*Combinatorics, Probability and Computing* (2011) 20, 921–938. © Cambridge University Press 2011
doi:10.1017/S096354831100037X

# Plünnecke's Inequality

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*Received 26 October 2010; revised 18 August 2011; first published online 5 October 2011*

Plünnecke's inequality is a standard tool for obtaining estimates on the cardinality of sumsets and has many applications in additive combinatorics. We present a new proof. The main novelty is that the proof is completed with no reference to Menger's theorem or Cartesian products of graphs. We also investigate the sharpness of the inequality and show that it can be sharp for arbitrarily long, but not for infinite commutative graphs. A key step in our investigation is the construction of arbitrarily long regular commutative graphs. Lastly we prove a necessary condition for the inequality to be attained.

## 1. Introduction

Plünnecke's inequality is among the most commonly used tools in additive combinatorics. It was discovered by Helmut Plünnecke in the late 1960s. The inequality puts bounds on the magnification ratios of a directed, layered graph $G$, which are defined as

$$D_i(G) = \min_{\emptyset \neq Z \subseteq V_0} \frac{|\operatorname{Im}^{(i)}(Z)|}{|Z|}.$$

$\operatorname{Im}^{(i)}(Z)$ is the $i$th out-neighbourhood of $Z$ and $V_0$ is the bottom layer of the graph. Plünnecke discovered that under some commutativity conditions on graphs, which have since been known as Plünnecke conditions and will be defined in Section 2, the sequence $D_i^{1/i}(G)$ is decreasing. The directed layered graphs that obey these conditions are called *commutative* (or Plünnecke) graphs. In particular Plünnecke proved [5] the following.

**Theorem 1.1 (Plünnecke).** *Let $G$ be a commutative graph with $D_h(G) = \Delta^h$. Then $D_i(G) \geqslant \Delta^i$ for all $1 \leqslant i \leqslant h$.*

The main objective of the paper is to present a new proof of Theorem 1.1.

Imre Ruzsa has simplified Plünnecke's proof in [6, 7]. Plünnecke's and Ruzsa's arguments have more similarities than differences as their backbone is the same. Ruzsa's simplified approach has become the standard way to prove the inequality and we will thus use it as the point of comparison with the present argument.

Ruzsa's argument relies on two key ingredients: Menger's theorem [3] and Cartesian products of graphs. While there are several variations in the literature [2, 4, 8, 9, 10], they all follow the original approach closely in first proving the special case when $D_h(G) = 1$ by applying Menger's theorem and then deducing the inequality by using Cartesian product of graphs. Here we present an elementary and more direct proof, which stays close to Plünnecke's and Ruzsa's argument for the special case, but uses neither of the two ingredients.

Completing the proof with no reference to Menger's or an equivalent theorem is noteworthy for two reasons. It shows that Plünnecke's inequality is a direct consequence of Plünnecke's conditions and little else. Therefore the bounds on the cardinality of sumsets that follow from it are also a direct consequence of commutativity of addition and little else. The second reason is that by avoiding Menger's theorem we are able to complete the proof without using Cartesian products of graphs. It has not been clear whether this very helpful tool is a necessary ingredient, and removing it makes the proof more transparent.

Despite its widespread use there has so far been no attempt to investigate whether Plünnecke's inequality is sharp. We answer this question for both finite and infinite commutative graphs.

**Theorem 1.2.** *For all $C \in \mathbb{Q}$ and $h \in \mathbb{Z}^+$ there exists a commutative graph with*

$$D_i(G) = C^i$$

*for all $1 \leqslant i \leqslant h$.*

**Theorem 1.3.** *Let $G$ be an infinite commutative graph. Then*

$$D_i(G) = C^i$$

*can hold if and only if $C = 1$.*

The extremal graphs for Plünnecke's inequality we present are all regular. It is natural to ask whether this condition is necessary. The final result of this paper is to show that in a way it is: every commutative graph where Plünnecke's inequality is attained must contain a regular commutative subgraph. The exact meaning of this assertion is explained in Section 5.

The remaining sections of the paper are organized as follows. In Section 2 we introduce commutative graphs and the notation used at the remainder of the paper. Section 3 is devoted to the proof of Plünnecke's inequality; an entirely self-contained argument is found in Sections 3.2 and 3.3. In Section 4 we prove Theorems 1.2 and 1.3. Finally, in Section 5 we deduce the existence of the regular subgraph in the case when all the quantities $D_i^{1/i}(G)$ are equal.

## 2. Commutative graphs

The material in this section can be found in any of the standard references [4, 8, 10]. The notation used, however, is slightly different.

### 2.1. Commutative graphs: definition and notation

$G$ will always be a directed layered graph with edge set $E(G)$ and vertex set $V(G) = V_0 \cup \cdots \cup V_h$, where the $V_i$ are the *layers* and $h$ is the *level* of the graph. For any $S \subseteq V_i$ we will write $S^c = V_i \backslash S$ for the complement of $S$ in $V_i$ and not in $V(G)$. We will furthermore assume that directed edges exist only between $V_i$ and $V_{i+1}$, and denote this set of edges by $E(V_i, V_{i+1})$.

In order to introduce Plünnecke's conditions we briefly recall that, given an integer $k$ and a bipartite undirected graph $G(X, Y)$, we say that a *one-to-$k$ matching* exits from $X$ to $Y$ if we can find distinct elements $\{y_x^i : x \in X \text{ and } 1 \leqslant i \leqslant k\}$ in $Y$ such that $xy_x^i \in E(G)$ for all $x \in X$ and $1 \leqslant i \leqslant k$. A one-to-one matching is referred to as a *matching*. We furthermore write Im and $\mathrm{Im}^{(-1)}$ for the out- and in-neighbourhoods.

Plünnecke's *upward* condition states that if $uv \in E(G)$, then there exists a matching from $\mathrm{Im}(v)$ to $\mathrm{Im}(u)$ (in the bipartite graph $G(\mathrm{Im}(u), \mathrm{Im}(v))$, where $xy$ is an undirected edge if and only if it is a directed edge in $G$). Plünnecke's *downward* condition states that if $vw \in E(G)$, then there exists a matching from $\mathrm{Im}^{-1}(v)$ to $\mathrm{Im}^{-1}(w)$ (in the bipartite graph $G(\mathrm{Im}^{-1}(v), \mathrm{Im}^{-1}(w))$, where $xy$ is an undirected edge if and only if it is a directed edge in $G$). A *commutative graph* is a directed layered graph that satisfies both properties. In the literature such graphs are sometimes referred to as *Plünnecke graphs*.

The most typical example is $G_+(A, B)$, the *addition graph* of two sets $A$ and $B$ in a commutative group. This is defined as the directed graph whose $i$th layer $V_i$ is $A + iB$, and a directed edge exists between $x \in V_{i-1}$ and $y \in V_i$ if and only if $y - x \in B$. When we take $A = \{0\}$ and $B = \{\gamma_1, \ldots, \gamma_n\}$, where 0 is the identity and $\gamma_i$ the generators of a free commutative group, we call $G_+(\{0\}, \{\gamma_1, \ldots, \gamma_n\})$ the *independent addition* graph on $n$ generators.

As usual, we write $d_H^+(v) = |\{w : vw \in E(H)\}|$ for the *out-degree* of a vertex $v$ in a graph $H$, and $d_H^-(v) = |\{u : uv \in E(H)\}|$ for its *in-degree*. Here $H$ will always be a subgraph of $G$. We write $d^+(v)$ and $d^-(v)$ for $d_G^+(v)$ and $d_G^-(v)$.

A *path* of length $\ell$ in $G$ is a sequence of vertices $v_0, v_1, \ldots, v_\ell$ such that $v_{i-1}v_i \in E(G)$ for all $1 \leqslant i \leqslant \ell$. For a subgraph $H$ of $G$ and $Z \subseteq V(H)$ we thus define

$$\mathrm{Im}_H^{(i)}(Z) = \{v \in V(H) : \exists \text{ path of length } i \text{ in } H \text{ that starts in } Z \text{ and ends in } v\}$$

and

$$\mathrm{Im}_H^{(-i)}(Z) = \{v \in V(H) : \exists \text{ path of length } i \text{ in } H \text{ that starts in } v \text{ and ends in } Z\}.$$

When the subscript is omitted we take $H$ to be $G$. When $i = 1$, and consequently $\mathrm{Im}^{(1)}(Z)$ is the out-neighbourhood of $Z$ in $H$, the superscript will be omitted. We can now formally define magnification ratios. As we have seen, the $i$th magnification ratio of $G$ is defined as

$$D_i(G) = \min_{\emptyset \neq Z \subseteq V_0} \frac{|\mathrm{Im}^{(i)}(Z)|}{|Z|}.$$

We will also write

$$\Delta = D_h^{1/h}(G).$$

For $X, Y \subseteq V(G)$ the *channel between X and Y* is the subgraph that consists of directed paths starting at $X$ and finishing in $Y$. For $Z \subseteq V_0$ the *channel of Z* is the channel between $Z$ and $V_h$.

A *separating set* in any subgraph $H$ is a set $S \subseteq V(H)$ that intersects all directed paths of maximum length in $H$.

### 2.2. Properties of commutative graphs

The following properties of commutative graphs are standard and will be used repeatedly.

(1) For $i > j$ and $X \subseteq V_i$, $Y \subseteq V_j$, the channel between $X$ and $Y$ is a commutative graph in its own right. An important special case is the channel of $Z \subseteq V_0$.

(2) For $uv \in E(G)$ Plünnecke's conditions imply $d^+(u) \geqslant d^+(v)$ and $d^-(u) \leqslant d^-(v)$.

(3) For commutative graphs $G$ and $H$ we define their *Cartesian product* $G \times H$ as follows. The $i$th layer of $G \times H$ is the Cartesian product of the $i$th layer of $G$ with the $i$th layer of $H$. As for the edges, $(u, x)(v, y) \in E(G \times H)$ if and only if $uv \in E(G)$ and $xy \in E(H)$. $G \times H$ is a commutative graph with $D_i(G \times H) = D_i(G)\, D_i(H)$. Vertex degrees are also multiplicative as $d^{\pm}_{G\times H}((u, x)) = d^{\pm}_G(u)\, d^{\pm}_H(x)$.

(4) We define the *inverse I* of a commutative graph $G$ as follows: the $i$th layer of $I$ is the $(h - i)$th layer of $G$ and $uv \in E(I)$ if and only if $vu \in E(G)$. One can informally think of $I$ as the graph consisting of all paths from $V_h$ to $V_0$. $I$ is always a commutative graph due to the symmetry of Plünnecke's conditions.

### 2.3. Hall's marriage theorem

We finish this introductory section by stating Hall's marriage theorem for bipartite graphs $G = G(X, Y)$. For any $x \in X$ we define its neighbourhood by

$$\Gamma(x) = \{y \in Y\ : xy \in E(G)\}$$

and the neighbourhood of $S \subseteq X$ by

$$\Gamma(S) = \bigcup_{x\in S} \Gamma(x).$$

It is clear that in order to have a one-to-$k$ matching from $X$ to $Y$ we need $|\Gamma(S)| \geqslant k\,|S|$ for all $S \subseteq X$. Philip Hall proved in 1935 that the converse is also true [1].

**Lemma 2.1 (Hall).** *Let $G(X, Y)$ be a bipartite graph. Then a one-to-k matching exists from X to Y if and only if*

$$|\Gamma(S)| \geqslant k\,|S| \quad \textit{for all } S \subseteq X.$$

## 3. Proof of Plünnecke's inequality

We begin our examination of Plünnecke's inequality with a new proof of Theorem 1.1. The proof is inspired by the work of Ruzsa that appeared in [6, 7] and in particular by an exposition of Ruzsa's argument due to Terence Tao [9]. However, there are crucial differences, as Menger's theorem and Cartesian products of graphs are not needed.

### 3.1. Outline of the Plünnecke–Ruzsa proof

The traditional proof of Theorem 1.1 can be split in two distinct parts. The first is to establish the special case when $\Delta = 1$. The key is the relation between magnification ratios and separating sets in the graph. By applying Menger's theorem Plünnecke proved the following powerful result.

**Proposition 3.1 (Plünnecke).** *Let $G$ be a commutative graph with $D_h(G) \geqslant 1$. Then there are $|V_0|$ vertex-disjoint paths from $V_0$ to $V_h$ in $G$, and therefore $D_i(G) \geqslant 1$ for all $i$.*

The duality between separating sets and vertex-disjoint paths is exploited fully. This poses a serious obstacle when trying to extend this idea for general values of $\Delta$, as Menger's theorem is no longer useful. Even for integer $\Delta \neq 1$ there is an example which shows that proving the following natural and plausible generalization would require ideas beyond those found in this paper.

**Question 3.2.** *Suppose that $D_h(G) \geqslant k^h$ for some integer $k$. Then there are $|V_0|$ vertex-disjoint trees, each having at least $k^i$ vertices in $V_i$.*

The second part of the proof is to overcome this obstacle by deducing the general case from Proposition 3.1. Ruzsa achieved this using the multiplicativity of magnification ratios. The quickest way to do this is by using some graphs we will introduce in Section 4. For any rational $q \leqslant \Delta$ there is a commutative graph $R_q$ with $D_i(R_q) = q^{-i}$ for all $i = 1, \ldots, h$. We know that $D_h(G \times R_q) = D_h(G)\, D_h(R_q) = (\Delta q^{-1})^h \geqslant 1$, and so

$$1 \leqslant D_i(G \times R_q) = D_i(G)\, D_i(R_q) = D_i(G) q^{-i}.$$

This implies that $D_i(G) \geqslant q^i$ for all rationals $q \leqslant \Delta$ and hence that $D_i(G) \geqslant \Delta^i$. For the reader's benefit we will note that the standard deduction uses independent addition graphs instead. In this context it is mandatory to take the product of $r$ copies of $G$ with suitably chosen independent addition graphs and then let $r \to \infty$.

Ruzsa's approach is elegant, but leaves one question unanswered: What is the precise role of Cartesian products in the proof and how does it allow us to use Proposition 3.1 in such a simple way when proving a generalization is tricky? A simple-minded approach is to see what the existence of the paths in $G \times R_q$ implies about $G$, but this yields a mere reformulation of Plünnecke's inequality. A more refined approach suggested by Tim Gowers is to work in a weighted version of $G$. In this setting Menger's theorem could be replaced by the max-flow min-cut theorem.

In fact Theorem 1.1 will be proved by focusing on the minimum cut in (the network generated by) $G$ without using any properties of a maximal flow. In doing so we will mirror Plünnecke's proof of Proposition 3.1 closely, but will introduce a further ingredient in Section 3.3 that allows us to apply his argument to all $\Delta$.

### 3.2. Weighted commutative graphs

The proof of Proposition 3.1 is built around the fact that when $\Delta = 1$ there is a very natural relation between separating sets in $G$ and magnification ratios. In order to make use of this observation for general $\Delta$ we need to work with weighted commutative graphs, *i.e.*, a commutative graph with a weight function

$$w : V(G) \mapsto \mathbb{R}^+.$$

We will eventually give every vertex in $V_i$ weight $\Delta^{-i}$. The reasons behind this choice will become apparent shortly, but different weights may be more suitable in other applications. It should be noted that this can be thought of as an alternative to taking a Cartesian product of $G$ with the $R_q$. We also need a notion of the weight of a set of vertices in $G$ and so we define the weight of any set $S \subseteq V(G)$ as

$$w(S) = \sum_{v \in S} w(v).$$

In what follows this will equal

$$\sum_{i=0}^{h} |S \cap V_i| \, C^{-i}$$

for a positive constant $C$. The heart of the proof of Proposition 3.1 is to 'pull down' any minimum separating set to one that lies entirely in $V_0 \cup V_h$. Plünnecke achieved this by applying Plünnecke's conditions to the paths given by Menger's theorem. The same can be done for weighted commutative graphs and, in fact, without any reference to Menger's or some other equivalent theorem. The following result demonstrates how powerful Plünnecke's conditions are.

**Lemma 3.3.** *Let $C$ be a positive real and let $G$ be a weighted commutative graph with vertex set $V_0 \cup V_1 \cup \cdots \cup V_h$ and $w(v) = C^{-i}$ for all $v \in V_i$. A separating set of minimum weight that lies entirely in $V_0 \cup V_h$ exists.*

Proving this lemma will be the main objective of the next subsection. For the time being let us quickly see how to deduce Theorem 1.1 from it.

**Corollary 3.4.** *Let $G$ a weighted commutative graph with vertex set $V_0 \cup V_1 \cup \cdots \cup V_h$ and $w(v) = \Delta^{-i} = D_h(G)^{-i/h}$ for all $v \in V_i$. The weight of any minimal separating set is $|V_0|$.*

**Proof.** By applying Lemma 3.3 we can assume that $S_0 \cup S_h$ is a separating set of minimum weight with $S_i \subseteq V_i$. We know that $\mathrm{Im}^{(h)}(S_0^c) \subseteq S_h$ and so $|S_h| \geqslant |\,\mathrm{Im}^{(h)}(S_0^c)| \geqslant D_h(G)|S_0^c|$. This in turn implies $w(S) = w(S_0) + w(S_h) = |S_0| + |S_h|\, D_h^{-1}(G) \geqslant |S_0| + |S_0^c| = |V_0|$. On the other hand $V_0$ is a separating set and hence $w(S) = |V_0|$ for any separating set of minimum weight. □

Plünnecke's inequality follows in a straightforward manner.

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**Proof of Theorem 1.1.** We consider any $Z \subseteq V_0$ in the weighted version of $G$, where each $v \in V_i$ has weight $\Delta^{-i}$. $Z^c \cup \mathrm{Im}^{(i)}(Z)$ is a separating set and thus

$$|V_0| \leqslant w(Z^c \cup \mathrm{Im}^{(i)}(Z)) = w(Z^c) + w(\mathrm{Im}^{(i)}(Z)) = |V_0| - |Z| + |\mathrm{Im}^{(i)}(Z)|\,\Delta^{-i}.$$

That is, $|\mathrm{Im}^{(i)}(Z)| \geqslant \Delta^i |Z|$. Taking the minimum over all non-empty $Z \subseteq V_0$ gives the lower bound on $D_i(G)$. □

### 3.3. Separating sets on weighted commutative graphs

We now turn to the proof of Lemma 3.3. The key is to make optimal use of separating sets of minimal weight. Instead of using vertex-disjoint paths we rely on the following elementary observation. Suppose that $S$ is a separating set of minimum weight. Then, for any $Z \subseteq S$,

$$w(\mathrm{Im}(Z)) \geqslant w(Z) \quad \text{and} \quad w(\mathrm{Im}^{-1}(Z)) \geqslant w(Z).$$

We begin by establishing the simplest case of Lemma 3.3 when $h = 2$ and the middle layer is the separating set. We will need to apply the following in the next section and therefore state it in slightly more general terms.

**Lemma 3.5.** *Let $C$ be a positive real and let $H$ be a commutative graph of level two with vertex set $U_0 \cup U_1 \cup U_2$. Suppose that, for all $S \subseteq U_1$,*

$$|\mathrm{Im}(S)| \geqslant C|S| \quad \text{and} \quad |\mathrm{Im}^{-1}(S)| \geqslant C^{-1}|S|.$$

*If $X_i$ is the set of vertices in $U_1$ that have in-degree equal to $i$ and $Y_i$ is set of vertices in $U_2$ that have in-degree equal to $i$, then*

$$C|X_i| = |Y_i|.$$

*Similarly, if $X_i'$ is the set of vertices in $U_1$ that have out-degree equal to $i$ and $Y_i'$ is the set of vertices in $U_0$ that have out-degree equal to $i$, then*

$$C^{-1}|X_i'| = |Y_i'|.$$

**Proof.** The sets $X_i$ form a partition of $U_1$. We partition $U_2$ into:

$$\begin{aligned} T_k &= \mathrm{Im}(X_k), \\ T_{k-1} &= \mathrm{Im}(X_{k-1}) \backslash T_k, \\ &\vdots \\ T_1 &= \mathrm{Im}(X_1) \backslash (T_2 \cup \cdots \cup T_k). \end{aligned}$$

Similarly we have a partition of $U_1$ into $X_1', \ldots, X_{k'}'$ and a partition of $U_0$ into:

$$\begin{aligned} T_{k'}' &= \mathrm{Im}^{-1}(X_{k'}'), \\ T_{k'-1}' &= \mathrm{Im}^{-1}(X_{k'-1}') \backslash T_{k'}, \\ &\vdots \\ T_1' &= \mathrm{Im}^{-1}(X_1') \backslash (T_2' \cup \cdots \cup T_{k'}'). \end{aligned}$$

See Figure 1.

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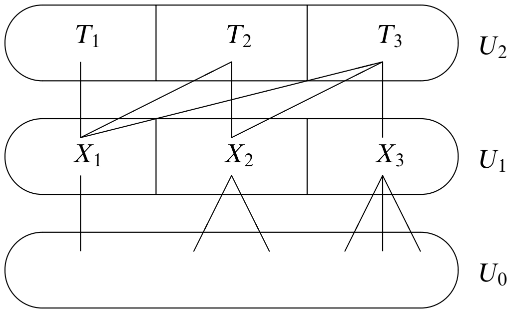


*Figure 1.* An illustration of the $k = 3$ case.

By the definition of the $T_i$ we have that

$$\mathrm{Im}(X_j \cup \cdots \cup X_k) = T_j \cup \cdots \cup T_k.$$

If we let $x_i = |X_i|$ and $t_i = |T_i|$, then the hypothesis on $H$ implies that

$$\sum_{i=j}^{k} t_i \geqslant C \sum_{i=j}^{k} x_i \qquad \text{for all} \quad 1 \leqslant j \leqslant k.$$

Adding these inequalities for $j = 1, \ldots, k$ gives

$$\sum_{i=1}^{k} i t_i \geqslant C \sum_{i=1}^{k} i x_i.$$

It follows from Plünnecke's downward condition and the definition of $T_i$ and $X_i$ that $d^-(v) \geqslant i$ for all $v \in T_i$. Hence

$$\begin{aligned} |E(U_0, U_1)| &= \sum_{i=1}^{k} |E(U_0, X_i)| \\ &= \sum_{i=1}^{k} i x_i \\ &\leqslant C^{-1} \sum_{i=1}^{k} i t_i \\ &\leqslant C^{-1} \sum_{i=1}^{k} |E(U_1, T_i)| \\ &= C^{-1} |E(U_1, U_2)|. \end{aligned}$$

We repeat the above calculation, this time using the second partition of $U_1$, and get

$$|E(U_1, U_2)| \leqslant C |E(U_0, U_1)|.$$

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Putting everything together yields

$$|E(U_0, U_1)| \leqslant C^{-1}|E(U_1, U_2)| \leqslant |E(U_0, U_1)|.$$

We must therefore have equality in every step, which implies that $Y_i = T_i$ and $Y_i' = T_i'$, as well as $C|X_i| = |Y_i|$ and $C^{-1}|X_i'| = |Y_i'|$. □

We now apply the lemma to 'pull down' minimal separating sets in the special, yet important, class of graphs of level two discussed in the beginning of the subsection.

**Lemma 3.6.** *Let $C$ be a positive real and let $H$ be a weighted commutative graph of level two with vertex set $U_0 \cup U_1 \cup U_2$ and $w(v) = C^{-i}$ for all $v \in V_i$. Suppose that $U_1$ is a separating set of minimum weight. Then so is $U_0$.*

**Proof.** For every $S \subseteq U_1$ both $S^c \cup \mathrm{Im}(S)$ and $S^c \cup \mathrm{Im}^{-1}(S)$ are separating sets. The minimality of $w(U_1)$ implies that

$$|\mathrm{Im}(S)| \geqslant C|S| \quad \text{and} \quad |\mathrm{Im}^{-1}(S)| \geqslant C^{-1}|S|.$$

We can therefore apply Lemma 3.5 to get

$$w(U_1) = C^{-1}|U_1| = C^{-1}\left|\bigcup_{i=1}^{k'} X_i'\right| = C^{-1}\sum_{i=1}^{k'}|X_i'| = \sum_{i=1}^{k'}|Y_i'| = \left|\bigcup_{i=1}^{k'} Y_i'\right| = |U_0| = w(U_0). \quad \square$$

We are finally able to prove Lemma 3.3, which will finish the proof of Theorem 1.1.

**Proof of Lemma 3.3.** Let $S$ be any separating set of minimum weight and $S_i = S \cap V_i$. Let $j \in \{0, 1, \ldots, h-1\}$ be maximal subject to $S_j \neq \varnothing$. We will show that, when $j > 0$, we can find another separating set of minimum weight that lies in $V_0 \cup \cdots \cup V_{j-1} \cup V_h$.

We work in a subgraph $H$ of level two consisting of all paths in $G$ that start in a suitably chosen $U_0 \subseteq V_{j-1}$ and end in a suitably chosen $U_2 \subseteq V_{j+1}$. $U_0$ consists of all vertices in $V_{j-1}$ that can be reached via paths in $G$ that successively pass from $S_0^c, \ldots, S_{j-1}^c$, and $U_2$ consists of all vertices in $V_{j+1}$ that lead to $S_h^c$. $S$ is a separating set of minimal weight and thus the middle layer $U_1$ equals $S_j$. In the weighted version of $H$, where vertices in $U_i$ have weight $C^{-i}$, $U_1$ is a separating set of minimum weight (if not, let $S_j'$ be a separating set of smaller weight and observe that $S_0 \cup \cdots \cup S_{j-1} \cup S_j' \cup S_h$ is then a separating set in $G$ of smaller weight than $S$). By Lemma 3.6, $U_0$ is also a separating set of minimum weight in $H$ and thus $S_0 \cup \cdots \cup S_{j-1} \cup U_0 \cup S_h$ is a separating set of minimum weight in $G$. □

Looking back at the proof of Plünnecke's inequality we realize that Plünnecke's conditions were not used directly. Instead we relied on two properties that follow from them: properties (1) and (2) in Section 2. It is clear that both are necessary in the proof. It is therefore natural to ask how different this pair of conditions is compared to Plünnecke's.

Ruzsa has already noted in [8] that the two sets of conditions are equivalent, and as a consequence the proof of Plünnecke's inequality requires the full strength of Plünnecke's conditions. This observation was left as an exercise and so we offer a quick explanation. Suppose that Plünnecke's, say upward, condition fails for an edge $uv$. It follows that there is no matching from $\mathrm{Im}(v)$ to $\mathrm{Im}(u)$ in the bipartite graph $G(\mathrm{Im}(v), \mathrm{Im}(u))$ where $xy$ is an edge if and only if $yx$ is an edge in $G$. By Lemma 2.1 we know there exists $S \subseteq \mathrm{Im}(v)$ such that $|\mathrm{Im}^{(-1)}(S)| < |S|$. Now consider the channel $H$ between $u$ and $S$. This is a commutative graph and $uv \in E(H)$, yet $d_H^+(u) = |\mathrm{Im}^{(-1)}(S)| < |S| = d_H^+(v)$.

Before moving on we prove a slight variation of Lemma 3.5, which will be useful in Section 5.

**Lemma 3.7.** *Let $C$ be a positive real and let $H$ be a commutative graph of level two with vertex set $U_0 \cup U_1 \cup U_2$. Suppose that for all $S \subseteq U_1$ we have*

$$|\mathrm{Im}^{-1}(S)| \geqslant C^{-1}|S| \quad \textit{and} \quad C|E(U_0, U_1)| = |E(U_1, U_2)|.$$

*Then $|U_1| = C|U_0|$.*

**Proof.** This is almost identical to what we have already seen. We partition $U_1$ and $U_0$ into, respectively, $X'_1, \ldots, X'_{k'}$ and $T'_1, \ldots, T'_{k'}$, as in the proof of Lemma 3.5. We have

$$\mathrm{Im}^{-1}(X'_j \cup \cdots \cup X'_{k'}) = T'_j \cup \cdots \cup T'_{k'}.$$

If we once again let $x'_i = |X'_i|$ and $t'_i = |T'_i|$, then the first hypothesis on $H$ implies that

$$C \sum_{i=j}^{k'} t'_i \geqslant \sum_{i=j}^{k'} x'_i \qquad \text{for all} \quad 1 \leqslant j \leqslant k'. \tag{3.1}$$

Adding the $k'$ inequalities gives

$$C \sum_{i=1}^{k'} i t'_i \geqslant \sum_{i=1}^{k'} i x'_i.$$

From Plünnecke's upward condition we know that $d^+(v) \geqslant i$ for all $v \in T'_i$, and in a similar fashion to the proof of Lemma 3.5 we get

$$|E(U_1, U_2)| \leqslant C|E(U_0, U_1)|.$$

The second condition on $H$ implies that equality must hold in every step. In particular, setting $j = 1$ on (3.1) gives

$$C|U_0| = C \sum_{i=1}^{k'} t'_i = \sum_{i=1}^{k'} x'_i = |U_1|. \qquad \square$$

## 4. Regular commutative graphs

We now turn to investigating the sharpness of Plünnecke's inequality and prove Theorems 1.2 and 1.3. For the former we construct arbitrarily long commutative graphs where $D_i^{1/i}(G)$ is constant. The latter will be proved by examining the growth of commutative graphs that originate at a singleton.

### 4.1. Regular commutative graphs

The two theorems are closely related with the existence of regular commutative graphs.

**Definition.** Let $C \in \mathbb{Q}^+$. $R_C$ is a *regular commutative graph of ratio C* whenever $d^-(v) = d$ and $d^+(v) = Cd$ for all $v \in V(G)$ and some $d \in \mathbb{Z}^+$.

It is easy to see why they are important in this context.

**Lemma 4.1.** *Let $C \in \mathbb{Q}^+$ and $i \leqslant h$ be positive integers. Suppose that G is a regular commutative graph of ratio C with vertex set $V_0 \cup \cdots \cup V_h$. Then*

$$D_i(G) = C^i$$

*and*

$$|V_i| = C^i |V_0|.$$

*Furthermore the inverse of G is an $R_{C^{-1}}$.*

**Proof.** Suppose that $d^- = d$ and $d^+ = Cd$ for all vertices of the graph. There are $Cd|Z|$ edges coming out from every $Z \subseteq V_0$. These edges land in at least $C|Z|$ vertices in $V_1$ and hence we get that $|\operatorname{Im}(Z)| \geqslant C|Z|$, and consequently that $D_1(G) \geqslant C$. Looking at $\operatorname{Im}^{(i)}(Z) = \operatorname{Im}(\operatorname{Im}^{(i-1)}(Z))$ we see that $|\operatorname{Im}^{(i)}(Z)| \geqslant C^i|Z|$ – and consequently that $D_i(G) \geqslant C^i$. Next we count the edges between $V_{i-1}$ and $V_i$ in two different ways to get

$$Cd\,|V_{i-1}| = |E(V_{i-1}, V_i)| = d\,|V_i|.$$

Hence $|V_i| = C^i|V_0|$, which shows that $D_i(G) \leqslant C^i$.

We know that the inverse of $G$ is a commutative graph. It is furthermore regular with ratio $C^{-1}$. □

To prove Theorem 1.2 it is therefore enough to construct arbitrarily long $R_C$ for all $C \in \mathbb{Q}^+$. Let us begin by two simple yet fundamental observations. It is enough to construct arbitrarily long $R_k$ for all positive integers $k$, because if we let $C = p/q$ be any rational, then the Cartesian product of an $R_p$ with the inverse of an $R_q$ is an $R_C$. A path is an infinite $R_1$, so from now on we will focus on $R_k$ for integer $k > 1$.

### 4.2. Arbitrarily long regular commutative graphs

Getting an $R_k$ of level two is not hard, but we will not present the simplest example as it cannot be extended to an $R_k$ of level three. We will instead inductively build arbitrarily long $R_k$. Our aim is to take an $R_k$ of level $h$ and add a layer from below in such as a way as to get an $R_k$ of level $h+1$. To achieve this we have to tweak the $R_k$ of level $h$ slightly by taking its Cartesian product with a suitably chosen commutative graph. The following graph has the desired properties.

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**Lemma 4.2.** *Let $k$ and $h$ be positive integers. There exists an $R_1$ of level $h$ that gives rise to a one-to-$k$ matching from the image of any $v \in U_0$ to $U_0$ itself. $U_0$ is the bottom layer of the graph.*

**Proof.** We work in $\mathbb{Z}_{2k^2}$ and consider the level $h$ addition graph $G_+(A,B)$ for $A = \mathbb{Z}_{2k^2}$ and

$$B = \{0, 1, 2, \ldots, k-1, k, 2k, 3k, \ldots, k^2\}.$$

This is an $R_1$. We define a map $\theta$ from the image of any $v \in U_0$ to $U_0^k$ by:

$$\begin{aligned}
\theta(v+0) &= \{v, v-1, v-2, \ldots, v-(k-1)\},\\
\theta(v+1) &= \{v+1, v+1-2k, v+1-3k, \ldots, v+1-k^2\},\\
\theta(v+2) &= \{v+2, v+2-2k, v+2-3k, \ldots, v+2-k^2\},\\
&\ \ \vdots\\
\theta(v+k-1) &= \{v+k-1, v+(k-1)-2k, v+(k-1)-3k, \ldots, v+(k-1)-k^2\},\\
\theta(v+k) &= \{v+k, v-k, v-2k, \ldots, v-(k-1)k\},\\
\theta(v+2k) &= \{v+2k, v+2k-1, v+2k-2, \ldots, v+2k-(k-1)\},\\
&\ \ \vdots\\
\theta(v+k^2) &= \{v+k^2, v+k^2-1, v+k^2-2, \ldots, v+k^2-(k-1)\}.
\end{aligned}$$

A routine check confirms that every element of $\theta(v+j)$ is indeed joined to $v+j$ in the graph. For example, $v+1$ is joined to $v+1$ as it equals $v+1-0$ and $v-k$ is joined to $v+k$ as it equals $v+k-2k$. A second routine check confirms that $\theta(v+b) \cap \theta(v+b') = \emptyset$ for all $v \in U_0$ and distinct $b, b' \in B$. In other words the graph yields a one-to-$k$ matching between the image of any $v \in U_0$ and $U_0$ itself, as claimed. □

We can now complete the inductive step by combining the above with Lemma 2.1 and the multiplicativity of degrees.

**Proposition 4.3.** *Suppose that an $R_k$ of level $h$ exists with the property that every vertex in the first layer is joined to every vertex in the second layer. Then an $R_k$ of level $h+1$ with the same property exists.* □

**Proof.** Suppose that $W_0, \ldots, W_h$ are the layers of $R_k$ with $|W_0| = d$. The defining properties of $R_k$ imply that $d^- = d$ and $d^+ = dk$. Let $R_1$ be the graph described in Lemma 4.2 with layers $U_0, \ldots, U_h$.

We let $G_h = R_k \times R_1$. This graph is a regular commutative graph of ratio $k$ whose bottom layer has size $|W_0 \times U_0| = 2dk^2$. Next we add a layer of size $2dk$ to the bottom and join every added vertex to the whole of $W_0 \times U_0$. Let $G_{h+1}$ be the resulting graph of level $h+1$, which is regular with in-degree $2dk$ and out-degree $2dk^2$. To complete the proof we show that $G_{h+1}$ is a commutative graph.

We only need to check Plünnecke's conditions involving the recently added bottom layer. The remaining layers pose no problem as they belong to $G_h$, which is commutative. The downward condition is immediate as the size of the bottom layer equals the in-degree. To check the upward condition we consider an edge $u(w,v)$, where $u$ lies in the bottom layer of $G_{h+1}$ and $(w,v)$ lies in the second layer, *i.e.*, $w \in W_0$ and $v \in U_0$. Plünnecke's condition requires finding a matching from $\mathrm{Im}_{G_{h+1}}((w,v)) = W_1 \times \mathrm{Im}_{R_1}(v)$ to $\mathrm{Im}_{G_{h+1}}(u) = W_0 \times U_0$.

With this in mind we turn our attention to the bipartite graph $(W_1 \times \mathrm{Im}_{R_1}(v), W_0 \times U_0)$ and aim to apply Lemma 2.1. We keep the same notation as in Section 2 and write $\Gamma(x)$ for the neighbourhood of $x$ in the bipartite graph, which is precisely $\mathrm{Im}^{-1}_{G_{h+1}}(x)$. Let $\pi_2$ be the projection onto $R_1$. For any $S \subseteq W_1 \times \mathrm{Im}_{R_1}(v)$ we have from Lemma 4.2

$$\begin{aligned}
|\Gamma(S)| &= \left| W_0 \times \bigcup_{x \in \pi_2(S)} \mathrm{Im}^{-1}_{R_1}(x) \right| \\
&\geqslant \sum_{x \in \pi_2(S)} |W_0|\,|\theta(x)| \\
&= |\pi_2(S)|\,|W_0|\,k \\
&= |\pi_2(S)|\,|W_1| \\
&\geqslant |S|.
\end{aligned}$$

Hence Hall's condition is satisfied, and as a consequence so is Plünnecke's. □

We construct arbitrarily long $R_k$ (and hence finish the proof of Theorem 1.2) as follows. We start with the two-layer (and hence non-commutative) graph consisting of a single vertex in $V_0$ joined to all $k$ vertices in $V_1$. A first application of Proposition 4.3 yields an $R_k$ of level two. Repeated applications yield an arbitrarily long $R_k$.

In light of Theorem 1.3 it should be noted that this construction does not lead to infinite regular commutative graphs as in each step the size of the bottom layer increases.

### 4.3. Infinite regular commutative graphs

The construction of arbitrarily long regular commutative graphs we have presented does not give infinite regular commutative graphs. This does not of course rule out their existence. In order to prove Theorem 1.3 we will examine how much a Plünnecke graph originating at a singleton can grow. Plünnecke's inequality gives $|V_h| \leqslant |V_1|^h$, but the growth is in fact far from exponential.

**Lemma 4.4.** *Let $G$ be an infinite commutative graph where $|V_0| = 1$ and $|V_1| = n$. Then*

$$|V_h| \leqslant \binom{n+h-1}{h}.$$

*The bound is best possible.*

**Proof.** We perform a double induction on $n$ and $h$. Let $A(n,h)$ be the maximum of $|V_h|$ taken over all commutative graphs with $|V_0| = 1$ and $|V_1| = n$. Take such a $G$ with $V_0 = \{u\}$ and $V_1 = \{v_1, \ldots, v_n\}$. Any element of $V_h$ can either be reached from a path passing from $v_1$ or exclusively via paths that pass from $\{v_2, \ldots, v_n\}$. In the former case the

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vertex lies in the commutative graph consisting of all paths that start in $v_1$. By Plünnecke's upward condition the second layer of this graph has at most $n$ elements and so there are at most $A(n, h-1)$ such vertices in $V_h$. 'In the latter case the vertex lies in the channel between $u$ and $V_h \backslash \mathrm{Im}^{(h)}(v_1)$.' The second layer of this graph is a subset of $\{v_2, \ldots, v_n\}$ and hence there are at most $A(n-1, h)$ such vertices in $V_h$. We have therefore proved that

$$A(n, h) \leqslant A(n, h-1) + A(n-1, h).$$

It follows from Plünnecke's condition that $A(1, h) = 1 = \binom{h}{h}$ for all $h$ and we know that $A(n, 1) = n = \binom{n}{1}$. The stated bound follows inductively from the well-known identity $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$, and is attained when $G$ is an independent addition graph on $n$ generators. □

Deducing Theorem 1.3 is straightforward.

**Proof of Theorem 1.3.** A path is an infinite commutative graph whose magnification ratios are all equal to one.

Let $1 \neq C \in \mathbb{Q}^+$ and let $G$ be a commutative graph where $D_i(G) = C^i$ for all $i$. We have to show that $G$ is finite.

When $C < 1$ we let $V_0$ be the bottom layer of $G$. The definition of magnification ratios implies that there exists $\emptyset \neq Z_i \subseteq V_0$ such that $|\mathrm{Im}^{(h)}(Z_i)| = C^i |Z_i|$. The quantity $C^i|Z_i|$ is a non-zero integer less than $C^i|V_0|$ and so $i$ cannot be arbitrarily large. When $C > 1$ we let $V_1$ be the second layer of $G$. Lemma 4.4 implies that

$$D_i(G) \leqslant \binom{|V_1| + i - 1}{i} = O(i^{|V_1|}),$$

which for large enough $i$ is less than $C^i$. □

## 5. Inverse theorem for Plünnecke's inequality

We conclude the paper by establishing a necessary condition for Plünnecke's inequality to be attained. We use some of the results in Section 3 to prove an inverse result for Theorem 1.1.

**Theorem 5.1.** *Let $C \in \mathbb{Q}^+$ and let $G$ be a commutative graph with $D_i(G) = C^i$ for all $i$. Then there exists $\emptyset \neq Z \subseteq V_0$ whose channel is a regular commutative graph of ratio $C$.*

The definition of 'channel of $Z$' can be found in Section 2.

### 5.1. Inverse theorem for Plünnecke's inequality

The first step in proving Theorem 5.1 is to identify $Z$. It turns out that choosing the smallest non-empty subset of $V_0$ that has a chance of working will do. We will later need the cardinalities of the various layers of the channel of such a $Z$.

**Lemma 5.2.** *Let $C \in \mathbb{Q}^+$ and suppose $G$ is a commutative graph with $D_i(G) = C^i$ for all $1 \leqslant i \leqslant h$. Let $\emptyset \neq Z \subseteq V_0$ be of minimal size subject to $|\operatorname{Im}(Z)| = C|Z|$ and let $H$ be the channel of $Z$ with vertex set $U_0 \cup \cdots \cup U_h$. Then $|U_i| = C^i|U_0|$ for all $1 \leqslant i \leqslant h$.*

**Proof.** For any $S \subseteq Z = U_0$ we have that $\operatorname{Im}^{(i)}(S) = \operatorname{Im}_H^{(i)}(S)$ so we will drop the subscript. Observe that $D_1(H) = C$. We use this to show that $D_i(H) = C^i$ for all $i$. Indeed

$$C^i = D_i(G) \leqslant D_i(H) \leqslant D_1^i(H) = C^i,$$

the first inequality following from the definition of magnification ratios, while the second follows from Theorem 1.1. Hence $|\operatorname{Im}^{(i)}(S)| = C^i|S|$ for some $S \subset U_0$. $\operatorname{Im}(S)^c \cup \operatorname{Im}^{(i)}(S)$ is a separating set in the weighted version of $H$, where as usual $w(v) = C^{-i}$ for all $v \in U_i$. By Corollary 3.4 we know that

$$\begin{aligned}|U_0| &\leqslant w(\operatorname{Im}(S)^c) + w(\operatorname{Im}^{(i)}(S)) \\ &= C^{-1}(|U_1| - |\operatorname{Im}(S)|) + C^{-i}|\operatorname{Im}^{(i)}(S)| \\ &= |U_0| - C^{-1}|\operatorname{Im}(S)| + |S|.\end{aligned}$$

Thus $|\operatorname{Im}(S)| \leqslant C|S|$. The minimality of $Z$ implies that $S = Z = U_0$. □

We proceed with the proof of Theorem 5.1. We will use Lemma 3.5 on page 927 repeatedly to show that $H$ has to in fact be regular.

**Proof of Theorem 5.1.** Similarly to above we let $\emptyset \neq Z \subseteq V_0$ be of minimal size subject to $|\operatorname{Im}(Z)| = C|Z|$. Our goal is to prove that its channel $H$ is a regular commutative graph of ratio $C$. We will not have to work in $G$ any further, so to keep the notation simple we will write Im and $\operatorname{Im}^{-1}$ instead of $\operatorname{Im}_H$ and $\operatorname{Im}_H^{-1}$. Note, however, that in general $\operatorname{Im}_G^{-1} \neq \operatorname{Im}_H^{-1}$.

We want to apply Lemma 3.5, so we let $U_0 \cup U_1 \cdots \cup U_h$ be the vertex set of $H$ with the usual weights $w(v) = C^{-i}$ for all $v \in U_i$. We partition $U_1$ into $X_1, \ldots, X_k$ (where $d^- \restriction X_i = i$) and $X'_1, \ldots, X'_{k'}$ (where $d^+ \restriction X'_i = i$). We also partition $U_0$ and $U_2$, respectively, into $Y'_1, \ldots, Y'_{k'}$ (where $d^+ \restriction Y'_i = i$) and $Y_1, \ldots, Y_k$ (where $d^- \restriction Y_i = i$). To check that the condition of Lemma 3.5 is satisfied we observe that $U_1$, which by Lemma 5.2 has weight $|U_0|$, is by Corollary 3.4 a separating set of minimum weight. For every $S \subseteq U_1$ both $S^c \cup \operatorname{Im}(S)$ and $S^c \cup \operatorname{Im}^{-1}(S)$ are separating sets. The minimality of $w(U_1)$ implies that

$$|\operatorname{Im}(S)| \geqslant C|S| \quad \text{and} \quad |\operatorname{Im}^{-1}(S)| \geqslant C^{-1}|S|.$$

Our first task will be to establish that $Y'_{k'} = U_0$ and that the out-degree in $U_0 \cup U_1$ is $k'$. Suppose not. Then

$$\bigcup_{i=1}^{k'-1} Y'_i$$

is both non-empty and not the whole of $U_0$. By the minimality of $Z$,

$$\left|\bigcup_{i=1}^{k'-1} X'_i\right| = \left|\operatorname{Im}\left(\bigcup_{i=1}^{k'-1} Y'_i\right)\right| > C\left|\bigcup_{i=1}^{k'-1} Y'_i\right|.$$

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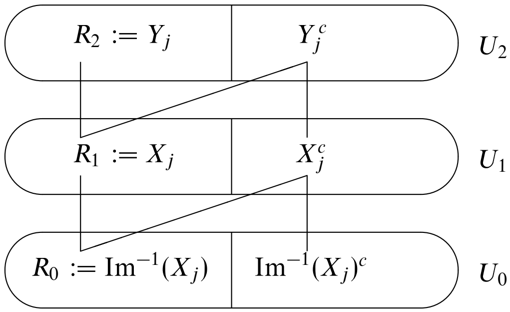


*Figure 2.* An illustration of how different parts of the graph are connected. Lines may correspond to multiple or no edges.

On the other hand, by Lemma 3.5 we know that

$$\left|\bigcup_{i=1}^{k'-1} X_i'\right| = \sum_{i=1}^{k'-1} |X_i'| = C \sum_{i=1}^{k'-1} |Y_i'| = C\left|\bigcup_{i=1}^{k'-1} Y_i'\right|.$$

So $Y_{k'}' = U_0$, and by Lemma 3.5 $|X_{k'}'| = C|Y_{k'}'| = |U_1|$, so $X_{k'}' = U_1$ and $d^+\restriction U_0 \cup U_1 = k'$.

Next we establish that $X_k = U_1$ and that the in-degree in $U_1 \cup U_2$ is $k$. Let $j$ be minimal subject to $Y_j \neq \emptyset$. Let $R$ be the channel between $Z = U_0$ and $Y_j$.

We observe that $\mathrm{Im}^{-1}(Y_j) = X_j$. This holds, as by Plünnecke's downward condition

$$\mathrm{Im}^{(-1)}(Y_j) \subseteq \bigcup_{i=1}^{j} X_i.$$

The choice of $j$ implies that $Y_i = \emptyset$, for $i < j$. By Lemma 3.5 we have $|X_i| = C^{-1}|Y_i| = 0$ for all $i < j$. Thus $\mathrm{Im}^{(-1)}(Y_j) = X_j$ as claimed.

Thus $R_0 = \mathrm{Im}^{-1}(X_j)$, $R_1 = X_j$ and $R_2 = Y_j$ are the layers of $R$ (see figure 2). We will apply Lemma 3.7 on page 930 to $R$ and so we need to check that the two conditions are satisfied. We begin with the second. We have $d_R^-(v) = d_H^-(v) = j$ for all $v \in R$ and by Lemma 3.5 that $|R_2| = |Y_j| = C|X_j| = C|R_1|$. Thus

$$|E(R_1, R_2)| = \sum_{w \in R_2} d^-(w) = |R_2|\, j = C|R_1|\, j = C|E(R_0, R_1)|.$$

For the first we observe that $\mathrm{Im}_R^{-1}(v) = \mathrm{Im}^{-1}(v)$ for all $v \in R$. We have seen above that $U_1$ is a separating set in $H$ of minimum weight and so we have that $|\mathrm{Im}_R^{-1}(S)| = |\mathrm{Im}^{-1}(S)| \geqslant C^{-1}|S|$ for all $S \subseteq R_1$. We can now apply Lemma 3.7 to get

$$|R_1| = C|R_0|. \tag{5.2}$$

On the other hand we know that $\mathrm{Im}(\mathrm{Im}^{-1}(X_j)^c) = X_j^c$, and so if $R_0 = \mathrm{Im}^{-1}(X_j) \neq U_0$, the minimality of $Z$ implies

$$|U_1| - |X_j| > C(|U_0| - |\mathrm{Im}^{-1}(X_j)| = |U_1| - C|\mathrm{Im}^{-1}(X_j)|,$$

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*i.e.*, that $C|R_0| > |R_1|$, which contradicts (5.2). We must therefore have $R_0 = U_0$. Hence $|X_j| = |R_1| = C|R_0| = C|U_0| = |U_1|$, *i.e.*, $X_j = U_1 = X_k$ and so $|Y_j| = C|X_j| = |U_2|$, *i.e.*, $Y_j = U_2 = Y_k$. In particular $d^- \restriction U_1 \cup U_2 = k$.

We therefore have regularity in the bottom three layers. We must check that the ratio of $k'$ to $k$ is $C$. This follows from counting the edges between $U_0$ and $U_1$ in two ways:

$$k'|U_0| = |E(U_0, U_1)| = k|U_1| = kC|U_0|.$$

The final step is to establish regularity for the remaining layers of $G$. We consider any $w \in U_2$. $d^+(w) \leqslant k' = Ck$, and so

$$C|E(U_1, U_2)| = C|U_1|Ck = Ck|U_2| \geqslant |E(U_2, U_3)|.$$

The fact that $|U_2| = C|U_1|$ follows from Lemma 5.2. Similarly $d^-(x) \geqslant k$ for any $x \in U_3$ and so

$$C|E(U_1, U_2)| = Ck|U_2| = k|U_3| \leqslant |E(U_2, U_3)|.$$

We must therefore have equality in each step and therefore $d^+(w) = Ck$ for all $w \in U_2$ and $d^-(x) = k$ for all $x \in U_3$. We repeat this step for all remaining layers to finish off the proof. □

**Remark.** An alternative way to prove Theorem 5.1 is to first establish the special case when $C = 1$ using Proposition 3.1, and then deduce the general case by the multiplicativity of magnification ratios and degrees. This time, independent addition graphs cannot work and we need to use regular commutative graphs.

Proposition 3.1 gives a sensible-looking necessary and sufficient condition for all magnification ratios of a commutative graph to be equal to one.

**Corollary 5.3.** *Let $G$ be a commutative graph and $V_0$ be its bottom layer. $D_i(G) = 1$ for all $i$ if and only if there exist $|V_0|$ vertex-disjoint paths of maximum length in $G$ and the channel of some $\emptyset \neq Z \subseteq V_0$ is an $R_1$.*

**Proof.** When $D_i(G) = 1$ for all $i$ we know from Proposition 3.1 that there are $|V_0|$ vertex-disjoint paths of maximum length in $G$, and we just proved the existence of a suitable non-empty $Z \subseteq V_0$. Conversely, the existence of the vertex-disjoint paths of maximum length guarantees that $D_i(G) \geqslant 1$ for all $i$ and Lemma 5.2 guarantees that $|\operatorname{Im}^{(i)}(Z)| = |Z|$ and hence $D_i(G) \leqslant 1$. □

Not a whole lot more can be said about the structure of such $G$. It is clear that the channel of $Z^c$ must have magnification ratios no smaller than one, and that is about it. For example, take any commutative graph $G$ of level two with vertex set $V_0 \cup V_1 \cup V_2$ and $D_i(G) \geqslant 1$ and any regular commutative graph $R$ with ratio one and vertex set $U_0 \cup U_1 \cup U_2$. Form a new graph $G'$ of level two by placing an edge between any $u \in V_i$ and any $v \in U_{i+1}$. $G'$ has magnification ratios equal to one as

$$|\operatorname{Im}^{(i)}_{G'}(S)| = |\operatorname{Im}^{(i)}_{G}(S \cap V_0)| + |\operatorname{Im}^{(i)}_{R}(S \cap U_0)| \geqslant |S \cap V_0| + |S \cap U_0| = |S|$$

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and

$$|\operatorname{Im}_{G'}^{(i)}(U_0)| = |\operatorname{Im}_R^{(i)}(U_0)| = |U_0|.$$

$G'$ is furthermore a commutative graph. The way $G$ is joined to $R$ means that for the upward condition we only need to worry about elements in $V_0$. Let $uv \in E(V_0, V_1)$. Then $\operatorname{Im}_{G'}(v) = \operatorname{Im}_G(v) \cup U_2$ and $\operatorname{Im}_{G'}(u) = \operatorname{Im}_G(u) \cup U_1$. We know from Plünnecke's condition that a matching exists from $\operatorname{Im}_G(v)$ to $\operatorname{Im}_G(u)$ and from Proposition 3.1 and Lemma 5.2 that a matching exists from $U_2$ to $U_1$. Putting the two matchings together gives a matching from $\operatorname{Im}_{G'}(v)$ to $\operatorname{Im}_{G'}(u)$. Next take $uv \in E(V_0, U_1)$, $\operatorname{Im}_{G'}(v) = \operatorname{Im}_R(v)$, and we know from Proposition 3.1 that there is a matching from $\operatorname{Im}_R(v)$ to $U_1 \subseteq \operatorname{Im}_{G'}(u)$ and hence from $\operatorname{Im}_{G'}(v)$ to $\operatorname{Im}_{G'}(u)$. Similar considerations show that the downward condition is satisfied.

## Acknowledgement

I would like to thank Tim Gowers for suggesting looking for a proof of Plünnecke's inequality that does not use Cartesian products of graphs and for sharing his insight. In particular, the idea of working with weighted commutative graphs was his. I would also like to thank Ben Green, Peter Keevash and Imre Ruzsa for many helpful suggestions.

## References

[1] Hall, P. (1935) On representatives of subsets. *J. London Math. Soc.* **10** 26–30.
[2] Malouf, J. L. (1995) On a theorem of Plünnecke concerning the sum of a basis and a set of positive density. *J. Number Theory* **54** 12–22.
[3] Menger, K. (1927) Zur allgemeinen Kurventheorie. *Fund. Math.* **10** 96–115.
[4] Nathanson, M. B. (1996) *Additive Number Theory: Inverse Problems and the Geometry of Subsets*, Springer.
[5] Plünnecke, H. (1970) Eine zahlentheoretische Anwendung der Graphtheorie. *J. Reine Angew. Math.* **243** 171–183.
[6] Ruzsa, I. Z. (1989) An application of graph theory to additive number theory. *Scientia Ser. A* **3** 97–109.
[7] Ruzsa, I. Z. (1990/1991) Addendum to: An application of graph theory to additive number theory. *Scientia Ser. A* **4** 93–94.
[8] Ruzsa, I. Z. (2009) Sumsets and structure. In *Combinatorial Number Theory and Additive Group Theory*, Springer.
[9] Tao, T. Additive combinatorics. Lecture notes 1, available online at `www.math.ucla.edu/~tao/254a.1.03w`.
[10] Tao, T. and Vu, V. H. (2006) *Additive Combinatorics*, Cambridge University Press.