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RESEARCH ARTICLE

On inactivity times of failed components of coherent system under double monitoring

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Abstract
This article discusses the stochastic behavior and reliability properties for the inactivity times of failed components in coherent systems under double monitoring. A mixture representation of reliability function is obtained for the inactivity times of failed components, and some stochastic comparison results are also established. Furthermore, some sufficient conditions are developed in terms of the aging properties of the inactivity times of failed components. Finally, some numerical examples are presented to illustrate the theoretical results.

1. Introduction

Coherent system is significant in reliability theory and survival analysis and was introduced in the classical monograph by Barlow and Proschan [5]. A system consisting of $n(n \geq 1)$ components is called a coherent system if there are no irrelevant components in the system (i.e., irrelevant components mean that their performance will not affect the performance of the system) and if the structure-function of the system is monotone in every component.

In order to investigate the reliability and structure of coherent systems, Samaniego [38] introduced the concept of system signature, which is a nice and crucial tool for engineering designers. Consider a coherent system consisting of n components with independent and identically distributed (i.i.d.) lifetimes $X_1, X_2, \ldots, X_n$ with absolutely continuous distribution function F. Denote $X_{i:n}$ the order statistic of random variables $X_1, X_2, \ldots, X_n$, and $\bar{F}_{i:n}(t)$ the survival function of $X_{i:n}$ $(i = 1, 2, \ldots, n)$. Let the lifetime of the coherent systems be T, and denote the reliability function of T by $\bar{F}_T$, then $\bar{F}_T$ can be written as

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \tag{1}$$

where $s_i = \mathbb{P}(T = X_{i:n})$, $i = 1, 2, \ldots, n$. The probability vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ is said to be the signature vector of the coherent system, which is not depend on the F and only related to the structure of the coherent system. Kochar *et al.* [22] defined the signature $\mathbf{s}$ as the probability vector with elements

$$s_i = \frac{\text{Numbers of orderings for which the } i\text{th faliure causes system failure}}{n!}, \quad i = 1, 2, \ldots, n.$$

Navarro and Rychlik [33] proved that Eq. (1) is still valid when the lifetimes of components in a coherent system has absolutely continuous exchangeable distribution. Marichal and Mathonet [26]

Table 1. *Coherent systems with four components and signatures of the form (2).*

$T = \tau(X_1, X_2, X_3, X_4)$	Signature
$X_{2:2} = \max(X_1, X_2)$ (2-parallel)	$(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$
$\max(X_2, \min(X_1, X_3))$ (consecutive 2-out-of-3: F)	$(0, \frac{1}{3}, \frac{5}{12}, \frac{1}{4})$
$\max(X_1, \min(X_2, X_3, X_4))$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$
$\max(X_1, \min(X_2, X_3), \min(X_3, X_4))$	$(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$
$X_{3:3} = \max(X_1, X_2, X_3)$ (3-parallel)	$(0, 0, \frac{1}{4}, \frac{3}{4})$
$\max(X_{2:3}, X_4)$	$(0, 0, \frac{3}{4}, \frac{1}{4})$
$\min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4))$ (consecutive 3-out-of-4: F)	$(0, 0, \frac{1}{2}, \frac{1}{2})$
$X_{4:4} = \max(X_1, X_2, X_3, X_4)$ (4-parallel)	$(0, 0, 0, 1)$

extended the concept of signatures of coherent systems to the case when the systems have dependent components. Coolen and Coolen-Maturi [10] presented the concept of the survival signature that extends the classical concept of system signatures to the case of systems with multiple types of components. It is well known that the computation of signature is a hard task. Recently, the calculation of signatures has attracted widespread attention among scholars. Navarro and Rubio [32] obtained an algorithm to compute the signature of coherent systems with n components. Da *et al.* [11] derived two basic formulas for calculating the signature of the coherent system which can be decomposed into two subsystems. Da *et al.* [12] proposed a novel algorithm to compute the signature of a coherent system with exchangeable components. Ding *et al.* [14] proposed a new method for comparing two coherent systems with heterogeneous components using survival signature. For more on the discussion and application of the signature, one can refer to Boland [8], Boland and Samaniego [9], Samaniego [39], Block *et al.* [7], Navarro *et al.* [35], and Guo *et al.* [18], and references therein.

The inactivity time of the coherent system is a significant topic in reliability theory and survival analysis. This issue has been widely studied by a lot of scholars under the different scenarios in the past decades. For instance, Zhao *et al.* [43] investigated stochastic monotone properties of the inactivity time on k-out-of-n systems with the independent and nonidentically distribution. Interested readers may refer to Navarro *et al.* [34], Li and Lu [23], Kayid and Ahmad [20], Asadi [3], Khaledi and Shaked [21], Li and Zhao [25], Li and Zhang [24], Goliforushani and Asadi [16], Ding *et al.* [13], Balakrishnan and Zhao [4], Gupta *et al.* [19], Navarro [31], Salehi and Tavangar [37], and Amini-Seresht *et al.* [1]. Besides, the inactivity times of components in the coherent system are also an important topic in many practical scenarios. To the best of our knowledge, only a few results in the literature consider the inactivity time of components in a coherent system. Interested readers may refer to Tavangar and Asadi [42], Goliforushani *et al.* [17], Nama and Asadi [28], and Tavangar [41].

For a coherent system consists of n i.i.d. components, assume that the system has the following signature vector

$$\mathbf{s} = (0, 0, \ldots, 0, s_i, \ldots, s_n), \quad i = 2, \ldots, n. \tag{2}$$

Obviously, the lifetime of the coherent system with signature vector of (2) only depend on $X_{k:n}(k = i, i+1, \ldots, n)$, that is, the kth ($k = 1, 2, \ldots, i-1$) failure of components would never cause the failure of system. In fact, for investigations of the signature vector (2) (refer to Table 1) can be referred to Navarro *et al.* [35], Navarro [29,30] and Goli [15], and references therein. As pointed in Goliforushani *et al.* [17] and Nama and Asadi [28], owing to the system is not usually checked continuously, after the kth ($k = 1, 2, \ldots, i-1$) failure of components, the system engineering designers might want to obtain some useful information about the history of the system, for instance, the average time of the inactivity times of these components. Therefore, the discussions of stochastic properties and aging properties of inactivity times about failed components might be significant and meaningful for system engineers and designers.

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Nama and Asadi [28] introduced the following conditional random variable

$$(t - X_{k:n} | T = t), \quad k = 1, \ldots, i-1.$$

and investigated the stochastic properties and aging properties of the inactivity times of failed components in coherent systems with signature vector (2).

As pointed in Goli [15], the functioning system is usually not checking continuously in a real environment, due to continuous inspection is impossible or expensive. Hence, a system is planned to be monitored twice at different times in a practical case. This article is mainly motivated by the results of Goliforushani *et al.* [17] and Nama and Asadi [28]. Considered that the system is twice monitored, exactly r components $(0 \leq r < i)$ have failed at time t_1, but the system is still operating, subsequently, the system has failed at time $t_2(> t_1)$, that is, we are interested in the following conditional inactivity times of failed components in the coherent system with the signature vector of form (2)

$$IX_{k:n}(t_1, t_2) = (t_2 - X_{k:n} | X_{r:n} = t_1, T = t_2), \quad \text{for } 0 \leq r < k < i.$$

In fact, $IX_{k:n}(t_1, t_2)$ is the inactivity time of failed component $X_{k:n}(k = r+1, \ldots, i-1)$ in the system at time $t_2(> t_1)$.

The remainder of this article rolled out as follows. In Section 2, we provide some definitions of stochastic orders and useful lemmas. In Section 3, we will give a mixture representation for the reliability function of the conditional inactivity time of the failed components, obtain several stochastic comparisons between two coherent systems, and some aging properties based on the proposed conditional random variable. Finally, some conclusions and remarks are made in Section 4.

2. Preliminaries

The term increasing and decreasing are used instead of monotone nondecreasing and monotone nonincreasing, respectively. For simplicity of the discussion, we denote $\mathbb{R}^+ = (0, +\infty)$.

Before introducing the results in this article, let us first recall some definitions and concepts of stochastic orders, which will be used in the sequel. Stochastic orders are a useful tool to provide stochastic comparisons of the random variables in reliability theory, risk theory, and economy finance. Throughout this article, all random variables are assumed to be positive and absolutely continuous.

Let X and Y be two random variables with distribution functions $F(x)$ and $G(x)$, survival functions $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, density functions $f(x)$ and $g(x)$, and hazard rate functions $h_X(x)$ and $h_Y(x)$, respectively.

Definition 1. *X is said to be smaller than Y in*

(i) the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}^+$;
(ii) the hazard rate order (denoted by $X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all $x \in \mathbb{R}^+$, or equivalently, if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}^+$;
(iii) the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}^+$; and
(iv) the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in \mathbb{R}^+$.

For two discrete probability distributions $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, $\boldsymbol{p}$ is said to be smaller than $\boldsymbol{q}$ in

(i) $\boldsymbol{p} \leq_{st} \boldsymbol{q}$, if $\sum_{j=i}^{n} q_j \geq \sum_{j=i}^{n} p_j$, $i = 1, 2, \ldots, n$;
(ii) $\boldsymbol{p} \leq_{hr} \boldsymbol{q}$, if $\sum_{j=i}^{n} q_j / \sum_{j=i}^{n} p_j$ is increasing in i;
(iii) $\boldsymbol{p} \leq_{rh} \boldsymbol{q}$, if $\sum_{j=1}^{i} q_j / \sum_{j=1}^{i} p_j$ is increasing in i;
(iv) $\boldsymbol{p} \leq_{lr} \boldsymbol{q}$, if q_i/p_i is increasing in i.

It is well known that the likelihood ratio order implies the (reversed) hazard rate order and the usual stochastic order, but the reversed statement is not true in general. For more comprehensive discussions

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on various stochastic orders and their applications, one may refer to the monographs by Shaked and Shanthikumar [40] and Belzunce *et al.* [6].

The following lemmas play a vital role in establishing the main results.

Lemma 1 (Arnold *et al.* [2]). *Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with distribution function $F(x_i)$ and density function $f(x_i)$, and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the order statistics obtained from this sample. Then, the conditional distribution of $X_{j:n}$, given that $X_{i:n} = x_i$ for $i < j$, is the same as the distribution of the $(j-i)$th order statistic obtained from a sample of size $n-i$ from a population whose distribution is simply $F(x)$ truncated on the left at x_i.*

Lemma 2 (Shaked and Shanthikumar [40]).

(i) Let $X_1, X_2, \ldots, X_n$ be n independent random variables, all with absolutely continuous distribution functions, all having the same support which is an interval of the real line, and all having differentiable densities. If $X_1 \le_{lr} X_2 \le_{lr} \cdots \le_{lr} X_n$, then $X_{k-1:n} \le_{lr} X_{k:n}, 2 \le k \le n$, and $X_{k-1:n-1} \le_{lr} X_{k:n}, 2 \le k \le n$.

(ii) If $X \le_{st} Y$ and g is any increasing [decreasing] function, then $g(X) \le_{st} [\ge_{st}] g(Y)$.

(iii) Let α and β be the real-valued function such that β is nonnegative, and α/β and β are increasing. If $X_1 \le_{hr} X_2$, then

$$E[\alpha(X_1)]E[\beta(X_2)] \le E[\alpha(X_2)]E[\beta(X_1)].$$

(iv) Let α and β be the real-valued function such that β is nonnegative, and α/β and β are decreasing. If $X_1 \le_{rh} X_2$, then

$$E[\alpha(X_1)]E[\beta(X_2)] \ge E[\alpha(X_2)]E[\beta(X_1)].$$

3. Main results

Let T be the lifetime of a coherent system with n i.i.d. components having lifetimes $X_1, X_2, \ldots, X_n$, which have the common distribution function F. Suppose that a coherent system has the signature vector of (2), we are interested in the scenario of the exactly $r (0 \le r < i)$ failed components of a coherent system at the observed time t_1, but the system is still operating at time t_1, and system failed at time $t_2 (> t_1)$. Then, the reliability function of $IX_{k:n}(t_1, t_2)$ can be represented by

$$\begin{aligned}
&\mathbb{P}(IX_{k:n}(t_1,t_2) > x)\\
&\quad = \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, T = t_2)\\
&\quad = \sum_{m=i}^{n} \mathbb{P}(t_2 - X_{k:n} > x, T = X_{m:n} \mid X_{r:n} = t_1, T = t_2)\\
&\quad = \sum_{m=i}^{n} \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, T = t_2, T = X_{m:n})\mathbb{P}(T = X_{m:n} \mid X_{r:n} = t_1, T = t_2)\\
&\quad = \sum_{m=i}^{n} \mathbb{P}(T = X_{m:n} \mid X_{r:n} = t_1, T = t_2)\mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2)\\
&\quad = \sum_{m=i}^{n} p_m(r, t_1, t_2) B_{k,r,m}(t_1, t_2), \qquad (3)
\end{aligned}$$

where

$$p_m(r, t_1, t_2) = \mathbb{P}(T = X_{m:n} \mid X_{r:n} = t_1, T = t_2),$$

and

$$B_{k,r,m}(t_1, t_2) = \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2).$$

Remark 1. *It should be noted that*

$$\begin{aligned} p_m(r,t_1,t_2) &= \mathbb{P}(T = X_{m:n} \mid X_{r:n} = t_1, T = t_2) \\ &= \frac{\mathbb{P}(T = X_{m:n}, X_{r:n} = t_1, T = t_2)}{\mathbb{P}(X_{r:n} = t_1, T = t_2)} \\ &= \frac{\mathbb{P}(T = X_{m:n})\mathbb{P}(X_{r:n} = t_1, X_{m:n} = t_2)}{\sum_{j=i}^{n} \mathbb{P}(T = X_{j:n})\mathbb{P}(X_{r:n} = t_1, X_{j:n} = t_2)} \\ &= \frac{s_m\mathbb{P}(X_{r:n} = t_1, X_{m:n} = t_2)}{\sum_{j=i}^{n} s_j\mathbb{P}(X_{r:n} = t_1, X_{j:n} = t_2)} \\ &= \frac{s_m\mathbb{P}(X_{m:n} = t_2 \mid X_{r:n} = t_1)\mathbb{P}(X_{r:n} = t_1)}{\sum_{j=i}^{n} s_j\mathbb{P}(X_{j:n} = t_2 \mid X_{r:n} = t_1)P(X_{r:n} = t_1)} \\ &= \frac{s_m f_{m|r}(t_2|t_1)}{\sum_{j=i}^{n} s_j f_{j|r}(t_2|t_1)}, \end{aligned}$$

where $f_{j|r}(t_2|t_1)$ denotes the density function of $X_{j:n}$ at t_2 given $X_{r:n} = t_1$. It follows from Lemma 1 that

$$f_{j|r}(t_2 \mid t_1) = f^{t_1}_{j-r:n-r}(x),$$

where $f^{t_1}_{j-r:n-r}(x)$ is the density function of the $(j-r)$th order statistics among $(n-r)$ i.i.d. random variables with distribution function $F_{t_1}(x) = (F(x) - F(t_1))/(1 - F(t_1))$ and density function $f_{t_1}(x) = f(x)/(1 - F(t_1))$, for any $x > t_1$. As a result, we have

$$\begin{aligned} p_m(r,t_1,t_2) &= \frac{s_m f^{t_1}_{m-r:n-r}(t_2)}{\sum_{j=i}^{n} s_j f^{t_1}_{j-r:n-r}(t_2)} \\ &= \frac{s_m \frac{(n-r)!}{(m-r-1)!(n-r-(m-r))!} f_{t_1}(t_2)(F_{t_1}(t_2))^{m-r-1}(1 - F_{t_1}(t_2))^{n-r-(m-r)}}{\sum_{j=i}^{n} s_j \frac{(n-r)!}{(j-r-1)!(n-r-(j-r))!} f_{t_1}(t_2)(F_{t_1}(t_2))^{j-r-1}(1 - F_{t_1}(t_2))^{n-r-(j-r)}} \\ &= \frac{s_m(m-r)\binom{n-r}{m-r}(F_{t_1}(t_2))^{m-r-1}(1 - F_{t_1}(t_2))^{n-r-(m-r)}}{\sum_{j=i}^{n} s_j(j-r)\binom{n-r}{j-r}(F_{t_1}(t_2))^{j-r-1}(1 - F_{t_1}(t_2))^{n-r-(j-r)}} \\ &= \frac{s_m C^n_{r,m}(\phi_{t_1}(t_2))^m}{\sum_{j=i}^{n} s_j C^n_{r,j}(\phi_{t_1}(t_2))^j}, \end{aligned}$$

where

$$\phi_{t_1}(t_2) = \frac{F_{t_1}(t_2)}{1 - F_{t_1}(t_2)} = \frac{F(t_2) - F(t_1)}{1 - F(t_2)}, \quad C^n_{r,m} = (m-r)\binom{n-r}{m-r}.$$

Remark 2. *Note that the vector of conditional signature*

$$\mathbf{p}(t_1,t_2) = (0,\ldots,0,p_i(r,t_1,t_2),\ldots,p_n(r,t_1,t_2))$$

depends on the underlying distribution function F. It follows from the representation (3) that the inactivity time $IX_{k:n}(t_1,t_2)$ of failed components under double monitoring can be represented as the mixture of the inactivity time $(t_2 - X_{k:n} \mid X_{r:n} = t_1, X_{m:n} = t_2)$ of the failed components with $X_{k:n}(k > r)$ in a $(n-m+1)$-out-of-n system, given that at time $t_1(t_1 \geq 0)$, exactly $r(r < i)$ components have failed, and at time t_2, the system have failed.

The following Lemma 3 gives the distribution function of conditional random variable $(t_2 - X_{k:n} \mid X_{r:n} = t_1, X_{m:n} = t_2)$.

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Lemma 3. *For $m = i, i+1, \ldots, n$ and $0 \le r < k < i$, we have*

$$(t_2 - X_{k:n} \mid X_{r:n} = t_1, X_{m:n} = t_2) \stackrel{d}{=} Z_{m-k:m-r-1}^{t_1,t_2},$$

where $\stackrel{d}{=}$ means equality in distribution, and $Z_{m-k:m-r-1}^{t_1,t_2}$ is the $(m-k)$th order statistics among $(m-r-1)$ i.i.d. random variables with the common distribution function $H_{t_1,t_2}(x) = (F(t_2-x) - F(t_1))/(F(t_2) - F(t_1))$.

Proof. The reliability function of $(t_2 - X_{k:n} \mid X_{r:n} = t_1, X_{m:n} = t_2)$ is given by

$$\begin{aligned} B_{k,r,m}(t_1,t_2) &= \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2) \\ &= \frac{\mathbb{P}(t_2 - X_{k:n} > x, X_{r:n} = t_1, X_{m:n} = t_2)}{\mathbb{P}(X_{r:n} = t_1, X_{m:n} = t_2)} \\ &= \sum_{j=k-r}^{m-r-1} \binom{m-r-1}{j} (H_{t_1,t_2}(x))^j (1 - H_{t_1,t_2}(x))^{m-r-1-j}, \end{aligned} \tag{4}$$

where $H_{t_1,t_2}(x) = (F(t_2-x) - F(t_1))/(F(t_2) - F(t_1))$. Thus, from (4), we obtain the equality in distribution

$$(t_2 - X_{k:n} \mid X_{r:n} = t_1, X_{m:n} = t_2) \stackrel{\mathrm{d}}{=} Z_{m-k:m-r-1}^{t_1,t_2},$$

which proves the desired result. □

The following Example 1 gives a method of how to use (3) and (4) to calculate the survival function of the inactivity time of failed components in a coherent system.

Example 1. *Suppose that X_1, X_2, X_3, and X_4 are i.i.d. random variables, consider a coherent system with lifetime $T = \max(X_{2:3}, X_4)$. Then, its signature vector is $\mathbf{s} = (0, 0, \frac{3}{4}, \frac{1}{4})$. Assume that exactly one failed component of the coherent system at time t_1, but the system failed at time $t_2 (> t_1)$. The inactivity time of failed component $X_{2:4}$ is given by*

$$IX_{2:4}(t_1,t_2) = (t_2 - X_{2:4} \mid X_{1:4} = t_1, T = t_2).$$

For $r = 1, k = 2$, it follows from (3) and Lemma 3 that the reliability function of $IX_{2:4}(t_1,t_2)$ can be written as

$$\mathbb{P}(t_2 - X_{2:4} > x | X_{1:4} = t_1, T = t_2) = \sum_{m=3}^{4} p_m(1,t_1,t_2) B_{2,1,m}(t_1,t_2),$$

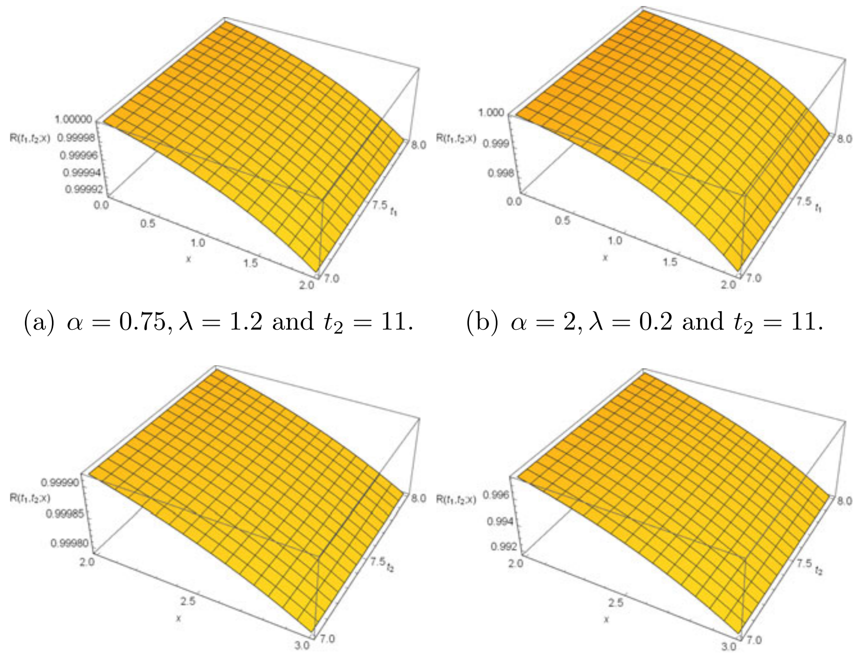
where

$$p_3(1,t_1,t_2) = \frac{6\bar{F}(t_2)}{5\bar{F}(t_2) + \bar{F}(t_1)}, \quad p_4(1,t_1,t_2) = \frac{\bar{F}(t_1) - \bar{F}(t_2)}{5\bar{F}(t_2) + \bar{F}(t_1)}, \tag{5}$$

$$B_{2,1,3}(t_1,t_2) = \mathbb{P}(t_2 - X_{2:4} > x \mid X_{1:4} = t_1, X_{3:4} = t_2) = 1 - \frac{\bar{F}(t_2 - x) - \bar{F}(t_2)}{\bar{F}(t_1) - \bar{F}(t_2)} \tag{6}$$

and

$$B_{2,1,4}(t_1,t_2) = \mathbb{P}(t_2 - X_{2:4} > x | X_{1:4} = t_1, X_{4:4} = t_2) = 1 - \left(\frac{\bar{F}(t_2 - x) - \bar{F}(t_2)}{\bar{F}(t_1) - \bar{F}(t_2)} \right)^2. \tag{7}$$



(a) $\alpha = 0.75, \lambda = 1.2$ and $t_2 = 11$. (b) $\alpha = 2, \lambda = 0.2$ and $t_2 = 11$.

(c) $\alpha = 0.75, \lambda = 1.2$ and $t_1 = 0.8$. (d) $\alpha = 2, \lambda = 0.2$ and $t_1 = 0.8$.

Figure 1. *The graphs of $R(t_1, t_2; x)$.*

Furthermore, suppose X_i follows the Weibull distribution (refer to Marshall and Olkin [27]), that is, $\bar{F}(x) = e^{-\lambda^\alpha x^\alpha}, x \in \mathbb{R}^+$, then according to (5), (6), and (7), we have

$$p_3(1, t_1, t_2) = \frac{6e^{-\lambda^\alpha t_2^\alpha}}{5e^{-\lambda^\alpha t_2^\alpha} + e^{-\lambda^\alpha t_1^\alpha}}, \quad p_4(1, t_1, t_2) = \frac{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}{5e^{-\lambda^\alpha t_2^\alpha} + e^{-\lambda^\alpha t_1^\alpha}},$$

$$B_{2,1,3}(t_1, t_2) = \mathbb{P}(t_2 - X_{2:4} > x \mid X_{1:4} = t_1, X_{3:4} = t_2) = \frac{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha (t_2 - x)^\alpha}}{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}},$$

and

$$B_{2,1,4}(t_1, t_2) = \mathbb{P}(t_2 - X_{2:4} > x \mid X_{1:4} = t_1, X_{4:4} = t_2) = 1 - \left(\frac{e^{-\lambda^\alpha (t_2 - x)^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}\right)^2.$$

Therefore, the reliability function of $IX_{2:4}(t_1, t_2)$ can be rewritten as

$$\begin{aligned} R(t_1, t_2; x) &= \mathbb{P}(t_2 - X_{2:4} > x \mid X_{1:4} = t_1, T = t_2) \\ &= \left(\frac{6e^{-\lambda^\alpha t_2^\alpha}}{5e^{-\lambda^\alpha t_2^\alpha} + e^{-\lambda^\alpha t_1^\alpha}}\right)\left(\frac{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha (t_2 - x)^\alpha}}{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}\right) \\ &\quad + \left(\frac{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}{5e^{-\lambda^\alpha t_2^\alpha} + e^{-\lambda^\alpha t_1^\alpha}}\right)\left(1 - \left(\frac{e^{-\lambda^\alpha (t_2 - x)^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}\right)^2\right) \\ &= \frac{4e^{-\lambda^\alpha (t_1^\alpha + t_2{}^\alpha)} - 4e^{-\lambda^\alpha (t_2^\alpha + (t_2 - x)^\alpha)} + e^{-2\lambda^\alpha t_1{}^\alpha} - e^{-2\lambda^\alpha (t_2 - x)^\alpha}}{4e^{-\lambda^\alpha (t_1^\alpha + t_2^\alpha)} - 5e^{-2\lambda^\alpha t_2{}^\alpha} + e^{-2\lambda^\alpha t_1{}^\alpha}}. \end{aligned}$$

Figure 1 displays the graphs of $R(t_1, t_2; x)$.

The following Theorem 1 compares the conditional inactivity times $(t_2 - X_{k:n} \mid X_{r:n} = t_1, X_{m:n} = t_2)$ of the failed components in two $(n - m + 1)$-out-of-n systems in terms of the likelihood ratio order.

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Theorem 1. *For* $0 \le r < k < l < m$*, we have*

$$(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, X_{l:n} = t_2) \le_{lr} (t_2 - X_{k:n} \,|\, X_{r:n} = t_1, X_{m:n} = t_2).$$

Proof. Combining Lemma 2(i) with Lemma 3, it holds that

$$Z^{t_1,t_2}_{l-k:l-r-1} \le_{\mathrm{lr}} Z^{t_1,t_2}_{m-k:m-r-1}.$$

□

Remark 3. *Theorem 1 shows that* $(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, X_{l:n} = t_2)$ *is increasing in* l *in the sense of the likelihood ratio order.*

The following result gives some sufficient conditions for the stochastic orders of the inactivity time of failed components in two coherent systems with i.i.d. components but having different structures.

Theorem 2. *Let* $T_j = \tau_j(X_1, X_2, \ldots, X_n)$ *be the lifetime of two coherent systems, where* $X_1, X_2, \ldots, X_n$ *are i.i.d. random variables with common distribution function* F*. Suppose two system* j *has signature vector* $\mathbf{s}^{(j)} = (0, \ldots, 0, s_i^{(j)}, \ldots, s_n^{(j)})$ *and conditional signature vector* $\mathbf{p}^{(j)}(t_1, t_2) = (0, \ldots, 0, p_i^{(j)}(r, t_1, t_2), \ldots, p_n^{(j)}(r, t_1, t_2))$*,* $j = 1, 2$*. Then, for any* $0 \le t_1 < t_2$*, we have*

(i) If $\mathbf{p}^{(1)}(t_1, t_2) \le_{st} \mathbf{p}^{(2)}(t_1, t_2)$*, then*

$$(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_1 = t_2) \le_{st} (t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_2 = t_2);$$

(ii) If $\mathbf{p}^{(1)}(t_1, t_2) \le_{hr} \mathbf{p}^{(2)}(t_1, t_2)$*, then*

$$(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_1 = t_2) \le_{hr} (t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_2 = t_2);$$

(iii) If $\mathbf{p}^{(1)}(t_1, t_2) \le_{rh} \mathbf{p}^{(2)}(t_1, t_2)$*, then*

$$(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_1 = t_2) \le_{rh} (t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_2 = t_2);$$

(iv) If $\mathbf{p}^{(1)}(t_1, t_2) \le_{lr} \mathbf{p}^{(2)}(t_1, t_2)$*, then*

$$(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_1 = t_2) \le_{lr} (t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_2 = t_2).$$

Proof. The proof of the theorem follows from the representation (3), Theorem 1, and Lemma 2. □

Remark 4. *It is noted that Theorem 2 extends Thm.* 3.1 *of Nama and Asadi [28] to the stochastic orders of the inactivity times of failed components under double monitoring.*

3.1. Aging properties of the failed components

In this subsection, we investigate some aging properties of the inactivity time of failed components. Before obtaining the main result, we need to give the following Lemma 4.

Lemma 4. *Suppose that* $X_1, X_2, \ldots, X_n$ *are i.i.d. random variables with common distribution function* F*. If* F *is concave, then for* $0 \le r < k < i$*,* $B_{k,r,m}(t_1, t_2)$ *is increasing in* $t_2 > t_1$*.*

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Proof. For any $x \in \mathbb{R}^+$, and $0 \le t_1 < t_2$, note that

$$\begin{aligned} B_{k,r,m}(t_1,t_2) &= \mathbb{P}(t_2 - X_{k:n} > x | X_{r:n} = t_1, X_{m:n} = t_2) \\ &= \sum_{j=k-r}^{m-r-1} \binom{m-r-1}{j} (H_{t_1,t_2}(x))^j (1 - H_{t_1,t_2}(x))^{m-r-1-j} \\ &= \int_0^{H_{t_1,t_2}(x)} (k-r) \binom{m-r-1}{k-r} u^{k-r-1} (1-u)^{m-k-1}\,\mathrm{d}u, \end{aligned} \tag{8}$$

where $H_{t_1,t_2}(x) = (F(t_2 - x) - F(t_1))/(F(t_2) - F(t_1))$. Taking the partial derivative of $H_{t_1,t_2}(x)$ with respect to t_2, we have

$$\frac{\partial H_{t_1,t_2}(x)}{\partial t_2} \overset{\text{sgn}}{=} f(t_2 - x)[F(t_2) - F(t_1)] - f(t_2)[F(t_2 - x) - F(t_1)]. \tag{9}$$

Notice that the concavity of F implies (9) is nonnegative, that is, $(F(t_2 - x) - F(t_1))/(F(t_2) - F(t_1))$ is increasing in t_2, thus, it follows from (8) that $B_{k,r,m}(t_1, t_2)$ is increasing in t_2. □

The following Theorem 3 proves that when the distribution F of the components of coherent system is concave function, then $\mathbb{P}(IX_{k:n}(t_1, t_2) > x)$ is an increasing function in t_2.

Theorem 3. *Consider a coherent system consisting of n i.i.d. components with lifetimes $X_1, X_2, \ldots, X_n$, suppose that the system has signature vector $\mathbf{s} = (0, 0, \ldots, 0, s_i, \ldots, s_n)$. If the common distribution function F of $X_1, X_2, \ldots, X_n$ is concave, then $\mathbb{P}(IX_{k:n}(t_1, t_2) > x)$ is increasing in $t_2 > t_1$.*

Proof. First, taking the partial derivative of reliability function of $IX_{k:n}(t_1, t_2)$ with respect to t_2 gives rise to

$$\begin{aligned} &\frac{\partial \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, T = t_2)}{\partial t_2} \\ &= \frac{\partial \sum_{m=i}^n p_m(r, t_1, t_2) \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2)}{\partial t_2} \\ &= \sum_{m=i}^n \frac{\partial p_m(r, t_1, t_2)}{\partial t_2} \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2) \\ &\quad + \sum_{m=i}^n p_m(r, t_1, t_2) \frac{\partial \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2)}{\partial t_2}. \end{aligned} \tag{10}$$

Since F is concave, it follows from Lemma 4 that the second term in (10) is nonnegative. Observe that

$$\begin{aligned} \frac{\partial p_m(r, t_1, t_2)}{\partial t_2} &= \frac{\partial \left(\frac{s_m C^n_{r,m} (\phi_{t_1}(t_2))^m}{\sum_{j=i}^n s_j C^n_{r,j} (\phi_{t_1}(t_2))^j} \right)}{\partial t_2} \\ &= \frac{\frac{\partial \phi_{t_1}(t_2)}{\partial t_2} \sum_{j=i}^n s_m s_j C^n_{r,m} C^n_{r,j} (\phi_{t_1}(t_2))^{m+j-1} (m-j)}{(\sum_{j=i}^n s_j C^n_{r,j} (\phi_{t_1}(t_2))^j)^2}. \end{aligned}$$

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For convenience, let $h(m) = \mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, X_{m:n} = t_2)$, then

$$\begin{aligned}
&\sum_{m=i}^{n} \frac{\partial p_m(r,t_1,t_2)}{\partial t_2} h(m) \\
&= \sum_{m=i}^{n} \frac{\frac{\partial \phi_{t_1}(t_2)}{\partial t_2} \sum_{j=i}^{n} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}(m-j)}{(\sum_{j=i}^{n} s_j C_{r,j}^n (\phi_{t_1}(t_2))^j)^2} h(m) \\
&\overset{\text{sgn}}{=} \sum_{m=i}^{n} \sum_{j=i}^{n} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}(m-j)h(m) \\
&= \sum_{m=i}^{n} \sum_{j=i}^{m} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}(m-j)h(m) \\
&\quad + \sum_{m=i}^{n} \sum_{j=m}^{n} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}(m-j)h(m) \\
&= \sum_{m=i}^{n} \sum_{j=i}^{m} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}(m-j)h(m) \\
&\quad - \sum_{m=i}^{n} \sum_{j=i}^{m} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}(m-j)h(j) \\
&= \sum_{m=i}^{n} \sum_{j=i}^{m} s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}(t_2))^{m+j-1}[(m-j)(h(m)-h(j))].
\end{aligned} \tag{11}$$

Hence, from Theorem 1, we conclude that $h(m)$ is increasing in m, which implies $(m-j)(h(m)-h(j)) \geq 0$, that is, (11) is nonnegative. It follows that (10) is nonnegative, which finishes the proof. □

Remark 5. *It should be pointed out that Theorem 3 generalizes Thm.* 3.2 *in Nama and Asadi [28] to the case of coherent system under double monitoring.*

3.2. *Stochastic comparison between the failed components*

In this subsection, we compare the inactivity times of the failed components for two coherent systems. Suppose that X is a random variable with distribution $F_X(\cdot)$, for given $t_1 < t_2$, we define

$$\gamma_X(t_1, t_2; x) = \frac{F_X(t_2 - x) - F_X(t_1)}{F_X(t_2) - F_X(t_1)}, \quad \text{for any } 0 \leq x < t_2 - t_1.$$

In fact, $\gamma_X(t_1, t_2; x)$ is just the distribution of random variable $[t_2 - X \mid t_1 < X \leq t_2]$.

The following theorem gives a sufficient condition for the usual stochastic order of the inactivity time of failed components in two coherent systems which have two different sets of i.i.d components but have a common structure.

Theorem 4. *Assume that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are i.i.d. random variables with distribution functions F and G, respectively. Let $T_1 = \tau(X_1, X_2, \ldots, X_n)$ and $T_2 = \tau(Y_1, Y_2, \ldots, Y_n)$ be the lifetimes of two coherent systems with common signature vector* $\mathbf{s} = (0, \ldots, 0, s_i, \ldots, s_n)$. *If $X_1 \leq_{hr} Y_1$, and for given $t_1 < t_2$, $\gamma_{X_1} \geq \gamma_{Y_1}$ $(0 \leq x < t_2 - t_1)$, then*

$$(t_2 - X_{k:n} \mid X_{r:n} = t_1, T_X = t_2) \geq_{st} (t_2 - Y_{k:n} | Y_{r:n} = t_1, T_Y = t_2).$$

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Proof. For $m = i, \ldots, n$ and $0 \leq t_1 < t_2$, let

$$p_m^X(r, t_1, t_2) = \frac{s_m C_{r,m}^n (\phi_{t_1}^X(t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi_{t_1}^X(t_2))^j}, \quad p_m^Y(r, t_1, t_2) = \frac{s_m C_{r,m}^n (\phi_{t_1}^Y(t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi_{t_1}^Y(t_2))^j},$$

where $\phi_{t_1}^X(t_2) = \bar{F}(t_1)/\bar{F}(t_2) - 1$ and $\phi_{t_1}^Y(t_2) = \bar{G}(t_1)/\bar{G}(t_2) - 1$. According to (3) and Lemma 3, the reliability functions of inactivity times $IX_{k:n}(t_1, t_2)$ and $IY_{k:n}(t_1, t_2)$ are

$$\mathbb{P}(t_2 - X_{k:n} > x \mid X_{r:n} = t_1, T_X = t_2) = \sum_{m=i}^n \frac{s_m C_{r,m}^n (\phi_{t_1}^X(t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi_{t_1}^X(t_2))^j} \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,X} > x),$$

and

$$\mathbb{P}(t_2 - Y_{k:n} > x \mid Y_{r:n} = t_1, T_Y = t_2) = \sum_{m=i}^n \frac{s_m C_{r,m}^n (\phi_{t_1}^Y(t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi_{t_1}^Y(t_2))^j} \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x),$$

respectively, where $Z_{m-k:m-r-1}^{t_1,t_2,X}$ is the $(m-k)$th order statistics among $(m-r-1)$ i.i.d. random variables with common distribution function $\gamma_X(t_1, t_2; x)$, and $Z_{m-k:m-r-1}^{t_1,t_2,Y}$ is the $(m-k)$th order statistics among $(m-r-1)$ i.i.d. random variables with common distribution function $\gamma_Y(t_1, t_2; x)$.

Thus, to prove the desired result, we need to show that

$$\sum_{m=i}^n \frac{s_m C_{r,m}^n (\phi_{t_1}^Y(t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi_{t_1}^Y(t_2))^j} \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x) \leq \sum_{m=i}^n \frac{s_m C_{r,m}^n (\phi_{t_1}^X(t_2))^m}{\sum_{j=i}^n s_j C_{r,j}^n (\phi_{t_1}^X(t_2))^j} \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,X} > x),$$

which is equivalent to

$$\begin{aligned} &\sum_{m=i}^n \sum_{j=i}^n s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}^X(t_2))^m (\phi_{t_1}^Y(t_2))^j \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,X} > x) \\ &\quad - \sum_{m=i}^n \sum_{j=i}^n s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}^X(t_2))^j (\phi_{t_1}^Y(t_2))^m \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x) \geq 0, \end{aligned}$$

that is,

$$\begin{aligned} &\sum_{m=i}^n \sum_{j=i}^n s_m s_j C_{r,m}^n C_{r,j}^n (\phi_{t_1}^X(t_2))^m (\phi_{t_1}^Y(t_2))^j [\mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,X} > x) - \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)] \\ &\quad + \sum_{m=i}^n \sum_{j=i}^n s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x) \\ &\quad \times [(\phi_{t_1}^X(t_2))^m (\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j (\phi_{t_1}^Y(t_2))^m] \geq 0. \end{aligned} \tag{12}$$

Note that $\gamma_{X_1} \geq \gamma_{Y_1}$, it follows from (8) that

$$\begin{aligned} &\mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,X} > x) - \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x) \\ &\quad = \int_{\gamma_{Y_1}}^{\gamma_{X_1}} (k-r) \binom{m-r-1}{k-r} u^{k-r-1} (1-u)^{m-k-1} \, \mathrm{d}u \geq 0. \end{aligned}$$

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which means that the first term in the left side of (12) is nonnegative. Observe that the second term in the left side of (12)

$$
\begin{aligned}
&\sum_{m=i}^{n}\sum_{j=i}^{n} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m] \\
&\quad = \sum_{m=i}^{n}\sum_{j=m}^{n} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m] \\
&\qquad + \sum_{m=i}^{n}\sum_{j=i}^{m-1} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m] \\
&\quad = \sum_{m=i}^{n}\sum_{j=m}^{n} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m] \\
&\qquad + \sum_{j=i}^{n}\sum_{m=j}^{n} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m] \\
&\quad = \sum_{m=i}^{n}\sum_{j=m}^{n} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m] \\
&\qquad + \sum_{m=i}^{n}\sum_{j=m}^{n} s_m s_j C_{r,m}^n C_{r,j}^n \mathbb{P}(Z_{j-k:j-r-1}^{t_1,t_2,Y} > x)[(\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m - (\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j] \\
&\quad = \sum_{m=i}^{n}\sum_{j=m}^{n} s_m s_j C_{r,m}^n C_{r,j}^n [\mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x) - \mathbb{P}(Z_{j-k:j-r-1}^{t_1,t_2,Y} > x)] \\
&\qquad \times [(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m]. \qquad (13)
\end{aligned}
$$

From Theorem 1, for any $m < j$, it holds that

$$\mathbb{P}(Z_{m-k:m-r-1}^{t_1,t_2,Y} > x) - \mathbb{P}(Z_{j-k:j-r-1}^{t_1,t_2,Y} > x) \le 0. \qquad (14)$$

Due to $X_1 \le_{\mathrm{hr}} Y_1$ means $(\phi_{t_1}^Y(t_2))^{j-m} - (\phi_{t_1}^X(t_2))^{j-m} \le 0$, it is obvious that

$$
\begin{aligned}
&(\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^j - (\phi_{t_1}^X(t_2))^j(\phi_{t_1}^Y(t_2))^m \\
&\quad = (\phi_{t_1}^X(t_2))^m(\phi_{t_1}^Y(t_2))^m[(\phi_{t_1}^Y(t_2))^{j-m} - (\phi_{t_1}^X(t_2))^{j-m}] \le 0. \qquad (15)
\end{aligned}
$$

According to (14) and (15), the nonnegativity of (13) is verified, which yields the result. □

Remark 6. *Theorem 4 provides a sufficient condition for the usual stochastic order holds between two coherent systems having the same structure but with two heterogeneous sets of i.i.d. components. This extends Thm. 3.3 of Nama and Asadi [28] to the scenario of double monitoring.*

It should be pointed out that verify $\gamma_{X_1} \ge \gamma_{Y_1}$ may be very difficult. Consider that $X_1 \le_{\mathrm{lr}} Y_1$ implies $X_1 \le_{\mathrm{hr}} Y_1$, and according to Thm. 1.C.5 of Shaked and Shanthikumar [40], $X_1 \le_{\mathrm{lr}} Y_1$ is equivalent to

$$\frac{F(t_2 - x) - F(t_1)}{F(t_2) - F(t_1)} \ge \frac{G(t_2 - x) - G(t_1)}{G(t_2) - G(t_1)}, \quad \text{for all } t_1 < t_2 \text{ and } 0 \le x < t_2 - t_1.$$

Hence, from Theorem 4, we can obtain the following corollary.

Corollary 1. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables with distribution functions F and G, respectively. Suppose that $T_1 = \tau(X_1, X_2, \ldots, X_n)$ and $T_2 = \tau(Y_1, Y_2, \ldots, Y_n)$ be the lifetimes*

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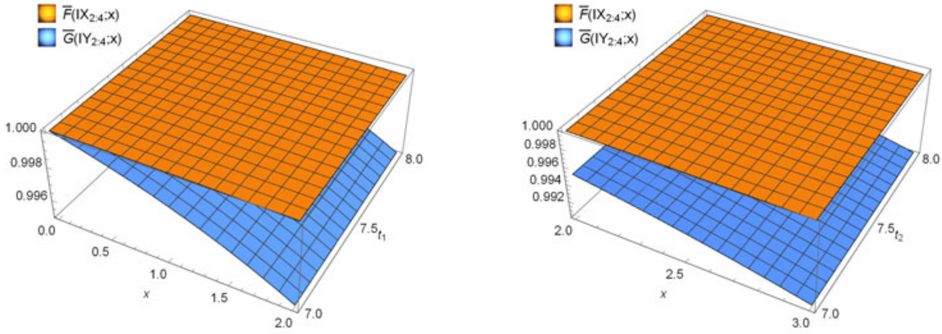

(a) (b)

Figure 2. *The reliability functions of $IX_{2:4}$ and $IY_{2:4}$.*

of two coherent systems with common signature vector $\mathbf{s} = (0, \ldots, 0, s_i, \ldots, s_n)$. *If* $X_1 \le_{lr} Y_1$, *for all* $0 \le x < t_2 - t_1$, *then*

$$(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_X = t_2) \ge_{st} (t_2 - Y_{k:n} \,|\, Y_{r:n} = t_1, T_Y = t_2).$$

The following example illustrates the result of Corollary 1.

Example 2. *Under the set-up of Example 1, suppose that* X_1 *and* Y_1 *have the Weibull distribution, that is,* $\bar{F}(\lambda_1, \alpha_1; x) = e^{-\lambda_1^{\alpha_1} x^{\alpha_1}}$, $\bar{G}(\lambda_2, \alpha_2; x) = e^{-\lambda_2^{\alpha_2} x^{\alpha_2}}$. *Set* $\lambda_1 = 0.9$, $\lambda_2 = 0.5$, $\alpha_1 = 0.8$, *and* $\alpha_2 = 0.8$. *Due to* $g(\lambda_2, \alpha_2; x)/f(\lambda_1, \alpha_1; x) = 0.624859e^{0.344817x^{0.8}}$ *is increasing in* $x \in \mathbb{R}^+$, *obviously,* $X_1 \le_{lr} Y_1$.

(i) Taking $t_2 = 11$, *Figure 2(a) plots the reliability functions of* $IX_{2:4}$ *and* $IY_{2:4}$, *it is easy to see that* $\bar{F}(IX_{2:4}; x) \ge \bar{G}(IY_{2:4}; x)$;
(ii) Setting $t_1 = 0.8$, *Figure 2(b) displays the reliability functions of* $IX_{2:4}$ *and* $IY_{2:4}$, *apparently,* $\bar{F}(IX_{2:4}; x) \ge \bar{G}(IY_{2:4}; x)$.

Therefore, it follows from (i) and (ii) that verified the validity of Corollary 1.

In combination with the results of Theorem 2(i) and Corollary 1, it is easy to obtain the following result in terms of the usual stochastic order.

Corollary 2. *Let* $X_1, X_2, \ldots, X_n$ *and* $Y_1, Y_2, \ldots, Y_n$ *be i.i.d. random variables with common distribution function* F *and* G, *respectively. Suppose that* $T_1 = \tau_1(X_1, X_2, \ldots, X_n)$, $T_2 = \tau_2(X_1, X_2, \ldots, X_n)$, *and* $T_3 = \tau_2(Y_1, Y_2, \ldots, Y_n)$ *be the lifetimes of three coherent systems, where* T_1 *has the signature vector* $\mathbf{s}^{(1)} = (0, \ldots, 0, s_i^{(1)}, \ldots, s_n^{(1)})$, *and* T_2 *and* T_3 *have common signature vector* $\mathbf{s}^{(2,3)} = (0, \ldots, 0, s_i^{(2,3)}, \ldots, s_n^{(2,3)})$. *If* $\mathbf{p}^{(1)} \le_{st} \mathbf{p}^{(2)}$ *and* $X_1 \ge_{lr} Y_1$, *for any* $t_1 < t_2$, *then*

$$\begin{aligned}(t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_1 = t_2) &\le_{st} (t_2 - X_{k:n} \,|\, X_{r:n} = t_1, T_2 = t_2)\\ &\le_{st} (t_2 - Y_{k:n} \,|\, Y_{r:n} = t_1, T_3 = t_2).\end{aligned}$$

3.3. *Some properties of* $p_m(r, t_1, t_2)$

In this subsection, we concentrate on $p_m(r, t_1, t_2)$ in (3). The following theorems give some stochastic properties of conditional signature vector

$$\mathbf{p}(t_1, t_2) = (0, \ldots, 0, p_i(r, t_1, t_2), \ldots, p_n(r, t_1, t_2)).$$

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Next, Theorem 5 gives a sufficient condition of the usual stochastic order for conditional signature vector $\mathbf{p}(t_1, t_2)$.

Theorem 5. *Suppose* $\mathbf{p}(t_1, t_2)$ *be the conditional signature vector of a coherent system. If* $t_2 \le t_3$, *then* $\mathbf{p}(t_1, t_2) \le_{st} \mathbf{p}(t_1, t_3)$.

Proof. To reach the desired result, it is sufficient to show that

$$\frac{\sum_{j=m}^{n} s_j C_{r,j}^n (\phi_{t_1}(t_2))^j}{\sum_{l=i}^{n} s_l C_{r,l}^n (\phi_{t_1}(t_2))^l} \le \frac{\sum_{j=m}^{n} s_j C_{r,j}^n (\phi_{t_1}(t_3))^j}{\sum_{l=i}^{n} s_l C_{r,l}^n (\phi_{t_1}(t_3))^l}, \quad \text{for all } m = i, i+1, \ldots, n,$$

which is equivalent to

$$\begin{aligned}
&\sum_{j=m}^{n}\sum_{l=i}^{n} s_j s_l C_{r,j}^n C_{r,l}^n [(\phi_{t_1}(t_3))^j (\phi_{t_1}(t_2))^l - (\phi_{t_1}(t_3))^l (\phi_{t_1}(t_2))^j] \\
&\quad = \sum_{j=m}^{n}\sum_{l=i}^{m-1} s_j s_l C_{r,j}^n C_{r,l}^n (\phi_{t_1}(t_3))^l (\phi_{t_1}(t_2))^l [(\phi_{t_1}(t_3))^{j-l} - (\phi_{t_1}(t_2))^{j-l}] \\
&\qquad + \sum_{j=m}^{n}\sum_{l=m}^{n} s_j s_l C_{r,j}^n C_{r,l}^n [(\phi_{t_1}(t_3))^j (\phi_{t_1}(t_2))^l - (\phi_{t_1}(t_3))^l (\phi_{t_1}(t_2))^j] \ge 0. \qquad (16)
\end{aligned}$$

For $t_1 < t_2 < t_3$ and $l < j$, due to the increasing property of $\phi_{t_1}(t_2)$ in t_2 and the second term of (16) equals zero, we obtain the nonnegativity of (16), thus the result is proved. □

The following result indicates that two signature vectors are ordered with respect to the likelihood ratio order, and then, the corresponding conditional signature vectors are also stochastic ordering.

Theorem 6. *Consider two coherent systems consisting of i.i.d. components with common distribution function F. Suppose that systems have the signature vector* $\mathbf{s}^{(j)} = (0, \ldots, 0, s_i^{(j)}, \ldots, s_n^{(j)})$, $j = 1, 2$. *If* $\mathbf{s}^{(1)} \le_{lr} \mathbf{s}^{(2)}$, *then* $\mathbf{p}^{(1)}(t_1, t_2) \le_{st} \mathbf{p}^{(2)}(t_1, t_2)$.

Proof. Note that, $\mathbf{p}^{(1)}(t_1, t_2) \le_{\mathrm{st}} \mathbf{p}^{(2)}(t_1, t_2)$ implies

$$\frac{\sum_{j=m}^{n} s_j^{(1)} C_{r,j}^n (\phi_{t_1}(t_2))^j}{\sum_{l=i}^{n} s_l^{(1)} C_{r,l}^n (\phi_{t_1}(t_2))^l} \le \frac{\sum_{j=m}^{n} s_j^{(2)} C_{r,j}^n (\phi_{t_1}(t_2))^j}{\sum_{l=i}^{n} s_l^{(2)} C_{r,l}^n (\phi_{t_1}(t_2))^l}, \quad \text{for any } m = i, i+1, \ldots, n,$$

which is equivalent to

$$\begin{aligned}
&\sum_{j=m}^{n}\sum_{l=i}^{n} C_{r,j}^n C_{r,l}^n (\phi_{t_1}(t_2))^j (\phi_{t_1}(t_2))^l (s_j^{(2)} s_l^{(1)} - s_j^{(1)} s_l^{(2)}) \\
&\quad = \sum_{j=m}^{n}\sum_{l=i}^{m-1} C_{r,j}^n C_{r,l}^n (\phi_{t_1}(t_2))^j (\phi_{t_1}(t_2))^l (s_j^{(2)} s_l^{(1)} - s_j^{(1)} s_l^{(2)}) \\
&\qquad + \sum_{j=m}^{n}\sum_{l=m}^{n} C_{r,j}^n C_{r,l}^n (\phi_{t_1}(t_2))^j (\phi_{t_1}(t_2))^l (s_j^{(2)} s_l^{(1)} - s_j^{(1)} s_l^{(2)}) \ge 0. \qquad (17)
\end{aligned}$$

One can easily show that the second term in the right side of (17) equals zero. It holds from $\mathbf{s}^{(1)} \le_{\mathrm{lr}} \mathbf{s}^{(2)}$ that $(s_j^{(2)} s_l^{(1)} - s_j^{(1)} s_l^{(2)}) \ge 0\,(l < j)$, we conclude that the first term in the left side of (17) is nonnegative. This completes the proof. □

As one anonymous reviewer suggested, whether the result $\mathbf{p}^{(1)}(t_1, t_2) \le_{\mathrm{st}} \mathbf{p}^{(2)}(t_1, t_2)$ of Theorem 6 could be enhanced or not? Unfortunately, the following Example 3(ii) gives a partial negative answer.

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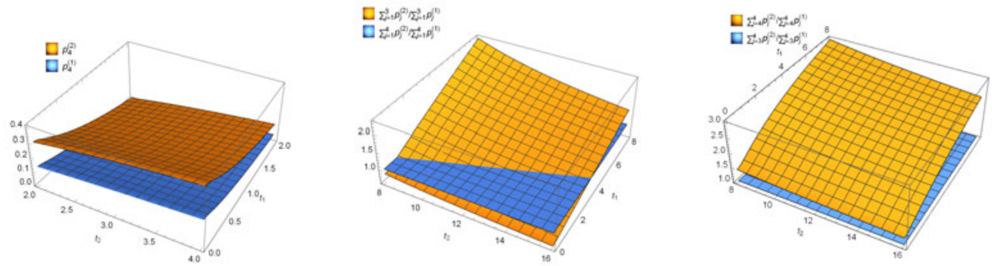

(a) (b) (c)

Figure 3. *(a) Plots of the elements $p_4^{(1)}$ and $p_4^{(2)}$ of $\mathbf{p}^{(1)}(t_1,t_2)$ and $\mathbf{p}^{(2)}(t_1,t_2)$; (b) plots of $\sum_{j=1}^{3} p_j^{(2)}/\sum_{j=1}^{3} p_j^{(1)}$ and $\sum_{j=1}^{4} p_j^{(2)}/\sum_{j=1}^{4} p_j^{(1)}$; and (c) plots of $\sum_{j=4}^{4} p_j^{(2)}/\sum_{j=4}^{4} p_j^{(1)}$ and $\sum_{j=3}^{4} p_j^{(2)}/\sum_{j=3}^{4} p_j^{(1)}$.*

Example 3. *Assume that X_1, X_2, X_3, and X_4 are i.i.d. random variables. Consider two coherent systems with lifetime $T_1 = \max(X_{2:3}, X_3)$ and $T_2 = \min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4))$, which have the signature vectors $\mathbf{s}^{(1)} = (0, 0, \frac{3}{4}, \frac{1}{4})$ and $\mathbf{s}^{(2)} = (0, 0, \frac{1}{2}, \frac{1}{2})$, respectively. Obviously, $\mathbf{s}^{(1)} \le_{lr} \mathbf{s}^{(2)}$. Suppose that there is one failed component in systems at time t_1, and the systems just failed at time $t_2 (> t_1)$. It is easy to compute the conditional signature vector of two systems that can be written as*

$$\mathbf{p}^{(1)}(t_1,t_2) = \left(0, 0, \frac{6\bar{F}(t_2)}{5\bar{F}(t_2)+\bar{F}(t_1)}, \frac{\bar{F}(t_1)-\bar{F}(t_2)}{5\bar{F}(t_2)+\bar{F}(t_1)}\right)$$

and

$$\mathbf{p}^{(2)}(t_1,t_2) = \left(0, 0, \frac{2\bar{F}(t_2)}{\bar{F}(t_1)+\bar{F}(t_2)}, \frac{\bar{F}(t_1)-\bar{F}(t_2)}{\bar{F}(t_1)+\bar{F}(t_2)}\right),$$

respectively. Let X_1 follow Frechet distribution, that is, $\bar{F}(\alpha,\beta;x) = 1 - e^{-(\frac{x}{\beta})^{-\alpha}}$, $x \in \mathbb{R}^+$.

(i) Taking $\alpha = 0.5, \beta = 1.2$, according to Figure 3(a), $p_4^{(1)} \le p_4^{(2)}$ implies that $\mathbf{p}^{(1)}(t_1,t_2) \le_{st} \mathbf{p}^{(2)}(t_1,t_2)$, hence, the validity of Theorem 6 be confirmed;

(ii) Setting $\alpha = 0.9, \beta = 0.75$, as we can see in Figure 3(b), the graphs of $\sum_{j=1}^{3} p_j^{(2)}/\sum_{j=1}^{3} p_j^{(1)}$ and $\sum_{j=1}^{4} p_j^{(2)}/\sum_{j=1}^{4} p_j^{(1)}$ are crossing, which means neither $\mathbf{p}^{(1)}(t_1,t_2) \nleq_{rh} \mathbf{p}^{(2)}(t_1,t_2)$ nor $\mathbf{p}^{(1)}(t_1,t_2) \ngeq_{rh} \mathbf{p}^{(2)}(t_1,t_2)$. Thus, the usual stochastic order of Theorem 6 cannot be enhanced to the stronger reversed hazard rate order;

(iii) Taking $\alpha = 0.9, \beta = 0.75$, as we can see in Figure 3(c), the graphs of $\sum_{j=4}^{4} p_j^{(2)}/\sum_{j=4}^{4} p_j^{(1)} \ge \sum_{j=3}^{4} p_j^{(2)}/\sum_{j=3}^{4} p_j^{(1)}$ implies $\mathbf{p}^{(1)}(t_1,t_2) \le_{hr} \mathbf{p}^{(2)}(t_1,t_2)$.

Remark 7. *On the one hand, we cannot find some examples that only satisfy the hazard rate (reversed hazard rate/usual stochastic) order while do not satisfy the likelihood ratio order, hence, whether weaken the condition the likelihood ratio order of Theorem 6 is left as an open question. On the other hand, we illustrate that the usually stochastic order of Theorem 6 cannot be enhanced to the reversed hazard rate order by a negative Example 3(ii), but Example 3(iii) shows that the hazard rate order is still valid. Unfortunately, we cannot provide rigorous mathematical proof. Therefore, these are very interesting topics, which are worth further discussions.*

The following Theorem 7 provides some properties of $p_m(r, t_1, t_2)$, $m = i, \ldots, n$.

Theorem 7. *(i) For a given t_2, $p_i(r, t_1, t_2)$ is increasing in t_1; for a given t_1, $p_i(r, t_1, t_2)$ is decreasing in t_2, and $\lim_{t_2 \to \infty} p_i(r, t_1, t_2) = 0$;*

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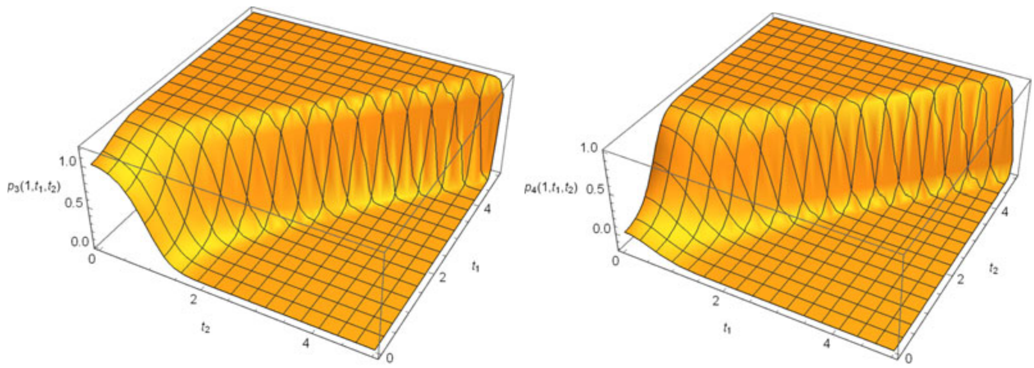

(a) (b)

Figure 4. *The graphs of (a) $p_3(1, t_1, t_2)$ and (b) $p_4(1, t_1, t_2)$.*

(ii) For a given t_2, $p_n(r, t_1, t_2)$ is decreasing in t_1; for a given t_1, $p_n(r, t_1, t_2)$ is increasing in t_2, and $\lim_{t_2\to\infty} p_n(r, t_1, t_2) = 1$.

Proof. (i) For a given t_1 and $m = i,\ i+1, \ldots, n$, when $0 \le t_1 < t_2$, it holds that

$$p_m(r, t_1, t_2) = \frac{s_m C^n_{r,m}(\phi_{t_1}(t_2))^m}{\sum_{l=i}^n s_l C^n_{r,l}(\phi_{t_1}(t_2))^l} = \frac{s_m C^n_{r,m}}{\sum_{l=i}^n s_l C^n_{r,l}(\phi_{t_1}(t_2))^{l-m}}.$$

It can be seen clearly that $\phi_{t_1}(t_2)$ is increasing in t_2, which implies $(\phi_{t_1}(t_2))^{l-i}$ is increasing in t_2 for $l = i,\ i+1, \ldots, n$, thus, $p_i(r, t_1, t_2)$ is decreasing in t_2, and $\lim_{t_2\to\infty} p_i(r, t_1, t_2) = 0$. Similarly, $p_i(r, t_1, t_2)$ is increasing in t_1, for a given t_2.

The proof of part(ii) is similar to that of part(i), so we omit it here for brevity. □

Example 4. *Under the set-up of Example 1, the inactivity time of failed component $X_{2:4}$ at t_2 is given by*

$$(t_2 - X_{2:4} \mid X_{1:4} = t_1, T = t_2).$$

Suppose X_i follows the Weibull distribution, that is, $\bar{F}(x) = e^{-\lambda^\alpha x^\alpha}$, $x \in \mathbb{R}^+$, we have

$$p_3(1, t_1, t_2) = \frac{6e^{-\lambda^\alpha t_2^\alpha}}{5e^{-\lambda^\alpha t_2^\alpha} + e^{-\lambda^\alpha t_1^\alpha}}, \quad p_4(1, t_1, t_2) = \frac{e^{-\lambda^\alpha t_1^\alpha} - e^{-\lambda^\alpha t_2^\alpha}}{5e^{-\lambda^\alpha t_2^\alpha} + e^{-\lambda^\alpha t_1^\alpha}}.$$

Set $\alpha = 2$ and $\lambda = 1.5$, Figure 4(a) shows that $p_3(1, t_1, t_2)$ is decreasing in t_2, $\lim_{t_2\to\infty} p_3(1, t_1, t_2) = 0$, and $p_3(1, t_1, t_2)$ is increasing in t_1; Figure 4(b) indicates that $p_4(t_1, t_2)$ is increasing in t_2, $\lim_{t_2\to\infty} p_4(1, t_1, t_2) = 1$, and $p_4(1, t_1, t_2)$ is decreasing in t_1. Hence, the validity of Theorem 7 is verified.

4. Concluding remarks

In this article, we investigate the stochastic behavior and reliability properties for the inactivity times of failed components in coherent systems under double monitoring, given that at time t_1, there are r components have failed, but the system has failed at time $t_2(> t_1)$. A mixture representation of reliability function is established for the inactivity times of failed components $X_{k:n}$, and some stochastic comparison results are established on the inactivity times of failed components under double checking. Besides, some sufficient condition is also developed in terms of the aging properties of the inactivity times of failed components and some properties obtained for the conditional signature vector.

In this article, we assume that the coherent system with independent identical components. As described by Rychlik [36], the components of the coherent system may be statistically dependent. One

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may also consider other scenarios of the components of the coherent system are dependent, which can be an interesting area for future research.

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