Classification of Geometries with Projective Metric.

By D. M. Y. SOMMERVILLE, D.Sc.

(Read and Received 14th January 1910).

§1. In the Cayley-Klein projective metric it is ordinarily assumed that the measure of angles, plane and dihedral, is always elliptic, *i.e.* in a sheaf of planes or lines there is no actual plane or line which makes an infinite angle with the others. With this restriction there are only three* kinds of geometry—Parabolic, Hyperbolic and Elliptic, *i.e.* the geometries of Euclid, Lobachevskij and Riemann ; and the form of the absolute is also limited. Thus in plane geometry the only degenerate form of the absolute which is possible is two coincident straight lines and a pair of imaginary points ; in three dimensions the absolute cannot be a ruled quadric, other than two coincident planes. If, however, this restriction as to angular measurement is removed, there are 9 systems of plane geometry and 27 in three dimensions ; for the measure of distance, plane angle and dihedral angle may be parabolic, hyperbolic, or elliptic.

It is the object of this paper to classify the geometries with reference to the form of the absolute, first in three dimensions and then generally in *n* dimensions.

§2. The points, lines, planes in the general geometry are represented by the points, lines, planes of ordinary Euclidean space. The points at infinity or *absolute points* are represented by the points of a quadric surface, the *absolute* : lines and planes at infinity or *absolute lines and planes* are represented by the tangent lines and planes of this surface. On every line there are two

---

* A distinction is not made here between the "antipodal" and the "polar" forms of elliptic geometry. The antipodal form (as in ordinary spherical geometry) does not in fact make its appearance at all in the projective metric, for two lines determine just one point, as two points determine just one line.

absolute points, and through every line there are two absolute planes. On every plane there is an assemblage of absolute points and lines, represented by the points and tangents of a conic ; and through every point there is an assemblage of absolute lines and planes, represented by the generators and tangent planes of a cone. The " distance " (*i.e.* distance, plane angle or dihedral angle) between two elements of the same kind is defined thus :

Two elements, P, Q (points, intersecting lines, planes), determine a one-dimensional geometric form PQ (range of points, pencil of lines, sheaf of planes), and in this form there are two absolute elements, X, Y. Then, (XY, PQ) representing the cross-ratio of the range, pencil or sheaf, we have

$$(PQ) = \text{const} \times \log(XY, PQ).$$

For points, lines and planes the constant has independent values, K, k, κ.

§3. We have the following forms of pairs of elements :

(1) X, Y real and not separating P, Q. (XY, PQ) is positive and $\log(XY, PQ)$ is real.

(2) X, Y real and separating P, Q. (XY, PQ) is negative and $\log(XY, PQ)$ is of the form $a + i\pi$. a is zero if P, Q are harmonic conjugates.

(3) X, Y imaginary. If the equations of P, Q are $P = 0$, $Q = 0$, the equations of X, Y will be of the form $P + (a + ib)Q = 0$, $P + (a - ib)Q = 0$. Then $(XY, PQ) = \dfrac{a + ib}{a - ib} = e^{2i\phi}$ where $\tan\phi = \dfrac{b}{a}$, and $\log(XY, PQ) = 2i\phi$.

(4) X = Y. (XY, PQ) = 1 and $\log(XY, PQ) = 0$.

(5) P = Q. (XY, PQ) = 1 and $\log(XY, PQ) = 0$.

(6) Q = Y ; (XY, PQ) = 0. Q = X ; (XY, PQ) = ∞. In either case $\log(XY, PQ) = \pm \infty$.

(7) X = Y = Q. (XY, PQ) is indeterminate.

If the absolute degenerates to two coincident planes, X = Y for any point-pair, but since the distance must in general be finite K must be infinite. In other cases K may be real or purely imaginary. Similarly for k and κ.

Two similar elements are *parallel* when their intersection is on the absolute.

$$X = Y \text{ and } (PQ) = 0.$$

Two similar elements are mutually *perpendicular* when they are harmonic conjugates with respect to the absolute.

$$(XY, PQ) = -1 \text{ and } (PQ) = i\pi C.$$

Distance is a periodic function. If $(XY, PQ) = a + ib$,

$$(PQ) = \text{Clog}(XY, PQ) = C(\log \sqrt{a^2 + b^2} + i\tan^{-1}\frac{b}{a} + 2i\pi n).$$

The period is thus $2i\pi C$.

In the ordinary (elliptic) measurement of angles the period is π and $k = \dfrac{i}{2}$. When two lines are at right angles $b = 0$, $a = -1$, and the angle $= k\pi i = \dfrac{\pi}{2}$. So that this value of the constant gives the ordinary circular measurement of angles. Compare Laguerre's expression for the angle between two lines u, u'; $\phi = \dfrac{i}{2}\log(uu', \omega\omega')$, where ω, ω' are the two isotropic lines through (uu'), *i.e.* the lines joining (uu') to the two circular points. Any other value of the constant simply corresponds to a different unit of measurement. For example, if $k = \dfrac{90i}{\pi}$ the unit angle is the degree.

§4. *The distance between two elements of a sheaf (or range) is a definite multiple of the distance between their polars.*

Let P, Q be two elements of a sheaf PQ, and let X, Y be the absolute elements of PQ; p, q the polars of P, Q, determining with PQ the elements P', Q' of the sheaf PQ. Let x, y be the absolute elements of the range pq.

Then $\qquad (XY, PP') = -1 = (XY, QQ')$,

and $\quad (xy, pq) = XY, P'Q') = (XY, P'P) \cdot (XY, PQ) \cdot (XY, QQ')$
$$= (XY, PQ).$$

But $\qquad (pq) = \text{clog}(xy, pq) \text{ and } (PQ) = \text{Clog}(XY, PQ)$

Hence $\qquad\qquad\qquad \dfrac{(pq)}{c} = \dfrac{(PQ)}{C}.$

The distance between dissimilar elements can be expressed in terms of the three constants, K, k, κ.

For example, the angle between a line a and a plane a may be defined thus. a cuts a in a point O. Take any other point P on a and find the line PM perpendicular to a (*i.e.* join P to the pole of a, cutting a in M). Then the angle $\widehat{aa}$ is a definite multiple of the angle POM. Or it may be defined in this way. Let the polar of a cut a in S. Then the angle $\widehat{aa}$ is a definite multiple of the angle between the planes a and Sa. Or, let the polar of O cut a and OM in points A, B. Then the angle $\widehat{aa}$ is a definite multiple of the distance (AB). These multiples are taken in such a way that the angle between an actual plane and an actual line is real if they intersect in an actual point.

Two non-intersecting lines a, b have two distances which may be defined thus. Let a', b' be the polars of a, b. Then there are two lines which cut the four lines a, b, a', b', and these are perpendicular to both a and b. The lengths of these common perpendiculars, multiplied by an appropriate constant, are the two distances.

If the two lines and their polars belong to the same regulus of a ruled surface of the second order there are an infinite number of common perpendiculars and they are all equal.

For take any two lines of the other regulus, cutting a, a', b, b', in P, P', Q, Q' and P_1, $\dot{P_1}'$, Q_1, Q_1', and the absolute in X, Y and X_1, Y_1. Then $(PP', QQ') = (P_1, P_1', Q_1, Q_1')$; X, Y are the double points of the involution (PP', QQ') and X_1, Y_1 are the double points of the involution (P_1, P_1', Q_1, Q_1'). Hence PP'QQ'XY and $P_1 P_1' Q_1 Q_1' X_1 Y_1$ are projective ranges, and $(XY, PQ) = (X_1 Y_1, P_1 Q_1)$.

§5. Let us now investigate the different systems of geometry. We have three constants to fix, and any of them may be infinite, real or imaginary, hence there are 27 possible systems. These depend upon the form of the absolute and the conditions laid down with regard to the actual* and ideal elements. We shall make the following assumptions :

---

* The term "actual" here is opposed to "ideal," and is preferred to "real," which is opposed to "imaginary."

1. *An actual geometric form contains actual elements.*

2. *The distance between two actual elements of an actual geometric form is real.*

Having fixed upon one plane a as an actual plane, a line a in a as an actual line, and a point A in a as an actual point, all points at a real finite distance from A are actual points.* A line is actual if it makes a real angle with a, or if it makes a real angle with an actual line; and similarly for planes. The actual points are separated from the ideal points by the absolute. The actual elements of an actual sheaf (*e.g.* a sheaf of lines passing through an actual point and lying in an actual plane) are separated from the ideal elements by the two absolute elements of the sheaf.

The values of the constants are as follows :—

K, k or κ is infinite if the absolute degenerates to two coincident planes, lines or points.

K is real or imaginary, according as actual lines do or do not cut the absolute.

κ is real or imaginary, according as actual lines do or do not project the absolute.

k is real or imaginary according as actual points in actual planes do or do not project the section of the absolute.

When K, k, κ is infinite, real, imaginary, the measure of distance, plane angle, dihedral angle is parabolic, hyperbolic, elliptic. In ordinary geometry, in hyperbolic geometry, and in elliptic geometry the measure of angles, plane and dihedral, is elliptic; k and κ are both imaginary, while K is infinite, real, or imaginary.

§6. The forms of the absolute and the various geometries are discussed as follows :—

A. *Absolute a proper quadric.*

    I. *Imaginary.*

        K, k, κ all imaginary. Distances and angles are always real and periodic. (ELLIPTIC GEOMETRY.)

    II. *Real and not ruled.*

        The absolute divides space into an actual and an ideal region of points, lines, and planes, and possesses an

---

* It may happen that the harmonic conjugate A′ of A is at a real finite distance from A, but points on AA′ in the vicinity of A′ are ideal. In this case A′ is ideal.

interior and an exterior.   A line projects the quadric if it does not cut it.

1. Actual points within.   Actual lines and planes cut the quadric.   K real, *k* and *κ* imaginary. (HYPERBOLIC GEOMETRY.)

2. Actual points outside.   Actual lines and planes cut the quadric.   K and *k* real, *κ* imaginary.

3. Actual points outside.   Actual planes cut the quadric, but actual lines do not.   K imaginary, *k* and *κ* real.

4. Actual points outside.   Actual lines and planes do not cut the quadric.   K and *k* imaginary, *κ* real.

III. *Ruled.*

There is no point from which real tangent lines and planes may not be drawn to the quadric, and every plane cuts the quadric.   A line projects the quadric if it cuts it.

1. Actual lines cut the quadric.

Take any such line and draw an arbitrary plane through it, cutting the quadric in a conic S.   Let this plane be actual.   Then there are two cases.

(*a*) Points within S are actual.

K real, *k* imaginary, *κ* real.

(*b*) Points outside S are actual.

K, *k*, *κ* real.

2. Actual lines do not cut the quadric.

K imaginary, *k* real, *κ* imaginary.

In the case of a ruled quadric there are two systems of lines which do not cut the quadric, and these are separated by the quadric.   If, therefore, we fix upon one line as actual, all lines of the other system are ideal since they contain no actual points.   The absolute divides the points of space into two sets, and it is arbitrary which set we agree to take as actual.

B. *Absolute a simply degenerate quadric.*

I. *A cone, two coincident points.*   *κ* = ∞.

1. *Imaginary cone.*

K and *k* imaginary.

2. *Real cone.*

(*a*) Actual points within.   K real, *k* imaginary.

(*b*) Actual points outside. Actual lines cut the cone. K, *k* real.

(*c*) Actual points outside. Actual lines do not cut the cone. K imaginary, *k* real.

II. *Two coincident planes, proper conic.* K = ∞.

1. *Imaginary conic.*

   k, κ imaginary. (PARABOLIC GEOMETRY).

2. *Real conic.*

   (*a*) Actual lines pass within the conic.

   k real, κ imaginary.

   (*b*) Actual lines and planes pass outside the conic. *k* imaginary, κ real.

   (*c*) Actual lines pass outside, actual planes cut the conic. *k*, κ real.

III. *Two planes, two coincident lines, two points.* *k* = ∞.

1. *Imaginary planes, imaginary points.*

   K, κ imaginary.

2. *Imaginary planes, real points.*

   K imaginary, κ real.

3. *Real planes, imaginary points.*

   K real, κ imaginary.

4. *Real planes, real points.*

   K, κ real.

C. *Absolute a doubly degenerate quadric.*

I. *Two coincident planes, two coincident lines, two points.*

   K, *k* = ∞.

1. *Imaginary points.*   κ imaginary.

2. *Real points.*   κ real.

II. *Two coincident planes, two lines, two coincident points.*

   K, κ = ∞.

1. *Imaginary lines.*   *k* imaginary.

2. *Real lines.*   *k* real.

III. *Two planes, two coincident lines, two coincident points.*

   k, κ = ∞.

1. *Imaginary planes.*   K imaginary.

2. *Real planes.*   K real.

D. *Absolute a triply degenerate quadric.*

   Two coincident planes, two coincident lines, two coincident points. K, *k*, κ = ∞.

§7.   These 27 geometries are tabulated in the following scheme.

K = ∞. *Two coincident planes.*

	κ = ∞	κ real	κ imaginary
k = ∞	*Two coincident lines.* *Two coincident points.*	*Two coincident lines.* *Two real points.*	*Two coincident lines.* *Two imaginary points.*
k real	*Two real lines.* *Two coincident points.*	*Real conic.* (Actual lines pass outside conic; actual planes cut conic).	*Real conic.* (Actual lines pass within conic).
k imaginary	*Two imaginary lines.* *Two coincident points.*	*Real conic.* (Actual planes pass outside conic).	*Imaginary conic.* PARABOLIC GEOMETRY.

K real.

	$\kappa = \infty$	κ real	κ imaginary.
$k = \infty$	*Two real planes. Two coincident lines. Two coincident points.*	*Two real planes. Two coincident lines. Two real points.*	*Two real planes. Two coincident lines. Two imaginary points.*
k real	*Real cone.* (Actual points outside; actual lines cut cone).	*Ruled quadric.* (Actual lines cut quadric; actual points outside curve of section of actual plane).	*Real quadric, not ruled.* (Actual points outside; actual lines cut quadric).
k imaginary	*Real cone.* (Actual points within).	*Ruled quadric.* (Actual lines cut quadric; actual points within curve of section of actual plane).	*Real quadric, not ruled.* (Actual points within). HYPERBOLIC GEOMETRY.

K imaginary.

	$\kappa = \infty$	κ real	κ imaginary
$k = \infty$	*Two imaginary planes.* *Two coincident lines.* *Two coincident points.*	*Two imaginary planes.* *Two coincident lines.* *Two real points.*	*Two imaginary planes.* *Two coincident lines.* *Two imaginary points.*
k real	*Real cone.* (Actual points outside; actual lines do not cut cone).	*Real quadric, not ruled.* (Actual points outside; actual planes cut quadric; actual lines do not).	*Ruled quadric.* (Actual lines do not cut quadric).
k imaginary	*Imaginary cone.*	*Real quadric, not ruled.* (Actual planes do not cut quadric).	*Imaginary quadric.* ELLIPTIC GEOMETRY.

For two dimensions the discussion is similar but simpler, and the final table is

	K = ∞	K real	K imaginary
k = ∞	*Two coincident lines.* *Two coincident points.*	*Two real lines.* *Two coincident points.*	*Two imaginary lines.* *Two coincident points.*
k real	*Two coincident lines.* *Two real points.*	*Real conic.* (Actual points outside; actual lines cut conic).	*Real conic.* (Actual lines do not cut conic).
k imaginary	*Two coincident lines.* *Two imaginary points.* PARABOLIC GEOMETRY.	*Real conic.* (Actual points within), HYPERBOLIC GEOMETRY.	*Imaginary conic.* ELLIPTIC GEOMETRY.

§8. We shall now extend the investigation to space of n dimensions. Here there are n constants, k_0, k_1, ..., k_{n-1}, and therefore 3^n geometries.

The absolute takes the following forms:

A_0. *A proper hyperquadric of n dimensions ("n-quadric").*

    I. *Imaginary.*

    II. *Real and not ruled.*

    III. *Ruled.*

A_1. *Simply degenerate.*

    (r) *A hypercone of species r of n dimensions* ("(n, r)-cone").

        This is formed by joining the points of a proper $(n-r)$-quadric to the points of an $(r-1)$-flat (the *axis*), and in the axis is taken a proper $(r-1)$-quadric. An (n, n)-cone consists of two coincident $(n-1)$-flats with an $(n-1)$-quadric; an $(n, n-1)$-cone consists of two $(n-1)$-flats with an $(n-2)$-quadric; and an $(n, 0)$-cone is a proper n-quadric.

A_l. *l-ply degenerate.*

    $(r_1, r_2 \ldots, r_l)$. Two coincident $(r_1 - 1)$-, $(r_2 - 1)$-, ...$(r_l - 1)$-flats $(r_1 < r_2 < \ldots < r_l)$. An (n, r_l)-cone with an (r_l, r_{l-1})-cone in its axis, and an (r_{l-1}, r_{l-2})-cone in the axis of the second hypercone, and so on, and finally an $(r_1 - 1)$-quadric in the axis of the last hypercone. ·

The number of geometries with non-degenerate absolute is 2^n. With an l-ply degenerate absolute with the symbol $(r_1, r_2, \ldots, r_l)$ there are 2^{n-l} geometries; and there are $_nC_l$ different l-ply degenerate absolutes.

§9. The geometries with non-degenerate absolute are classified— first, partly according to the nature of the absolute; second, partly according to the nature of the section of the absolute by an "actual" $(n-1)$-flat. If the absolute is real and not ruled the section may be real and not ruled or imaginary; if the absolute is ruled the section may be real and not ruled or ruled; if the absolute is imaginary the section is imaginary. Thirdly, the geometries are classified partly according to the nature of the section by an actual $(n-2)$-flat, and so on.

If the absolute is not ruled the tangent $(n-1)$-flats through an $(n-2)$-flat are imaginary or real according as the $(n-2)$-flat does or

does not cut the absolute. If the absolute is ruled the tangent $(n-1)$-flats through a non-intersecting $(n-2)$-flat are imaginary, but those through an intersecting $(n-2)$-flat may be either real or imaginary.

To prove these statements consider first an unruled quadric. The tangent $(n-1)$-flats through an $(n-2)$-flat are determined by the $(n-2)$-flat and the points in which the polar line of the $(n-2)$-flat cuts the quadric. The polar line of an intersecting $(n-2)$-flat is the intersection of the $(n-1)$-flats which touch the quadric at its intersection. These have only one real point in common with the quadric, and hence the polar line does not cut the quadric. On the other hand, the polar line of a non-intersecting $(n-2)$-flat is the locus of the poles of the $(n-2)$-flat with respect to the section of the quadric by $(n-1)$-flats through the $(n-2)$-flat. Now some of these $(n-1)$-flats cut the quadric, and the corresponding poles lie within the section : the polar line therefore cuts the quadric.

Suppose next that the quadric is ruled,* and consider a non-intersecting $(n-2)$-flat. Then all the $(n-1)$-flats through it cut the quadric in unruled sections, for a $(p-1)$-flat always cuts a ruled quadric in R_p in a real section. Hence the tangent $(n-1)$-flats must be imaginary for they would cut the quadric in hypercones. But the $(n-1)$-flats through an intersecting $(n-2)$-flat may cut the quadric in sections either ruled or unruled. If one section is unruled then the section by the $(n-2)$-flat must be unruled ; but all the sections cannot be unruled, for then there would be no lines

---

* The discussion for a ruled quadric is not complete, as in space of n dimensions, R_n, there are ruled quadrics of different *ranks*, viz., in R_{2p-1} or R_{2p} ruled quadrics may contain lines, planes, 3-flats, ... or $(p-1)$-flats. At any stage these may become imaginary, so that there are quadrics of rank $p-1$ down to 0 (unruled) and -1 (imaginary). Central quadrics of each rank exist. If the equation of a central quadric in R_{n-1} be written in homogeneous coordinates $\Sigma a_r x_r^2 = 0$, where k of the coefficients are positive and $n-k$ negative, the rank is $\frac{1}{2}\{n - \mid 2k-n \mid \} - 1$. For some of the properties of ruled quadrics see *Bertini, Introduzione alla geometria proiettiva degli iperspazi*, Pisa, 1907.

The discussion in the text is sufficient, however, for the classification. The theorems relating to the tangent $(n-1)$-flats through an $(n-2)$ flat may be regarded merely as existence-theorems, the circumstances under which they are true not being completely discussed.

belonging to the surface passing through any of the points of the section by the $(n-2)$-flat. The limits between the ruled and the unruled sections are the tangent $(n-1)$-flats, which are therefore real. On the other hand, if the section by the $(n-2)$-flat is a ruled quadric of the same rank as the absolute, so also is every section through the $(n-2)$-flat. The tangent $(n-1)$-flats, if they were real, would meet the absolute in hypercones, and the $(n-2)$-flat would cut the hypercones in ruled quadrics of *lower* rank unless it contained the vertex. Hence either the tangent $(n-1)$-flats are imaginary or the $(n-2)$-flat is itself a tangent to the quadric.

I. If the absolute is imaginary there is one geometry, since all the sections are imaginary. The constants k_0, ..., k_{n-1} are all imaginary. (Elliptic geometry).

II. The number of geometries with a real unruled absolute is $n+1$. This may be proved by induction. Assume that there are r sub-classes of geometries according to the nature of the section by an actual intersecting $(r-1)$-flat. Then if the section by an actual r-flat is real, the section by an actual $(r-1)$-flat may be real or imaginary. In the former case we have r sub-classes and in the latter 1. But for the section by an intersecting line there are two sub-classes, according as actual points are within or outside the section by an actual plane. If actual r-flats cut the absolute while actual $(r-1)$-flats do not $(n>r>1)$, k_0, ..., k_{r-2}, k_{r+1}, ..., k_{n-1} are all imaginary, while k_{r-1} and k_r are real; if actual $(n-1)$-flats do not cut the absolute k_0, ..., k_{n-2} are imaginary while k_{n-1} is real; and if actual points lie within the absolute (Hyperbolic geometry) k_0 is real and k_1, ..., k_{n-1} are imaginary. The values of the constants are therefore tabulated as follows :—

$$\begin{array}{cccccc}
k_0 & k_1 & k_2 & & k_{n-2} & k_{n-1} \\
r & i & i & \ldots\ldots & i & i \\
r & r & i & \ldots\ldots & i & i \\
i & r & r & \ldots\ldots & i & i \\
& & & \ldots\ldots\ldots\ldots\ldots\ldots & & \\
i & i & i & \ldots\ldots & r & r \\
i & i & i & \ldots\ldots & i & r \\
\end{array}$$

III. If the absolute is ruled there are the remaining $2^n - (n+2)$ geometries. If the section by an actual $(n-1)$-flat is an unruled

quadric this gives n geometries. For these the constants are as follows :—

k_0	k_1	k_2		k_{n-3}	k_{n-2}	k_{n-1}
r	i	i	 i	i	r
r	r	i	 i	i	r
i	r	r	 i	i	r
....
i	i	i	...... r	r	r
i	i	i	 i	r	i

If the section by an actual $(n-1)$-flat is ruled there are two cases according as an actual $(n-2)$-flat does or does not project the absolute. Corresponding to each of these cases there will be $\frac{1}{2}(2^n - n - 2 - n) = 2^{n-1} - (n+1)$ geometries. In the first case k_{n-1} is real, in the second case imaginary; the values of the other constants are the same in the two cases, and are just those which correspond to a ruled quadric in space of $n-1$ dimensions. The next classification is according to whether the section by an actual $(n-2)$-flat is ruled or unruled. The latter case gives $(n-1)$ geometries. The former gives again two cases according as an actual $(n-3)$-flat does or does not project the section of the absolute by the actual $(n-1)$-flat, and each gives $2^{n-2} - n$ geometries; in the first set k_{n-2} is real, in the second imaginary. Continue this classification down to the section by a ruled 3-flat. This gives two cases according as an actual plane does or does not project the section of the absolute by an actual 4-flat, and each gives $2^3 - 5 = 3$ geometries. These are classified lastly according as :—

1. An actual line cuts the absolute, and actual points lie within the curve of section of an actual plane. (k_0 real, k_1 imaginary, k_2 real).

2. An actual line cuts the absolute, and actual points lie outside the curve of section of an actual plane. (k_0 real, k_1 real, k_2 real).

3. An actual line does not cut the absolute. (k_0 imaginary, k_1 real, k_2 imaginary).

§10. The geometries with an l-ply degenerate absolute with symbol $(r_1, r_2, ..., r_l)$ are classified first according to the nature of the hypercones, real or imaginary, and then according to the nature of the sections. $k_{n-r_i} = \infty$, $(i = 1, 2, ..., l)$.

A real (n, r)-cone is, in general, cut by an $(n-1)$-flat in a real $(n-1, r-1)$-cone, for the $(n-1)$-flat cuts the axis in an $(r-1)$-flat and every line through any point cuts the $(n-1)$-flat.   Hence a real (n, r)-cone is, in general, cut by any p-flat $(p \geqslant n-r+1)$ in a real $(p, p+r-n)$-cone.   If $p \leqslant n-r$ the section is a real p-quadric.

With the (n, r_t)-cone we have then the following cases:—

1. The hypercone imaginary. $k_0, k_1, \ldots, k_{n-r_t-1}$ all imaginary.

2. The hypercone real and cut by an actual $(n-r_t)$-flat in a real hyperquadric, with reference to which the geometries are classified into $2^{n-r_t}-1$ classes, and $k_0, k_1, \ldots, k_{n-r_t-1}$ take all combinations of real and imaginary values except all imaginary.

If $n-r_t = 2$ the hyperquadric is a proper conic.   The classification of the geometries with respect to this is as follows:—

1. The conic imaginary. k_0, k_1 imaginary.
2. Real points within. k_0 real, k_1 imaginary.
3. Real lines cut the conic, real points outside. k_0, k_1 real.
4. Real lines do not cut the conic. k_0 imaginary, k_1 real.

There are thus 2^2 cases, which agrees with the general result for quadrics.

With the first hypercone there are thus 2^{n-r_t} classes.

With the second hypercone we have the following cases:—

1. The hypercone imaginary. $k_{n-r_t+1}, k_{n-r_t+2}, \ldots, k_{n-r_{t-1}-1}$ all imaginary.

2. The hypercone real and cut by an actual $(r_t - r_{t-1} - 1)$-flat lying in the $(r_t - 1)$-flat in a real hyperquadric, with reference to which the geometries are classified into $2^p - 1$ classes, where $p = r_t - r_{t-1} - 1$.

   $k_{n-r_t+1}, k_{n-r_t+2}, \ldots, k_{n-r_{t-1}-1}$ take all combinations of real and imaginary values except all imaginary.

There are thus 2^p classes according to the nature of the second hypercone and its sections.

Finally in the axis of the (r_2, r_1)-cone there is a hyperquadric, real or imaginary, with reference to which the geometries are classified into 2^{r_1-1} classes. $k_{n-r_1+1}, k_{n-r_1+2}, \ldots, k_n$ take all combinations of real and imaginary values.

Hence altogether there are

$$2^{n-rl} \cdot 2^{r_l - r_{l-1} - 1} \ldots 2^{r_3 - r_2 - 1} \cdot 2^{r_2 - r_1 - 1} \cdot 2^{r_1 - 1} = 2^{n-l}$$

geometries with an l-ply degenerate absolute with a given symbol.

§11. This completes the discussion of the general case.

A word may be said in elucidation of the method of determining the values of the constants in the case of a degenerate absolute. For simplicity suppose the absolute to be simply degenerate with $k_{n-4} = \infty$. It therefore has two coincident 3-flats and consists of an $(n, 4)$-cone with a proper quadric in its axis. The values of k_0, k_1, ..., k_{n-5} are determined with reference to a real $(n-4)$-quadric; the values of k_{n-3}, k_{n-2}, k_{n-1} with reference to a 3-quadric. An $(n-1)$-flat, an $(n-2)$-flat and an $(n-3)$-flat cut the 3-flat containing this quadric in a plane, a line and a point respectively, and then k_{n-3}, k_{n-2}, k_{n-1}, are considered just as if they referred to point-pairs, plane and dihedral angles. Thus if the quadric is real and not ruled, and actual $(n-3)$-flats pass within it, the absolute $(n-1)$-flats of a sheaf of $(n-1)$-flats through an actual $(n-2)$-flat are imaginary, and so also are the absolute $(n-2)$-flats of a sheaf of $(n-2)$-flats through an actual $(n-3)$-flat and lying in an actual $(n-1)$-flat, but the absolute $(n-3)$-flats of a sheaf of $(n-3)$-flats through an $(n-4)$-flat and lying in an actual $(n-2)$-flat are real, since these are determined by the $(n-4)$-flat and the real points in which the $(n-2)$-flat cuts the quadric; *i.e.* k_{n-1} and k_{n-2} are imaginary, while k_{n-3} is real.