On the Choice of Algebra for Quantization

Benjamin H. Feintzeig*†

In this article, I examine the relationship between physical quantities and physical states in quantum theories. I argue against the claim made by Arageorgis that the approach to interpreting quantum theories known as Algebraic Imperialism allows for "too many states." I prove a result establishing that the Algebraic Imperialist has very general resources that she can employ to change her abstract algebra of quantities in order to rule out unphysical states.

1. Introduction. To construct a quantum theory, one performs a procedure known as "quantization." Quantization can be thought of as having two steps: first, one constructs an abstract C*-algebra to represent the physical quantities, or observables, of the system, which must obey the canonical commutation relations, and second, one finds a representation of that algebra in the bounded operators on some Hilbert space. Much philosophical attention has been directed at this second step, as it is now well known that in many cases of physical interest, including quantum field theories and quantum statistical theories in the thermodynamic limit, there is not a unique Hilbert space representation of the algebra. This leads to two general interpretive options. The *Hilbert Space Conservative* claims that we must pick one particular Hilbert space representation of our algebra (out of the many competing ones) and use this one as our quantum theory. But, the *Algebraic Imperialist* asserts that we do not need any Hilbert space representations to interpret our theory and instead can think of our physical theory as comprised by the abstract algebra itself.

The purpose of this article is to argue that the methods of the Algebraic Imperialist have a particular virtue, which I call *adaptability*. By this, I mean

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*To contact the author, please write to: Department of Philosophy, University of Washington, Savery Hall, Seattle, WA 98195-3550; e-mail: bfeintze@uw.edu.

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that one can make small but systematic changes in the abstract algebraic framework to deal with problems that arise.

The particular problem that I am concerned with confronting from the algebraic perspective here is how to construct a quantum theory that allows for the correct space of physically possible states. In many physical theories, one can find models that appear pathological or unphysical. And indeed, it has been claimed that one of the downfalls of the algebraic framework for quantum theories is that it allows for "too many states." And so, the argument goes, one is forced to use Hilbert space methods to restrict the physical state space under consideration.

I show in this article, however, that one need not appeal to Hilbert space methods to perform this reduction of the state space. Instead, one can reduce the physical state space using algebraic methods alone. I present a result that establishes precise conditions under which this reduction of the state space can be performed. This result provides a general strategy through which to use algebraic methods to construct quantum theories.

In slightly more detail, the Algebraic Imperialist advocates using the collection of states on the abstract algebra of the canonical commutation relations as the state space of our quantum theory. But there are states on this algebra that some researchers describe as unphysical. These unphysical states often cannot be represented by any density operator in the relevant Hilbert space; this means they are ruled out by the Hilbert Space Conservative, who countenances as possible only states that appear in her favored Hilbert space representation. So it seems that the Algebraic Imperialist allows for extra states that Hilbert space methods can excise from the theory.

The response I want to suggest here, on behalf of the Algebraic Imperialist, is that we have been considering the *wrong algebra*, at least for the purposes of assessing the space of physical states. There is often a different algebra that one can use, which has exactly the physical state space we were looking for. Moreover, I will show that there is a completely general and systematic way of changing the abstract algebra—the one we use to implement the canonical commutation relations—in order to reduce the physical state space. Importantly, this general procedure is appropriate for the Algebraic Imperialist because it does not use Hilbert space methods. This allows the Algebraic Imperialist a new freedom in constructing quantum theories, and it leads to a new perspective on the issues surrounding quantization. In this sense, the argument of this article can be thought of as an argument in favor of Algebraic Imperialism because it shows how much more one can do with algebraic methods than previously thought.

But the arguments that I present need not come attached to the interpretive position of Imperialism; the results of this article are important for anyone who wants to use a quantization procedure to construct quantum theories, including the Hilbert Space Conservative. Before one can even begin asking about the necessity of Hilbert space representations, one must grapple with the question of how to construct an appropriate C*-algebra in the first step of the quantization procedure. According to the Imperialist, this is where all of the interesting physical development happens in quantization, but even the Conservative needs to have some algebra in mind before she can take its representations. Different options for the abstract algebra of observables appear in the physics literature, but so far only one of these options has been discussed in the philosophical literature to my knowledge. As such, there has been no systematic discussion of how to choose between different algebras. One contribution of this article, independent of the dispute between Imperialists and Conservatives, will be to provide tools for choosing the appropriate algebra.

I want to note that I will not in this article make a particular recommendation as to which algebra we ought to use as the algebra of physical observables in any specific case. I will comment briefly on the physical, mathematical, and methodological issues that lead us to the algebra that is standardly used to implement the canonical commutation relations and on why I think we ought to consider other algebras, too. But what I mostly hope to show is that the question of which algebra to use is inextricably tangled with the question of what states are physical, in such a way that any progress toward answering one question yields progress for answering the other. I will have to save more detailed analysis of which physical states spaces (and hence which algebras) are appropriate for future work.

The article is organized as follows. Section 2 briefly presents mathematical preliminaries concerning both abstract algebraic methods and Hilbert space representations. Section 3 presents the objection that Imperialism allows for "too many" states, which we take up in section 4. There, we present a general result concerning the reduction of the state space of an abstract algebra and show that it specifically allows the Algebraic Imperialist at least all of the same resources that the Hilbert Space Conservative has for state space reduction. Section 5 concludes with a discussion of the significance of the results and further open questions.

2. Mathematical Preliminaries. The bounded observables of a physical theory carry the structure of a *C*-algebra*.[1] This means that one may add and multiply observables and multiply observables by scalars. In addition, a C*-algebra carries an operation of involution that is a generalization of

1. In this section, we present only the minimal technical background required to state the results of sec. 4. For more on operator algebras, see Sakai (1971), Kadison and Ringrose (1997), and Landsman (1998). For more on algebraic quantum theory, see Emch (1972), Haag (1992), Wald (1994), and Bratteli and Robinson (1996). For philosophical introductions, see Halvorson (2006) and Ruetsche (2011).

complex conjugation. A C*-algebra $\mathfrak{A}$ comes equipped with a norm, which is required to satisfy the C*-identity:

$$\| A^* A \| = \| A \|^2,$$

for all $A \in \mathfrak{A}$. The norm defines a topology, called the *norm topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges to A in the norm topology if and only if (iff)

$$\| A_i - A \| \to 0,$$

where the convergence is now in the standard topology on $\mathbb{R}$.[2] The C*-algebra $\mathfrak{A}$ is required to be complete with respect to this topology in the sense that for every Cauchy net $\{A_i\} \subseteq \mathfrak{A}$ there is an $A \in \mathfrak{A}$ such that $A_i \to A$ in the norm topology. Here, a net $\{A_i\} \subseteq \mathfrak{A}$ is Cauchy just in case

$$\| A_i - A_j \| \to 0.$$

Standard results in the theory of normed vector spaces tell us that every normed vector space has a unique completion.[3]

Since $\mathfrak{A}$ is a vector space, we can also consider the dual space $\mathfrak{A}^*$ of bounded (i.e., norm continuous) linear functionals $\rho : \mathfrak{A} \to \mathbb{C}$. A *state* on a C*-algebra $\mathfrak{A}$ is just a particular kind of element of the dual space $\mathfrak{A}^*$, namely, one that is positive and normalized.[4]

The dual space $\mathfrak{A}^*$ can be used to define an alternative to the norm topology on $\mathfrak{A}$, called the *weak topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges in the weak topology to $A \in \mathfrak{A}$ iff for every $\rho \in \mathfrak{A}^*$,

$$\rho(A_i) \to \rho(A),$$

where the convergence is now in the standard topology on $\mathbb{C}$. The weak topology is the coarsest topology on $\mathfrak{A}$ with respect to which all of the linear functionals in $\mathfrak{A}^*$ are continuous.

A C*-algebra $\mathfrak{A}$ need not be complete with respect to its weak topology; there may be nets $\{A_i\} \subseteq \mathfrak{A}$ that are Cauchy in the sense that

$$\rho(A_i - A_j) \to 0,$$

2. One could restrict attention here to sequences because the norm topology is second countable, but for the weak topologies considered later, which are not second countable, one must work with arbitrary nets.

3. A complete normed vector space is called a Banach space. A C*-algebra is thus a Banach algebra whose norm is, in a certain sense, compatible with multiplication and involution.

4. A functional $\rho \in \mathfrak{A}^*$ is *positive* if $\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and *normalized* if $\| \rho \| = 1$.

for every $\rho \in \mathfrak{A}^*$ without the net having a limit point $A \in \mathfrak{A}$ such that $A_i \to A$ in the weak topology. However, a C*-algebra can always be completed in its weak topology to form its *bidual* $\mathfrak{A}^{**}$, as follows.[5]

The bidual carries a topology known as the *weak* topology*, which is a natural generalization of the weak topology on $\mathfrak{A}$. The weak* topology is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}^{**}$ converges in the weak* topology to $A \in \mathfrak{A}^{**}$ iff for every $\rho \in \mathfrak{A}^*$,

$$A_i(\rho) \to A(\rho).$$

The weak* topology on the bidual $\mathfrak{A}^{**}$ corresponds precisely to the extension of the condition of convergence for the weak topology on $\mathfrak{A}$ to the larger algebra $\mathfrak{A}^{**}$. In particular, the weak* topology on $\mathfrak{A}^{**}$ is the coarsest topology on $\mathfrak{A}^{**}$ that makes every linear functional in $\mathfrak{A}^*$ continuous. One can show that the bidual $\mathfrak{A}^{**}$ is complete with respect to the weak* topology.

Moreover, the original C*-algebra $\mathfrak{A}$ is canonically embedded in its bidual by $A \in \mathfrak{A} \mapsto \widehat{A} \in \mathfrak{A}^{**}$, with $\widehat{A}$ defined by

$$\widehat{A}(\rho) = \rho(A),$$

for all $\rho \in \mathfrak{A}^*$. With respect to this embedding, $\mathfrak{A}$ is dense in $\mathfrak{A}^{**}$ in the weak* topology, so the bidual $\mathfrak{A}^{**}$ can be understood as the completion of $\mathfrak{A}$ in its weak topology, which is the subspace topology of the weak* topology on $\mathfrak{A}^{**}$.

Importantly, the algebraic approach can be translated back into the familiar Hilbert space formalism for quantum mechanics. A *representation* of a C*-algebra $\mathfrak{A}$ is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism into the bounded linear operators on $\mathcal{H}$.[6]

We can use the Hilbert space structure of a representation $(\pi, \mathcal{H})$ to induce a new topology on the algebra $\pi(\mathfrak{A})$. The *ultraweak topology* is characterized through the following condition for convergence: a net $\pi(A_i)$ converges to $\pi(A)$ in the ultraweak topology if for every density operator ρ on $\mathcal{H}$,

$$Tr(\pi(A_i)\rho) \to Tr(\pi(A)\rho).$$

The ultraweak topology represents a notion of convergence of expectation values by certain states, namely, the density operator states.

A state $\omega \in \mathfrak{A}^*$ has a density operator representative in the representation $(\pi, \mathcal{H})$ just in case there is a density operator ρ_ω such that $\omega(A) = Tr(A\rho_\omega)$ for all $A \in \mathfrak{A}$. In general, there may be states on $\mathfrak{A}$ without density operator representatives in a given representation $(\pi, \mathcal{H})$. In other words, the density op-

5. See Feintzeig (2017b) for more on the completion of a C*-algebra into its bidual.

6. One of the most fundamental results in the theory of C*-algebras, known as the GNS Theorem (see Kadison and Ringrose 1997), tells us that every C*-algebra has Hilbert space representations.

erator states on a representation may not exhaust the states on the abstract algebra $\mathfrak{A}$.[7] However, we know that a state $\omega \in \mathfrak{A}^*$ has a density operator representative in a given representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ just in case ω is ultraweakly continuous in that representation. We use these facts in our discussion of appropriate state spaces below.

3. Interpretive Options. In this section, I explicate the objection that Algebraic Imperialism allows for "too many states." I show that this question is tangled with the choice of algebra of quantum observables in section 3.2, and I briefly illustrate these issues in a simple example in section 3.3.

3.1. Physical States. The procedure of quantization can be understood as involving two steps. First, one finds a quantum algebra of observables $\mathfrak{A}$. Second, one chooses a representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ on some Hilbert space. The interpretive debate between Algebraic Imperialism and Hilbert Space Conservatism concerns just whether this second step is necessary.[8] According to the Algebraic Imperialist, it is not: a quantum theory (or, at least its kinematics) can be captured by the abstract algebra $\mathfrak{A}$ in the sense that the physical quantities of a system can be adequately represented by the elements of $\mathfrak{A}$—or perhaps by weak limits in $\mathfrak{A}^{**}$—and the physical states can be represented by states on $\mathfrak{A}$. But, according to the Hilbert Space Conservative, we need to pick a representation $(\pi, \mathcal{H})$. Then we represent the physical quantities of our system by elements of $\pi(\mathfrak{A})$—or perhaps by ultraweak limits—and the physical states by density operators on $\mathcal{H}$.

The objection that has been leveled at Algebraic Imperialism that I want to discuss is that the Imperialist allows for "too many" states, in the sense that many states on the abstract algebra $\mathfrak{A}$ are unphysical (see Arageorgis 1995). Taking a Hilbert space representation allows us to consider an appropriate collection of physical states by focusing our attention on only the states that have density operator representatives in our chosen representation. So the fact that Hilbert space methods provide resources to rule out unphysical states is supposed to count in favor of the Hilbert Space Conservative.

Before we can see what these unphysical states are, we need to specify the algebra of observables we are using—after all, different algebras will in general have different state spaces. The algebra of observables that gives rise to the unphysical states at issue is known as the *Weyl Algebra*. This algebra puts the canonical commutation relations between position and momentum ob-

7. This follows immediately from the presence of unitarily inequivalent representations of an algebra, as discussed in Ruetsche (2011).

8. For more on Algebraic Imperialism and Hilbert Space Conservatism, see Ruetsche (2002, 2003, 2006, 2011).

servables in bounded form by considering only exponentiated forms of those observables.[9]

The states that one might consider unphysical come in many varieties. For example, the Weyl algebra allows for *nonregular states* (Beaume et al. 1974; Halvorson 2001, 2004), which fail to satisfy a continuity condition and in doing so fail to allow one to simultaneously define both position and momentum observables from the Weyl operators. These nonregular states do not have density operator representatives in the usual Hilbert space representation (the Schrödinger representation) of the Weyl algebra. It is a standard move in algebraic quantum theory to restrict attention only to regular states on the Weyl algebra and their representations to rule out these nonregular states.

But there are many other states on the Weyl algebra that one might consider ruling out as unphysical. Arageorgis (1995) mentions a proposal that we ought to restrict attention to *locally definite* states, which vanish on smaller and smaller regions of space-time. Or perhaps we ought to restrict attention to *Hadamard* states, which allow for an appropriately well-defined stress-energy observable (see Wald 1994). And Halvorson (2006) takes up a suggestion by Doplicher, Haag, and Roberts that we use only what he calls *DHR* states, which differ only locally from the vacuum. In addition, we might want to restrict attention to states accessible from *Fock states*, that is, states that allow for an interpretation in terms of particle states with creation and annihilation operators (see Petz 1990, chap. 4).

The results that I present below in section 4 are immediately applicable to two of the suggestions just considered: regular states and states accessible from Fock states. The reason, as I will remark later, is that these states form the folium of an irreducible Hilbert space representation of the Weyl algebra, and I will show explicitly that the results of section 4 apply to the folium of any irreducible Hilbert space representation of any algebra. For the purposes of this article, I will not analyze any of the other suggestions in detail because they require substantially more mathematical apparatus. However, the results that I present in the next section are meant to be sufficiently general to apply to any of these suggestions for appropriate physical state spaces, and I hope further work can make these connections explicit for locally definite states, Hadamard states, and DHR states.[10]

9. The simplest Weyl operators take the form $U(a) \coloneqq e^{iaQ}$ and $V(b) \coloneqq e^{ibP}$ for position Q, momentum P, and constants $a, b \in \mathbb{R}$ (at least in the usual Schrödinger representation). For more on the Weyl algebra, see Petz (1990) and Clifton and Halvorson (2001).

10. Specifically, the methods of DHR theory in some ways resemble the reduction of the state space developed in sec. 4, and the state space of the final theory is given in terms of the states on a particular Hilbert space representation. So prospects for relating the DHR theory to the methods of this article are hopeful.

One might already wonder whether the Algebraic Imperialist really needs a response to the "too many states" objection. Given that there is not a clear consensus as to what the physical states of any quantum theory are, one might think that it is not worth putting stock in an objection that relies on the assumption that at least some states are unphysical. Still, the objection is worth considering for two reasons. First, I think there are situations in which the Weyl algebra is not the natural choice of an algebra of observables for the Algebraic Imperialist, and at least part of the motivation for this claim is that there are states on the Weyl algebra that are viewed as pathological and very rarely used (see sec. 3.3). Addressing unphysical states in general can help us understand this particular case. Second, I think that the tools that one can develop in response, if one takes the "too many states" objection seriously, have the potential to lead to progress in our understanding of a wide variety of issues surrounding quantization. It may very well prove useful for current and future physics to have tools for systematically constructing, analyzing, and interpreting algebras and state spaces. So the objection has at least instrumental value in that it leads to new techniques and interesting questions, some of which I will be able to point to at the end of this article.

Furthermore, I should stress that it is not my purpose here to judge whether any of the proposals cited above provide an adequate specification of the states that we ought to deem physical. I mention these concrete proposals only to show that others have expressed an interest in restricting the space of quantum states. My goal in this article is only to show that however one wants to specify the collection of physical states, there is an intimate relationship between this state space and the algebra of observables of a quantum theory.

3.2. Physical Algebras. The fact that the abstract Weyl algebra allows for so many supposedly unphysical states has been taken by some (e.g., Arageorgis 1995) as an argument against Algebraic Imperialism. However, I propose that this only gives us reason to use a different algebra. Even if one particular algebra allows for unphysical states, this does not imply that all abstract algebras allow for unphysical states. There are other options for the abstract algebra of observables.

To motivate this approach, I want to note that the Weyl algebra does not have a particularly stable privileged status. To get the Weyl algebra, one starts with the canonical commutation relations between position and momentum observables, then applies these to particular bounded observables (namely, exponentiated position and momentum), and finally completes the resulting algebra in norm. First, the approach of using bounded observables to generate the Weyl algebra is largely for technical convenience. One can approach quantization using unbounded operators (Dubin, Hennings, and Smith 2000), but one then needs to keep track of distinct domains for these unbounded operators. Some (Kay and Wald 1991; Wald 1994) have preferred to approach field quantization in terms of a *-algebra rather than a C*-algebra, which corresponds to choosing a collection of possibly unbounded observables. While there are still open questions concerning the relationship between the unbounded approach following Wightman's axiomatic formulation (see Haag 1992) and the bounded approach, these are beyond the scope of this article. All I wish to point out is that proponents of the unbounded approach already reject the Weyl algebra. In what follows in this article, I consider only bounded approaches in terms of C*-algebras because one can demonstrate the kinds of choices one has to make in that context already. It might also be of interest to investigate whether the results of section 4 can be generalized to arbitrary *-algebras.

Second, using the particular bounded observables defined as exponentiated position and momentum and completing the algebra in norm are also choices made largely to simplify technical machinery. Using the Weyl unitaries—those exponentiated position and momentum operators—allows one to put the canonical commutation relations in a particularly simple form so that they fully define the multiplication relation for the algebra. As I show below, there are other bounded operators, namely, the compact operators, that one could also use to implement the canonical commutation relations. Further, one might consider completing an algebra in other topologies apart from that defined by a norm, as Feintzeig (2017b) considers completing a C*-algebra in its weak topology. So the Weyl algebra, which we arrive at through norm completion, is not the only option.

Finally, it is worth pointing out that the canonical commutation relations we use to generate the Weyl algebra are not a priori principles we must hold onto at all costs. Surely, they form some of the core tenets of quantum theory by underlying the uncertainty relations, but in doing so they constitute empirical hypotheses, which are open to revision just like any other principle of quantum theory. Alternative commutation relations will not play any role in the discussion that follows; as we will see, all of the algebras I consider in this article are modifications of the Weyl algebra so that they, in some sense, implement the same commutation relations. I do not think this poses a problem for this article, as my goal is not to justify the commutation relations but instead to take them on as empirical hypotheses made by all quantum theories. I do, however, think this point raises an important concern—in addition to the considerations raised here concerning the choice of algebra, there is also much philosophical work to be done to analyze the canonical commutation relations (in their many forms), perhaps by applying the approach of Alfsen and Shultz (2001) to explicate the physical content of the algebraic structure in terms of the constraints it places on the state space of an algebra. Such work on the significance of the canonical commutation relations is beyond the scope of this article. But noticing that one might have occasion to use different commutation relations helps motivate the approach of considering alternative algebras of observables.

Some of the choices made in defining an algebra of observables may affect the physical content of a quantum theory, and some may not. I submit that one of the ways (but not the only way) to probe the physical content—or perhaps better, the *physical consequences*—of using a particular algebraic structure is to analyze the collection of states that algebraic structure allows. If two algebras differ in their state spaces, and if one wants to countenance these differing states as physically significant and employ them for physical purposes, then the choice of algebra becomes significant for physics as well. By using the phrase 'physical consequences', I mean to signal that one may not intend in choosing a particular algebra to assert that all of the states on that algebra are to be regarded as physically significant. In practice, working mathematical physicists use the conditions of the previous section to restrict attention to certain states after they have chosen an algebra. But some (e.g., Halvorson 2004) have started from the premise that we do regard all states on an algebra as physically significant. So it is at least worth investigating to what extent one can hold onto this premise and make a principled choice of algebra that leads to an appropriate collection of physical states.

As an aside: given that one may have worries about whether there are any principled approaches to choosing the collections of physical states and physical quantities, it is worth pointing out that there is hope on the horizon. One can use tools developed recently for taking the classical limit of a quantum theory via deformation quantization with continuous fields of C*-algebras (Landsman 1998, 2006) to try to provide a principled way of picking out the physical algebra and physical state space. The principled approach I have in mind is to abide by the following guideline: the classical limit of a quantum theory should lead to a physical algebra and physical state space for the classical theory. This provides substantive constraints on the algebras and state spaces we use for our quantum theories. While not providing a complete and ultimate solution, this guideline at least has the advantage of reducing questions about quantum theories to questions about classical theories, with the hopes that we have a better understanding of how to answer these questions in the classical case. I mention this proposal only to point out future directions for research and also potential applications of the tools developed in this article. Further discussion of this principled approach requires additional technical tools and is beyond the scope of this article.

3.3. Example: Regular States and Compact Operators. In the next section, I present general algebraic results that allow us to change the algebra implementing the canonical commutation relations in order to restrict its physical state space. But before I present these results, I want to note that we already have a procedure, at least in the case of a simple system with fi-

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nitely many degrees of freedom,[11] for eliminating unphysical states by choosing an appropriate algebra. For these simple systems, it is standard to restrict attention to only regular states, in part because a result known as the Stone–von Neumann Theorem tells us that there is a unique irreducible representation of the Weyl algebra in which all of the regular states (and, it turns out, only the regular states) have density operator representatives.[12] This representation is just the usual Schrödinger representation, and in this representation the ultraweak closure of the Weyl algebra is the algebra $\mathcal{B}(\mathcal{H})$. So this leads us back to the familiar setting for nonrelativistic quantum mechanics, where every (self-adjoint) element of $\mathcal{B}(\mathcal{H})$ is considered as a physically significant observable and every physical state has a density operator representative.[13]

However, for these simple systems with finitely many degrees of freedom, there is another path one can take to get the same theory without using Hilbert space methods.[14] Instead of choosing the Weyl algebra, one can directly use the *algebra of compact operators* on a separable Hilbert space (i.e., $L^2(\mathbb{R}^n)$).[15] Then one immediately finds that the state space of this algebra is equivalent to the collection of regular states. One way to see this is to notice that the compact operators have a unique irreducible representation,[16] so one has an immediate analog of the Stone–von Neumann Theorem. The bidual of the algebra of compact operators is its ultraweak closure in this representation, the familiar algebra $\mathcal{B}(\mathcal{H})$. And there are no nonregular or otherwise unphysical states on the algebra of compact operators that cannot be represented as density operators on this representation.

Thus, one can construct the same quantum theory that we get through the Weyl algebra and the Stone–von Neumann Theorem by instead using the compact operators and forgoing the need to restrict attention to some subspace of states. This procedure provides a route for the Algebraic Imperialist to arrive at the same collection of physical states the Hilbert Space Conservative arrives at. The key to this procedure is an auspicious choice of algebra

11. Specifically, the procedure will work for quantizing a classical system with finitely many degrees of freedom and simply connected phase space $\mathbb{R}^{2n}$.

12. A representation $(\pi, \mathcal{H})$ of a C*-algebra $\mathfrak{A}$ is irreducible if the only subspaces of $\mathcal{H}$ that $\pi(\mathfrak{A})$ leaves invariant are $\{0\}$ and $\mathcal{H}$.

13. See Petz (1990) or Summers (1999) for more on the Stone–von Neumann Theorem and the Schrödinger representation of the Weyl algebra.

14. For more on this algebraic approach to regular states, see Feintzeig (2017a).

15. For example, one might arrive at this algebra through the prescription known as Berezin quantization (Landsman 1998, 2006). If one is worried that one needs a Hilbert space to define this algebra, note that Landsman (1990) provides a way to understand this construction from a purely C*-algebraic point of view.

16. See theorem 10.4.6 of Kadison and Ringrose (1997, 751).

of observables for our quantum theory. So, I suggest that one might pay closer attention to the choice of algebra in quantization procedures in other cases when constructing quantum theories. The next section shows that if one takes seriously the importance of this choice of algebra, then one can develop powerful algebraic tools generalizing this procedure for representing physical systems with specified state spaces.

4. Algebraic Adaptability. The purpose of this section is to develop a general response on behalf of the Algebraic Imperialist to the objection that the abstract algebra allows for "too many states." First, in section 4.1 we prove a general result providing necessary and sufficient conditions under which one can find a C*-algebra with a restricted state space. This allows the Algebraic Imperialist far more flexibility than previously thought in constructing a theory with the appropriate state space. Next, in section 4.2 we illustrate how this result can be applied to transform any algebra sufficiently similar to the Weyl algebra to the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a separable Hilbert space. This shows that the Algebraic Imperialist has at least as much power as the Hilbert Space Conservative to limit the collection of states she deems physical—the Algebraic Imperialist always has the option to choose the collection of states with density operator representatives in a given Hilbert space representation as her privileged collection of states and then apply the results here to find a new algebra with precisely that state space. As a corollary, we show that the results here apply when the space of physical states is chosen to be the collection of regular states or the collection of particle states accessible from a Fock state.

4.1. General Algebraic Results. Suppose that we have a C*-algebra $\mathfrak{A}$ and some preferred subset of its dual space $V \subseteq \mathfrak{A}^*$. This section establishes necessary and sufficient conditions for the existence of an algebra inheriting "the same" algebraic relations but with V as its entire dual space.

First we will need a condition to ensure that V can be dual to a space supporting an appropriate C*-norm. To that end, define $\|\cdot\|_V : \mathfrak{A} \to \mathbb{R}$ as follows for all $A \in \mathfrak{A}$:

$$\| A \|_V = \sup_{\omega \in V; \| \omega \| = 1} |\omega(A)|.$$

With this definition, consider the following condition on V:

1) For all $A, B \in \mathfrak{A}$, $\| AB \|_V \leq \| A \|_V \| B \|_V$.

This is a minimal technical condition to ensure V can be the dual space to a C*-algebra: it ensures that V can be dual to an algebra whose multiplication operation is norm continuous.

We need one more piece of background before we can present the main result. The following piece of apparatus allows us to ensure that V has enough structure to support algebraic operations inherited from $\mathfrak{A}$, as we will make precise in a moment. For any $\omega \in \mathfrak{A}^*$, define a relation $\sim_\omega$ on $\mathfrak{A}$ by

$$A \sim_\omega B \text{ iff for all } C \in \mathfrak{A}, \omega(AC) = \omega(BC) \text{ and } \omega(CA) = \omega(CB).$$

Two observables A and B satisfy this relation $\sim_\omega$ just in case ω assigns the same expectation value to A and B, and if whenever we concatenate another measurement represented by the observable C, then ω yields the same expectation value for a measurement of A and C as it does for B and C, at least when the A–C and B–C measurements are performed in the same relative order.

Now define a relation $\sim_V$ on $\mathfrak{A}$ by

$$A \sim_V B \text{ iff for all } \omega \in V, A \sim_\omega B.$$

Two observables A and B satisfy this relation $\sim_V$ just in case each state in V assigns the same expectation values to A as to B, and similarly for concatenating another measurement C, as above.

One easily checks that for any $\omega \in \mathfrak{A}^*$, the relation $\sim_\omega$ is an equivalence relation, and similarly for any $V \subseteq \mathfrak{A}^*$, the relation $\sim_V$ is an equivalence relation. Now consider the following condition:

2) V is *maximal* in the sense that for all $\omega \in \mathfrak{A}^*$, if $A \sim_\omega B$ whenever $A \sim_V B$ for $A, B \in \mathfrak{A}$, then $\omega \in V$.[17]

Condition 2 guarantees that V is as big as possible, by containing all states compatible with the states in V.

Here, a state $\omega \in \mathfrak{A}^*$ is compatible with the states in V just in case it judges equivalent any two observables that all of the states in V judge equivalent. Intuitively, picking a collection of physical states tells us which observables are really the same or distinct according to whether those observables could possibly be assigned different expectation values by the physical states. If two observables are always assigned the same expectation value, then one is hard pressed to think of them as distinct observables, but if two observables are at least sometimes assigned different expectation values, then it is easier to see those observables as distinct. Now, there may be a state ω that

17. If condition 1 is satisfied, then condition 2 holds iff for all $\omega \in \mathfrak{A}^*$, if $\omega(A) = 0$ whenever $A \in I$, then $\omega \in V$, where $I = \{A \in \mathfrak{A} : \omega(A) = 0 \text{ for all } \omega \in V\}$. In other words, V is the collection of all functionals *absolutely continuous* with respect to all of the functionals in V. One could take this statement as the definition of maximality instead, and the results in this article would hold exactly as stated.

judges as distinct all the observables the states in V judge distinct and judges equivalent all the observables the states in V judge equivalent. In this case, we might have just as much reason to countenance the state ω as physically significant as we have for the states in V. If one believes this, then one is likely to say that we have just forgotten to include ω in our specification of V. The maximality condition ensures that we do not forget to include any such states.

As we will see, condition 2 is used to ensure that V is "big enough" to support the algebraic operations of $\mathfrak{A}$. This can be seen as a technical redundancy because we can generate a maximal subspace from any other subspace $V_0 \subseteq \mathfrak{A}^*$ by the following proposition.[18]

Proposition 1. Let $\mathfrak{A}$ be a C*-algebra and let $V_0 \subseteq \mathfrak{A}^*$. Let

$$V = \{\omega \in \mathfrak{A}^* : \text{ for all } A, B \in \mathfrak{A}, \text{ if } A \sim_{V_0} B, \text{ then } A \sim_\omega B\}.$$

Then V is maximal in the sense of condition 2 above.

In this case, we will say that the vector space V is *generated by* V_0. Given a collection of states in V_0, V is the smallest maximal collection of states containing V_0, and it is the collection of all states compatible with those in V_0 in the sense above. Now we are ready to present our main result, which says that the conditions listed above are necessary and sufficient for reducing the state space of our algebra.

Theorem 1. Let $\mathfrak{A}$ be a C*-algebra and let $V \subseteq \mathfrak{A}^*$. Then there exists a C*-algebra $\mathfrak{B}$ and a surjective *-homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{B}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{B}^* \mapsto (\omega \circ f) \in V$ iff conditions 1 and 2 are satisfied.

The main idea of the proof (contained in the appendix) is to construct the C*-algebra $\mathfrak{B}$ by taking the quotient of the original algebra $\mathfrak{A}$ by the equivalence relation $\sim_V$; in other words, $\mathfrak{B} = \mathfrak{A}/\sim_V$. Moreover, we can show that the algebra we get through this construction is, in a certain sense, unique.

Theorem 2. Let $(\mathfrak{B}, f)$ be the pair given by theorem 1. For any other C*-algebra $\mathfrak{C}$ and surjective *-homomorphism $g : \mathfrak{A} \to \mathfrak{C}$ such that $\mathfrak{C}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{C}^* \mapsto (\omega \circ g) \in V$, there is a *-isomorphism $\alpha : \mathfrak{B} \to \mathfrak{C}$ such that $\alpha \circ f = g$.

These theorems show that one can specify the physical state space however one likes, as long as it satisfies conditions 1 and 2. Whenever these con-

18. Proofs of all results are contained in the appendix.

ditions are satisfied, and only when these conditions are satisfied, we can find a new algebra $\mathfrak{A}/\sim_V$ that inherits the algebraic relations of $\mathfrak{A}$ through the surjective *-homomorphism f, and this new algebra has precisely the physical state space V as its entire state space.

4.2. Example: $\mathcal{B}(\mathcal{H})$. In this section we deal with the same simple quantum theories of systems with finitely many degrees of freedom considered in section 3.3. Such a theory has as physical states the density operators on a Hilbert space $\mathcal{H}$ with observables given by all self-adjoint operators in $\mathcal{B}(\mathcal{H})$. But suppose that we have chosen some other algebra of quantum observables. Let $\mathfrak{A}$ be any C*-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the constants and separates density operator states in the sense that for any two density operators ρ_1, ρ_2 on $\mathcal{H}$, there is an $A \in \mathfrak{A}$ such that $Tr(A\rho_1) \neq Tr(A\rho_2)$. The Schrödinger representation of the Weyl algebra mentioned in the previous section is an example of just such an algebra. Here we show that there is a natural way to transform $\mathfrak{A}$ into all of $\mathcal{B}(\mathcal{H})$ using purely algebraic methods.

Why do we need such a procedure, given that we can just take the ultraweak closure of the algebra $\mathfrak{A}$ in the natural inclusion representation on $\mathcal{H}$ to get all of $\mathcal{B}(\mathcal{H})$?[19] Of course, we could obtain all of $\mathcal{B}(\mathcal{H})$ in this way, but there is something unsatisfying about this approach from the algebraic perspective. Why should we complete $\mathfrak{A}$ in the ultraweak topology when there may be states on $\mathfrak{A}$ that are not ultraweakly continuous? There is a sense in which completing $\mathfrak{A}$ in the ultraweak topology does not respect the algebraic structure (or really the state space) of $\mathfrak{A}$, because the completion of $\mathfrak{A}$ in its abstract weak topology (as opposed to the ultraweak topology) to form the bidual $\mathfrak{A}^{**}$ in fact leads us to a much larger algebra than $\mathcal{B}(\mathcal{H})$.[20]

I will show that theorem 1 provides a general procedure to transform $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$. To see this, notice that there is a sense in which $\mathfrak{A}$ is too small and a sense in which it is too large. The algebra $\mathfrak{A}$ is too small in the sense that it may not contain many elements of $\mathcal{B}(\mathcal{H})$ like projections (as is the case with the Weyl algebra). But $\mathfrak{A}$ is too large in the sense that it may allow for states that are not ultraweakly continuous and so cannot be represented by density operators on $\mathcal{H}$. As such, we will first enlarge $\mathfrak{A}$ to $\mathfrak{A}^{**}$ to obtain all of the missing operators including the projections on $\mathcal{H}$. Then we will restrict attention to a collection of physical states by applying theorem 1.

19. Notice that the natural inclusion representation of the algebra $\mathfrak{A}$ on $\mathcal{H}$ is irreducible and employ proposition 1.21.9 of Sakai (1971, 52).

20. More precisely, the algebra $\mathcal{B}(\mathcal{H})$ will in general be properly embedded in $\mathfrak{A}^{**}$, which one can see directly in the universal representation of $\mathfrak{A}$ (see Feintzeig 2017b).

Let V_0^Q be the collection of bounded linear functionals on $\mathfrak{A}$ that are ultraweakly continuous on $\mathcal{H}$. The following proposition shows that reducing $\mathfrak{A}^{**}$ by theorem 1 with this preferred collection of states brings us back to the usual setting of $\mathcal{B}(\mathcal{H})$.

> **Proposition 2.** Let V_Q be the vector subspace of bounded linear functionals on $\mathfrak{A}^{**}$ generated by V_0^Q. Then the C*-algebra $\mathfrak{A}^{**}/\sim_{V_Q}$ in theorem 1 is *-isomorphic to $\mathcal{B}(\mathcal{H})$.

This shows that one can generate $\mathcal{B}(\mathcal{H})$ by first adding to $\mathfrak{A}$ the weak limit observables in $\mathfrak{A}^{**}$ and then reducing the algebra by theorem 1 to restrict attention to only states with density operator representatives on $\mathcal{H}$. Again, the first step of enlarging the algebra to $\mathfrak{A}^{**}$ is necessary because there may be many bounded operators outside $\mathfrak{A}$. In particular, we cannot reduce $\mathfrak{A}$ directly to the compact operators (recall that $\mathcal{B}(\mathcal{H})$ is the bidual and hence ultraweak closure of the compact operators) because there may be compact operators that are not in $\mathfrak{A}$, as in the case of representations of the Weyl algebra.

This shows that the Algebraic Imperialist always has the resources to reduce the physical state space of her algebra in at least all of the ways the Hilbert Space Conservative can. Where the Hilbert Space Conservative would take an irreducible representation of the algebra and restrict attention to density operator states in that representation, the Algebraic Imperialist can just choose the vector subspace of $\mathfrak{A}^*$ generated by those states (i.e., the density operator states in that representation) as her privileged subspace and directly apply theorem 1 to obtain a new algebra with precisely the right state space.

Moreover, we can immediately apply proposition 2 to two of the candidates for physical state spaces considered in section 3.1. First, consider the Weyl algebra $\mathcal{W}(\mathbb{R}^{2n})$ over a finite-dimensional and simply connected phase space $\mathbb{R}^{2n}$ with the standard symplectic form. Let V_0^R be the vector space of linear combinations of regular states on $\mathcal{W}(\mathbb{R}^{2n})$, where the linear combinations are taken according to the vector space operations in $\mathcal{W}(\mathbb{R}^{2n})^*$ (not according to the Hilbert space operations of some particular representation). Since the Schrödinger representation on the Hilbert space $\mathcal{H}_S$ is irreducible and faithful, it follows that $\mathcal{W}(\mathbb{R}^{2n})$ is *-isomorphic to a C*-subalgebra of $\mathcal{B}(\mathcal{H}_S)$ that separates density operator states. And, since the Schrödinger representation is regular, every regular state on $\mathcal{W}(\mathbb{R}^{2n})$ is ultraweakly continuous on $\mathcal{H}_S$ (see Petz 1990), which means V_0^R is the collection of bounded linear functionals on $\mathcal{W}(\mathbb{R}^{2n})$ that are ultraweakly continuous in the Schrödinger representation on $\mathcal{H}_S$. In other words, V_0^R is just the physical state space V_0^Q defined above for the particular case of the Weyl algebra $\mathcal{W}(\mathbb{R}^{2n})$ in the Schrödinger representation on $\mathcal{H}_S$. Thus, we have the following immediate corollary of proposition 2.

Corollary 1. Let V_R be the vector subspace of bounded linear functionals on $\mathcal{W}(\mathbb{R}^{2n})^{**}$ generated by V_0^R. Then the C*-algebra $\mathcal{W}(\mathbb{R}^{2n})^{**}/\sim_{V_R}$ in theorem 1 is *-isomorphic to $\mathcal{B}(\mathcal{H}_S)$.

This shows that theorem 1 can be applied to the Weyl algebra with the collection of physical states as the regular states.

Next, consider the Weyl algebra for a possibly infinite dimensional phase space $\mathcal{M}$ (e.g., a field system), which we will denote $\mathcal{W}(\mathcal{M})$. We will say that a state is accessible from a Fock state ω (see Petz 1990, chap. 4) if it is a vector state in the Fock representation for ω that is obtained by some linear combination of creation operators acting on the vacuum vector representing ω.[21] Intuitively, a Fock state is one that can be represented as a superposition of particle configurations obtained by creating particles in the vacuum state ω. As is well known, this collection of vectors spans the Fock space, that is, the Hilbert space $\mathcal{H}_F$ of the Fock representation (Petz 1990, 34). Since the Fock representation for ω is faithful and irreducible (Petz 1990, theorem 4.7, 34), we know that $\mathcal{W}(\mathcal{M})$ is *-isomorphic to a C*-subalgebra of $\mathcal{B}(\mathcal{H}_F)$ that separates density operator states.

Now, in order to apply theorem 1, we need to find a closed vector subspace of $\mathcal{W}(\mathcal{M})^*$ containing the physical states. Let V_0^F be the smallest closed vector subspace of $\mathcal{W}(\mathcal{M})^*$ containing the states accessible from the Fock state ω. To generate this vector space V_0^F, we need to allow for mixtures and arbitrary linear combinations of the states accessible from the Fock state ω and then close the resulting vector subspace in the norm topology on $\mathcal{W}(\mathcal{M})^*$. Allowing for mixtures (convex combinations) of states accessible from the Fock state ω yields that all finite rank density operator states on the Fock space $\mathcal{H}_F$ are in V_0^F, while allowing for arbitrary linear combinations yields that all finite rank operators on $\mathcal{H}_F$ are in V_0^F. From basic facts in the theory of Hilbert space operators (see Reed and Simon 1980), we know that the finite rank operators are dense in the collection of all trace class operators on $\mathcal{H}_F$ with the trace norm, which is identical to the standard supremum norm on those operators, considered as linear functionals in $\mathcal{W}(\mathcal{M})^*$ by the usual trace prescription. Thus, V_0^F is the collection of all trace class operators on $\mathcal{H}_F$ and hence is the collection of all bounded linear functionals on $\mathcal{W}(\mathcal{M})$ that are ultraweakly continuous on the Fock space $\mathcal{H}_F$. In other words, V_0^F is just the physical state space V_0^Q defined above for the particular case of the Weyl algebra $\mathcal{W}(\mathcal{M})$ in the Fock representation on $\mathcal{H}_F$. Thus, we have the following immediate corollary of proposition 2.

21. The Fock representation is just the GNS representation for ω. For more on the GNS representation and vector states, see Kadison and Ringrose (1997).

Corollary 2. Let V_F be the vector subspace of bounded linear functionals on $\mathcal{W}(\mathcal{M})$ generated by V_0^F. Then the C*-algebra $\mathcal{W}(\mathcal{M})^{**}/\sim_{V_F}$ in theorem 1 is *-isomorphic to $\mathcal{B}(\mathcal{H}_F)$.

This shows that theorem 1 can be applied to the Weyl algebra with the collection of physical states as those accessible from a Fock state.

One might worry at this point about the status of the collection of states V we deem physically significant in order to apply theorem 1. The goal of this article was to show that one can find purely algebraic resources for choosing an appropriate algebra of physical quantities. But it seems that the specification of a space of physical states V is an extra resource we need to avail ourselves of. And in the previous examples the state space that we use consists of the folium of a Hilbert space representation, so one might doubt that the methods of this section provide purely algebraic tools.

In response, I want to note that my purpose in this article is only to show that however one specifies the collection of physically significant states, we can find an algebra with that collection of states as its full state space (as long as it satisfies a few constraints). It is a natural question whether one needs a Hilbert space representation to specify V, but that is not a question I can deal with fully in this article. It may be helpful, however, to make a number of remarks concerning the above examples. First, one can define the regular states in a way completely free of reference to any Hilbert space representation by simply looking at the expectation values assigned by states to one-parameter families of Weyl unitaries. It turns out this definition specifies the folium of a particular Hilbert space representation, but this can be understood as a consequence of the continuity assumptions. In the same vein, although we have used a Fock representation to specify the states accessible from a Fock state, this does not imply that one needs a Hilbert space representation to define this collection of states. It is a further substantive question whether one can give an algebraic specification of these particle-like states, especially given that the definition of Fock states on their own does not require the use of any Hilbert space representations. For example, can one understand these particle-like states to constitute the smallest collection of states containing a Fock state and satisfying conditions 1 and 2?

As may be clear by now, I am not interested in specifying a rigid set of rules that the Algebraic Imperialist must abide by when specifying a collection of physical states. I think asking for such a set of rules in some sense misses the point. The point is that the use of Hilbert space methods has been distracting and obscuring in some cases by confusing conditions of technical convenience with conditions that have physical content. The tools developed in this article constitute just one step toward untangling these conditions. The point is not that one has reason for rejecting Hilbert space methods altogether but that too much focus on the use of Hilbert space methods or even debate about whether Hilbert space methods are useful can distract from the application of alternative algebraic methods that have promise for clarifying issues of physical interpretation that have confused researchers working in the Hilbert space approach.

5. Discussion. I have argued that theorem 1 gives a general tool for the Algebraic Imperialist to use to respond to the "too many states" objection. I have shown that there is a very general procedure the Algebraic Imperialist can use to rid herself of unphysical states. The result in proposition 2 shows that this procedure can apply to any collection of physical states that consists of all and only the density operator states on some irreducible Hilbert space representation. Specifically, this procedure applies to the regular states on the Weyl algebra, which exhaust the density operator states in the Schrödinger representation, and to the states accessible from a Fock state, which can be thought of as particle states obtained through particle creation in a particular vacuum state. But it is still an open question whether the Algebraic Imperialist can apply this procedure to other existing candidates for physical state spaces. In other words, it is still left to be shown that the rest of the standard proposals for physical state spaces (e.g., the Hadamard states, locally definite states, and DHR states) actually satisfy conditions 1 and 2 of theorem 1. Answering this question would inform us about precisely how much freedom the Imperialist has in responding to the objection that she allows for "too many states."

The results of this article show that the Algebraic Imperialist has at least as much flexibility for restricting the state space of a quantum theory as the Hilbert Space Conservative does. Proposition 2 shows that when the physical states themselves form the space of density operator states on an irreducible Hilbert space representation, then it is possible to reduce the algebra to one with an appropriate state space. But I have also claimed that the flexibility or adaptability the Algebraic Imperialist gains through theorem 1 is a virtue of the algebraic point of view. As such, one ought to ask whether theorem 1 gives us more freedom for reducing the state space than the Hilbert Space Conservative has. In other words, are there any subspaces of states V satisfying 1 and 2 that do not form the space of density operator states of some Hilbert space representation? If not, then the procedure I have outlined for reducing the state space of an abstract algebra works in exactly the same cases that the Conservative's procedure would work. Thus, I have not yet made the case that this virtue I have brought to our attention—adaptability of the algebra and state space—is a virtue of Imperialism over Conservatism. My results do show that the Imperialist can deal with the objection that she allows for "too many" states, but I admit that this may just bring the Imperialist in line with the Hilbert Space Conservative.

Even with these open questions, I believe the results of this article have significance for the interpretation of algebraic quantum theories. They show

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that considerations of the physical content of the abstract algebra, its states, and topologies can lead us to new technical and conceptual tools beyond those of Hilbert space representations. I think the tools developed here have the potential to prove useful for understanding and interpreting quantum theories. I think the fact that the perspective taken in this article leads to precise technical and conceptual questions whose answers would appear to have philosophical significance shows the promise inherent in this approach. I can only hope that others find these tools to be as useful as I do.

Appendix

Proofs of Results

Here we prove the results of section 4. The arguments rely on some technical notions not defined in the body of the article; for explicit definitions, see Kadison and Ringrose (1997). First, we prove the results of section 4.1.

Proposition 1. Let $\mathfrak{A}$ be a C*-algebra and let $V_0 \subseteq \mathfrak{A}^*$. Define V by

$$V = \{\omega \in \mathfrak{A}^*: \text{ for all } A, B \in \mathfrak{A}, \text{ if } A \sim_{V_0} B, \text{ then } A \sim_\omega B\}.$$

Then V is maximal in the sense of condition 2; that is, for all $\omega \in \mathfrak{A}^*$,

$$\text{if } A \sim_\omega B \text{ whenever } A \sim_V B \text{ for all } A, B \in \mathfrak{A}, \text{then } \omega \in V.$$

Proof. Suppose $\omega \in \mathfrak{A}^*$ is such that $A \sim_\omega B$ whenever $A \sim_V B$ for all A, $B \in \mathfrak{A}$. Let $A, B \in \mathfrak{A}$ be such that $A \sim_{V_0} B$. Let $\rho \in V$. Then, by the definition of V, $A \sim_\rho B$. Since this holds for all $\rho \in V$, it follows that $A \sim_V B$. This implies by the assumption on ω that $A \sim_\omega B$. Again, by the definition of V, it follows that $\omega \in V$. QED

To prove theorem 1, we will need the following lemma.

Lemma 1. Suppose V satisfies 1 and let

$$I = \{A \in \mathfrak{A} : \omega(A) = 0 \text{ for all } \omega \in V\}.$$

Then I is a closed two-sided ideal.

Proof. First, I is an additive subgroup of $\mathfrak{A}$ since for all $\omega \in V$ and all A, $B \in I$,

$$\omega(A + B) = \omega(A) + \omega(B) = 0.$$

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Next, I is a two-sided ideal since for all $A \in I$, $C \in \mathfrak{A}$, and $\omega \in V$, it follows from condition 1 and the fact that $\| A \|_V = 0$ that

$$|\omega(AC)| \leq \| \omega \| \, \| A \|_V \, \| C \|_V = 0;$$

$$|\omega(CA)| \leq \| \omega \| \, \| C \|_V \, \| A \|_V = 0.$$

Finally, we show that I is closed. Suppose $A_n \in I$ and $\| A - A_n \| \to 0$ for $A \in \mathfrak{A}$. Then

$$|\omega(A - A_n)| \leq \| \omega \| \, \| A - A_n \| \to 0.$$

But we also know that for all n,

$$\omega(A - A_n) = \omega(A) - \omega(A_n) = \omega(A),$$

and hence the sequence $\omega(A - A_n)$ does not depend on n, from which it follows that $\omega(A) = 0$. QED

Theorem 1. Let $\mathfrak{A}$ be a C*-algebra and let $V \subseteq \mathfrak{A}^*$. Then there exists a C*-algebra $\mathfrak{B}$ and a surjective *-homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{B}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{B}^* \mapsto (\omega \circ f) \in V$ iff conditions 1 and 2 are satisfied.

Proof. First, we show that the conditions 1 and 2 are sufficient. Let $I = \{A \in \mathfrak{A} : A \sim_V 0\}$ and define a relation $\sim_I$ on $\mathfrak{A}$ by

$$A \sim_I B \text{ iff } A = B + C \text{ for some } C \in I.$$

We show that $A \sim_I B$ iff $A \sim_V B$. If $A \sim_V B$, then one can show $(B - A) \in I$, so $B = A + (B - A)$, and hence $A \sim_I B$. If $A \sim_I B$, then there is a $C \in I$ such that $A = B + C$. For any $\omega \in V$ and any $D \in \mathfrak{A}$

$$\omega(AD) = \omega(BD) + \omega(CD) = \omega(BD);$$

$$\omega(DA) = \omega(DB) + \omega(DC) = \omega(DB).$$

Hence, $A \sim_V B$. We will denote by $[A]$ the equivalence class of all $B \in \mathfrak{A}$ such that $A \sim_V B$ and $A \sim_I B$. By theorem 1.8.2 of Dixmier (1977, 20), we know that $\mathfrak{B} := \mathfrak{A}/I = \mathfrak{A}/\sim_V$ is a C*-algebra, and $f : \mathfrak{A} \to \mathfrak{B}$ defined by $f(A) = [A]$ for all $A \in \mathfrak{A}$ is a surjective *-homomorphism.

Now suppose that $\omega \in \mathfrak{A}^*$ is such that $\omega(C) = 0$ for all $C \in I$. Then for any $A, B \in \mathfrak{A}$ such that $A \sim_V B$, we know that $A = B + C$ for some $C \in I$, and hence for all $D \in \mathfrak{A}$,

$$\omega(AD) = \omega(BD) + \omega(CD) = \omega(BD),$$

$$\omega(DA) = \omega(DB) + \omega(DC) = \omega(DB),$$

from which it follows that $A \sim_\omega B$. Since V is maximal, $\omega \in V$. Since all $\omega \in V$ assign $\omega(C) = 0$ for all $C \in I$, it follows that

$$V = \{\omega \in \mathfrak{A}^* : \omega(C) = 0 \text{ for all } C \in I\},$$

and by proposition 2.11.8 of Dixmier (1977, 63), we know that $\mathfrak{B}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{B}^* \mapsto (\omega \circ f) \in V$.

Now we show that conditions 1 and 2 are necessary. Suppose $f : \mathfrak{A} \to \mathfrak{B}$ is a surjective *-homomorphism such that $\mathfrak{B}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{B}^* \mapsto (\omega \circ f) \in V$. We immediately know that V satisfies condition 1 because multiplication in the C*-algebra $\mathfrak{B}$ is norm continuous.

Let

$$I = \{C \in \mathfrak{A} : f(C) = 0\}.$$

Then $A/I \cong \mathfrak{B}$ by corollary 1.8.3 of Dixmier (1977, 21). Again, by proposition 2.11.8 of Dixmier (1977, 63) we know that

$$V = \{\omega \in \mathfrak{A}^* : \omega(C) = 0 \text{ for all } C \in I\}.$$

Suppose $\omega \in \mathfrak{A}^*$ is such that for all $A, B \in \mathfrak{A}$, if $A \sim_V B$, then $A \sim_\omega B$. For any $C \in I$, we know $C \sim_V 0$, which implies $C \sim_\omega 0$, and hence $\omega(C) = \omega(0) = 0$. It follows that $\omega \in V$, which shows that V satisfies condition 2. QED

Theorem 2. Let $(\mathfrak{B}, f)$ be the pair given by theorem 1. For any other C*-algebra $\mathfrak{C}$ and surjective *-homomorphism $g : \mathfrak{A} \to \mathfrak{C}$ such that $\mathfrak{C}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{C}^* \mapsto (\omega \circ g) \in V$, there is a *-isomorphism $\alpha : \mathfrak{B} \to \mathfrak{C}$ such that $\alpha \circ f = g$.

Proof. Suppose $\mathfrak{C}$ is a C*-algebra with surjective *-homomorphism $g : \mathfrak{A} \to \mathfrak{C}$ such that $\mathfrak{C}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{C}^* \mapsto (\omega \circ g) \in V$. Then define $\alpha : \mathfrak{B} \to \mathfrak{C}$ by $\alpha([A]) = g(A)$ for all $[A] \in \mathfrak{B}$ (one requires the axiom of choice to choose a representative A for each $[A] \in \mathfrak{B}$). One easily checks that α is well defined (because $A \sim_V B$ implies $\| g(A) - g(B) \| = 0$, which implies $g(A) = g(B)$) and a *-isomorphism. Furthermore, it follows immediately that $\alpha \circ f = g$. QED

Now we prove the results of section 4.2. Here, $\mathfrak{A}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the constants and separating density operator states. Let π denote the representation $\pi(A) = \pi_U(A)P$ of $\mathfrak{A}$ on $\mathcal{H}_U$, where $(\pi_U, \mathcal{H}_U)$ is the universal representation and P is the projection determined through theorem 10.1.12 of Kadison and Ringrose (1997) associated with the inclusion mapping of $\mathfrak{A}$ on $\mathcal{H}$. To prove proposition 2, we will need the following lemma characterizing P.

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Lemma 2. P is the projection onto the span of all subspaces of $\mathcal{H}_U$ carrying a subrepresentation of π_U quasi equivalent to π.

Proof. Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be representations of $\mathfrak{A}$. Let $P_1, P_2 \in \overline{\pi_U(\mathfrak{A})}$ be their associated central projections. First, we show that if π_1, π_2 are quasi equivalent, then $P_1 = P_2$. The representation $\varphi_1 : A \mapsto \pi_U(A)P_1$ is quasi equivalent to π_1, and $\varphi_2 : A \mapsto \pi_U(A)P_2$ is quasi equivalent to π_2. So if π_1, π_2 are quasi equivalent, then so are φ_1 and φ_2. By theorem 10.3.3.ii of Kadison and Ringrose (1997, 736), the central carriers of P_1 and P_2 are equal; that is, $C_{P_1} = C_{P_2}$. Since P_1, P_2 are central projections, $P_1 = C_{P_1} = C_{P_2} = P_2$.

Next, we show that if π_1, π_2 are disjoint, then $P_1P_2 = 0$. If π_1, π_2 are disjoint, then so are φ_1 and φ_2—for suppose there were a subrepresentation of φ_1 quasi equivalent to a subrepresentation of φ_2. Then we would have a subrepresentation of π_1 quasi equivalent to a subrepresentation of π_2 (by composition of the relevant *-isomorphisms; see Kadison and Ringrose 1997, theorem 10.3.4, 737). Hence, by theorem 10.3.3 of Kadison and Ringrose, $P_1P_2 = C_{P_1}C_{P_2} = 0$. QED

Proposition 2. Let V_Q be the vector subspace of $\mathfrak{A}^{***}$ generated by

$$V_0^Q = \{\omega \in \mathfrak{A}^* : \omega \text{ is ultraweakly continuous on } \mathcal{H}\}.$$

Then the C*-algebra $\mathfrak{A}^{**}/\sim_{V_Q}$ in theorem 1 is *-isomorphic to $\mathcal{B}(\mathcal{H})$.

Proof. First, notice that by proposition 10.1.14 of Kadison and Ringrose (1997, 722), V_0^Q is the collection of functionals $\omega \in \mathfrak{A}^*$ such that $\omega = P\omega$, where $P\omega$ is defined as in Kadison and Ringrose (1997, 721–22). Let $\overline{\pi(\mathfrak{A})}$ denote the ultraweak closure of $\pi(\mathfrak{A})$.

Define $j : (\mathfrak{A}^{**}/\sim_{V_Q}) \to \overline{\pi(\mathfrak{A})}$ by $j([A]) = \tilde{\pi}_U(A)P$ for any $A = \mathfrak{A}^{**}$, where $\tilde{\pi}_U$ is the unique weakly continuous extension of π_U to $\mathfrak{A}^{**}$. This map j is well defined because for any $A, B \in \mathfrak{A}^{**}$, $A \sim_{V_Q} B$ implies $\tilde{\pi}_U(A)P = \tilde{\pi}_U(B)P$.

The map j is onto: for every $\hat{A} \in \overline{\pi(\mathfrak{A})}$, $\hat{A} = j([A])$ for $A = \tilde{\pi}_U^{-1}(\alpha^{-1}(\hat{A}))$, where $\alpha : \overline{\pi_U(\mathfrak{A})}P \to \overline{\pi(\mathfrak{A})}$ is the *-isomorphism provided by theorem 10.1.12 of Kadison and Ringrose (1997). Furthermore, j is one to one: if $\tilde{\pi}_U(A)P = 0$ for $A = \mathfrak{A}^{**}$, then for all $\omega \in \mathfrak{A}^*$ such that $\omega = P\omega$, we know that $\omega(A) = \omega(\tilde{\pi}_U(A)P) = 0$, and hence $[A] = [0]$.

Since j obviously preserves algebraic operations, it follows that j is a *-isomorphism, and since π is quasi equivalent to the inclusion mapping of $\mathfrak{A}$ on $\mathcal{H}$ by theorem 10.1.12 of Kadison and Ringrose (1997), it follows that $\mathfrak{A}^{**}/\sim_{V_Q}$ is *-isomorphic to $\mathcal{B}(\mathcal{H})$. QED

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