THE JOURNAL OF SYMBOLIC LOGIC
Volume 84, Number 3, September 2019

ON THE NUMBER OF COUNTABLE MODELS OF A COUNTABLE NSOP$_1$ THEORY WITHOUT WEIGHT ω

BYUNGHAN KIM

Abstract. In this article, we prove that if a countable non-$\aleph_0$-categorical NSOP$_1$ theory with nonforking existence has finitely many countable models, then there is a finite tuple whose own preweight is ω. This result is an extension of a theorem of the author on any supersimple theory.

§1. Introduction. In this article, T always is a complete theory in a language $\mathcal{L}$, and recall that $I(\omega, T)$ denotes the number of nonisomorphic countable models of T. We extend the following theorem of the author for supersimple theories to the context of NSOP$_1$ theories.

FACT 1.1 ([6]). *If T is countable and supersimple, then $I(\omega, T)$ is either 1 or infinite.*

As it is well known, Fact 1.1 is an extension of Lachlan's result in [8] for superstable theories. Later, Pillay pointed out that the following described in [3] is implicit in the proof of Lachlan's result.

FACT 1.2. *Assume countable T is stable and $1 < I(\omega, T) < \omega$. Then there is a finite tuple whose own preweight is ω.*

The author indeed proved the same Fact 1.2 for simple theories, which directly implies Fact 1.1 since a supersimple theory cannot have a type of finite tuple whose weight is ω.

Our main theorem in this note is the extension of Fact 1.2 for NSOP$_1$ theories.

THEOREM 1.3. *Assume countable T is NSOP$_1$ holding nonforking existence. If $1 < I(\omega, T) < \omega$, then there is a finite tuple whose own preweight is ω.*

Now we recall basic facts and terminology for this note. As usual we work in a large saturated model. Unless said otherwise, $a, b, c, \dots$ are *finite* tuples, $A, B, C, \dots$ are small sets, and $M, N, \dots$ are elementary submodels from the saturated model. That $a \equiv_A b$ means a, b have the same type over A; and for tuples a_i $(i < \kappa)$, $a_{<j}$ denotes $\{a_i \mid i < j\}$. The following (until Fact 1.6) can be found in the literature on simple theories, for example, in [7].

DEFINITION 1.4. (1) A formula $\varphi(x, a_0)$ *divides* over A if there is an A-indiscernible sequence $\langle a_i \mid i < \omega\rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is

Received September 18, 2018.
2010 *Mathematics Subject Classification.* 03C45, 03C50.
Key words and phrases. NSOP$_1$ theories, semi-isolation, preweight.

0022-4812/19/8403-0013
DOI:10.1017/jsl.2019.47

inconsistent. A formula *forks* over A if the formula implies some finite disjunction of formulas, each of which divides over A. A type divides/forks over A if the type implies a formula which divides/forks over A. We write $A \perp_B C$ if for any finite $a \in A$, $\mathrm{tp}(a/BC)$ does not fork over B.

(2) We say T is *stable* if nonforking holds uniqueness over models: For any $M \subseteq A$ and $p(x) \in S(M)$, there is a unique extension $q(x) \in S(A)$ of p which does not fork over M.

(3) We say T is *simple* if nonforking satisfies local character: For any a and A, there is $A_0 \subseteq A$ with $|A| \leq |T|$ such that $a \perp_{A_0} A$. Any stable theory is simple. We say T is *supersimple* if for any a and A, there is finite $A_0 \subseteq A$ such that $a \perp_{A_0} A$; and T is *superstable* if T is stable and supersimple.

(4) An A-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ is said to be *Morley* over A (or A-*Morley*) if $a_i \perp_A a_{<i}$ for each $i < \omega$.

Fact 1.5. *The following hold in any T.*

(1) (*Extension*) *If $a \perp_A B$ then for any C there is $a' \equiv_{AB} a$ such that $a' \perp_A BC$.*
(2) (*Base monotonicity*) *If $A \perp_B CD$ then $A \perp_{BC} D$.*
(3) (*Left transitivity*) *If $B \perp_C D$ and $A \perp_{BC} D$, then $AB \perp_C D$. Hence for a sequence $\langle c_i \mid i < \kappa \rangle$, if $c_i \perp_A c_{<i}$ holds for each $i < \kappa$, then $c_{\geq i} \perp_A c_{<i}$ for all $i < \kappa$.*

Fact 1.6. *Assume T is simple. Then the following hold.*

(1) (*Existence*) *For any a and A, we have that $a \perp_A A$. Equivalently, for any a_0 and A, there is an A-Morley sequence $\langle a_i \mid i < \omega \rangle$.*
(2) *A formula divides over a set iff the formula forks over the set.*
(3) *$\varphi(x, a_0)$ divides over A iff for some/any Morley sequence $\langle a_i \mid i < \omega \rangle$ over A, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*
(4) (*Symmetry*) *For any A, B, C we have $A \perp_B C$ iff $C \perp_B A$.*
(5) (*Transitivity*) *For any $B \subseteq C \subseteq D$, if $A \perp_B C$ and $A \perp_C D$, then we have $A \perp_B D$.*
(6) (*Type-amalgamation over a model*) *Assume $A_0 \perp_M A_1$, $c_0 \equiv_M c_1$, and $c_i \perp_M A_i$ for $i = 0, 1$. Then there is $c \equiv_{MA_i} c_i$ such that $c \perp_M A_1 A_2$.*

Recently, Kaplan and Ramsey proved in [4] and [5] that all the properties in Fact 1.6 (except base monotonicity in Fact 1.5) still hold *over models* in NSOP_1 theories with respect to *Kim-independence*. The 1-strong order property (SOP_1) is introduced by Shelah in [10], and a nice criterion for SOP_1 is given in [1] and [4].

Definition 1.7 ([10]).

(1) We say T has *SOP_1* if there are formula $\varphi(x, y)$ and tuples c_α ($\alpha \in 2^{<\omega}$) such that, for each $\beta \in 2^\omega$, $\{\varphi(x, c_{\beta \restriction m}) \mid m \in \omega\}$ is consistent; and $\{\varphi(x, c_{\alpha^\frown\langle 1 \rangle}), \varphi(x, c_\gamma)\}$ is inconsistent whenever $\alpha^\frown\langle 0 \rangle \trianglelefteq \gamma \in 2^{<\omega}$, that is, $\alpha^\frown\langle 0 \rangle$ is an initial segment of γ.
(2) We say T is *$NSOP_1$* if T does not have SOP_1. Any *simple* theory is NSOP_1.

Fact 1.8 ([1,4]). *T has SOP_1 iff there are a sequence $\langle a_i c_i \mid i < \omega \rangle$ and a formula $\varphi(x, y)$ such that*

https://doi.org/10.1017/jsl.2019.47 Published online by Cambridge University Press

(1) $a_i \equiv_{(ac)_{<i}} c_i$ *for each* $i < \omega$,
(2) $\{\varphi(x, a_i) \mid i < \omega\}$ *is consistent, while*
(3) $\{\varphi(x, c_i) \mid i < \omega\}$ *is* k*-inconsistent for some* $k \geq 2$.

Definition 1.9 (Assume T satisfies nonforking existence over A, i.e., for any c, $c \perp_A A$). A formula $\varphi(x, a_0)$ *Kim-divides* over A if there is an A-Morley sequence $\langle a_i \mid i < \omega\rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. A formula *Kim-forks* over A if the formula implies some finite disjunction of formulas, each of which Kim-divides over A. A type Kim-divides/forks over A if the type implies a formula which Kim-divides/forks over A. We write $B \perp^K_A C$ if for any finite $b \in B$, $\operatorname{tp}(b/AC)$ does not Kim-fork over B. Obviously $B \perp_A C$ implies $B \perp^K_A C$. Due to Fact 1.6(3), T is simple then $\perp = \perp^K$.

An A-indiscernible sequence $\langle b_i \mid i < \omega\rangle$ is called $\perp^K$-*Morley* over A (in $p(x)$) if $b_i \perp^K_A b_{<i}$ holds for each $i < \omega$ (and $p(x) = \operatorname{tp}(b_i/A)$).

Note that nonforking existence holds over any model since any type over a model is finitely satisfiable over the model.

Fact 1.10 ([4]). *Let T be NSOP$_1$.*

(1) (*Kim's lemma for* $\perp^K$ *over a model*) $\varphi(x, a_0)$ *Kim-divides over* M *iff for any Morley sequence* $\langle a_i \mid i < \omega\rangle$ *over* M, $\{\varphi(x, a_i) \mid i < \omega\}$ *is inconsistent.*
(2) *A formula Kim-divides over a model iff the formula Kim-forks over the model.*
(3) (*Extension for* $\perp^K$ *over a model*) *If* $a \perp^K_M B$ *then for any* C *there is* $a' \equiv_{MB} a$ *such that* $a' \perp^K_M BC$.
(4) (*Symmetry for* $\perp^K$ *over a model*) *For any* A, C *we have* $A \perp^K_M C$ *iff* $C \perp^K_M A$.
(5) (*Type-amalgamation for* $\perp^K$ *over a model*) *Assume* $A_0 \perp^K_M A_1$, $c_0 \equiv_M c_1$, *and* $c_i \perp^K_M A_i$ *for* $i = 0, 1$. *Then there is* $c \equiv_{MA_i} c_i$ *such that* $c \perp^K_M A_1 A_2$.

In a joint work [2], it is now proved that Fact 1.10 still holds over any set as far as nonforking existence holds. Due to Fact 1.6(1), the class of NSOP$_1$ theories with nonforking existence fully contains that of simple theories. Moreover all the typical nonsimple NSOP$_1$ examples described in [4] (namely, the random parameterized equivalence relations, ω-free PAC fields, and an infinite dimensional vector space over an algebraically closed field equipped with a symmetric alternating bilinear form) have nonforking existence. Even we conjecture that nonforking existence holds in any NSOP$_1$ T.

Fact 1.11 ([2]). *Assume T is NSOP$_1$ with nonforking existence* (*Fact* 1.6(1)).

(1) (*Kim's lemma for* $\perp^K$) $\varphi(x, a_0)$ *Kim-divides over* A *iff for any Morley sequence* $\langle a_i \mid i < \omega\rangle$ *over* A, $\{\varphi(x, a_i) \mid i < \omega\}$ *is inconsistent.*
(2) *A formula Kim-divides over some set iff the formula Kim-forks over the set.*
(3) (*Extension for* $\perp^K$) *If* $p(x)$ *is a type over* B *which does not Kim-forks over* A, *then there is a completion* $q(x) \in S(AB)$ *which does not Kim-fork over* A. *In particular if* $a \perp^K_A B$ *then for any* C *there is* $a' \equiv_{AB} a$ *such that* $a' \perp^K_A BC$.

https://doi.org/10.1017/jsl.2019.47 Published online by Cambridge University Press

(4) *(Symmetry for* $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$*) For any* A, B, C *we have* $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K_B C$ *iff* $C \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K_B A$.

(5) *(Chain condition for* $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$*) Let* $a \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K_A b_0$*, and let* $I = \langle b_i \mid i < \omega \rangle$ *be* $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$*-Morley over* A*. Then there is* $a' \equiv_{Ab_0} a$ *such that* $a' \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K_A I$ *and* I *is* $a'A$*-indiscernible.*

From now on for simplicity, *we assume that any NSOP*$_1$ *theory in this note has nonforking existence.*

In addition to Fact 1.11, type-amalgamation over sets for Lascar types are proved in [2] for any NSOP$_1$ theory, but we omit to state it as we will not use the property. Instead we will use Fact 1.11(5).

Remark 1.12. (1) Assume T is NSOP$_1$ and let $p(x) \in S(A)$. Then that '$\langle x_i \mid i < \omega \rangle$ is a sequence of realizations of p such that $x_i \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K_A x_{<i}$ for each $i < \omega$' is A-type-definable by $\bigwedge_{i<\omega} p(x_i) \cup \Gamma(x_0, x_1, \dots)$ where

$$\Gamma(x_0, x_1, \dots) := \{\neg\varphi(x_0, \dots, x_n, x_{n+1}) \in \mathcal{L}(A) \mid \varphi(x_0, \dots, x_n, a) \text{ Kim-divides over } A \text{ for some/any } a \models p\}.$$

Hence clearly that $\langle x_i \mid i < \omega \rangle$ is a $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$-Morley sequence over A in p is A-type-definable as well.

(2) Notice that contrary to simple theory context, that $\langle c_i \mid i < \omega \rangle$ is $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$-Morley over A in NSOP$_1$ T *need not* imply

$$c_i \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K_A \{c_j \mid j \neq i\}$$

for all $i \in \omega$, since base monotonicity for $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$ does not hold.

Now we are ready to talk about the notion of weight.

Definition 1.13. Assume T is NSOP$_1$. We say a finite tuple c (or its type) has *own preweight* ω if there are $b_i \equiv c$ $(i < \omega)$ such that $c \not\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K b_i$, and $b_i \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K b_{<i}$ for all $i < \omega$.

For more development of the weight notion in simple theories, see [7]. As pointed out in Remark 1.12(2), in Definition 1.13, $\{b_i \mid i < \omega\}$ need not be fully $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$-independent.

Recall that T is supersimple iff there do not exist c and sets A_i $(i < \omega)$ such that $A_i \subseteq A_{i+1}$ and $c \not\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}_{A_i} A_{i+1}$ for any i. Since $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}} = \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K$ in simple T, if T is supersimple then due to transitivity there is no $p(x) \in S(\emptyset)$ whose own preweight is ω.

Example 1.14. (1) Consider the typical stable but nonsuperstable theory. Namely, T is the theory in $\mathcal{L} = \{E_i(x, y) \mid i < \omega\}$ saying that each binary E_i is an equivalence relation only having infinitely many infinite classes such that for each $j > i$, E_j is finer than E_i and each E_i-class contains infinitely many E_j-classes. Notice that T is a small (i.e., $S(\emptyset)$ is countable) non-$\aleph_0$-categorical theory. But there is no finite tuple whose own preweight is ω.

(2) Due to our Theorem 1.3, a necessary condition for an NSOP$_1$ theory to have $1 < I(\omega, T) < \omega$ is that T should be small and having a finite tuple with own preweight ω. Herwig constructed such an example of a stable theory [3].

https://doi.org/10.1017/jsl.2019.47 Published online by Cambridge University Press

§2. Kim-forking and isolation. In order to prove Theorem 1.3, we will take the similar pattern of the proof for Fact 1.1 in [6].

We first recall Pillay's notion of semi-isolation ([3, 9]), and figure out its relationship with Kim-forking in NSOP_1 theories. We say $\mathrm{tp}(b/a)$ is *semi-isolated* if there is a formula $\varphi(x, a)$ in $\mathrm{tp}(b/a)$ such that $\models \varphi(x, a) \rightarrow \mathrm{tp}(b)$.

FACT 2.1. (1) *If* $\mathrm{tp}(b/a)$ *is isolated, then* $\mathrm{tp}(b/a)$ *is semi-isolated.*
(2) *If* $\mathrm{tp}(c/b)$ *and* $\mathrm{tp}(b/a)$ *are semi-isolated, then* $\mathrm{tp}(c/a)$ *is semi-isolated.*

We give a proof of the the following folklore for self-containedness.

FACT 2.2. *Suppose that* $\mathrm{tp}(b/a)$ *is isolated, whereas* $\mathrm{tp}(a/b)$ *is nonisolated. Then* $\mathrm{tp}(a/b)$ *is nonsemi-isolated.*

PROOF. Let $\mathrm{tp}(b/a)$ be isolated by $\varphi(x, a)$ (*). To lead a contradiction assume that $\psi(b, y)$ semi-isolates $\mathrm{tp}(a/b)$. Now since $\mathrm{tp}(a/b)$ is nonisolated, there is an $\mathcal{L}$-formula $\phi(x, y)$ such that $\varphi(b, y) \wedge \psi(b, y) \wedge \phi(b, y)$ and $\varphi(b, y) \wedge \psi(b, y) \wedge \neg\phi(b, y)$ are both consistent, while both imply $\mathrm{tp}(a)$. Hence $\varphi(x, a) \wedge \phi(x, a)$ and $\varphi(x, a) \wedge \neg\phi(x, a)$ are both consistent, contradicting (*). ⊣

The following is the key proposition describing a relationship between isolation and Kim-dividing in NSOP_1 theories.

PROPOSITION 2.3. *Assume that T is $NSOP_1$. Let $a \equiv b$. Assume* $\mathrm{tp}(b/a)$ *is semi-isolated, but* $\mathrm{tp}(a/b)$ *is nonsemi-isolated. Then* $a \not\mathop{\smile\!\!\!\!\!|}^K b$.

PROOF. Suppose not, so that $a \mathop{\smile\!\!\!\!\!|}^K b$.

CLAIM 2.4. *There is $c \models q = \mathrm{tp}(a)$ such that $b \mathop{\smile\!\!\!\!\!|}^K ac$ and $ba \equiv cb$: Choose $c_0 \models q$ such that $ba \equiv c_0 b$. Hence $a \mathop{\smile\!\!\!\!\!|}^K b$ and $b \mathop{\smile\!\!\!\!\!|}^K c_0$. Now $\mathrm{tp}(a/b)$ does not Kim-divide over $\emptyset$. Thus by the definition of Kim-dividing and compactness, for any infinite κ, there is some Morley sequence $I = \langle b_i \mid i < \kappa \rangle$ with $b = b_0$ such that I is a-indiscernible. Moreover by symmetry for $\mathop{\smile\!\!\!\!\!|}^K$ (Fact 1.11(4)), we have $c_0 \mathop{\smile\!\!\!\!\!|}^K b$. Hence $\mathrm{tp}(c_0/b)$ does not Kim-divide over $\emptyset$, and again by the definition of Kim-dividing, $\bigcup_{i<\kappa} p(x, b_i)$ is consistent where $p(x, b_0) = p(x, b) = \mathrm{tp}(c_0/b)$. Choose c_0' realizing $\bigcup_{i<\kappa} p(x, b_i)$ so that $c_0 b \equiv c_0' b$ and $c_0' b' \equiv c_0' b$ for any $b' \in I$ (*). Now take $\kappa = (2^{|T|})^+$. Then by the pigeonhole principle, there is a subsequence $I' = \langle b_i' \mid i < \kappa \rangle$ of I such that $\mathrm{tp}(b_i'/ac_0')$ is fixed for any i. Then since I' is Morley as well, we have $ac_0' \mathop{\smile\!\!\!\!\!|}^K b_0'$, by Fact 1.11(1),(2). Note now that $Ia \equiv I'a$. Hence there is c such that $c_0' b_0' I' a \equiv cb_0 Ia = cbIa$. Therefore by symmetry, we have $b \mathop{\smile\!\!\!\!\!|}^K ac$, and due to (*), $ba \equiv c_0 b \equiv c_0' b \equiv c_0' b_0' \equiv cb$, as wanted. We have proved Claim 1.*

Now put $c_0 b_0 a_0 = cba$. We can find $c_i b_i a_i \equiv cba$ $(i < \omega)$ such that $a_i c_{i-1} \equiv ba$ (**), and $(cba)_i \mathop{\smile\!\!\!\!\!|}^K (cba)_{<i}$, as follows. Assume we have found such $c_i b_i a_i$ for $i < k$. We want to find $c_k b_k a_k$ holding the conditions. Since $c_{k-1} \equiv a_0$, there is a_k such that $a_k c_{k-1} \equiv ba$. Now since $b \mathop{\smile\!\!\!\!\!|}^K a$, we have $a_k \mathop{\smile\!\!\!\!\!|}^K c_{k-1}$. Hence by extension for $\mathop{\smile\!\!\!\!\!|}^K$ (Fact 1.11(3)), we can assume that $a_k \mathop{\smile\!\!\!\!\!|}^K (cba)_{<k}$. By symmetry we have $(cba)_{<k} \mathop{\smile\!\!\!\!\!|}^K a_k$. Then again by extension, there is $c_k b_k$ such that $c_k b_k a_k \equiv c_0 b_0 a_0$ and $(cba)_{<k} \mathop{\smile\!\!\!\!\!|}^K c_k b_k a_k$. By symmetry, $c_k b_k a_k \mathop{\smile\!\!\!\!\!|}^K (cba)_{<k}$ as wanted.

We now let $\varphi(x, a)$ be a formula semi-isolating $\mathrm{tp}(b/a)$.

https://doi.org/10.1017/jsl.2019.47 Published online by Cambridge University Press

CLAIM 2.5. *The collection of formulas* $\{\varphi(c_i, x) \wedge \varphi(x, a_i) \mid i < \omega\}$ *is* 2-*inconsistent*: *If it were not* 2-*inconsistent, then there is* d *such that* $\varphi(d, a_j)$ *and* $\varphi(c_i, d)$ *for some* $j > i$. *Therefore clearly* $\mathrm{tp}(d/a_j)$ *and* $\mathrm{tp}(c_i/d)$ *are both semi-isolated, and hence again by Fact* 2.1(2), *so does* $\mathrm{tp}(c_i/a_j)$. *Now since* $\mathrm{tp}(a_j/a_{i+1})$ *is semi-isolated by* (**), *once more Fact* 2.1(2) *implies* $\mathrm{tp}(c_i/a_{i+1})$ *is semi-isolated. But since* $\mathrm{tp}(c_i a_{i+1}) = \mathrm{tp}(ab)$, *it leads a contradiction. Hence the claim is proved.*

Now by compactness applying to the type-definability described in Remark 1.12(1), there is some $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequence $\langle c_0' b_0' a_0' \mid i < \omega\rangle$ over A such that $c_0' b_0' a_0' = cba$ and $\{\varphi(c_i', x) \wedge \varphi(x, a_i') \mid i < \omega\}$ is 2-inconsistent. Note now that $b \models \varphi(c, x) \wedge \varphi(x, a)$. Then due to the chain condition for $\mathop{\smile\!\!\!\!\!\!|}^K$ in Fact 1.11, we must have $b \not\mathop{\smile\!\!\!\!\!\!|}^K ac$, contradicting Claim 1. Therefore we must have $a \not\mathop{\smile\!\!\!\!\!\!|}^K b$. ⊣

COROLLARY 2.6. *Assume that* T *is* $NSOP_1$, *and we let* $a \equiv b$. *If* $\mathrm{tp}(b/a)$ *is isolated, and* $\mathrm{tp}(a/b)$ *is nonisolated, then* $a \not\mathop{\smile\!\!\!\!\!\!|}^K b$.

§3. Proof of Theorem 1.3.

In this section, T is *countable and non-*$\aleph_0$*-categorical.* A proof of the following fact can be found for example in [3] or [6].

FACT 3.1 (Folklore). *Suppose that* $I(\omega, T)$ *is finite. Then there is a tuple* a *and a prime model* M *over* a *such that* $p(x) := \mathrm{tp}(a)$ *is nonisolated and all the types of finite tuples are realized in* M. *Moreover there is a tuple* b *in* M *such that* $b \equiv a$ *and* $\mathrm{tp}(a/b)$ *is nonisolated.*

We are ready to prove Theorem 1.3. We keep the notation in Fact 3.1. Assume further that T is NSOP_1.

CLAIM 3.2. *There are two realizations* a_1, a_0 *of* p *in* M *such that* $a_1 \mathop{\smile\!\!\!\!\!\!|} a_0$, *and both* $\mathrm{tp}(a_0/a_1), \mathrm{tp}(a_1/a_0)$ *are nonisolated.*

PROOF. Due to nonforking existence and extension, there is $c \models p$ such that $c \mathop{\smile\!\!\!\!\!\!|} ab$, and hence $c \mathop{\smile\!\!\!\!\!\!|}^K ab$. Now, by Fact 2.2, $\mathrm{tp}(a/b)$ is nonsemi-isolated. Hence, by Fact 2.1, either $\mathrm{tp}(a/c)$ or $\mathrm{tp}(c/b)$ must be nonisolated. Since $c \mathop{\smile\!\!\!\!\!\!|}^K ab$, if $\mathrm{tp}(a/c)$ is nonisolated then so is $\mathrm{tp}(c/a)$, by Corollary 2.6. The same holds when $\mathrm{tp}(c/b)$ is nonisolated. Now choose $a_1 a_0$ in M such that $a_1 a_0 \equiv ca$ or cb. ⊣

We continue the proof with the selected tuples. Note now that $\mathrm{tp}(a/a_0), \mathrm{tp}(a/a_1)$ are both nonisolated (†), since if say $\mathrm{tp}(a/a_0)$ were isolated, then M is prime over a_0 and so $\mathrm{tp}(a_1/a_0)$ would be isolated, a contradiction. Therefore again by Corollalry 2.6, we have $a \not\mathop{\smile\!\!\!\!\!\!|}^K a_0$ and $a \not\mathop{\smile\!\!\!\!\!\!|}^K a_1$. We are ready to claim the following which says that p has its own preweight ω, so finishes our proof of Theorem 1.3.

CLAIM 3.3. *There is a set* $\{a_u \mid u \in X,\ a_u \models p\}$ *where*

$$X = \{u \in 2^{<\omega} \mid u = 0^{m+1} = \overbrace{0\ldots0}^{m+1} \text{ or } 0^m 1 \text{ for some } m < \omega\}$$

such that for each $m < \omega$,

(1) $a_1 a_0 a \equiv a_{0^m 1} a_{0^{m+1}} a_{0^m}$,
(2) $a_{0^m 1} \mathop{\smile\!\!\!\!\!\!|} \{a_u \mid u \in X \text{ and } 0^{m+1} \trianglelefteq u\}$,
(3) $a_{0^m 1} \mathop{\smile\!\!\!\!\!\!|}^K a_1 a_{01} \ldots a_{0^{m-1}1}$, *and*
(4) $a \not\mathop{\smile\!\!\!\!\!\!|}^K a_u$ *for all* $u \in X$.

https://doi.org/10.1017/jsl.2019.47 Published online by Cambridge University Press

We prove the claim using induction. Given $k < \omega$, assume that we have selected $A = \{a_u \mid u \in X, |u| \leq k+1\}$ satisfying above (1)–(4) for each $m \leq k$. Note that $a_1 a_0 a$ satisfies the initial condition for $k = 0$. We will find appropriate $a_{0^{k+1}1}, a_{0^{k+2}} \models p$ holding (1)–(4) for $k+1$.

First choose $d_1 = a_{0^{k+1}1}$, $d_0 = a_{0^{k+2}} \models p$ such that $d_1 d_0 a_{0^{k+1}} \equiv a_1 a_0 a$. Now since A satisfies (1) with $m = k$, we have that $a_1 a_0 \equiv a_{0^k 1} a_{0^{k+1}}$ and so $a_{0^k 1} \mathop{\smile\!\!\!\!\!|} a_{0^{k+1}}$. Hence due to nonforking extension (Fact 1.5(1)), by possibly moving $d_1 d_0$ while fixing $a_{0^k 1} a_{0^{k+1}}$ we can additionally assume the chosen d_1, d_0 satisfy that $a_{0^k 1} \mathop{\smile\!\!\!\!\!|} d_1 d_0 a_{0^{k+1}}$. Now we iterate this process. Namely, since A satisfies (2) for $m = 0, \dots, k-1$ as well, again by nonforking extension we can further assume (by iteratively moving $d_1 d_0$ while fixing A pointwise) that $a_{0^{k-1}1} \mathop{\smile\!\!\!\!\!|} d_1 d_0 a_{0^{k+1}} a_{0^k 1} a_{0^k}$, $a_{0^{k-2}1} \mathop{\smile\!\!\!\!\!|} d_1 d_0 a_{0^{k+1}} a_{0^k 1} a_{0^k} a_{0^{k-1}1} a_{0^{k-1}}, \dots, a_1 \mathop{\smile\!\!\!\!\!|} d_1 d_0 (A \setminus \{a_1\})$. Therefore with this choice of $a_{0^{k+1}1} a_{0^{k+2}} = d_1 d_0$, (2) also holds on $A a_{0^{k+1}1} a_{0^{k+2}}$ for each $m \leq k+1$. The rest can be shown with the tuples $a_{0^{k+1}1} a_{0^{k+2}}$.

Namely, by (2), for each $m \leq k+1$, we have

$$a_{0^m 1} \mathop{\smile\!\!\!\!\!|} \{a_{0^n 1} \mid m < n \leq k+1\}.$$

Thus by Fact 1.5, we have that for each $n \leq k+1$,

$$\{a_{0^m 1} \mid m < n\} \mathop{\smile\!\!\!\!\!|} a_{0^n 1},$$

and hence

$$\{a_{0^m 1} \mid m < n\} \mathop{\smile\!\!\!\!\!|}^K a_{0^n 1}$$

holds since $\mathop{\smile\!\!\!\!\!|} \Rightarrow \mathop{\smile\!\!\!\!\!|}^K$. Then by symmetry for $\mathop{\smile\!\!\!\!\!|}^K$ (Fact 1.11(4)), it follows that

$$a_{0^n 1} \mathop{\smile\!\!\!\!\!|}^K \{a_{0^m 1} \mid m < n\}.$$

We have shown that (3) holds on $A a_{0^{k+1}1} a_{0^{k+2}}$. Now for a_1, we already know $a \not\mathop{\smile\!\!\!\!\!|}^K a_1$, For other a_u $(u \in X)$, due to (1) and Fact 2.1, we have that $\mathrm{tp}(a_u/a)$ is semi-isolated. However $\mathrm{tp}(a/a_u)$ is nonsemi-isolated since if it were so then again by Fact 2.1, $\mathrm{tp}(a/a_0)$ is semi-isolated contradicting above (†) and Fact 2.2. Therefore by Fact 2.3, $a \not\mathop{\smile\!\!\!\!\!|}^K a_u$. We have proved (4) and so complete the proof of Theorem 1.3.

Acknowledgment. The author has been supported by Samsung Science Technology Foundation under Project Number SSTF-BA1301-03.

REFERENCES

[1] A. Chernikov and N. Ramsey, *On model-theoretic tree properties*. ***Journal of Mathematical Logic***, vol. 16 (2016), no. 2, 1650009.

[2] J. Dobrowolski, B. Kim, and N. Ramsey, *Independence over arbitrary sets in NSOP$_1$ theories*, preprint.

[3] B. Herwig, *Weight ω in stable theories with few types*, this Journal, vol. 60 (1995), pp. 353–373.

[4] I. Kaplan and N. Ramsey, *On Kim-independence*. ***Journal of the European Mathematical Society***, to appear.

[5] ———, *Transitivity of Kim-independence*, preprint, 2019, arXiv:1901.07026.

[6] B. Kim, *On the number of countable models of a countable supersimple theory*. ***Journal of the London Mathematical Society (2)***, vol. 60 (1999), pp. 641–645.

[7] ———, ***Simplicity Theory***, Oxford University Press, Oxford, 2014.

https://doi.org/10.1017/jsl.2019.47 Published online by Cambridge University Press

[8] A. H. LACHLAN, *On the number of countable models of a countable superstable theory*, ***Logic, Methodology and Philosophy of Science IV*** (L. Henkin, editor), North-Holland, Amsterdam, 1973, pp. 45–56.

[9] A. PILLAY, *Countable models of stable theories*. ***Proceedings of the American Mathematical Society***, vol. 89 (1983), pp. 666–672.

[10] S. SHELAH, *Toward classifying unstable theories*. ***Annals of Pure and Applied Logic***, vol. 80 (1996), pp. 229–255.

DEPARTMENT OF MATHEMATICS
YONSEI UNIVERSITY
50 YONSEI-RO SEODAEMUN-GU
SEOUL 03722, REPUBLIC OF KOREA
E-mail: bkim@yonsei.ac.kr