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TERTILES AND THE TIME CONSTANT

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Abstract

We consider planar first-passage percolation and show that the time constant can be bounded by multiples of the first and second tertiles of the weight distribution. As a consequence, we obtain a counter-example to a problem proposed by Alm and Deijfen (2015).

Keywords: First-passage percolation; quantiles; time constant

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1. A theorem and a counter-example

In first-passage percolation on the square lattice, weights ω_e are assigned independently to the edges according to some distribution F on $[0, \infty)$. The resulting weighted graph induces a random (pseudo-)metric T on $\mathbb{Z}^2$ as follows: For all $x, y \in \mathbb{Z}^2$, let

$$T(x, y) := \inf \left\{ \sum_{e\in\pi} \omega_e : \pi \text{ is a self-avoiding path connecting } x \text{ to } y \right\}.$$

Let Y denote the minimum of four independent variables distributed as F. When $\mathbb{E}[Y] < \infty$ the limit $\mu := \lim_{n\to\infty} \frac{1}{n} T\big((0, 0), (n, 0)\big)$ exists almost surely as a consequence of Kingman's subadditive ergodic theorem. However, as is well known, the limit exists in probability for all weight distributions. See, e.g, [2] for background and details.

Upper bounds for μ can be expressed in terms of moments involving F; see, e.g., [6]. The moral of this note is that moments are in general poor estimates of μ. We provide bounds in terms of the first and second tertiles. A related lower bound has previously been obtained by Cox [3]. Our proof is much shorter. Let $t_q := \inf\{t \ge 0 : F(t) \ge q\}$.

Theorem 1. *For any F, the time constant* μ *satisfies* $\frac{1}{100} t_{1/3} \le \mu \le 2t_{2/3}$.

In $d \ge 2$ dimensions the arguments can be adapted to give $\frac{1}{4} t_{1/2d} \le \mu \le d t_{p_c(d)}$, where $p_c(d) \sim 1/d$ is the critical probability for oriented percolation on $\mathbb{Z}^d$.

Proof. The connoisseur will note that the upper bound is immediate from the 'flat edge' of Durrett and Liggett [4]. Spelling things out, let A_n denote the event that there exists a path of length n connecting the origin to a point in $\{(x, y) \in \mathbb{Z}^2 : x + y = n, x \ge n/2, y \ge 0\}$ having total weight at most $nt_{2/3} + M$. Similarly, let A'_n denote the event that there is a path of length n connecting $(n, 0)$ to $\{(x, x) \in \mathbb{Z}^2 : 0 \le x \le n/2\}$ having total weight at most $nt_{2/3} + M$. Since $2/3$ exceeds the critical probability for oriented percolation on $\mathbb{Z}^2$ (see [5]), standard

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results in percolation theory (see [4]) show that for large M we have $\mathbb{P}(A_n) = \mathbb{P}(A'_n) \geq 3/4$, uniformly in n. On the intersection $A_n \cap A'_n$, which occurs with probability at least $1/2$, we have $T\big((0, 0), (n, 0)\big)$ bounded by $2nt_{2/3} + 2M$, thus implying that $\mu \leq 2t_{2/3}$.

The lower bound is inspired by an argument explored by Smythe and Wierman [6], who in turn cite Hammersley. Given $\delta > 0$, let N_n denote the number of self-avoiding walks of length n starting at the origin that have fewer than δn edges with $\omega_e \geq t_{1/3}$. The number of self-avoiding walks of length n is at most 2.7^n for large n (see [6, p. 24]). For a given path of length n, the number of edges with $\omega_e \geq t_{1/3}$ is binomially distributed with parameters n and $p \geq 2/3$. Let X be binomial with parameters n and $2/3$. For $\beta > 0$ Markov's inequality gives

$$\mathbb{P}(X < \delta n) = \mathbb{P}\big(n - X > (1-\delta)n\big) \leq \mathrm{e}^{-\beta(1-\delta)n}\mathbb{E}\big[\mathrm{e}^{\beta(n-X)}\big] = \big[\tfrac{1}{3}\mathrm{e}^{\beta\delta} + \tfrac{2}{3}\mathrm{e}^{-\beta(1-\delta)}\big]^n.$$

Set $\beta = 5$ and $\delta = 1/100$. Monotonicity of the binomial distribution then gives

$$\mathbb{P}(N_n \geq 1) \leq \mathbb{E}[N_n] \leq 2.7^n\,\mathbb{P}(X < \delta n) \leq 2.7^n \cdot .36^n = .972^n.$$

On $\{N_n = 0\}$ we have $T\big((0, 0), (n, 0)\big) \geq \frac{n}{100}t_{1/3}$, so $\mu \geq \frac{1}{100}t_{1/3}$, as required. □

We have noted that for many common distributions $2t_{2/3}$ exceeds the mean of F. Nevertheless, it is curious that one may disregard as much as a third of the mass of a distribution and yet produce general upper and lower bounds on μ. A more careful analysis should be able to increase the fraction $1/3$, and decrease $2/3$, arbitrarily close to $1/2$. A conversation with Michael Damron led to the following examples, suggesting that the median cannot be used to obtain general upper nor lower bounds on μ: Let $(F_n)_{n\geq1}$ put mass $1/2$ at 1, be fully supported on [0,1], and converging weakly to the balanced Bernoulli distribution. Let $(F'_n)_{n\geq1}$ put mass $1/2$ at 1, be fully supported on $[1, \infty)$, with mass $1/2$ diverging in the limit. In all cases the median is 1. By continuity, the time constant for F_n tends to zero as $n \to \infty$. We believe further that the time constant for F'_n tends to infinity with n, although this requires an argument.

Based on simulations, it was suggested in [1, p. 668], and restated in [2, Question 12], that $\mu \geq \mathbb{E}[Y]$ should hold in great generality, at least when F puts no mass at zero. The above upper bound on μ provides a fairly general counter-example: Let F be any distribution on $[0, \infty)$ that puts mass at least $2/3$ on [0,1] and mass at least $1/6$ on $[3888, \infty)$. Then $\mu \leq 2$, whereas an easy calculation gives $\mathbb{E}[Y] \geq 3$.

References

[1] Alm, S. E. and Deijfen, M. (2015). First passage percolation on $\mathbb{Z}^2$: A simulation study. *J. Stat. Phys.* **161**, 657–678.

[2] Auffinger, A., Damron, M. and Hanson, J. (2017). *50 Years of First-Passage Percolation* (Univ. Lect. Ser. **68**). American Mathematical Society, Providence, RI.

[3] Cox, J. T. (1980). The time constant of first-passage percolation on the square lattice. *Adv. Appl. Prob.* **12**, 864–879.

[4] Durrett, R. and Liggett, T. M. (1981). The shape of the limit set in Richardson's growth model. *Ann. Prob.* **9**, 186–193.

[5] Liggett, T. M. (1995). Survival of discrete time growth models, with applications to oriented percolation. *Ann. Appl. Prob.* **5**, 613–636.

[6] Smythe, R. T. and Wierman, J. C. (1978). *First-Passage Percolation on the Square Lattice* (Lect. Notes Math. **671**). Springer, Berlin.