 and κ_2 are third-order differential invariants, κ_3 is fourth order and κ_4 is fifth order. The invariants κ_i, $i = 1, \dots, 4$, form a generating system of independent differential invariants for generic curves of Grassmannians.

3.3. Invariant evolutions of Grassmannian curves

Once a group-based moving frame has been found, lemma 3 in [14] provides us with a classical moving frame (an invariant curve in the frame bundle over the curve). This classical frame can be used to write the most general form of an invariant evolution of curves in M, i.e. an evolution for which the group takes solutions to solutions. The following theorem explains what that evolution is in our case. The theorem is a direct consequence of the results in [14].

THEOREM 3.7. *Let $\rho = g^{-1}$ be the left-moving frame obtained in § 3.1, and let B be the matrix defining g. Then, the most general form of an evolution of Grassmannian curves, invariant under the action (3.1), is given by*

$$u_t = u_1 B^{-1} \boldsymbol{r} B, \tag{3.19}$$

where $\boldsymbol{r}$ is any 2×2 matrix depending on κ_i, $i = 1, \dots, 4$, and their derivatives.

This expression is the particular form in our example of the more general formula for $|1|$-graded Lie algebras $\rho_{-1}^{-1}(\rho_{-1})_t = \mathrm{Ad}(\rho_0)(\boldsymbol{r})$, where $\boldsymbol{r} \in \mathfrak{g}_{-1}$ is an element of $\mathfrak{g}_{-1}$ depending on differential invariants of the curve, and where $\rho = \rho_{-1}\rho_0\rho_1$ is a left-moving frame factored out following the gradation, with ρ_{-1} defining our section (for more details see [14]). In our case

$$\rho_{-1} = \begin{pmatrix} I & u \\ \mathbf{0} & I \end{pmatrix}, \qquad \rho_0 = \begin{pmatrix} u_1 B^{-1} & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} I & \mathbf{0} \\ -Z & I \end{pmatrix}.$$

The matrix $\boldsymbol{r}$ has a Hamiltonian interpretation that will be explained in the next section. Because of the Hamiltonian interpretation, the following theorems are of interest to us. They describe the behaviour of evolutions as non-Schwarzian invariants κ_3, κ_4 vanish, with $\boldsymbol{r}$ having the same normal form as $S(u)$. Since $BS(u)B^{-1} = i_3 = -2K_{12}$, the particular choice $\boldsymbol{r} = K_{12}$ corresponds to the flow

$$u_t = u_1 B^{-1} K_{12} B = -\tfrac{1}{2} u_1 S(u),$$

which is a generalization of the well-known *KdV–Schwarzian evolution* to Grassmannian flows.

THEOREM 3.8. *Assume that $S(u)$ has real eigenvalues and that $\boldsymbol{r}$ is a diagonal matrix. Then, the level set $\kappa_3 = 0$ is invariant under evolution (3.19).*

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For general diagonal $\boldsymbol{r}$*, and in particular when* $\boldsymbol{r} = K_{12}$*, the level set* $\kappa_4 = 0$ *is not invariant under evolution (3.19).*

Assume that $S(u)$ *has complex eigenvalues, and that*

$$\boldsymbol{r} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is in normal form. Then, for general $\boldsymbol{r}$ *(and in particular if* $\boldsymbol{r} = K_{12}$*), the flow blows up as* $\kappa_3 \to 0$*. The level set* $\kappa_4 = \kappa_3 = 0$ *is, therefore, not preserved.*

The relevance of the second part of this proposition will be better understood in our next section. The situation described in this theorem is very similar to the one for spinor curves [15] for which it was proved that *any* choice of local moving frame would induce this situation.

Proof. Let $u(t, x)$ be a flow solution of (3.19) with $\boldsymbol{r}$ diagonal. Theorem 9 in [14] (or a straightforward calculation) shows that $N = \rho^{-1}\rho_t$ is of the form

$$N = \begin{pmatrix} N_{11} & \boldsymbol{r} \\ N_{21} & N_{22} \end{pmatrix}.$$

Furthermore, since $\mathrm{d}/\mathrm{d}x$ and $\mathrm{d}/\mathrm{d}t$ commute, the compatibility of $\rho_x = \rho K$ and $\rho_t = \rho N$ implies that

$$K_t = N_x + [K, N].$$

This breaks up into a number of individual equations (according to the gradation of the algebra), namely

$$0 = \boldsymbol{r}_x + N_{22} - N_{11} + [K_{11}, \boldsymbol{r}], \tag{3.20}$$

$$0 = (N_{11} - N_{22})_x + [K_{11}, N_{11} - N_{22}] - \boldsymbol{r}K_{21} - K_{21}\boldsymbol{r} + 2N_{21}, \tag{3.21}$$

$$(K_{11})_t = (N_{11})_x + [K_{11}, N_{11}] - \boldsymbol{r}K_{21} + N_{21}, \tag{3.22}$$

$$(K_{21})_t = (N_{21})_x + K_{21}N_{11} - N_{22}K_{21} + [K_{11}, N_{21}]. \tag{3.23}$$

Real case. Assume now that $S(u)$ has real eigenvalues and that

$$\boldsymbol{r} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

is diagonal. Let us define

$$N_{11} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \qquad N_{22} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then, using (3.20) we obtain

$$N_{11} - N_{22} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{pmatrix} = \begin{pmatrix} (r_1)_x & r_2 - r_1 \\ \kappa_3(r_1 - r_2) & (r_2)_x \end{pmatrix}. \tag{3.24}$$

Using (3.21) we have

$$N_{21} = \begin{pmatrix} -\frac{1}{2}r_1'' + \kappa_1 r_1 & (r_1 - r_2)' + \kappa_4(r_1 - r_2) \\ (\frac{1}{2}(\kappa_3)' - \kappa_4\kappa_3)(r_1 - r_2) + \kappa_3(r_1 - r_2)' & -\frac{1}{2}r_2'' + r_2\kappa_2 \end{pmatrix}. \tag{3.25}$$

From here, (3.23) becomes

$$\begin{pmatrix} (\kappa_1)_t & 0 \\ 0 & (\kappa_2)_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}r_1''' + (r_1\kappa_1)' + \kappa_1 r_1' & \kappa_1 b_1 - \kappa_2 b_2 + \frac{3}{2}(r_1 - r_2)'' + \kappa_2 r_2 - \kappa_1 r_1 \\ \kappa_2 c_1 - \kappa_1 c_2 & -\frac{1}{2}r_2''' + (r_2\kappa_2)' + \kappa_2 r_2' \end{pmatrix} + X_1, \tag{3.26}$$

where X_1 is a matrix whose explicit expression can be found directly, and whose relevant property is that it is well defined for κ_3 and κ_4 sufficiently small; X_1 becomes strictly upper triangular as $\kappa_3 \to 0$, and $X_1 \to 0$ as $\kappa_3, \kappa_4 \to 0$.

Therefore, (3.26) determines completely the evolution of κ_1 and κ_2. Notice that the choice $r_i = \kappa_i$ produces a decoupled system of KdV equations in κ_1 and κ_2 in the limit. From (3.24) we have that, as $\kappa_3 \to 0$, $c_1 = c_2$, which, together with (3.26), gives us, as $\kappa_3 \to 0$ (and as long as $\kappa_1 - \kappa_2 \neq 0$),

$$c_1 = c_2 = 0.$$

Equations (3.24) and (3.26) also provide the explicit expressions for b_1 and b_2:

$$b_1 = -\frac{3}{2}\frac{(r_1 - r_2)''}{\kappa_1 - \kappa_2} + \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} r_1 - 2\frac{\kappa_2}{\kappa_1 - \kappa_2} r_2 + Y_1, \tag{3.27}$$

$$b_2 = -\frac{3}{2}\frac{(r_1 - r_2)''}{\kappa_1 - \kappa_2} + 2\frac{\kappa_1}{\kappa_1 - \kappa_2} r_1 - \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} r_2 + Y_2, \tag{3.28}$$

where $Y_i \to 0$ as $\kappa_4 \to 0$. With these values, (3.22) proves the proposition. The entry $(2, 1)$ of (3.22) shows $(\kappa_3)_t = 0$ as $\kappa_3 \to 0$ (independently of κ_4), and hence $\kappa_3 = 0$ is preserved by the evolution. Finally, the $(1, 2)$ entry of (3.22) will determine the value of $d_1 - a_1$, and we know that $a_1 - a_2 = (r_1)_x$, $d_1 - d_2 = (r_2)_x$ and $a_1 + a_2 + d_1 + d_2 = 0$ (from the algebra condition). These relations completely determine all values of N. If we further assume that $\kappa_4 \to 0$, after straightforward calculations the remaining values show that

$$(\kappa_4)_t = \tfrac{1}{2}b_1'' + \tfrac{1}{4}(r_1'' - r_2'')$$

and so $\kappa_4 = 0$ is not, in general, preserved, even in the case $r_i = \kappa_i$.

Complex case. Assume now that $S(u)$ has complex eigenvalues, and assume that

$$\boldsymbol{r} = \begin{pmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{pmatrix}.$$

Using the same equations as in the real case, we obtained the values

$$N_{11} - N_{22} = \begin{pmatrix} r_1' + r_2 & -r_2' - 2\kappa_3 r_2 \\ r_2' - 2\kappa_3 r_2 & r_1' - r_2 \end{pmatrix},$$

$$N_{21} = \begin{pmatrix} -\frac{1}{2}r_1'' - r_2' + r_1\kappa_1 - r_2\kappa_2 & \frac{1}{2}r_2'' - (r_1\kappa_2 + r_2\kappa_1) \\ -\frac{1}{2}r_2'' + r_2\kappa_1 + r_1\kappa_2 & -\frac{1}{2}r_1'' + r_2' + r_1\kappa_1 - r_2\kappa_2 \end{pmatrix} + Z_1,$$

where

$$Z_1 = \begin{pmatrix} -2\kappa_3\kappa_4 r_2 + \kappa_3 r_2 & -r_2\kappa_4 + \kappa_3(r_2' + 2\kappa_3 r_2) + (r_2\kappa_3)' \\ -r_2\kappa_4 + \kappa_3(r_2' - 2\kappa_3 r_2) + (\kappa_3 r_2)' & 2\kappa_3\kappa_4 r_2 - \kappa_3 x r_2 \end{pmatrix}.$$

The t-evolution of K_{12} in (3.20) imposes some conditions on the entries of left-hand side, namely the $(1,1)$ and $(2,2)$ entries must be equal, and the $(1,2)$ and $(2,1)$ entries must be the negative of each other. This leads to solving for $c_1 + b_1$ and for $a_1 - d_1$, which are determined by

$$\kappa_2(c_1 + b_1) = z_1, \qquad \kappa_2(a_1 - d_1) = z_2,$$

where z_1 and z_2 are well defined as κ_3 and κ_4 vanish, and they both vanish in the limit. Recall that the trace of N is zero. Therefore, $a_1 + d_1 + a + 2 + d_2 = 2(a_1 + d_1) - 2r_1' = 0$. From here $a_1 + d_1 = r_1'$ and, in the limit, $a_1 = \frac{1}{2}r_1'$. From here we obtain the evolutions for κ_1 and κ_2, namely

$$\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}D^3 + D\kappa_1 + \kappa_1 D & -D\kappa_2 - \kappa_2 D \\ D\kappa_2 + \kappa_2 D & -\frac{1}{2}D^3 + D\kappa_1 + \kappa_1 D \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad (3.29)$$

where, again, y_1 and y_2 are well defined and vanish as κ_3, κ_4 vanish. In the limit, the evolution becomes a complexly coupled system of KdV equations, evolution that also appears in the conformal case. Still, such a limit creates a singularity in the flow. A final condition is imposed by the evolution of K_{11} in (3.22), where the sum of the $(1,2)$ and $(2,1)$ entries vanishes. The condition gives us an equation that allows us to finally solve for c_1 and b_1 (we can solve for all the other ones with the data we have up to this point):

$$(c_1 + b_1)' + 2\kappa_3(b_1 - c_1) - 2(d_1 - a_1)\kappa_4 - 2\kappa_4 r_2 + 2\kappa_3 r_2' + 2(\kappa_3 r_2)' = 0.$$

The expressions for $d_1 - a_1$ and $c_1 + b_1$ tell us that b_1 and c_1 are not well defined in the limit. Therefore, the t-evolution of κ_4 also blows up as $\kappa_3 \to 0$. Finally, one can check that the level set $\kappa_3 = 0$ is preserved and $(\kappa_3)_t = 0$ as $\kappa_4 \to 0$. We leave the rest of the details to the reader. □

4. Geometric Hamiltonian structures

There are two well-known Poisson structures defined on the space of loops on the dual of a semisimple Lie algebra. If $\mathcal{H}, \mathcal{F} \colon \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ are two operators, their Poisson brackets are given by new operators on $\mathcal{L}\mathfrak{g}^*$ defined as

$$\{\mathcal{H}, \mathcal{F}\}(L) = \int_{S^1} \left\langle B\left(\frac{\delta\mathcal{H}}{\delta L}\right)_x + \mathrm{ad}^*\left(\frac{\delta\mathcal{H}}{\delta L}\right)(L), \frac{\delta\mathcal{F}}{\delta L}\right\rangle \mathrm{d}x, \qquad (4.1)$$

$$\{\mathcal{H}, \mathcal{F}\}_0(L) = \int_{S^1} \left\langle \mathrm{ad}^*\left(\frac{\delta\mathcal{H}}{\delta L}\right)(L_0), \frac{\delta\mathcal{F}}{\delta L}\right\rangle \mathrm{d}x, \qquad (4.2)$$

where B is any bilinear identification of $\mathfrak{g}$ with its dual and where $L_0 \in \mathfrak{g}$ is any constant element. The elements $\delta\mathcal{H}/\delta L, \delta\mathcal{F}/\delta L \in \mathfrak{g}$ are the variational derivatives at L and $\langle\cdot,\cdot\rangle$ is the invariant pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. The usual choice is to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the Killing form, or, for example, the trace of the product

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if $\mathfrak{g} \subset \mathrm{gl}(n,\mathbb{R})$. With these standard choices, if $\mathfrak{g} \subset \mathrm{gl}(n,\mathbb{R})$, then the dual to the entry (i,j) is given by the entry (j,i) (or a multiple of it).

For a given choice of normalization equations, let us denote the space of Maurer–Cartan matrices by $\mathcal{K}$. Marí Beffa [14] showed that, locally around a generic curve $u(x) \in G/H$ with G semisimple, the space $\mathcal{K}$ can be written as a quotient $U/\mathcal{L}H$, where $U \subset \mathcal{L}\mathfrak{g}^*$ is open. She also showed that the Poisson bracket (4.1) could then be reduced to the quotient to produce what she called a geometric Poisson structure, a Hamiltonian structure on the space of differential invariant of curves. The Poisson bracket (4.2) cannot always be reduced, but its reduction usually indicates the existence of a curve evolution inducing a completely integrable system on its invariants. In this case, the integrable system is bi-Hamiltonian with respect to both reduced brackets. She also showed in [14] how, given *any* evolution Hamiltonian with respect to the geometric Poisson bracket, one could always find a geometric realization as invariant flows in G/H. The coefficients of the realization were explicitly related to the Hamiltonian functional.

Both the reduction of (4.1) and the geometric realizations of Hamiltonian evolutions can be found explicitly in most cases, under some minimal conditions. For example, the reduction process can be described in the following terms: given $h, f\colon \mathcal{K} \to \mathbb{R}$, we extend them to functionals $\mathcal{H}, \mathcal{F}\colon \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ so that the extensions are constant on the $\mathcal{L}H$-leaves along $\mathcal{K}$. This condition is infinitesimally described by stating

$$\left(\frac{\delta\mathcal{H}}{\delta L}\right)_x + \mathrm{ad}(K)\left(\frac{\delta\mathcal{H}}{\delta L}\right) \in \boldsymbol{h}^0, \tag{4.3}$$

with $K \in \mathcal{K}$. If two such extensions can be found, then the reduced Poisson bracket is given by $\{h, f\}_R(\boldsymbol{k}) = \{\mathcal{H}, \mathcal{F}\}(K)$, where K is the Maurer–Cartan matrix defining the generating invariants $\boldsymbol{k}$. The infinitesimal condition usually allows us to solve explicitly for $\delta\mathcal{H}/\delta L$.

Perhaps the most interesting part is the direct relation between invariant evolutions and evolutions that are Hamiltonian with respect to the reduction of (4.1). The following theorem can be found in [14].

THEOREM 4.1. *Assume an evolution*

$$\boldsymbol{k}_t = F(\boldsymbol{k}, \boldsymbol{k}_x, \boldsymbol{k}_{xx}, \dots) \tag{4.4}$$

is Hamiltonian with respect to the geometric Poisson bracket obtained when reducing (4.1) to $\mathcal{K}$*. Assume* f *is its Hamiltonian functional and* $\mathcal{F}$ *is an extension constant on the* $\mathcal{L}H$*-leaves along* $\mathcal{K}$*. Assume* $\rho = \rho_{-1}\rho_0\rho_1$ *according to a* $|1|$*-grading of the algebra and assume we can identify* ρ_{-1} *with a non-degenerate curve* u *in* G/H*. Let* $\boldsymbol{r} = ((\delta\mathcal{F}/\delta L)(K))_{-1}$*. Then*

$$\rho_{-1}^{-1}(\rho_{-1})_t = \mathrm{Ad}(\rho_0)\boldsymbol{r} \tag{4.5}$$

is a geometric realization for (4.4).

This direct relation, together with theorem 3.8, tells us that the reduction of (4.1) cannot be further restricted to the level set $\kappa_3 = \kappa_4 = 0$. On the other hand, such a reduction was possible for the conformal sphere of signature $(n, 0)$, and, when reduced, one would obtain bi-Hamiltonian structures for a complexly coupled system of KdV equations on κ_1 and κ_2 [15]. The level set $\kappa_3 = \kappa_4 = 0$ was also preserved by invariant evolutions whenever $\boldsymbol{r}$ was in the same normal form as $S(u)$. To achieve this restriction in the conformal case, one needed to use *natural conformal moving frames*. In our next section we study the local geometry of conformal curves on the sphere of signature $(2,2)$, followed by the definition of natural frames. Natural moving frames are easier to understand geometrically in the conformal picture than in the Grassmannian one, although they correspond algebraically, so our strategy is to shift the geometric knowledge we have in the conformal case to the Grassmannian one.

5. The local geometry of conformal curves

In this section we shall describe the information analogous to that presented in the previous section for the case of the conformal Möbius sphere of signature $(2,2)$. Although the case of the conformal Möbius sphere of signature $(n,0)$ was studied in [12], the change of signature and the need to relate it to the Grassmannian case urge us to use normalization equations, a different approach from that in the original paper. The normalization constants will be matched to the ones used for the Grassmannian case, while trying to reproduce the results in [15] for the general conformal case. Although the isomorphism Φ in § 2 guarantees our process, we need to have explicit descriptions to relate it to the natural frame.

Following the gradation of $\mathfrak{so}(3,3)$ described in § 2, an element $g \in \mathrm{SO}(3,3)$ can be locally factored as

$$g = \begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ -\frac{1}{2}\|z\|^2 & -z^{\mathrm{T}}J & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & -y^{\mathrm{T}}J & -\frac{1}{2}\|y\|^2 \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha \in \mathbb{R}$, $\Theta \in O(2,2)$, $O(2,2)$ is represented as the group that preserves

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and where $\|v\|^2 = v^{\mathrm{T}}Jv$ (the notation is deceiving since it can be negative). We shall also define $\langle v, w\rangle = v^{\mathrm{T}}Jw$. If we identify $\mathrm{SO}(3,3)/P_C$ with the section $\alpha = 1$, $z = 0$, $\Theta = I$, that is, with

$$g_u = \begin{pmatrix} 1 & -u^{\mathrm{T}}J & -\frac{1}{2}\|u\|^2 \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix},$$

then $g \cdot u$ is completely determined by the relation

$$g g_u = g_{g\cdot u} h$$

with $h \in P_C$. The relation uniquely determines

$$g \cdot u = \frac{\Theta(u+y) - \frac{1}{2}\alpha\|u+y\|^2 z}{\alpha^{-1} - z^{\mathrm{T}} J \Theta(u+y) + \frac{1}{4}\alpha\|z\|^2\|u+y\|^2}. \tag{5.1}$$

5.1. Group-based moving frame for conformal generic curves

Let us denote denominator and numerator by $g \cdot u = F/M$. The normalization process below uses normalization constants that are coordinated with those of the Grassmannian. Consider the vectors $e_1 = (1,0,0,1)^{\mathrm{T}}$, $e_2 = (0,1,1,0)^{\mathrm{T}}$, $e_3 = (0,1,-1,0)^{\mathrm{T}}$, $e_4 = (1,0,0,-1)^{\mathrm{T}}$ so that $\|e_1\|^2 = \|e_2\|^2 = -\|e_3\|^2 = -\|e_4\|^2 = 2$.

Step 1 (zeroth normalization equation). This equation is given by

$$g \cdot u = i_0 = 0,$$

which can be readily solved by choosing $u + y = 0$ or $y = -u$.

Step 2 (first normalization equation). Since

$$g \cdot u_1 = \frac{F_1}{M} - \frac{F}{M}\frac{M_1}{M},$$

when substituting $g \cdot u = F/M = 0$ we get that the first normalization equation is

$$g \cdot u_1 = \frac{F_1}{M} = \alpha\Theta u_1 = i_1 = e_3.$$

This determines $\alpha^2\|u_1\|^2 = -2$ and imposes conditions on Θ. If $\|u_1\|^2 > 0$ (notice the abuse of notation since $\|\cdot\|^2$ can be negative), then this choice is not possible and both Grassmannian and conformal cases will need to be renormalized to adjust to the situation (by, for example, choosing $e_2 = i_1$ and

$$i_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the Grassmannian case). There is no problem doing so and we obtain an analogous result.

Step 3 (econd normalization equation). Since

$$g \cdot u_2 = \frac{F_2}{M} - 2\frac{F_1}{M}\frac{M_1}{M} - \frac{F}{M}\left(\frac{M_2}{M} - 2\frac{M_1^2}{M^2}\right),$$

after substituting previous normalizations we obtain

$$g \cdot u_2 = \alpha\Theta u_2 + 2z + 2(z^{\mathrm{T}} J e_3)e_3 = i_2 = 0.$$

This choice of i_2 allows us to solve for z in terms of Θ (if $z = (z_1, z_2, z_3, z_4)^{\mathrm{T}}$, then $(z_1, z_3, z_2, z_4)^{\mathrm{T}} = -\frac{1}{2}\alpha\Theta u_2$). With this value for z, one can check directly that $\|z\|^2 = \frac{1}{4}\|u_2\|^2 - \frac{1}{4}\alpha^2\langle u_1, u_2\rangle^2$ and $z^{\mathrm{T}} J e_3 = \frac{1}{2}\alpha\langle u_1, u_2\rangle$.

STEP 4 (third normalization equation). Given that

$$g \cdot u_3 = \frac{F_3}{M} - 3\frac{M_1}{M}\left(\frac{F_2}{M} - 2\frac{F_1 M_1}{M^2}\right) - 3\frac{M_2}{M}\frac{F_1}{M} - \frac{F}{M}\left(\frac{M_3}{M} - 6\frac{M_2 M_1}{M^2} + 6\frac{M_1^3}{M^3}\right),$$

substituting previous normalizations yields the equation

$$\alpha\Theta u_3 - 3\alpha^2\langle u_1, u_2\rangle z - 6(z^{\mathrm{T}} J e_3)^2 e_3 - 3\|z\|^2 e_3 = i_3.$$

If we write z and e_3 in terms of Θ using previous normalization equations, we arrive at the third normalization equation

$$\alpha\Theta\left(u_3 - 3\frac{\langle u_1, u_2\rangle}{\|u_1\|^2}u_2 + \frac{3}{2}\frac{\|u_2\|^2}{\|u_1\|^2}u_1\right) = \alpha\Theta G_3 = i_3, \tag{5.2}$$

where the vector G_3 is defined uniquely by the equation.

As expected, this equation has rank 2. In fact,

$$\langle i_3, i_1\rangle = -2\left(\frac{\langle u_1, u_3\rangle}{\|u_1\|^2} - 3\frac{\langle u_1, u_2\rangle^2}{\|u_1\|^4} + \frac{3}{2}\frac{\|u_2\|^2}{\|u_1\|^2}\right) = -2I_1 \tag{5.3}$$

(we define I_1 through this equation) and $\|i_3\|^2 = -2(I_2 + I_1^2)$, where

$$I_2 = \frac{\langle u_3, u_3\rangle}{\|u_1\|^2} - 6\frac{\langle u_1, u_2\rangle\langle u_2, u_3\rangle}{\|u_1\|^4} - \frac{\langle u_1, u_3\rangle^2}{\|u_1\|^4} + 6\frac{\langle u_1, u_2\rangle^2\langle u_1, u_3\rangle}{\|u_1\|^6} + 9\frac{\langle u_1, u_2\rangle^2\langle u_2, u_2\rangle}{\|u_1\|^6} - 9\frac{\langle u_1, u_2\rangle^4}{\|u_1\|^8}. \tag{5.4}$$

Both I_1 and I_2 appeared in [12] for the case of signature $(n+1, 1)$, and in [15] they were called *differential invariants of Schwarzian or projective type*. If ϕ is a change of parameter, one can check that

$$\phi^* I_1 = (\phi')^2 I_1 \circ \phi + S(\phi), \qquad \phi^* I_2 = (\phi')^4 I_2 \circ \phi,$$

where $S(\phi)$ denotes the Schwarzian derivative of ϕ. The invariant $I_2^{1/4}$ is used to define the conformal arc length.

Now we need to make a choice for i_3. The value of i_3 in the Grassmannian case depends on whether the Schwarzian derivative of u has real or complex eigenvalues. Here we shall make two different choices to match the Grassmannian choices. We shall also call them the real and complex cases, although this relation is only apparent through its connection to the Grassmannian.

Real case. In this case we shall choose

$$i_3 = \begin{pmatrix} 0 \\ * \\ * \\ 0 \end{pmatrix}. \tag{5.5}$$

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That is, we shall require $\alpha\Theta G_3$ to lie on the plane generated by e_2 and e_3. Assume $i_3 = \hat{k}_1 e_2 + \hat{k}_2 e_3$. Given that $\alpha\Theta u_1 = e_3$, this will determine the value of $\Theta^{-1}e_3$ and that of $\Theta^{-1}e_2$ (with $\Theta \in O(2,2)$ being guaranteed by the entries that are not normalized). We need to find one more piece of information about $\Theta^{-1}e_1$ or e_4 to completely determine Θ. We can accomplish that in the fourth normalization equation, where only one more equation needs to be normalized.

Complex case. In the complex case we shall choose

$$i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_4$$

so that we shall have determined $\Theta^{-1}e_3$ and $\Theta^{-1}e_4$. Again, one more normalization in the fourth-order equation will completely determine Θ.

STEP 5 (fourth normalization equation). As done with the third equation, we shall need to differentiate once more and substitute the previous normalizations. Details are dull and do not add information, so we shall leave them up to the interested reader. After substituting the values for z and e_3 in terms of $\alpha\Theta$, once more the fourth normalization equation will look like

$$\alpha\Theta G_4 = i_4,$$

where G_4 is a vector depending exclusively on derivatives of u.

Real case. In this case we shall require i_4 to belong to the subspace generated by e_2, e_3 and e_4; this forces the coefficient of e_1 to vanish, which is the last normalization equation we need to use to complete the determination of the moving frame ρ. Notice that these choices can be accomplished generically (to be sure, we would need to find G_4, which is straightforward but long and space consuming so we shall not include it in the paper).

Complex case. In this case we shall require the same condition on i_4, which will also imply that the e_1 coefficient of ΘG_4 will vanish. This normalization determines $\Theta^{-1}e_2$ additionally to $\Theta^{-1}e_3$ and $\Theta^{-1}e_4$, so the frame is now completely determined.

5.2. Grassmannian Maurer–Cartan matrix associated to the moving frame

In this section we shall use proposition 3.5 to find the explicit form of the Maurer–Cartan matrix K associated to the moving frame ρ that we just found. Let $v \in \mathfrak{so}(3,3)$ and assume we split it as

$$v = \begin{pmatrix} v_\alpha & -v_{-1}^{\mathrm{T}} J & 0 \\ v_1 & v_0 & v_{-1} \\ 0 & -v_1^{\mathrm{T}} J & -v_\alpha \end{pmatrix}. \tag{5.6}$$

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Then, the prolonged zero-, first-, second- and third-order infinitesimal actions of the algebra on the manifold are given by

$$
\begin{aligned}
v \cdot u &= v_0 \cdot u - \tfrac{1}{2} v_1 \|u\|^2 + v_{-1}, \\
v \cdot u_1 &= v_0 u_1 + v_\alpha u_1 + R_1, \\
v \cdot u_2 &= v_\alpha u_2 + v_0 u_2 - \|u_1\|^2 v_1 + 2 u_1 v_1^{\mathrm{T}} J u_1 + R_2, \\
v \cdot u_3 &= v_\alpha u_3 + v_0 u_3 - 3\langle u_1, u_2\rangle v_1 + 3(v_1^{\mathrm{T}} J u_1) u_2 + 3(v_1^{\mathrm{T}} J u_2) u_1 + R_3,
\end{aligned}
$$

where R_i all contain terms that vanish as u vanishes. Let K be the Maurer–Cartan matrix in the conformal case and assume we split K as done in (5.6). Using these equations for the infinitesimal action in proposition 3.5, we get the following information on K:

$$
\begin{aligned}
K \cdot i_0 &= K_{-1} \\
&= i_1 \\
&= e_3, \\
K \cdot i_1 &= K_0 i_1 + K_\alpha i_1 \\
&= (K_0 + K_\alpha) e_3 \\
&= 0, \\
K \cdot i_2 &= 2K_1 - 2(K_1^{\mathrm{T}} J e_3) e_3 \\
&= i_3, \\
K \cdot i_3 &= K_0 i_3 + K_\alpha i_3 \\
&= i_4 - (i_3)_x.
\end{aligned}
$$

The first equation determines K_{-1} and the second tells us that $K_\alpha = 0$ and

$$
K_0 = \begin{pmatrix} a & b & b & 0 \\ c & 0 & 0 & -b \\ c & 0 & 0 & -b \\ 0 & -c & -c & -a \end{pmatrix}, \tag{5.7}
$$

where the entries still need to be determined. The last equation will determine the value of K_1, depending on whether we are in a real or a complex case, as expected. In the real case, if $i_3 = \hat{k}_1 e_2 + \hat{k}_2 e_3$, then

$$
K_1 = \begin{pmatrix} 0 & k_1 & k_2 & 0 \end{pmatrix}^{\mathrm{T}},
$$

where $k_1 = \frac{1}{2}(\hat{k}_1 - \hat{k}_2)$ and $k_2 = \frac{1}{2}(\hat{k}_1 + \hat{k}_2)$.

In the complex case, if $i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_4$, then

$$
K_1 = \begin{pmatrix} -k_2 & k_1 & -k_1 & k_2 \end{pmatrix}^{\mathrm{T}}
$$

where $k_i = -\frac{1}{2}\hat{k}_i$. These are combinations of the classical invariants I_1, I_2.

Our last recurrence relation completely determines K_0. Indeed, in the real case $K_0 i_3 = i_4 - (i_3)_x$ implies that $K_0 i_3$ has vanishing e_1 component. If K_0 is given as in (5.7), this condition implies $c = b = k_4$ and $a = k_3$. In the complex case,

$K_0 i_3 = i_4 - (i_3)_x$ implies also that the e_1 component of $K_0 e_3$ vanishes, which in this case implies $a = 0$. Notice that one can go back in our calculations and obtain an explicit formula for these invariants, if needed. We just proved the following theorem.

THEOREM 5.1. *Let ρ be the left conformal moving frame found in the previous subsection. Then $\rho^{-1}\rho_x = K$ where K is of the form*

$$K = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & k_3 & k_4 & k_4 & 0 & 0 \\ k_1 & k_4 & 0 & 0 & -k_4 & 1 \\ k_2 & k_4 & 0 & 0 & -k_4 & -1 \\ 0 & 0 & -k_4 & -k_4 & -k_3 & 0 \\ 0 & 0 & -k_2 & -k_1 & 0 & 0 \end{pmatrix}$$

in the real case, and

$$K = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ -k_2 & 0 & k_3 & k_3 & 0 & 0 \\ k_1 & k_4 & 0 & 0 & -k_3 & 1 \\ -k_1 & k_4 & 0 & 0 & -k_3 & -1 \\ k_2 & 0 & -k_4 & -k_4 & 0 & 0 \\ 0 & -k_2 & k_1 & -k_1 & k_2 & 0 \end{pmatrix}$$

in the complex case. The invariants k_1 and k_2 generate all the differential invariants of third order, while k_3 and k_4 generate independent invariants of higher order. Explicit algebraic formulae can be found for each one of the invariants in terms of derivatives of u.

6. Natural frames for conformal and Grassmannian curves

The concept of a natural frame appeared in [2]; Bishop's idea was that, although widely used, classical Serret–Frenet frames in Riemannian geometry were by no means the only way to frame Riemannian curves, and often not the most convenient one. Bishop showed that one could, for example, find a classical moving frame (which he called a *natural moving frame*) for which the derivative of any of the vectors in the frame, other than the unit tangent, will be in the tangential direction (thus, it is often called the *parallel frame*). Physically, the moving frame does not record any rotational movement on the normal plane. That is, if $\nu = (T, T_1, \dots, T_{n-1})$ is a classical Serret–Frenet moving frame, and T is the unit tangent, then

$$\nu_x = \nu \begin{pmatrix} 0 & -\kappa_1 & 0 & \cdots & 0 \\ \kappa_1 & 0 & -\kappa_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_{n-2} & 0 & -\kappa_{n-1} \\ 0 & \cdots & 0 & \kappa_{n-1} & 0 \end{pmatrix},$$

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while if $\eta = (T, N_1, \dots, N_{n-1})$ is the natural frame, then

$$\eta_x = \eta \begin{pmatrix} 0 & -k_1 & -k_2 & \cdots & -k_{n_1} \\ k_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

A main characteristic of this natural frame is that their vectors are *non-local*, that is, they cannot be written as an algebraic expression of derivatives of the curve. In fact, one needs to solve a linear differential equation to find the natural frame.

The concept of the Riemannian natural moving frame was translated into the conformal picture in [12]. In the case of the Möbius sphere of signature $(n+1, 1)$ one may find a classical moving frame $(T, T_1, \dots, T_{n-1})$ using a normalizing Riemannian-type process [15]. In this process, and unlike in the Riemannian case, T had no role and the role of generating the rest of the frame though differentiation was carried out by T_1 instead of the tangent (T_1 is a vector of differential order 3 related to G_3 in (5.2)). In the conformal case Marí Beffa [15] defined the classical *conformal natural moving frame* to be the frame for which the derivatives of all vectors other than T and T_1 are conformally in the direction of T_1. In terms of the group-based Maurer–Cartan matrix, if ρ_C is the moving frame associated to the classical Serret–Frenet frame $(T, T_1, \dots, T_{n-1})$ as in [15] and ρ_N is the one associated to $(T, N_1, \dots, N_{n-1})$, the natural frame, then

$$(\rho_C)_x = \rho_C \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ k_1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ k_2 & 0 & 0 & -k_3 & 0 & \cdots & 0 \\ 0 & 0 & k_3 & 0 & -k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \cdots & \cdots & 0 & -k_n & 0 \\ 0 & 0 & \cdots & 0 & k_n & 0 & 0 \\ 0 & k_1 & k_2 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

while

$$(\rho_N)_x = \rho_N \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ k_1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ k_2 & 0 & 0 & -\kappa_3 & \cdots & -\kappa_n & 0 \\ 0 & 0 & \kappa_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \kappa_n & 0 & \cdots & 0 & 0 \\ 0 & k_1 & k_2 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{6.1}$$

As in the Riemannian case, the conformal natural frame is non-local and one needs to solve a linear differential equation to find it. More interestingly, Marí Beffa [15] showed that if the geometric Poisson bracket is to be written using a classical Serret–Frenet equation, the Poisson structure does not restrict to the submanifold

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$k_i = 0$, $i = 3, \ldots$, while when written in terms of natural ones the restriction to $\kappa_i = 0$, $i = 3, \ldots$, can be carried out to obtain a complexly coupled system of KdV structures. The restriction is bi-Hamiltonian and one indeed obtains a geometric realization of a complexly coupled system of KdV equations for k_1 and k_2 associated to this restriction. Our hope is that, once we find the equivalent definition of the natural frame for our conformal and Grassmannian manifolds, we shall also be able to obtain restrictions and realizations for both cases.

The definition of natural frames is not well suited to our algebraic approach of describing moving frames. Therefore, we shall try to describe natural frames algebraically in a way that can be readily applied to our situation. Notice that, if one has a group-based left-moving frame, the factor of the frame that acts linearly on the manifold (in our case ρ_0) represents a classical frame [14]. The map $M \to M$ taking u to $\rho_0 \cdot u$ is linear and so represented by an element of $\mathrm{GL}(n, \mathbb{R})$. That element has in its columns a classical moving frame. As shown in [12], the way these elements behave under differentiation will be reflected in K_0, according to the gradation.

Let us first have a good look at the conformal $(n+1, 1)$ case; the normalization constants for the frame ρ_N in (6.1) are $i_1 = (1, \ldots, 0)^{\mathrm{T}}$ and $i_3 = (k_1, k_2, 0, \ldots, 0)^{\mathrm{T}}$ (they correspond to the K_{-1} and K_1 components of K, respectively). If Θ is analogous to ours for the $(n+1, 1)$ signature, this choice of constant implies that $\Theta u_1 = \|u_1\|^2 e_1$ and $\Theta G_3 = k_1 e_1 + k_2 e_2$; that is, the first column of the classical moving frame $\alpha^{-1}\Theta^{-1}$ will be the unit tangent and the second will be a combination of the tangent with G_3. The fact that all other derivatives are in the direction of the second vector is reflected in the form of K_0, namely

$$K_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\kappa_3 & \cdots & -\kappa_n \\ 0 & \kappa_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \kappa_n & 0 & \cdots & 0 \end{pmatrix}.$$

Algebraically, $K_0 e_i$ is a multiple of e_2 for any $i = 3, \ldots$.

In our signature $(2, 2)$ conformal sphere, $i_1 = e_3$, $i_2 = 0$, for the real case $i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_2$ and for the complex case $i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_4$. Therefore, in the real case we shall say a moving frame is natural if its associated matrix K_0 satisfies $K_0 e_3 = 0$ and $K_0 e_i$ is a multiple of e_2 for $i = 1, 4$. In the complex case, we substitute e_2 with e_4. Condition $K_0 e_3 = 0$ implies

$$K_0 = \begin{pmatrix} a & b & b & 0 \\ c & 0 & 0 & -b \\ c & 0 & 0 & -b \\ 0 & -c & -c & -a \end{pmatrix},$$

while

$$K_0 e_1 = \begin{pmatrix} a \\ c-b \\ c-b \\ -a \end{pmatrix}, \qquad K_0 e_2 = \begin{pmatrix} 2b \\ 0 \\ 0 \\ -2c \end{pmatrix}, \qquad K_0 e_4 = \begin{pmatrix} a \\ c+b \\ c+b \\ a \end{pmatrix}.$$

From here, in the real case we need $a = 0$, while in the complex case we need $b = c$.

THEOREM 6.1. *In the real case, there exists a conformal moving frame ρ_N (one we shall call the natural moving frame), such that*

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \kappa_3 & \kappa_3 & 0 & 0 \\ k_1 & \kappa_4 & 0 & 0 & -\kappa_3 & 1 \\ \kappa_2 & \kappa_4 & 0 & 0 & -\kappa_3 & -1 \\ 0 & 0 & -\kappa_4 & -\kappa_4 & 0 & 0 \\ 0 & 0 & -k_2 & -k_1 & 0 & 0 \end{pmatrix}.$$

In the complex case, the natural moving frame also exists and is given by a matrix of the form

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & \kappa_3 & \kappa_4 & \kappa_4 & 0 & 0 \\ k_1 & \kappa_4 & 0 & 0 & -\kappa_4 & 1 \\ \kappa_2 & \kappa_4 & 0 & 0 & -\kappa_4 & -1 \\ 0 & 0 & -\kappa_4 & -\kappa_4 & -\kappa_3 & 0 \\ 0 & 0 & -k_2 & -k_1 & 0 & 0 \end{pmatrix}.$$

Proof. Given any two left-moving frames ρ_1 and ρ_2, $\rho_1 = \rho_2 g$ with g depending on differential invariants ($g = \rho_2^{-1}\rho_1$). If K_1 and K_2 are their two Maurer–Cartan matrices, then they are related by the gauge

$$g^{-1}g_x + g^{-1}K_2 g = K_1.$$

Therefore, we need to find an invariant g gauging K as in theorem 5.1 to K_N above. One can directly check that, in the real case, $\rho_N = \rho g$, where

$$g = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}$$

and where γ is the solution to the differential equation $\gamma_x = -k_3\gamma$, that is, $\gamma = \exp(-\int k_3)$. The invariants k_3 and k_4 change accordingly into κ_3 and κ_4.

In the complex case it is only slightly more complicated:

$$g = \begin{pmatrix} \gamma & \eta & \eta & \gamma - 1 \\ -\eta & \gamma & \gamma - 1 & -\eta \\ -\eta & \gamma - 1 & \gamma & -\eta \\ \gamma - 1 & \eta & \eta & \gamma \end{pmatrix}$$

with

$$1 + \frac{\gamma}{\eta}\frac{\gamma - 1}{\eta} = 0 \quad \text{and} \quad \frac{\gamma}{\eta} = -\tan\left(\frac{1}{2}\int (k_3 + k_4)\right).$$

Again, the invariants k_4 and k_3 transform accordingly. □

These two transformations $k_3, k_4 \to \kappa_3, \kappa_4$ are generalizations of the well-known *Hasimoto transformation* for Euclidean geometry to the conformal case. The Hasimoto transformation was proven to be a map from classical Frenet to natural moving frames [10].

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These moving frames can be translated into the Grassmannian picture using the isomorphism. For completeness we shall describe natural frames and also how to find them in the Grassmannian picture. Notice that, even though in the conformal case one can think of geometric ways to define natural frames, this is far less intuitive and not at all obvious in the Grassmannian manifold. It is an interesting question whether or not this can be done for any flat parabolic manifold.

THEOREM 6.2. *In the real case, there exists a left Grassmannian moving frame (the natural frame) such its Maurer–Cartan matrix is given by*

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} 0 & \kappa_3 & 1 & 0 \\ \kappa_4 & 0 & 0 & 1 \\ k_1 & 0 & 0 & \kappa_3 \\ \mathbf{0} & k_2 & \kappa_4 & 0 \end{pmatrix}.$$

In the complex case, the natural moving frame also exists and it satisfies

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} \kappa_3 & \kappa_4 & 1 & 0 \\ \kappa_4 & -\kappa_3 & 0 & 1 \\ k_1 & -k_2 & \kappa_3 & \kappa_4 \\ k_2 & k_1 & \kappa_4 & -\kappa_3 \end{pmatrix}.$$

We do not need to prove this theorem; it is true due to the existence of the isomorphism. Still, we can explicitly find the relation to the moving frame we found previously. In the real case, $\rho_N = \rho g$, where

$$g = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma^{-1} & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}$$

with $\gamma = \exp(-\int k_4)$. In the complex case

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta = -\int k_4$.

7. Invariant evolutions and bi-Hamiltonian equations in terms of natural frames: coupled and decoupled systems of KdV equations

Finally, in this section we shall show that Hamiltonian structures (4.1) and (4.2) both reduce to the submanifold of $\mathcal{K}$ defined by $\kappa_3 = \kappa_4 = 0$ to produce a bi-Hamiltonian decoupled system of two KdV structures in the real case, and a bi-Hamiltonian complexly coupled system of KdV equations in the complex case. This result also implies the existence of geometric realizations for these completely integrable systems. These geometric realizations are best understood as a limit case: as $\kappa_3, \kappa_4 \to 0$, the evolution of k_1 and k_2 becomes completely integrable. Indeed, κ_3,

κ_4, in principle, cannot geometrically vanish, as seen in the previous definition of natural frames (in the real case, k_3 will need to blow up for κ_3 to vanish). This is a surprising geometric realization of KdV systems in the Grassmannian case, but it also shows an unexpected result, namely the existence of geometric realizations of decoupled systems of KdV equations in the conformal case. Only the one for complexly coupled systems of KdV was previously known. This realization can be readily found in a general conformal sphere. For this, we merely need to choose the appropriate normalization i_3 as in (5.5) in the work found in [15]. That leads to the appropriate K_1-form for the matrix.

From now on we shall assume that we are working with the natural Maurer–Cartan matrices. To describe the reduction and the reduced brackets, we shall follow § 4. We shall also assume that we are identifying the dual to the Lie algebra with the Lie algebra itself using the trace. That means that the dual to the entry a_{ij} is the entry a_{ji}. In this section, both the reduction of the structures and their relation to geometric realization are explained. Assume we have a Hamiltonian functional $f\colon \mathcal{K} \to \mathbb{R}$ defined on the space of Grassmannian differential invariants, i.e. in the space $C^\infty(S^1)\times C^\infty(S^1)\times C^\infty(S^1)\times C^\infty(S^1)$. Assume $\mathcal{F}$ is an extension which is constant on the leaves of the subgroup N, and assume

$$\frac{\partial \mathcal{F}}{\partial L}(K) = \begin{pmatrix} F_2 & F_{-1} \\ F_1 & F_3 \end{pmatrix}, \quad K = \begin{pmatrix} K_2 & I \\ K_1 & K_2 \end{pmatrix}. \tag{7.1}$$

Real case. In this case

$$K_1 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 0 & \kappa_3 \\ \kappa_4 & 0 \end{pmatrix}.$$

If we define $f_i = \partial f(\boldsymbol{k})/\partial k_i$, then

$$F_{-1} = \begin{pmatrix} f_1 & \alpha \\ \beta & f_2 \end{pmatrix}, \quad F_2 = \begin{pmatrix} a & \frac{1}{2}f_4 + f \\ \frac{1}{2}f_3 + e & b \end{pmatrix}, \quad F_3 = \begin{pmatrix} c & \frac{1}{2}f_4 - f \\ \frac{1}{2}f_3 - e & -a-b-c \end{pmatrix} \tag{7.2}$$

and F_1 is arbitrary. In this case, the parabolic subgroup P_G is formed by matrices of the block form

$$\begin{pmatrix} * & \mathbf{0} \\ * & * \end{pmatrix}.$$

Therefore, $\mathfrak{n}^0 = \mathfrak{p}_G^0$ can be identified with matrices with block form

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} \end{pmatrix}.$$

Substituting (7.1) into (4.3) we get

$$F_{-1}' + F_3 - F_2 + [K_2, F_{-1}] = \mathbf{0}, \tag{7.3}$$

$$F_2' + [K_2, F_2] - F_{-1}K_1 + F_1 = \mathbf{0}, \tag{7.4}$$

$$F_3' + [K_3, F_3] + K_1F_{-1} - F_1 = \mathbf{0}. \tag{7.5}$$

We can readily solve for $F_2 - F_3$ using (7.3) and for F_1 using (7.4) minus (7.5). The rest of the entries are uniquely determined by the entries in which f_i' are located

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in the equations. Solving, we get

$$\begin{aligned}
\alpha &= \frac{1}{k_2 - k_1}(f_4' - 2\kappa_3 D^{-1}(\kappa_4 f_4 - \kappa_3 f_3)),\\
\beta &= \frac{1}{k_2 - k_1}(f_3' + 2\kappa_4 D^{-1}(\kappa_4 f_4 - \kappa_3 f_3)),\\
2a &= f_1' + \kappa_3\beta - \alpha\kappa_4 + D^{-1}(\kappa_4 f_4 - \kappa_3 f_3),\\
2c &= -f_1' - \kappa_3\beta + \alpha\kappa_4 + D^{-1}(\kappa_4 f_4 - \kappa_3 f_3),\\
2b &= f_2' - \kappa_3\beta + \alpha\kappa_4 - D^{-1}(\kappa_4 f_4 - \kappa_3 f_3),\\
2f &= \alpha' + \kappa_3(f_2 - f_1),\\
2e &= \beta' + \kappa_4(h_1 - h_2).
\end{aligned}$$

Using these results, we can calculate the reduction of (4.1) to the space of differential invariants. If f and h are two such functionals, then the geometric Hamiltonian structure is defined as

$$\{f, h\}(\boldsymbol{k}) = \int_{S^1} \operatorname{tr}\left(\left(\frac{\delta\mathcal{F}}{\delta L}(K)\right)' + \left[K, \frac{\delta\mathcal{F}}{\delta L}(K)\right]\right)\frac{\delta\mathcal{H}}{\delta L}(K)\,\mathrm{d}x,$$

where $\delta\mathcal{H}(K)/\delta L$ is calculated similarly to the variational derivative of $\mathcal{F}$. In [14] Marí Beffa proved that this bracket is always a Poisson bracket. To show that the bracket can be further restricted to the submanifold $\kappa_3 = \kappa_4 = 0$, we shall check that, if f depends only on k_1 and k_2 (that is, $f_3 = f_4 = 0$), while h depends only on κ_3 and κ_4 (that is, $h_2 = h_1 = 0$), then $\{f, h\}(\boldsymbol{k}) = 0$ along the submanifold $\kappa_3 = \kappa_4 = 0$. After this step, we can calculate the bracket of two functionals that depend only on k_1 and k_2.

First of all, notice that condition (4.3) implies that

$$\{f, h\}(\boldsymbol{k}) = \int_{S^1} \operatorname{tr}((F_1' + K_1F_2 - F_3K_1)H_{-1})\,\mathrm{d}x.$$

If a functional f satisfies $f_3 = f_4 = 0$, then we have $\alpha = \beta = e = f = 0$ and $a = \frac{1}{2}f_1' + C$, $b = \frac{1}{2}f_2' - C$, $c = -\frac{1}{2}f_1' + C$, where C is a possible constant that will end up being cancelled out in the calculation of the bracket. These values imply that whenever $\kappa_3 = \kappa_4 = 0$ the matrix $F_1' + K_1F_2 - F_3K_1$ will be diagonal. On the other hand, if $h_1 = h_2 = 0$, then H_{-1} will have a vanishing diagonal and, therefore, $\{f, h\}(\boldsymbol{k}) = 0$.

Finally, if both f and h depend on k_1, k_2 only, then, when $\kappa_3 = \kappa_4 = 0$,

$$F_1' + K_1F_2 - F_3K_1 = \begin{pmatrix} -\frac{1}{2}f_1'' + k_1f_1 & 0 \\ 0 & -\frac{1}{2}f_2'' + k_2f_2 \end{pmatrix}_{x+} \begin{pmatrix} k_1f_1' & 0 \\ 0 & k_2f_2' \end{pmatrix},$$

and therefore

$$\{f, h\}(\boldsymbol{k}) = \int_{S^1} \operatorname{tr}(F_1' + K_1F_2 - F_3K_1)H_{-1}\,\mathrm{d}x = \int_{S^1} \begin{pmatrix} \dfrac{\delta f}{\delta k_1} & \dfrac{\delta f}{\delta k_2} \end{pmatrix} \mathcal{P} \begin{pmatrix} \dfrac{\delta h}{\delta k_1} \\ \dfrac{\delta h}{\delta k_2} \end{pmatrix} \mathrm{d}x,$$

where

$$\mathcal{P}=\begin{pmatrix}-\frac{1}{2}D^3+k_1D+Dk_1 & 0\\ 0 & -\frac{1}{2}D^3+k_2D+Dk_2\end{pmatrix},$$

which defines a decoupled Hamiltonian structure for the KdV equations. One can also check directly that, if we substitute the values of (7.1) in (4.2) for the choice

$$L_0=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ I & \mathbf{0}\end{pmatrix},$$

the resulting bracket is given by

$$\{f,h\}_0(k_1,k_2)=\frac{1}{2}\int_{S^1}\begin{pmatrix}\dfrac{\delta f}{\delta k_1} & \dfrac{\delta f}{\delta k_2}\end{pmatrix}\mathcal{P}_0\begin{pmatrix}\dfrac{\delta h}{\delta k_1}\\ \dfrac{\delta h}{\delta k_2}\end{pmatrix}\,\mathrm{d}x,$$

where $\mathcal{P}_0$ is the second Hamiltonian structure for a decoupled system of KdV equations, that is

$$\mathcal{P}_0=\begin{pmatrix}D & 0\\ 0 & D\end{pmatrix}.$$

One can also check that the bracket of two Hamiltonians depending on (k_1,k_2) only and (κ_3,κ_4) only vanishes when κ_3 and κ_4 vanish.

Complex case. In the complex case the same procedure is followed but with different matrices K_1, K_2, F_1, F_{-1}, F_2 and F_3. In this case

$$K_1=\begin{pmatrix}k_1 & -k_2\\ k_2 & k_1\end{pmatrix},\qquad K_2=\begin{pmatrix}\kappa_3 & \kappa_4\\ -\kappa_3 & \kappa_4\end{pmatrix}$$

and

$$F_{-1}=\begin{pmatrix}\frac{1}{2}f_1+\alpha & \frac{1}{2}f_2+\beta\\ -\frac{1}{2}f_2+\beta & \frac{1}{2}f_1-\alpha\end{pmatrix},$$

$$F_2=\begin{pmatrix}\frac{1}{4}f_3+a & \frac{1}{4}f_4+f\\ \frac{1}{4}f_4+b & -\frac{1}{4}f_3+c\end{pmatrix},$$

$$F_3=\begin{pmatrix}\frac{1}{4}f_3-a & \frac{1}{4}f_4+e\\ \frac{1}{4}f_4+d & -\frac{1}{4}f_3-c\end{pmatrix}.$$

In this case, condition (4.3) has the same equations as the real case, but this results in different values for our unknowns:

$$\alpha=-\frac{1}{4k_2}f_4'+\frac{\kappa_3}{k_2}D^{-1}(\kappa_3f_4-\kappa_4f_3),$$

$$\beta=\frac{1}{4k_2}f_3'+\frac{\kappa_4}{k_2}D^{-1}(\kappa_3f_4-\kappa_4f_3),$$

$$2a=\alpha'+\tfrac{1}{2}f_1'-\kappa_4f_2,$$

$$2c=-\alpha'+\tfrac{1}{2}f_1'+\kappa_4f_2,$$

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$$\begin{aligned} b+d &= D^{-1}(\kappa_3 f_4 - \kappa_4 f_3),\\ b-d &= \beta' - \tfrac{1}{2} f_2' + \kappa_3(f_2 - 2\beta) + 2\kappa_4\alpha,\\ f+e &= -D^{-1}(\kappa_3 f_4 - \kappa_4 f_3),\\ f-e &= \beta' + \tfrac{1}{2} f_2' + \kappa_3(f_2 + 2\beta) - 2\kappa_4\alpha. \end{aligned}$$

As before, if $f_3 = f_4 = 0$ and $h_1 = h_2 = 0$, one can check straightforwardly that $\{f,h\}(k_1,k_2) = 0$. The calculations are only slightly longer than those for the real case. Also, if both f and h satisfy $f_3 = f_4 = h_3 = h_4 = 0$, then

$$\begin{aligned} &H_1' + K_1 H_2 - H_3 K_1\\ &\quad = \frac{1}{2}\begin{pmatrix} -\frac{1}{2} f_1''' + (k_1 f_1)' + k_1 f_1' + (k_2 f_2)' + k_2 f_2' & -\frac{1}{2} f_2''' + (k_1 f_2)' + k_1 f_2' - (k_2 f_1)' - k_2 f_1' \\ \frac{1}{2} f_2''' - (k_1 f_2)' - k_1 f_2' + (k_2 f_1)' + k_2 f_1' & -\frac{1}{2} f_1''' + (k_1 f_1)' + k_1 f_1' + (k_2 f_2)' + k_2 f_2' \end{pmatrix}. \end{aligned}$$

Putting all of these values together in (4.1), we obtain the restricted bracket

$$\{f,h\}(k_1,k_2) = \int_{S^1} \begin{pmatrix} \dfrac{\delta f}{\delta k_1} & \dfrac{\delta f}{\delta k_2} \end{pmatrix} \mathcal{P} \begin{pmatrix} \dfrac{\delta h}{\delta k_1} \\ \dfrac{\delta h}{\delta k_2} \end{pmatrix} \,\mathrm{d}x,$$

where

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} -\frac{1}{2} D^3 + Dk_1 + k_1 D & Dk_2 + k_2 D \\ Dk_2 + k_2 D & \frac{1}{2} D^3 - Dk_1 - k_1 D \end{pmatrix},$$

the well-known Hamiltonian structure for a complexly coupled system of KdV equations. This structure was obtained in [15]. Accordingly, we can guess that the choice of the third normalization in [15] is in a different prolonged orbit from the real case we found here, and that a choice in the same orbit will result in a decoupled system of KdV equations. Once more, if we substitute our values in (4.2) for the same choice of L_0 as in the real case, we obtain a second Poisson structure $\mathcal{P}_0$, namely

$$\mathcal{P}_0 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}.$$

This is known to be the second Hamiltonian structure for a complexly coupled KdV system. As before, one can also check that the bracket of Hamiltonians depending on (k_1, k_2) only and (κ_3, κ_4) only vanishes when κ_3 and κ_4 vanish.

The interest of these different structures is that (4.1) reduces always to a Geometric Poisson bracket that is directly linked to a geometric realization of the Hamiltonian evolution. The relation is given as in (4.5). Therefore, we have almost finished the proof of the following theorem.

THEOREM 7.1. *Assume $u(t,x)$ describes an evolution of a curve of Grassmannian planes in $\mathbb{R}^4$ solution of the equation*

$$u_t = u_1 S(u) = u_3 - \tfrac{3}{2} u_2 u_1^{-1} u_2. \tag{7.6}$$

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Then, if $S(u)$ has real eigenvalues k_1 and k_2, as $\kappa_3, \kappa_4 \to 0$, the curvatures k_1 and k_2 satisfy a decoupled system of KdV equations. If $S(u)$ has complex eigenvalues, $k_1 \pm k_2\mathrm{i}$, then k_1 and k_2 satisfy a complexly coupled system of KdV equations. In both cases, the Poisson structures (4.1) and (4.2) reduce to the space $\kappa_3 = \kappa_4 = 0$ to produce a bi-Hamiltonian pencil for decoupled KdV equations or complexly coupled KdV equations, depending on the case.

Proof. Assume

$$\left(\frac{\delta\mathcal{H}}{\delta L}(K)\right)_{-1} = g_r, \quad \text{with } g_r = \begin{pmatrix} \mathbf{0} & \boldsymbol{r} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\boldsymbol{r}$ is a matrix of differential invariants and $\mathcal{H}$ is the appropriate extension of a Hamiltonian functional h, and where by $(\cdot)_{-1}$ we indicate the projection on the tangent to the manifold G/P, as represented by the subspace $\mathfrak{g}_{-1}$ of the Lie algebra $\mathfrak{g}$. Then, from [14], the evolution

$$\rho_{-1}^{-1}(\rho_{-1})_t = \mathrm{Ad}(\rho_0)g_r$$

induces the h-hamiltonian evolution on the differential invariants. If we choose

$$h(k_1, k_2) = \frac{1}{2}\int_{S^1} (k_1^2 + k_2^2)\,\mathrm{d}x,$$

then

$$h_1 = k_1, \quad h_2 = k_2 \quad \text{and} \quad H_{-1} = \boldsymbol{r} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Calculating (4.5) with the natural moving frame, we obtain

$$\begin{pmatrix} I & -u \\ \mathbf{0} & I \end{pmatrix}\begin{pmatrix} \mathbf{0} & u_t \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} u_1 B^{-1} & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \boldsymbol{r} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} B u_1^{-1} & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix},$$

which results in

$$u_t = u_1 B^{-1}\boldsymbol{r}B.$$

If we now look at the third normalization equations, notice that i_3 is either

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

in the real case, or

$$\begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}$$

in the complex case. In either situation, if we choose $\boldsymbol{r} = i_3$, then we get $B^{-1}i_3B = S(u)$, since the third normalization equation in both cases reads $BS(u)B^{-1} = i_3$.

The choice of h clearly results in the completely integrable systems stated by the theorem. □

Notice that this immediately implies that (7.6) preserves the level set $\kappa_3 = \kappa_4 = 0$: a fact that can also be found directly using the techniques in § 3.3.

The last results in this paper are proven choosing a different, but also non-local, moving frame. This moving frame corresponds to the Laguerre–Forsyth canonical form of the Serret–Frenet equations for ρ, and it was linked to Grassmannians in [18]. The invariants generated by this last Grassmannian moving frame evolve following a non-commutative KdV equation. When reduced to these invariants, both brackets (4.1) and (4.2) become the bi-Hamiltonian structure for non-commutative KdV equations.

THEOREM 7.2. *There exists a (non-local) Grassmannian moving frame such that its associated Maurer–Cartan matrix is given by*

$$\begin{pmatrix} \mathbf{0} & I \\ \hat{K} & \mathbf{0} \end{pmatrix},$$

where the entries of $\hat{K}$ are independent and generating differential invariants. Both structures (4.1) and (4.2), with the choice

$$L_0 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} \end{pmatrix},$$

reduce to $\mathcal{K}$ as represented by these invariants to produce a bi-Hamiltonian structure for the non-commutative KdV equation. Furthermore, the evolution

$$u_t = u_1 \hat{K}$$

induces a non-commutative KdV equation on $\hat{K}$.

Proof. Since the technique is identical to the previous cases, we shall describe the first part of the calculations without additional explanations. If we gauge (3.17) or (3.18) by an element of the form

$$g = \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \Theta \end{pmatrix},$$

the result is

$$\begin{pmatrix} \Theta^{-1}\Theta_x + \Theta^{-1}K_0\Theta & I \\ \Theta^{-1}K_1\Theta & \Theta^{-1}\Theta_x + \Theta^{-1}K_0\Theta \end{pmatrix}.$$

Therefore, choosing Θ to be the solution of $\Theta_x = -K_0\Theta$ results in the choice $\hat{K} = \Theta^{-1}K_1\Theta$. Notice that the invariants k_1 and k_2 are generated by the trace and determinant of $\hat{K}$.

We now calculate the reduction of the Poisson brackets. Assume that

$$\frac{\delta \mathcal{H}}{\delta L} = \begin{pmatrix} A & \boldsymbol{h} \\ C & B \end{pmatrix}$$

is the derivative of an appropriate extension of h with $\boldsymbol{h} = \delta h/\delta \kappa$. Then, (4.3) results in the values

$$\begin{aligned} A &= \tfrac{1}{2}\boldsymbol{h}_x - \tfrac{1}{2}D^{-1}(\hat{K}\boldsymbol{h} - \boldsymbol{h}\hat{K}), \\ B &= -\tfrac{1}{2}\boldsymbol{h}_x - \tfrac{1}{2}D^{-1}(\hat{K}\boldsymbol{h} - \boldsymbol{h}\hat{K}), \\ C &= -\tfrac{1}{2}\boldsymbol{h}_{xx} + \tfrac{1}{2}(\hat{K}\boldsymbol{h} + \boldsymbol{h}\hat{K}). \end{aligned}$$

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Using this extension, the reduction of (4.1) is given by

$$\{h,f\}_R(\kappa) = \int \operatorname{tr}\biggl((C_x + \hat{K}A - B\hat{K})\frac{\delta f}{\delta\kappa}\biggr)\,\mathrm{d}x = \int \boldsymbol{h}^{\mathrm{T}}\mathcal{P}\boldsymbol{f}\,\mathrm{d}x,$$

where

$$2\mathcal{P}\boldsymbol{h} = -\boldsymbol{h}_{xxx} + (\boldsymbol{h}\hat{K} + \hat{K}\boldsymbol{h})_x + \hat{K}\boldsymbol{h}_x + \boldsymbol{h}_x\hat{K} - \hat{K}D^{-1}(\hat{K}\boldsymbol{h} - \boldsymbol{h}\hat{K}) + D^{-1}(\hat{K}\boldsymbol{h} - \boldsymbol{h}\hat{K})\hat{K}.$$

Choosing

$$L_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

in (4.2) and using the expression for the extension $\delta\mathcal{H}/\delta L$, we have that the reduction of (4.2) is given by

$$\{h,f\}_0(\kappa) = \int \boldsymbol{h}^{\mathrm{T}} D\boldsymbol{f}\,\mathrm{d}x.$$

These two structures are those appearing in [16] as bi-Hamiltonian structures for non-commutative KdV equations. The calculation of the geometric realization is identical to the previous cases.

There is one last point that needs to be checked. An arbitrary gauge by a matrix of invariants of a moving frame does result in a new moving frame, but the entries of its Maurer–Cartan matrix do not necessarily generate all other invariants. Indeed, the proof in [9] shows that if a moving frame is found using normalization constants (as in the first frame we found), then these entries do indeed generate all other invariants. But, although it may be true, a general theorem for any moving frame has not yet been proved. Therefore, we need to check that such is the case here.

We know that k_1 and k_2 are generated by the entries of $\hat{K}$. Lengthy but straightforward calculations show that if

$$\varTheta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \hat{K} = \begin{pmatrix} \hat{\kappa}_1 & \hat{\kappa}_2 \\ \hat{\kappa}_3 & \hat{\kappa}_4 \end{pmatrix},$$

then

$$\frac{a'a}{bb'} = -\frac{\hat{\kappa}_3}{\hat{\kappa}_2}, \qquad \frac{a'b}{ab'} = \frac{k_2^2 - k_1^2}{\hat{\kappa}_1 k_1 - \hat{\kappa}_4 k_2} + 1.$$

This implies that $n_1 = a'/b'$ and $n_2 = a/b$ are functionally generated by $\hat{\kappa}_i$, $i = 1, 2, 3, 4$. Further calculations show that $\kappa_3\kappa_4 = -n_1'n_2'$ and

$$\frac{\kappa_3'}{\kappa_3}n_2' - 2\frac{n_2'}{n_1 - n_2} = n_2'',$$

yielding that both κ_3 and κ_4 are also formally functionally generated by the entries of $\hat{K}$. □

Our final corollaries state the existence of a conformal level set for a decoupled system of KdV equations. It also shows that the signature $(2,2)$ conformal Poisson brackets are equivalent to the bi-Hamiltonian structure for non-commutative KdV 2×2 equations. Although this is immediate under the isomorphism, it was not previously obvious.

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Corollary 7.3. *There exists a conformally invariant evolution of curves (written in terms of a natural classical moving frame) such that it preserves* $\kappa_3 = \kappa_4 = 0$*. As* $\kappa_3, \kappa_4 \to 0$*, the evolution induces a decoupled system of KdV equations on* k_1 *and* k_2*.*

Corollary 7.4. *Both brackets (4.1) and (4.2) reduce to the space of conformal differential invariants to produce a bi-Hamiltonian structure equivalent to that of the non-commutative KdV equation that appears in [16].*

The interested reader can find the exact equations for these geometric realizations using the isomorphism with the corresponding Grassmannian evolution.

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