APPROXIMATELY MULTIPLICATIVE DECOMPOSITIONS OF NUCLEAR MAPS

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Abstract

We expand upon work from many hands on the decomposition of nuclear maps. Such maps can be characterised by their ability to be approximately written as the composition of maps to and from matrices. Under certain conditions (such as quasidiagonality), we can find a decomposition whose maps behave nicely, by preserving multiplication up to an arbitrary degree of accuracy and being constructed from order-zero maps (as in the definition of nuclear dimension). We investigate these conditions and relate them to a W*-analogue.

1. Introduction

The germinal idea of this investigation was the definition of *nuclear dimension* in [15, Definition 2.1], which quantifies the *completely positive approximation property* (CPAP) of nuclear maps (although this notion goes back even further to the *decomposition rank* in [14]).

DEFINITION 1.1 [4, Ch. 2]. A contractive completely positive map $\pi : A \to B$ between C*-algebras (or to a von Neumann algebra B) is called *nuclear* (respectively *weakly nuclear*) if there exist a net (F_n) of finite-dimensional C*-algebras and contractive completely positive maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B$ such that $(\varphi_n \circ \psi_n)$ converges to π in the point-norm (respectively point-σ-weak) topology. If π is the identity, then we say that A itself is *nuclear*; if π is an inclusion, we say that A is *exact*.

Methods for strengthening the completely positive approximation property were first explored in [13]. This was synthesised with results of [2] on quasidiagonal nuclear C*-algebras into [3], which was built upon in [6]. This paper builds further by providing a partial answer to the final question in [6].

The case of nuclear A was examined in [13, Theorem 1.4], showing that we may then find $\varphi_n : F_n \to A$ (as in Definition 1.1) that are convex combinations of finitely many contractive completely positive order zero maps. From [3, Theorem 3.1], the net

$(\psi_n : A \to F_n)$ may be chosen to be approximately order zero. Furthermore, drawing from [2], if both A and all of its traces are quasidiagonal, then (ψ_n) may be chosen to be approximately multiplicative. In fact, [3, Theorem 2.2] shows that the converse is true as well.

Using the result from [13] as a starting point, Carrión and Schafhauser [6] began the process of generalising to nonnuclear A: provided that the nuclear map π is order zero, we may find (ψ_n) that is approximately order zero and φ_n that are convex combinations of contractive completely positive order zero maps [6, Theorem 1]. Additionally, approximate multiplicativity of (ψ_n) is attainable for weakly nuclear π with quasidiagonal A [6, Proposition 3].

The question is, then, 'What is necessary or sufficient to get approximate multiplicativity for nuclear π?' We give two necessary conditions.

LEMMA 1.2. *Let $\pi : A \to B$ be a *-homomorphism between C*-algebras A, B that admits an approximately multiplicative norm-decomposition (see Definition 2.5) and let τ be a trace on $\pi(A)$. Then $\tau \circ \pi$ is a quasidiagonal trace on A.*

THEOREM 1.3. *Let $\pi : A \to B \subseteq B^{**}$ be a *-homomorphism that admits an approximately multiplicative σ-strong*-decomposition (see Definition 2.2). Then the inclusion $\pi(A) \subseteq B$ is nuclear.*

We also reach a full characterisation in the exact case, which stems from a more general sufficient condition given in Proposition 2.9.

THEOREM 1.4. *Let A be an exact C*-algebra and $\pi : A \to B$ be a *-homomorphism to another C*-algebra B. Then π admits an approximately multiplicative norm-decomposition if and only if it is nuclear and quasidiagonal and $\tau \circ \pi$ is a quasidiagonal trace on A for every trace τ on $\pi(A)$.*

We shall use the fact that the σ-strong* topology on a von Neumann algebra N agrees on bounded subsets with the topology generated by seminorms

$$\|x\|_\rho := \sqrt{\rho\left(\frac{x^*x + xx^*}{2}\right)}$$

for a separating family of normal states ρ on N (see, for example, [1, Proposition III.2.2.19]).

For simplicity, we shall assume that our C*-algebras are separable and unital.

2. Results

Our results are only possible through use of the following as yet unpublished result, which is something of a folklore theorem (see [7]). We thank the authors of [5] for allowing us to reproduce the proof here.

THEOREM 2.1 [5]. *Let $\theta, \pi : A \to N$ be weakly nuclear *-homomorphisms from a C*-algebra A to a finite von Neumann algebra N that agree on traces (that is,*

$\tau \circ \theta = \tau \circ \pi$ *for every trace τ on N). Then θ and π are strong* approximately unitarily equivalent.*

PROOF. Let τ_0 be a normal trace on N. Replacing N with $\pi_{\tau_0}(N)$ (where π_τ is the Gelfand–Naimark–Segal representation corresponding to τ_0) if necessary, we may assume that τ_0 is faithful. We need to show that θ and π are unitarily equivalent as maps into the tracial ultrapower N_τ^ω of N with respect to τ_0.

Define $\mu : A \to \mathbb{M}_2(N_\tau^\omega)$ by

$$\mu(a) = \begin{bmatrix} \theta(a) & 0 \\ 0 & \pi(a) \end{bmatrix}.$$

Then μ is a weakly nuclear *-homomorphism. We claim that $M := \pi(A)''$ is hyperfinite.

Indeed, let $\psi_n : A \to F_n$ and $\varphi_n : F_n \to M$ be contractive completely positive maps for finite-dimensional C*-algebras F_n such that $(\varphi_n \circ \psi_n)$ converges in the point-weak* topology to μ. Fix a unital normal representation $M \subseteq \mathbb{B}(\mathcal{H})$ and define maps $\eta, \eta_n : A \otimes_{\mathrm{alg}} M' \to \mathbb{B}(\mathcal{H})$ by $\eta(a \otimes b) = \mu(a)b$ and $\eta_n(a \otimes b) = \varphi_n \circ \psi_n(a)b$, so that (η_n) converges to η in the point-weak* topology. The η_n are continuous with respect to the minimal tensor product since they factor through $F_k \otimes M'$ and hence so is η. A conditional expectation of $\mathbb{B}(\mathcal{H})$ onto M' is then provided by [4, Proposition 3.6.5], confirming that M is injective and hence hyperfinite by Connes' theorem [9, Theorem 6].

Define projections

$$p_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in $\mathbb{M}_2(N_\tau^\omega) \cap \pi(A)'$. Then $\tau(p_1 x) = \tau(p_2 x)$ for every normal trace τ and every $x \in \pi(A)$ and hence also for every $x \in \pi(A)''$. By [8, Lemma 4.5], $\tau(p_1) = \tau(p_2)$ for every trace τ on $\mathbb{M}_2(N_\tau^\omega) \cap \pi(A)'$. Thus, there is a unitary $u = [u_{i,j}] \in \mathbb{M}_2(N_\tau^\omega) \cap \pi(A)'$ such that $u^* p_1 u = p_2$ and hence $u_{1,1} = 0$ and $u_{1,2}^* u_{1,2} = 1_{N_\tau^\omega}$. Therefore, $u_{1,2}$ is unitary since N_τ^ω is finite, and $\theta(a)u_{1,2} = u_{1,2}\pi(a)$ since $u \in \pi(A)'$.   □

Most of the proof of the following lemma is borrowed from [3]; improvements are due to mixing in material from [6] and Theorem 2.1. To properly state our results, we need (the W* half of) the paper's central definition.

DEFINITION 2.2. Let $\pi : A \to N$ be a *-homomorphism from a C*-algebra A to a von Neumann algebra N. An *approximately multiplicative σ-strong*-decomposition* of π is a net of unitary completely positive maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} N$ for finite-dimensional C*-algebras F_n such that:

(i) $\varphi_n \circ \psi_n(x) \to \pi(x)$ in the σ-strong* topology for all $x \in A$;
(ii) $\|\psi_n(x)\psi_n(y) - \psi_n(xy)\| \to 0$ for all $x, y \in A$;
(iii)   every φ_n is a *-homomorphism.

LEMMA 2.3. *Let A be a quasidiagonal C*-algebra and $\pi : A \to N$ be a weakly nuclear *-homomorphism such that, for every trace τ on $\pi(A)$, the trace $\tau \circ \pi$ is quasidiagonal. Then π admits an approximately multiplicative σ-strong*-decomposition.*

REMARK 2.4. Note that the 'quasidiagonal $\tau \circ \pi$' condition is satisfied if either every trace on A or every trace on $\pi(A)$ is quasidiagonal. As foreshadowed in Lemma 1.2, this property is essential.

PROOF OF LEMMA 2.3. Let $\mathcal{F} \subset A$ be a finite set of contractions, $\mathcal{S}$ a finite set of normal states on N and $\epsilon > 0$. Define a normal state $\rho = \sum_{\rho' \in \mathcal{S}} \rho' / |\mathcal{S}|$, so that $\|b\|_{\rho'}^2 \leq |\mathcal{S}| \|b\|_\rho^2$ for every $\rho' \in \mathcal{S}$ and $b \in N$. Thus, we need only make reference to ρ in the proof, rather than the entire set $\mathcal{S}$.

Note that N may be decomposed into $N_1 \oplus N_\infty$ for von Neumann algebras N_1, N_∞ that are respectively finite and properly infinite. Similarly, $\pi = \pi_1 \oplus \pi_\infty$ for weakly nuclear *-homomorphisms $\pi_1 : A \to N_1$ and $\pi_\infty : A \to N_\infty$. We shall deal with each summand separately by assuming that it composes all of N.

Case 1. N is properly infinite. We begin by following the proof of [6, Proposition 3]. By the assumption of weak nuclearity, there are $k \in \mathbb{Z}^+$ and unitary completely positive maps $A \xrightarrow{\psi'} \mathbb{M}_k \xrightarrow{\dot{\varphi}'} N$ such that $\|\dot{\varphi}' \circ \psi'(x) - \pi(x)\|_\rho \leq \epsilon$ for every $x \in \mathcal{F}$. Since A is separable, we may fix a faithful unital representation $A \subseteq \mathbb{B}(\mathcal{H})$ such that $\mathcal{H}$ is separable and A contains no nonzero compact operators. Voiculescu's theorem [11, Theorem II.5.3] provides an isometry $v : \mathbb{C}^k \to \mathcal{H}$ such that $\|v^* x v - \psi'(x)\| \leq \epsilon$ for every $x \in \mathcal{F}$. Likewise, quasidiagonality provides a finite-rank projection $p \in \mathbb{B}(\mathcal{H})$ such that $\|pv - p\| \leq \epsilon$ and $\|px - xp\| \leq \epsilon$ for every $x \in \mathcal{F}$. Define unitary completely positive maps $A \xrightarrow{\psi} \mathbb{B}(p\mathcal{H}) \xrightarrow{\dot{\varphi}} N$ by $\psi(a) = pap$ and $\dot{\varphi}(T) = \dot{\varphi}'(v^* T v)$. Thus, for all $x, y \in \mathcal{F}$,

$$\|\psi(x)\psi(y) - \psi(xy)\| \leq \|p\| \|xp - px\| \|yp\| \leq \epsilon,$$
$$\|\dot{\varphi} \circ \psi(x) - \pi(x)\|_\rho \leq \|\dot{\varphi}'\| \|v^* \psi(x) v - \psi'(x)\| + \epsilon$$
$$\leq \|v^* p - v^*\| \|xpv\| + \|v^* x\| \|pv - v\| + 2\epsilon \leq 4\epsilon.$$

We now follow the proof of [3, Lemma 2.4], beginning on page 50 with the explanation that the reference itself is following the proof of [12, Proposition 2.2]. Our properly infinite assumption finally kicks in, allowing us to find a unital embedding $\iota : \mathbb{B}(p\mathcal{H}) \to N$. By [12, Proposition 2.1], there is an isometry $w \in N$ such that $\dot{\varphi}(T) = w^* \iota(T) w$ for all $T \in \mathbb{B}(p\mathcal{H})$. Then [12, page 167] shows how w may be approximated by a unitary u so that

$$\|\mathrm{Ad}(u^*) \circ \iota(T) - \dot{\varphi}(T)\|_\rho \leq \|T\|\epsilon \leq \epsilon$$

for every $T \in \psi(\mathcal{F}) \subset \mathbb{B}(p\mathcal{H})$. Therefore, this case is proven by defining the *-homomorphism $\varphi = \mathrm{Ad}(u^*) \circ \iota$.

Case 2. N is finite. This proof follows exactly in the footsteps of that of [3, Lemma 2.5], until its last two paragraphs. At that point we note that θ is nuclear by definition:

we have finite-dimensional unitary completely positive maps $\widetilde{\psi}_n := \bigoplus_{k=1}^n \psi_k$ mapping $A \to \bigoplus_{k=1}^n F_k$ and *-homomorphisms $\widetilde{\phi}_n : \bigoplus_{k=1}^n F_k \to N^\omega$ given by

$$\widetilde{\phi}_n((T_1, \ldots, T_n)) = q((\phi_1(T_1), \ldots, \phi_{n-1}(T_{n-1}), \phi_n(T_n), \phi_n(T_n), \ldots)_n).$$

Thus, $\theta = \lim_\omega \phi_n \circ \psi_n = \lim_n \widetilde{\phi}_n \circ \widetilde{\psi}_n$. This allows us to use Theorem 2.1 to deduce that θ is strong* approximately unitarily equivalent to π^ω, and the proof is finished using Kirchberg's ϵ-test as in the original.   □

Just as in [6] and the other preceding papers, our interest in this form of W*-decomposition is due solely to its relationship with the following C*-counterpart.

DEFINITION 2.5. Let $\pi : A \to B$ be a *-homomorphism between C*-algebras A, B. An *approximately multiplicative norm-decomposition* of π is a sequence of unitary completely positive maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B$ for finite-dimensional C*-algebras F_n such that:

(i) $\|\varphi_n \circ \psi_n(x) - \pi(x)\| \to 0$ for all $x \in A$;
(ii) $\|\psi_n(x)\psi_n(y) - \psi_n(xy)\| \to 0$ for all $x, y \in A$;
(iii) every φ_n is a convex combination of unitary completely positive order zero maps.

PROOF OF LEMMA 1.2. The composition of a trace with an order-zero map is again (rescalable into) a trace [15, Corollary 3.4], so each $\tau \circ \varphi_n$ is a trace on the finite-dimensional C*-algebra F_n. By assumption, the ψ_n are approximately multiplicative and $\tau \circ \varphi_n \circ \psi_n \to \tau \circ \pi$ in the weak* topology. Therefore, $\tau \circ \pi$ is quasidiagonal.   □

PROOF OF THEOREM 1.3. Let $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B^{**}$ be an approximately multiplicative σ-strong*-decomposition. By Arveson's extension theorem, we have conditional expectations $E_n : B^{**} \to \varphi_n(F_n)$. Moreover, by Alaoglu's theorem, we may pass (E_n) to a subsequence that converges to a linear map $E : B^{**} \to B^{**}$ in the point-σ-weak topology (see [4, Theorem 1.3.7]).

Fix $\epsilon > 0$, $a \in A$ and a normal functional $\eta \in B^*$. Then $\eta \circ E$ is also a normal functional, so by the previous paragraph there exists $m \in \mathbb{N}$ such that $n > m$ implies that each of the following hold:

$$|\eta(E_n(\pi(a)) - E(\pi(a)))| \le \epsilon/4,$$
$$|\eta \circ E(\pi(a) - \varphi_n \circ \psi_n(a))| \le \epsilon/4,$$
$$|\eta(E(\varphi_n \circ \psi_n(a)) - E_n(\varphi_n \circ \psi_n(a)))| \le \epsilon/4,$$
$$|\eta(\varphi_n \circ \psi_n(a) - \pi(a))| \le \epsilon/4.$$

Combined,

$$|\eta(E_n \circ \pi(a) - \pi(a))| \le \epsilon$$

and therefore the inclusion $\pi(A) \subseteq B^{**}$ is weakly nuclear.

We now use a Hahn–Banach argument akin to the proof (as in [4, Proposition 2.3.6]) that if A^{**} is semidiscrete, then A is nuclear. Let X be the set of all maps from $\pi(A)$ to B^{**} of the form $\varphi \circ \psi$ for two unitary completely positive maps $\psi : \pi(A) \to F$ and $\varphi : F \to B^{**}$ and finite-dimensional F. Then X is convex.

Indeed, let $w \in (0, 1)$ and $\varphi_0 \circ \psi_0, \varphi_1 \circ \psi_1 \in X$. Then it follows that $\psi := \psi_0 \oplus \psi_1 : \pi(A) \to F_0 \oplus F_1$ is unitary completely positive, as is the map $\varphi : F_0 \oplus F_1 \to B^{**}$ given by $\varphi((M_0, M_1)) = w\varphi_0(M_0) + (1 - w)\varphi_1(M_1)$. Thus, $w\varphi_0 \circ \psi_0 + (1 - w)\varphi_1 \circ \psi_1 = \varphi \circ \psi \in X$.

Fix $\mathcal{F} = \{b_i \mid i \in [1, k]\} \subset \pi(A)$. By weak nuclearity, the k-tuple $(b_i)_{i=1}^k \in \bigoplus_{i=1}^k B^{**}$ is in the weak-closure of the set $\{(\varphi \circ \psi(b_i))_{i=1}^k \mid \varphi \circ \psi \in X\}$ and hence also in its norm-closure by the Hahn–Banach theorem. Therefore, there is a map $\varphi \circ \psi \in X$ such that

$$\max_i \|b_i - \varphi \circ \psi(b_i)\| = \|(b_i)_{i=1}^k - (\varphi \circ \psi(b_i))_{i=1}^k\| < \epsilon. \qquad \square$$

PROPOSITION 2.6. *Let $\pi : A \to B$ be a *-homomorphism between C*-algebras A, B. Then π admits an approximately multiplicative norm-decomposition if and only if $\pi : A \to B^{**}$ admits an approximately multiplicative σ-strong*-decomposition.*

PROOF. $\Rightarrow$: We retread the proof of Lemma 2.3, beginning by fixing normal state ρ, finite $\mathcal{F} \subset A$ and $\epsilon > 0$.

We may skip the first paragraph of the properly infinite case, instead using the assumed approximately multiplicative norm-decomposition to provide $A \xrightarrow{\psi} F \xrightarrow{\dot{\varphi}} B$ that satisfy the necessary inequalities. We now need only use [12] to find a unitary $u \in N$ so that $\varphi = \mathrm{Ad}(u^*) \circ \iota$ approximates $\dot{\varphi}$ for a unital embedding $\iota : F \to B^{**}$.

The finite case does not require that A be quasidiagonal. Of course, property (i) of our norm-decomposition witnesses the nuclearity of π and hence $\pi : A \to B^{**}$ is weakly nuclear. Therefore, Lemma 1.2 allows us finish this case, and this direction.

$\Leftarrow$: This is a perturbation of [13, Theorem 1.4]. Let $A \xrightarrow{\psi_n} F_n \xrightarrow{\dot{\varphi}_n} B^{**}$ be an approximately multiplicative σ-strong*-decomposition. Using [13, Lemma 1.1], we can find order-zero unitary completely positive maps $\varphi_n : F_n \to B$ (note the range) such that $\varphi_n \circ \psi_n(x) \to \pi(x)$ in the σ-weak topology for every $x \in A$. We once again conclude with a Hahn–Banach argument. $\qquad \square$

With the equivalence of these decompositions, we hope to find a way to characterise their existence. An important component seems to be that π is a quasidiagonal *-homomorphism.

DEFINITION 2.7. A *-homomorphism $\pi : A \to B$ is *quasidiagonal* if it factors through a quasidiagonal C*-algebra D. That is, there exist *-homomorphisms $\pi_1 : A \to D$ and $\pi_2 : D \to B$ such that π_1 is surjective and $\pi = \pi_2 \circ \pi_1$.

DEFINITION 2.8. An extension

$$0 \longrightarrow \ker \pi \longrightarrow A \longrightarrow B \longrightarrow 0$$

is called *locally split* if every finite subset $\mathcal{G} \subset B$ of contractions admits a unitary completely positive *local lifting* $\lambda : \mathrm{span}(\mathcal{G}) \to A$ such that $\pi \circ \lambda(b) = b$ for every $b \in \mathrm{span}\,\mathcal{G}$.

For exact A, every extension $0 \to \ker \pi \to A \to B \to 0$ is locally split (see, for example, [4, Proposition 9.1.4]). However, it is of note that the full power of exactness is not required for the following result.

PROPOSITION 2.9. *Let A be a C*-algebra, $\pi : A \to B$ be a nuclear, quasidiagonal *-homomorphism to another C*-algebra B and D, π_1, π_2 be as in Definition 2.8. Further, suppose that the trace $\tau \circ \pi$ is quasidiagonal for every trace τ on $\pi(A)$ and that the extension*

$$0 \longrightarrow \ker \pi_1 \longrightarrow A \longrightarrow D \longrightarrow 0$$

is locally split. Then π admits an approximately multiplicative decomposition.

PROOF. Fix $\epsilon > 0$ and a finite subset of contractions $\mathcal{G} \subset D$. Let $\lambda : \mathrm{span}(\mathcal{G}) \to A$ be a local lifting. By nuclearity, there are $i \in \mathbb{N}$ and unitary completely positive maps $A \xrightarrow{\psi} \mathbb{M}_i \xrightarrow{\dot{\varphi}} B$ such that $\|\dot{\varphi} \circ \psi(a) - \pi(a)\| < \epsilon$ for every $a \in \lambda(\mathcal{G})$. Arveson's extension theorem then provides a unitary completely positive $\psi' : D \to \mathbb{M}_i$ that agrees with $\psi \circ \lambda$ on $\mathrm{span}(\mathcal{G})$. Thus, for any $d \in \mathcal{G}$,

$$\|\dot{\varphi} \circ \psi'(d) - \pi_2(d)\| = \|\dot{\varphi} \circ \psi(\lambda(d)) - \pi(\lambda(d))\| < \epsilon.$$

From this, we conclude that π_2 is nuclear. This allows us to apply the properly infinite case of Lemma 2.3 to get approximately multiplicative decompositions of $\pi_2 : D \to B^{**}$. This case may then be extended to work for A itself by replacing the resultant ψ_n with $\psi_n \circ \pi_1$. The finite case goes through for A unmodified, resulting in approximately multiplicative decompositions of $\pi : A \to B^{**}$. □

However, exactness does provide us with the converse as stated in Theorem 1.4.

PROOF OF THEOREM 1.4. As mentioned, Proposition 2.9 already provides the backward direction, so we need now only address the forward one. Let $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B$ be the approximately multiplicative norm-decomposition. The decomposition itself witnesses the nuclearity of π. The quasidiagonality of every $\tau \circ \pi$ was shown in Lemma 1.2. The quasidiagonality of π itself is a consequence of [10, Theorem 4.8]; for the convenience of the reader, we present a distillation of the proof.

Since A is exact, we may treat it as a C*-subalgebra of some C*-algebra C such that the inclusion is nuclear. Using Arveson's extension theorem, we may treat the domains of the ψ_n as C. They induce a unitary completely positive map Ψ from C to an ultraproduct $\prod_\omega F_n$, where $(\psi_n(a))_n$ is a representative sequence of $\Psi(a)$. Note that the approximate multiplicativity of (ψ_n) makes $\Psi|_A$ a *-homomorphism. Likewise, we have a contractive completely positive map Φ from $\prod_\omega F_n$ to the ultrapower B^ω given by $\Phi((T_n)_n) = (\varphi_n(T_n))_n$. Thus, $\Phi \circ \Psi(a) = (\varphi_n \circ \psi_n(a))_n = (\pi(a))_n$, which we

may identify with $\pi(a)$ itself by treating B as a C*-subalgebra of B^ω through constant sequences. Thus, $\Phi|_{\Psi(A)}$ must also be a *-homomorphism.

Fix $\epsilon > 0$ and a finite subset of contractions $\mathcal{G} \subset \Psi(A)$. Also, let $\lambda : \mathrm{span}(\mathcal{G}) \to A$ be a local lifting of Ψ. By nuclearity of $A \subseteq C$, there exist a finite-dimensional C*-algebra G and contractive completely positive maps $A \xrightarrow{\theta} G \xrightarrow{\xi} C$ such that, for every $d \in \mathcal{G}$,

$$\|(\Psi \circ \xi) \circ (\theta \circ \lambda)(d) - d\| = \|\Psi \circ \xi \circ \theta \circ \lambda(d) - \Psi \circ \lambda(d)\| \leq \|\xi \circ \theta \circ \lambda(d) - \lambda(d)\| < \epsilon.$$

Another use of Arveson's extension theorem yields $\theta' : \Psi(A) \to G$ with restriction $\theta'|_{\mathcal{G}} = \theta \circ \lambda$.

Thus, the inclusion $\Psi(A) \subseteq \prod_\omega F_n$ is nuclear. By the Choi–Effros lifting theorem [4, Theorem C.3], this inclusion lifts to a contractive completely positive map to $\prod_n F_n$. Therefore, $\Psi(A)$ is quasidiagonal (see, for example, [4, Exercise 7.1.3]). □

REMARK 2.10. It seems probable to the author that this theorem may be strengthened to show that π admits an approximately multiplicative decomposition if and only if it factors through a quasidiagonal C*-algebra D via $A \xrightarrow{\pi_1} D \xrightarrow{\pi_2} B$ such that π_2 admits an approximately multiplicative decomposition. All that is needed for this is to show that the trace $\tau \circ \pi_2$ is quasidiagonal for every trace τ on $\pi_2(D) = \pi(A)$. This is of course satisfied if every trace on B is quasidiagonal, but the common theme of these results has been moving requirements away from the C*-algebras and onto the *-homomorphism itself.

Acknowledgments

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