Canad. Math. Bull.Vol. 35 (1), 1992 pp. 136–144

REGULARITY OF THE INTERFACES IN THE STEFAN PROBLEM WITH A MUSHY REGION

HONG-MING YIN

ABSTRACT. This paper deals with the Stefan-type problem with a zone of coexistence of both phases. We formulate the problem in the enthalpy form and show that the interfaces between the liquid and the mushy, the mushy and the solid phase are smooth. Our approach is to study the structures of the level sets of the solution via Sard's Lemma and the implicit function theorem.

1. **Introduction.** The Stefan problem with mushy regions has been considered by many authors (cf. Fasano [7], Lacey [10], Meirmarov [11], Primicerio [14], Rubinstein [15], etc.). Recently, M. Bertsch et al. [1] formulated the problem as a degenerate parabolic equation and proved the existence and the uniqueness of the solution in a certain weak sense. Moreover, in [2] and [3], the continuity of the interfaces between the liquid, the mushy and the solid phase was obtained by means of the comparison principle. In this paper we shall improve the regularity of the interfaces. More precisely, we consider the one dimensional problem in the enthalpy form:

$$\frac{\partial A(u)}{\partial t} = u_{xx} + g(u)_x + f(u), \quad \text{in } Q_T, \tag{1.1}$$

$$u(0,t) = f_1(t) > 0, \quad 0 \le t \le T, \tag{1.2}$$

$$u(1,t) = f_2(t) < 0, \quad 0 \le t \le T, \tag{1.3}$$

$$A\big(u(x,0)\big) = A\big(u_0(x)\big), \quad 0 \le x \le 1, \tag{1.4}$$

where $Q_T = (0,1) \times (0,T]$, while

$$A(u) = \begin{cases} \int_0^u a_1(\xi)\,d\xi, & \text{if } u > 0, \\ [0,1], & \text{if } u = 0, \\ \int_0^u a_2(\xi)\,d\xi - 1, & \text{if } u < 0. \end{cases}$$

and

$$g(u) = \begin{cases} \int_0^u g_1(\xi)\,d\xi, & \text{if } u \ge 0, \\ \int_0^u g_2(\xi)\,d\xi, & \text{if } u < 0. \end{cases}$$

The weak solution is defined as in Oleinik [13]:

Received by the editors April 2, 1990; revised: November 28, 1990.
AMS subject classification: 35R35, 80A20.
© Canadian Mathematical Society 1992.

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

DEFINITION. A continuous function $u(x,t)$ defined on $\bar{Q}_T$ is said to be a *weak solution* of the problem (1.1)–(1.4) if

$$\begin{aligned}(1.5)\qquad &\int_{Q_T}[A(u)\psi_t + u\psi_{xx} - g(u)\psi_x + f(u)\psi]\,dx\,dt\\ &= \int_0^T [f_2(t)\psi_x(1,t) - f_1(t)\psi_x(0,t)]\,dt - \int_0^1 A\big(u_0(x)\big)\psi(x,0)\,dx\end{aligned}$$

for any test function

$$\psi(x,t)\in X = \{\psi\in C^{2,1}(\bar{Q}_T):\psi(0,t)=\psi(1,t)=\psi(x,T)=0\}.$$

In the above system, $u(x,t)$ represents the temperature. The fundamental constitutive assumption is that the phase change will take place whenever $u(x,t)$ reaches the melting temperature $\big(u(x,t)=0\big)$. Assume that $u(x,t)$ is a weak solution. Let

$$\begin{aligned}Q_T^- &= \{(x,t)\in\bar{Q}_T: u(x,t)>0\};\\ M_T &= \{(x,t)\in\bar{Q}_T: u(x,t)=0\};\\ Q_T^+ &= \{(x,t)\in\bar{Q}_T: u(x,t)<0\}.\end{aligned}$$

Physically, Q_T^-, M_T and Q_T^+ are corresponding to the liquid, mushy and the solid phase, respectively. If M_T is composed of the graph of a smooth curve $x = s(t)$, it is easy to verify that $\big(u(x,t), s(t)\big)$ is the unique classical solution of the two-phase Stefan problem in one space dimension. Because of the effect of the heat source, the interior set M_T^0 may not be empty even though no mush exists at the initial moment (cf. [1]–[3]). Our main interest is to investigate the regularity property of the interfaces. We first use the argument developed in [5] (also see [4] and [16]) to establish the Lipschitz continuity of the interfaces. Then we show that the interfaces are smooth. For the several space dimension, the Lipschtiz continuity of the interface was established under the monotonicity conditions on the known data (cf. [12]).

2. **Assumptions and the main results.** We shall assume the following basic conditions:

H(1) $a_i(\xi), g_i(\xi)\in C^3(R^1)$ and there exists a positive constant a_0 such that $a_i(\xi)>a_0$ for all $\xi\in R^1$, $i=1,2$. Moreover, $f(0)=0$ and $f(u)$ is uniformly Lipschitz continuous.

H(2) $f_i(t)\in C^2[0,T]$ with $f_1(t)>0$ and $f_2(t)<0$ for all $t\in[0,T]$. $u_0(x)\in C^4[0,1]$ with the property $u_0(x)>0$ on $[0,a)$, $u_0(x)=0$ on $[a,b]$ and $u_0(x)<0$ on $(b,1]$, where $0<a\le b<1$. Moreover, $u_0'(a)<0$ and $u_0{}'(b)<0$. The consistency condition holds

$$f_1(0)=u_0(0),\quad f_2(0)=u_0(1).$$

REMARK. If $a<b$, this means that there exist three states at the initial moment.

We should point out that the condition $f(0)=0$ is crucial. This guarantees that no superheating or supercooling exists in the system. The example in [3] has been constructed with $f=8$ that the interface is not continuous. Our main result is

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

THEOREM 2.1. *Under the assumptions H(1)–H(2), there exist two Lipschitz continuous curves $x = s_1(t)$ and $x = s_2(t)$ with $0 < s_1(t) \le s_2(t) < 1$ such that*

$$Q_T^- = \{(x,t) \in \bar{Q}_T : 0 \le x < s_1(t)\};$$
$$M_T = \{(x,t) \in \bar{Q}_T : s_1(t) < x < s_2(t)\};$$
$$Q_T^+ = \{(x,t) \in \bar{Q}_T : s_2(t) < x \le 1\}.$$

Moreover, $s_1(t)$ and $s_2(t)$ belong to $C^\infty(0,T]$ if $g_i(\xi)$ and $f(\xi)$ are smooth, $(i = 1,2)$.

If we assume that $s_1(t), s_2(t), u(x,t)$ and $A\big(u(x,t)\big)$ are smooth, then we have the following parabolic-hyperbolic system:

$$a_1(u)u_t = u_{xx} + g_1(u)u_x + f(u), \quad Q_T^- \tag{2.1}$$
$$a_2(u)u_t = u_{xx} + g_2(u)u_x + f(u), \quad Q_T^+ \tag{2.2}$$
$$A\big(u(x,t)\big)_t = f(u(x,t)), \quad (x,t) \in M_T^0, \tag{2.3}$$
$$u(0,t) = f_1(t) > 0, \quad 0 \le t \le T, \tag{2.4}$$
$$u(1,t) = f_2(t) < 0, \quad 0 \le t \le T, \tag{2.5}$$
$$A\big(u(x,0)\big) = A\big(u_0(x)\big), \quad 0 \le x \le 1 \tag{2.6}$$
$$u\big(s_1(t),t\big) = u\big(s_2(t),t\big) = 0, \quad 0 \le t \le T, \tag{2.7}$$
$$A\big(u(s_1(t)+,t)\big)\dot{s}_1(t) = u_x\big(s_1(t)-,t\big), \quad 0 \le t \le T, \tag{2.8}$$
$$A\big(u(s_2(t)-,t)\big)\dot{s}_2(t) = u_x\big(s_2(t)+,t\big), \quad 0 \le t \le T, \tag{2.9}$$
$$s_1(0) = a, \; s_2(0) = b. \tag{2.10}$$

Conversely, if $s_1(t), s_2(t), u(x,t)$ and $A\big(u(x,t)\big)$ satisfy the equations (2.1)–(2.10), we can see by performing the integration by parts that $u(x,t)$ is a weak solution. However, the equations (2.3), (2.8)–(2.9) are only formal since $A\big(u(x,t)\big)$ is actually not continuous on M_T. But we have

COROLLARY. *If $f(u) \equiv 0$ and $a = b$, then $s_1(t) = s_2(t)$ and is smooth. Thus, the Stefan problem (1.1)–(1.4) has a unique classical solution globally.*

If we consider the periodic problem and assume that $f_1(t)$ and $f_2(t)$ are periodic with the period T without the initial condition (1.4), then we have (cf. [16])

THEOREM 2.2. *There exist two smooth periodic curves $x = s_1(t)$ and $x = s_2(t)$ with the period T such that the result of Theorem 2.1 holds under the conditions H(1)–H(2) and the additional restriction $f'(u) \le 0$.*

3. **The Proof.** Since we need to analyze the structures of level sets of both the weak and the approximate solutions, we give the construction of the approximation solution. For simplicity, we assume that $f(u)$ is smooth. Construct the following approximate problem:

$$\frac{\partial A_n(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial g_n(u)}{\partial x} + f(u) \quad \text{in } Q_T, \tag{3.1}$$

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

(3.2) $$u(0,t) = f_1(t), \quad t \in [0,T],$$

(3.3) $$u(1,t) = f_2(t), \quad t \in [0,T],$$

(3.4) $$u(x,0) = u_{0n}(x), \quad x \in [0,T]$$

where $A_n(\xi)$ and $g_n(\xi) \in C^3(R^1)$ and satisfy :

(i) $A_n'(\xi) > a_0/2 > 0$,

(ii) $A_n(\xi) = A(\xi)$ if $|\xi| \geq \frac{1}{n}$, $n = 1,2,\ldots$,

(iii) $A_n(\xi), g_n(\xi) \longrightarrow A(\xi), g(\xi)$ in L^2 as $n \longrightarrow +\infty$,

while that u_{0n} is constructed such that

(i) $u_{0n}(x) \in C^4[0,T]$ is uniformly convergent to $u_0(x)$ as $n \longrightarrow \infty$

(ii) $u_{0n}'(x) \leq d(n) < 0$ on $[a-\delta_0, b+\delta_0]$, and $\frac{u_{0n}''(x)+g_n'(u_{0n}(x))u_{0n}'(x)+f(u_n(x))}{u_{0n}'(x)}$ is uniformly bounded on $[a-\frac{1}{n}, b+\frac{1}{n}]$

(iii) the consistency conditions up to the fourth order hold at $(0,0)$ and $(1,0)$,

with a small constant $\delta_0 > 0$ and a negative number $d(n)$ which depends on n.

The property (ii) of u_{0n} can be done due to the assumption H(2). By the theory of parabolic equations, there exists a unique solution $u_n(x,t) \in C^{4,2}(\bar{Q}_T)$. The following result can be established by the analogous argument as [16] without the essential difficulty, we do not repeat it here.

LEMMA 3.1. *There exists a unique weak solution for the problem (1.1)–(1.4) under the hypotheses H(1) and H(2). Moreover,*

$$u(x,t) \in C^{\frac{1}{2},\frac{1}{4}}(\bar{Q}_T) \cap C^{4,2}\Big[Q_T^-/(0,0) \cup Q_T^+/(1,0)\Big].$$

$u(x,t)$ satisfies the equations (2.1)–(2.2) and (2.4)–(2.5) in the classical sense. Moreover,

$$u(x,0) = u_0(x) \textit{ for } x \in [0,a) \cup (b,1].$$

Let $u(x,t)$ be the weak solution of (1.1)–(1.4) and $u_n(x,t)$ the classical one of (3.1)–(3.4). Since the weak solution is continuous, we can define Q_T^-, Q_T^+ and M_T as in Section 1. Let ε and $-\varepsilon$ be small positive noncritical values of both $u(x,t)$ and $u_n(x,t)$ for all n. Sard's lemma tells us that the above ε exists. Let

$$\Gamma_\varepsilon^{(n)} = \{(x,t) \in \bar{Q}_T : u_n(x,t) = \varepsilon\},$$
$$\Gamma_{-\varepsilon}^{(n)} = \{(x,t) \in \bar{Q}_T : u_n(x,t) = -\varepsilon\},$$
$$\Gamma_\varepsilon = \{(x,t) \in \bar{Q}_T : u(x,t) = \varepsilon\},$$
$$\Gamma_{-\varepsilon} = \{(x,t) \in \bar{Q}_T : u(x,t) = -\varepsilon\}$$

Since ε is not a critical value of $u_n(x,t)$, we have that $\nabla u_n(x,t) = \{u_{nx}(x,t), u_{nt}(x,t)\} \neq \vec{0}$ for all $(x,t) \in \Gamma_\varepsilon^{(n)} \cup \Gamma_{-\varepsilon}^{(n)}$.

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

LEMMA 3.2. *The level sets* $\Gamma_{\varepsilon}^{(n)}$ *and* $\Gamma_{-\varepsilon}^{(n)}$ *consist of the graphs of the curves* $x = s_{\varepsilon}^{(n)}(t)$ *and* $x = s_{-\varepsilon}^{(n)}(t)$, *respectively, which enter at* $t = 0$ *and exit at* $t = T$. *These curves belong to* $C^1[0, T]$ *and have the following properties:*

(i) *These curves are monotonic increasing in the t-direction;*

(ii) *No open segment on these curves is parallel to the x-axis.*

Moreover, there exists a constant $r_n < 0$ *which may depend on n such that*

$$\frac{\partial u_n(x,t)}{\partial x} \leq r_n \text{ for all } (x,t) \in R_n(x,t),$$

where

$$R_n = \left\{ (x,t) : s_{\varepsilon}^{(n)}(t) \leq x \leq s_{-\varepsilon}^{(n)}(t),\ 0 \leq t \leq T \right\}.$$

PROOF. The first part of the Lemma can be obtained exactly as in [4] by the implicit function theorem along with the maximum principle. To prove the last result, we set $V(x,t) = u_{nx}(x,t), (x,t) \in \bar{Q}_T$. We claim that $V(s_{\pm\varepsilon}^{(n)}(t), t) < 0$ for $t \in [0, T]$. Indeed, let $Q_n^- = \left\{ (x,t) : 0 < x < s_{\varepsilon}^{(n)}(t), 0 < t \leq T \right\}$. Then, $u_n(x,t)$ is the classical solution of the following problem

$$A_n'(u_n)u_{nt} = u_{nxx} + g_n'(u_n)u_{nx} + f(u_n) \tag{3.5}$$

$$u_n(0,t) = f_1(t), \quad 0 \leq t \leq T, \tag{3.6}$$

$$u_n\left(s_{\varepsilon}^{(n)}(t), t\right) = \varepsilon, \quad 0 \leq t \leq T, \tag{3.7}$$

$$u_n(x,0) = u_{0n}(x), \quad 0 \leq x \leq a_n, \tag{3.8}$$

where $a_n = s_{\varepsilon}^{n}(0)$ is the point with $u_{0n}(a_n) = \varepsilon$.

Since $f_1(t) > 0$ on $[0, T]$ and $u_0(x) > 0$ on $[0, a)$, we can take ε small enough such that $f_1(t)$ and $u_{0n}(x)$ are greater than ε. Furthermore, we can assume that a_n satisfies $a - \delta_0 < a_n < a$. Since $f(0) = 0$, the maximum principle indicates that $u_n(x,t)$ attains its minimum at every point on the boundary $x = s_{\varepsilon}^{(n)}(t)$. Therefore, the strong maximum principle yields

$$u_{nx}(s_{\varepsilon}^{(n)}(t), t) < 0 \text{ for all } t \in [0, T].$$

Similarly, we can show that $V\left(s_{-\varepsilon}^{(n)}(t), t\right) < 0$ and there exists $b_n = s_{-\varepsilon}^{n}(0) \in (b, b + \delta_0)$ with the property $u_{0n}(b_n) = -\varepsilon$. By the construction, we have $V(x, 0) < 0$ on $[a_n, b_n]$. Hence $V(x,t)$ is strictly negative on the parabolic boundary of R_n. It is easy to see that $V(x,t)$ satisfies in R_n

$$A_n'(u_n)V_t = V_{xx} + g_n'(u_n)V_x + [-A_n''(u_n)u_{nt} + g_n''(u_n)u_{nx} + f'(u_n)]V \tag{3.9}$$

Consequently, the maximum principle implies the desired result.

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

LEMMA 3.3. *The level sets Γ_ε and $\Gamma_{-\varepsilon}$ consist of the graphs of the curves $x = s_\varepsilon(t)$ and $x = s_{-\varepsilon}(t)$, respectively, which enter at $t = 0$ and exit at $t = T$. These curves possess the same properties as those in Lemma 3.2. Moreover, there exists a constant $k_0 < 0$ such that*

$$u_x\big(s_{\pm\varepsilon}(t), t\big) < k_0 \text{ for } t \in [0, T].$$

PROOF. The first conclusion of the Lemma can be demonstrated as that in [6]. Let

$$Q_\varepsilon^- = \{(x,t) : 0 < x < s_\varepsilon(t), 0 < t \le T\}.$$

Note that $u(x, t)$ is a classical solution of the problem

$$a_1(u)u_t = u_{xx} + g_1(u)u_x + f(u), \text{ in } Q_\varepsilon^-, \tag{3.10}$$

$$u(0,t) = f_1(t) > 0, \quad 0 \le t \le T, \tag{3.11}$$

$$u\big(s_\varepsilon(t), t\big) = \varepsilon, \quad 0 \le t \le T, \tag{3.12}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le a^* = s_\varepsilon(0), \tag{3.13}$$

where $a^* \in (0, a)$ is a point where $u_0(a^*) = \varepsilon$. The strong maximum principle yields $u_x(s_\varepsilon(t), t) < 0$. The proof of the other part is similar. ■

Since $u(x, t)$ is smooth on Q_T^- and Q_T^+, by Lemma 3.3 we immediately have

COROLLARY 3.4. *There exists a constant $\delta > 0$ which depends only on known data such that*

$$u_x(x,t) \le \frac{k_0}{2} < 0, \ (x,t) \in R_\delta^- \cup R_\delta^+. \tag{3.4}$$

Moreover, $u_t(x, t)$ is also uniformly bounded on $R_\delta^- \cup R_\delta^+$, where

$$R_\delta^- = \{(x,t) \in \bar{Q} : |x - s_\varepsilon(t)| < \delta, \ t \in [0,T]\} \subset Q_T^-,$$
$$R_\delta^+ = \{(x,t) \in \bar{Q} : |x - s_{-\varepsilon}(t)| < \delta, \ t \in [0,T]\} \subset Q_T^+.$$ ■

Now we consider some special level sets of $u_n(x, t)$. Let

$$\Gamma_n = \Big\{(x,t) : u_n(x,t) = \frac{1}{n}\Big\};$$
$$\Gamma_{-n} = \Big\{(x,t) : u_n(x,t) = -\frac{1}{n}\Big\}.$$

Let n be large enough such that $0 < \frac{1}{n} < \varepsilon$. Then

$$\Gamma_n \cup \Gamma_{-n} \subset R_n.$$

Since $u_{nx}(x, t) < 0$ in R_n, we have the functions $x = s_n(t)$ and $x = s_{-n}(t)$ with $u_n\big(s_n(t), t\big) = \frac{1}{n}$ and $u_n\big(s_{-n}(t), t\big) = -\frac{1}{n}$. Moreover, the curves $s_n(t)$ and $s_{-n}(t)$ have the same features as $s_{\pm\varepsilon}^{(n)}(t)$. Define

$$K_n = \{(x,t) : s_n(t) < x < s_{-n}(t), 0 < t \le T\}.$$

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

It is clear that $K_n \subset R_n$. Let $U(x,t) = u_{nt}$, $(x,t) \in \bar{Q}_T$ and

$$W(x,t) = \frac{U(x,t)}{V(x,t)}, (x,t) \in R_n.$$

The Lemma 3.2 indicates that $W(x,t)$ is well-defined. We show that $W(x,t)$ is uniformly bounded.

LEMMA 3.5. *There exists a constant k_1 independent of n such that*

$$|W(x,t)| \leq k_1,\ (x,t) \in R_n.$$

PROOF. We define the operator

$$L = A_n'(u_n)\frac{\partial}{\partial t} - \Big[-\frac{1}{u_{nx}}\Big]\Big[\frac{\partial^2}{\partial x^2} + g_n'(u_n)\frac{\partial}{\partial x}\Big].$$

It can be seen by a direct computation that $W(x,t)$ satisfies

$$LW(x,t) = 0,\ \text{in } R_n$$

Hence, the maximum principle implies that

$$\max_{R_n} |W(x,t)| \leq \max_{\partial_p R_n} |W(x,t)|.$$

Since $u_n(x,t)$ converges $u(x,t)$ on $\bar{Q}_T$ uniformly, we have when n is large enough

$$s_\varepsilon^{(n)}(t) \subset R_\delta^-,\ s_{-\varepsilon}^{(n)}(t) \subset R_\delta^+$$

and

$$|u_{nt}(s_\varepsilon^{(n)}(t),t)| \leq 2\max_{R_\delta^-} |u_t(x,t)|,$$

$$|u_{nx}(s_{-\varepsilon}^{(n)}(t),t)| \leq -\frac{1}{2}\max_{R_\delta^+} |u_x(x,t)| \leq \frac{k_0}{4}$$

Moreover, observe that

$$W(x,0) = \frac{1}{A_n'\big(u_{0n}(x)\big)}\,\frac{u_{0n}''(x) + g_n'\big(u_{0n}(x)\big)u_{0n}'(x) + f\big(u_n(x)\big)}{u_{0n}'(x)}$$

is also uniformly bounded, it follows that $W(x,t)$ is uniformly bounded on the parabolic boundary $\partial_p R_n$ when n is large enough. ∎

Now $u_n\big(s_n(t),t\big) = \frac{1}{n}$ we have

$$\frac{ds_n(t)}{dt} = -W\big(s_n(t),t\big).$$

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

Since $K_n \subset R_n$,

$$\left|\frac{ds_n(t)}{dt}\right| = |-W(s_n(t),t)| \\ \leq \max_{R_n} |W(x,t)| \\ \leq \max_{\partial_p R_n} |W(x,t)| \leq C$$

where C is independent of n.

By a compactness argument, we have a subsequence (still denoted by $s_n(t)$) which converges to a function, denoted by $s_1(t) \in C_T^{0+1}[0,T]$. The uniqueness of the weak solution indicates that the whole sequence $s_n(t)$ must converge to $s(t)$. Since $u_n\big(s_n(t),t\big) = \frac{1}{n} \longrightarrow 0$, we have $u\big(s_1(t),t\big) = 0$.

To see the more regularity of the interfaces, we note that $u(x,t)$ satisfies the equation (2.1) in the classical sense. The regularity theory for parabolic equations yields that $u(x,t) \in C^{\infty,\infty}(Q_T^-)$ if $g_1(\xi)$ and $f(\xi)$ are smooth. Moreover, since $s_1(t)$ is Lipschitz continuous, we know that $u_x(x,t)$ is continuous up to the boundary $x = s_1(t)$. Let

$$Q_T^{\delta_1\delta_2} = \{(x,t) : \delta_1 < x \leq s_1(t) - \delta_2,\ 0 < t < T\},$$

where δ_1 and δ_2 are positive numbers. We integrate the equation (2.1) over the region $Q_T^{\delta_1\delta_2}$ and perform the integration by parts, we have:

$$\int_{\delta_1}^{s_1(t)-\delta_2} A\big(u(x,t)\big)\,dx - \int_{\delta_1}^{s_1(0)-\delta_2} A\big(u_0(x)\big)\,dx + \int_0^t A\Big(u\big(s_1(t)-\delta_2,t\big)\Big)\dot{s}_1(t)\,dt \\ = \int_0^t [u_x(s_1(t)-\delta_2,t) - u_x(\delta_1,t)]\,dt + \int_0^t\int_{\delta_1}^{s_1(\tau)-\delta_2} f(u)\,dx\,d\tau.$$

Note that $A(0-) = 0$ and $u_x(x,t)$ is continuous up to $s_1(t)$; by taking $\delta_2 \longrightarrow 0$, we obtain

$$\int_{\delta_1}^{s_1(t)} A\big(u(x,t)\big)\,dx - \int_{\delta_1}^{s_1(0)} A\big(u_0(x)\big)\,dx \\ = \int_0^t [u_x(s_1(t),t) - u_x(\delta_1,t)]\,dt + \int_0^t\int_{\delta_1}^{s_1(\tau)} f(u)\,dx\,dt.$$

It follows that $s_1(t)$ is differentiable.

It is not difficult to see that $u(x,t), s_1(t)$ is now a solution of the following free boundary problem:

$$a_1(u)u_t = u_{xx} + g(u)_x + f(u),\ \delta_1 < x < s(t),\ 0 < t < T; \\ u(\delta_1,t) \text{ is smooth, } u\big(s(t),t\big) = 0,\ 0 \leq t \leq T; \\ u_x\big(s(t),t\big) = u_x(\delta_1,t) + g\big(u(\delta_1,t)\big) + \int_{\delta_1}^{s_1(t)} f(u)\,dx,\ 0 \leq t \leq T.$$

Since $f(0) = 0$, we see that the boundary value of $u(x,t)$ at $x = s(t)$ is smooth. It is a Cauchy-type free boundary problem. The results of [6] implies that $s_1(t) \in C^\infty(0,T]$. The other part of Theorem 2.1 can be shown similarly. ■

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press

REMARK. When $f(u) \equiv 0$ and $a = b$, it is easy to see that

$$A_n(u_n(s_n(t),t)) + A_n(u_n(s_{-n}(t),t)) = A_n\left(\frac{1}{n}\right) + A_n\left(-\frac{1}{n}\right) \longrightarrow -1.$$

By the analogous computation as in [5], we can show $s_1(t) = s_2(t)$. For the proof of Theorem 2.2, the reader is referred to [5], [6] and [16].

ACKNOWLEDGMENT. The author wishes to thank Professor M. Chadam for his valuable comments.

REFERENCES

1. M. Bertsch, P. De Mottoni and L. A. Peletier, *Degenerate diffusion and the Stefan problem*, Nonlinear Analysis **8**(1984), 1311–1336.

2. ———, *The Stefan problem with heating: Appearance and disappearance of a mushy region*, Trans. A. M. S. **293**(1986), 677–691.

3. M. Bertsch and M. H. A. Klaver, *The Stefan problem with mushy regions: continuity of the interfaces*, Proceedings of Royal Society of Edinburgh **112A**(1989), 33–52.

4. J. R. Cannon and H. M. Yin, *On the existence of the weak solution and the regularity of the free boundary for a degenerate Stefan problem*, J. Diff. Eq. **73**(1988), 104–118.

5. ———, *A periodic free boundary problem arising in some diffusion chemical reaction processes*, Nonlinear Analysis, TMA **15**(1990), 639–648.

6. A. Fasano, M. Primicerio, *Cauchy-type free boundary problems for nonlinear parabolic equations*, Riv. Mat. Univ. Parma **5**(1980), 695–634.

7. ———, *A phase change model with a zone of coexistence of phases*, preprint.

8. A. Friedman, *Partial differential equations of parabolic type*. Prentice Hall, Englewood Cliffs, N.J. 1964.

9. ———, *The Stefan problem in several space variables*, Trans. Amer. Math. Soc. **133**(1968), 51–87.

10. A. A. Lacey and A. B. Tayer, *A mushy region in a Stefan problem*, IMA J. Appl. Math. **30**(1983), 303–313.

11. A. M. Meirmarov, *An example of nonexistence of a classical solution of the Stefan problem*, Dokl. A. N. SSSR **258**(1981), 564–566.

12. R. H. Nochetto, *A class of nondegenerate two-phase Stefan problems in several space variables*, Comm. P. D. E. **12**(1987), 21–45.

13. O. A. Oleinik, *A method of solution of the general Stefan problem*, Dokl. A.N. SSSR. **135**(1960), 1054–1057.

14. M. Primicerio, *Mushy regions in phase-change problems*, in Applied nonlinear functional analysis. (eds. R. Gorenflo, K. Hoffmann), Frankfurt, 1982.

15. L. Rubinstein, *On mathematical models for solid-liquid zones in a two-phase monocomponent system and in binary alloys*, in Free boundary problems theory and applications, Vol. III (eds. A. Fasano and P. Primicerio), Pitman, 1985.

16. H. M. Yin, *The classical solution of the periodic Stefan problem*, Journal of Partial Differential Equations **1**(1988), 43–60.

Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario L8S 4K1

Current Address:
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1

https://doi.org/10.4153/CMB-1992-020-4 Published online by Cambridge University Press