Glasgow Math. J. **45** (2003) 527–533. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089503001423. Printed in the United Kingdom

WHEN IS THE AUTOMORPHISM GROUP OF A VIRTUALLY POLYCYCLIC GROUP VIRTUALLY POLYCYCLIC?

BETTINA EICK

Institute of Geometry, Technische Universität Braunschweig, D-38106 Braunschweig, Germany
e-mail: beick@tu-bs.de

(Received 4 July, 2002; accepted 13 January, 2003)

Abstract. The automorphism group of a virtually polycyclic group G is either virtually polycyclic or it contains a non-abelian free subgroup. We describe conditions on the structure of G to decide which of the two alternatives occurs for $Aut(G)$.

2000 *Mathematics Subject Classification.* Primary 20F16, 20F28.

1. Introduction. A group G is virtually polycyclic (or polycyclic-by-finite) if it contains a polycyclic normal subgroup of finite index. The automorphism group $Aut(G)$ of a virtually polycyclic group G is finitely generated and embeds into $GL(n, \mathbb{Z})$ for some $n \in \mathbb{N}$. Thus, Tits' alternative yields that $Aut(G)$ is virtually polycyclic or it contains a non-abelian free subgroup.

Here we consider the question: *Given a virtually polycyclic group G, can we decide whether* $Aut(G)$ *is virtually polycyclic without explicitly determining* $Aut(G)$*?*

Each virtually polycyclic group G has a *characteristic semisimple series*; that is, a characteristic series whose factors are either finite or free abelian and rationally semisimple. We want to use the module structure of the free abelian factors in such a series to investigate $Aut(G)$. We say that a free abelian subfactor F of G is $\mathbb{R}$-*inhomogeneous* if F considered as a $\mathbb{Q}G$-module is a direct sum of pairwise non-isomorphic irreducible modules $F = F_1 \oplus \ldots \oplus F_r$ and each F_i either has Schur index $m_{\mathbb{Q}}(F_i) = 1$ or F_i is irreducible as a $\mathbb{R}G$-module. The module structure of G yields the following sufficient condition for $Aut(G)$ to be virtually polycyclic (see Section 3).

THEOREM 1.1. *Let G be virtually polycyclic. Then* $Aut(G)$ *is virtually polycyclic if each free abelian factor F in a characteristic semisimple series of G is* $\mathbb{R}$*-inhomogeneous.*

In certain cases, this sufficient condition is also necessary. We prove the following criterion to decide whether $Aut(G)$ is virtually polycyclic (see Section 4).

THEOREM 1.2. *Let G be virtually polycyclic and let F be a characteristic, free abelian and rationally semisimple subgroup of G which has an almost complement in G. We denote* $H = G/F$. *Then* $Aut(G)$ *is virtually polycyclic if and only if*

- *F is* $\mathbb{R}$*-inhomogeneous, and*
- $C_{Aut(H)}(H/C_H(F))$ *is virtually polycyclic.*

Thus if there exists a characteristic, free abelian and rationally semisimple subgroup F in G which is almost complemented, then the module structure of F and the automorphism group of its factor $Aut(G/F)$ determine whether $Aut(G)$ is virtually polycyclic. Note that F is almost complemented if there exists a subgroup U in G with $U \cap F = 1$ and $[G : UF] < \infty$.

Malfait and Szczepanski introduced the $\mathbb{R}$-inhomogeneous modules in [**3**] and used them to decide whether the automorphism group of a given Bieberbach group is virtually polycyclic. This is a special case of Theorem 1.2 as we show in Section 6. Section 6 also includes some applications of the above Theorems and an example to demonstrate that the condition of Theorem 1.1 is not necessary in general. Section 5 gives a further analysis of $\mathbb{R}$-inhomogeneous modules.

2. Preliminaries. Each virtually polycyclic group G has a characteristic series whose factors are either finite or free abelian and such a series can be determined readily without computing $Aut(G)$. (See [**5**] for background on polycyclic groups.) We observe in the following that such a series can be refined to a characteristic semisimple series of G. If F is a free abelian subfactor of G and K is a field extension of $\mathbb{Q}$, then we can consider $F_K = F \otimes K$ as a KG-module. We say that F is semisimple if $F_{\mathbb{Q}}$ is semisimple as a $\mathbb{Q}G$-module. Further, the radical $Rad_{\mathbb{Q}G}(F_{\mathbb{Q}})$ is defined as the intersection of all maximal submodules of $F_{\mathbb{Q}}$.

LEMMA 2.1. *Let F be a characteristic, free abelian subfactor of G. Then the radical $R = F \cap Rad_{\mathbb{Q}G}(F_{\mathbb{Q}})$ is an $Aut(G)$-invariant subgroup of F with semisimple factor F/R.*

Proof. $Aut(G)$ acts linearly on $F_{\mathbb{Q}}$ and the maximal submodules of $F_{\mathbb{Q}}$ are permuted under this action. Hence their intersection $Rad_{\mathbb{Q}G}(F_{\mathbb{Q}})$ is $Aut(G)$-invariant. Thus also R is $Aut(G)$-invariant. Since $F_{\mathbb{Q}}$ is finite dimensional, its radical factor is semisimple. Thus also F/R is semisimple. □

Lemma 2.1 yields that each virtually polycyclic group G has a characteristic semisimple series which can be determined without computing $Aut(G)$.

LEMMA 2.2. *Let G be a virtually polycyclic group and let $G = G_1 > \ldots > G_{l+1} = 1$ be a characteristic semisimple series of G with $I = \{i \in \{1, \ldots, l\} \mid G_i/G_{i+1}$ free abelian$\}$. Then $Aut(G)$ is virtually polycyclic if and only if $Aut(G)$ induces a virtually polycyclic group of automorphisms on G_i/G_{i+1} for each $i \in I$.*

Proof. Let $Aut(G) \xrightarrow{\psi} \prod_{i=1}^{l} Aut(G_i/G_{i+1}) \xrightarrow{\varphi} \prod_{i \in I} Aut(G_i/G_{i+1})$ be the natural homomorphisms. By Hall's theorem, $Ker(\psi)$ is nilpotent. Since the factors G_i/G_{i+1} for $i \notin I$ are finite, $Ker(\varphi)$ is finite. Hence $Ker(\psi\varphi)$ is virtually nilpotent. The image $Im(\psi\varphi)$ is virtually polycyclic if and only if $Im(\psi_i)$ is virtually polycyclic for each $i \in I$, where $\psi_i : Aut(G) \to Aut(G_i/G_{i+1})$ denotes the natural action. □

3. The module structure of a virtually polycyclic group. In this section we prove Theorem 1.1 and thus we obtain a sufficient condition on G to have a virtually polycyclic automorphism group $Aut(G)$. The proof is divided into a sequence of theorems as outlined in the following. We call a subgroup $K \leq GL(d, \mathbb{Z})$ semisimple or irreducible if the natural $\mathbb{Q}K$-module $\mathbb{Q}^d$ is semisimple or irreducible as a $\mathbb{Q}K$-module.

THEOREM 3.1. *Let $K \leq GL(d, \mathbb{Z})$ be virtually polycyclic and semisimple. We write $C = C_{GL(d,\mathbb{Z})}(K)$ and $N = N_{GL(d,\mathbb{Z})}(K)$. Then N/C is virtually abelian.*

Proof. It is well-known that a semisimple, virtually polycyclic subgroup K of $GL(d, \mathbb{Z})$ has a characteristic, abelian subgroup F of finite index. For example,

$Z(Fit(K))$ is of this type, see [**2**, 6.3 and 6.5]. Let $V \cong \mathbb{Q}^d$ be the natural $\mathbb{Q}K$-module. Since F has finite index in K, we have that V is semisimple as a $\mathbb{Q}F$-module. Thus $V_{\mathbb{C}}$ is semisimple as a $\mathbb{C}F$-module. Since $F \trianglelefteq N$, we can apply Lemma 1.12 of [**6**] and obtain that $[N : C_N(F)] < \infty$.

Let $U = C_N(F)$ and note that it suffices to prove that U/C is virtually abelian. By construction, the factor U/C embeds into $Aut(K)$. As F is central in U, we obtain that the image of this embedding is contained in $S = C_{Aut(K)}(F)$. The group S is virtually abelian, since $C_S(K/F)$ is a free abelian normal subgroup of finite index in S. Thus U/C is virtually abelian as desired. □

Next, we consider the centralizer of a semisimple subgroup $K \leq GL(d, \mathbb{Z})$. The following theorem is proved in [**3**] for finite groups K. The proof is based on the investigation in [**7**] of the unit group of a $\mathbb{Z}$-order in the centralizer algebra of K and it generalizes directly to semisimple groups K.

THEOREM 3.2. *Let $K \leq GL(d, \mathbb{Z})$ be semisimple. Then the centralizer $C_{GL(d,\mathbb{Z})}(K)$ is virtually polycyclic if and only if the natural module for K is $\mathbb{R}$-inhomogeneous.*

Theorems 3.1 and 3.2 yield the following as a direct corollary.

THEOREM 3.3. *Let $K \leq GL(d, \mathbb{Z})$ be virtually polycyclic and semisimple. Then the normalizer $N_{GL(d,\mathbb{Z})}(K)$ is virtually polycyclic if and only if the natural module for K is $\mathbb{R}$-inhomogeneous.*

Theorem 1.1 is a corollary of Theorem 3.3 and Lemma 2.2. To see this let F be a free abelian factor in a characteristic semisimple series of G. Then $Aut(F) \cong GL(d, \mathbb{Z})$ for some $d \in \mathbb{N}$ and G acts as a virtually polycyclic, semisimple group $K \leq GL(d, \mathbb{Z})$ on F. The image of $\psi_F : Aut(G) \to GL(d, \mathbb{Z})$ fulfills $Aut(G)^{\psi_F} \leq N_{GL(d,\mathbb{Z})}(K)$. Thus if F is $\mathbb{R}$-inhomogeneous for all free abelian factors F in the given series, then $Aut(G)^{\psi_F}$ is virtually polycyclic for all F by Theorem 3.3 and $Aut(G)$ is virtually polycyclic by Lemma 2.2.

4. The extension structure of a virtually polycyclic group. In this section we prove Theorem 1.2 and thus we obtain a necessary and sufficient condition for $Aut(G)$ to be virtually polycyclic for certain groups G. This investigation is based on the extension structure of G. As a first step, we recall the structure of the automorphism group of an extension in the following theorem. For further details and a proof we refer to [**4**].

THEOREM 4.1. *Let F be a characteristic, free abelian subgroup of G and set $H = G/F$. Let $H \to Aut(F) : h \mapsto \overline{h}$ be the action of H on F and let $P(G, F) = \{(\alpha, \nu) \in Aut(H) \times Aut(F) \mid \overline{h^\alpha} = \overline{h}^\nu$ for all $h \in H\}$ be the group of compatible pairs.*

(a) *Let $\psi : Aut(G) \to Aut(H) \times Aut(F)$ be the natural homomorphism induced by the action of $Aut(G)$ on H and F. Then $Ker(\psi)$ is abelian and $Im(\psi) \leq P(G, F)$.*

(b) *Let $\gamma \in H^2(H, A)$ be a cocycle corresponding to the extension G of F by H. Then using the natural action of $P(G, F)$ on $H^2(H, A)$ we obtain $Im(\psi) = Stab_{P(G,F)}(\gamma)$.*

This description of the automorphism group of an extension can be used to prove the desired Theorem 1.2 as we observe in the following.

THEOREM 4.2. *Let G be virtually polycyclic and let F be a characteristic, free abelian and semisimple subgroup of G.*

https://doi.org/10.1017/S0017089503001423 Published online by Cambridge University Press

(a) *Suppose that F has an almost complement in G. Then $Aut(G)$ is virtually polycyclic if and only if $P(G, F)$ is virtually polycyclic.*

(b) *Denote $H = G/F$ and let $H \to Aut(F) : h \mapsto \overline{h}$ be the action of H on F. Then $P(G, F)$ is virtually polycyclic if and only if $N_{Aut(F)}(\overline{H})$ and $C_{Aut(H)}(H/C_H(F))$ are virtually polycyclic.*

Proof. (a) Let $\gamma \in H^2(H, A)$ be a cocycle corresponding to the extension G of F by H. If F has an almost complement in G, then γ has finite order, see [**4**, 2.5]. Since $P(G, F)$ acts as a group of automorphisms on the finitely generated abelian group $H^2(H, F)$, we obtain that the orbit of γ under $P(G, F)$ is finite. By Theorem 4.1b) this yields that $Aut(G)$ is virtually polycyclic if and only if $P(G, F)$ is virtually polycyclic.

(b) Let $\gamma : P(G, F) \to Aut(F)$ be the natural projection on the second factor and denote $C = C_{Aut(F)}(\overline{H})$ and $N = N_{Aut(F)}(\overline{H})$. By the definition of $P(G, F)$, we obtain that $C \leq Im(\gamma) \leq N$. Since $F_{\mathbb{Q}}$ is semisimple, Theorem 3.1 applies and N is virtually polycyclic if and only if C is virtually polycyclic. In turn, we obtain that $Im(\gamma)$ is virtually polycyclic if and only if N is virtually polycyclic. Further, $Ker(\gamma) = \{(\alpha, id) \mid \overline{h^{\alpha}} = \overline{h} \text{ for all } h \in H\} \cong C_{Aut(H)}(\overline{H})$ and $\overline{H} \cong H/C_H(F)$. □

Theorem 1.2 can now be derived directly from Theorems 3.3 and 4.2.

5. An investigation of $\mathbb{R}$-inhomogeneous modules.

Let $K \leq GL(d, \mathbb{Z})$ be irreducible and denote by V the natural module for K, that is, $V = \mathbb{Q}^d$ is irreducible as a $\mathbb{Q}K$-module. In this section we investigate the property of being $\mathbb{R}$-inhomogeneous in more detail. We define

$$\begin{aligned} A &= C_{M_d(\mathbb{Q})}(K) \quad \text{its rational centralizer algebra,} \\ B &= C_{M_d(\mathbb{Z})}(K) \quad \text{its integral centralizer ring, and} \\ C &= C_{GL(d,\mathbb{Z})}(K) \quad \text{its centralizer.} \end{aligned}$$

By Schur's lemma A is a simple algebra, B is a $\mathbb{Z}$-order in A and C is the group of units in B. As shown in [**7**] and [**3**] we have

$$\begin{aligned} & V \text{ is } \mathbb{R}\text{-inhomogeneous} \\ \Leftrightarrow\ & A \text{ is commutative or a positive definite quaternion algebra over } \mathbb{Q} \\ \Leftrightarrow\ & C \text{ is virtually polycyclic.} \end{aligned}$$

We note that a basis of A can be obtained by solving a system of linear equation. Also, a basis of the centre $Z(A)$ can be determined using linear algebra and thus its dimension $[Z(A) : \mathbb{Q}]$ can be read off. The following lemma yields an approach to check whether A is commutative or a quaternion algebra over $\mathbb{Q}$. (Compare with [**1**] also.)

LEMMA 5.1. *Let $K \leq GL(d, \mathbb{Z})$ be irreducible and let V and A be as above.*
(a) *$dim(A) = zm^2$, where $m = m_{\mathbb{Q}}(V)$ and $z = [Z(A) : \mathbb{Q}]$.*
(b) *A is commutative if and only if $m = 1$.*
(c) *A is a quaternion algebra over $\mathbb{Q}$ if and only if $m = 2$ and $z = 1$.*

Proof. (a) Since $\mathbb{Q}K$ is a simple algebra, we have that $\mathbb{Q}K = M_r(D)$ for some division algebra D of dimension m^2 over $Z(A)$. The algebra D is anti-isomorphic to A and thus we obtain $dim(A) = zm^2$.

https://doi.org/10.1017/S0017089503001423 Published online by Cambridge University Press

(b) The algebra A is commutative if and only if $A = Z(A)$.

(c) A quaternion algebra over $\mathbb{Q}$ has dimension 4 and $\mathbb{Q}$ as its centre by definition. Vice versa, each simple algebra of dimension 4 with centre $\mathbb{Q}$ is a quaternion algebra. □

If $V_{\mathbb{R}}$ is irreducible, then $A_{\mathbb{R}} = A \otimes \mathbb{R}$ is a division algebra. By the classification of the finite-dimensional real division algebras, $A_{\mathbb{R}}$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the real quaternions. Thus A is either commutative or a positive definite quaternion algebra. Vice versa, if A is a positive definite quaternion algebra, then $A_{\mathbb{R}}$ is a division algebra and $V_{\mathbb{R}}$ is irreducible.

6. Sample applications. For practical applications of Theorem 1.2 it is useful to observe that a normal subgroup F has an almost complement in G if there exists a nilpotent normal subgroup Q/F in G/F which acts fixed-point freely on F. This is used in the following.

EXAMPLE 6.1. We consider the polycyclic group

$$G = \langle a, b, c, d, e \mid b^a = be, c^a = d, d^a = e, e^a = cd^4, c^b = c^2d, d^b = d^2e, e^b = cd^4e^2, \\ b^{a^{-1}} = bd^{-1}, c^{a^{-1}} = c^{-4}e, d^{a^{-1}} = c, e^{a^{-1}} = d, c^{b^{-1}} = d^{-2}e, d^{b^{-1}} = cd^4e^{-2}, \\ e^{b^{-1}} = c^{-2}d^{-7}e^4 \ (c, d, e \text{ commute})\rangle$$

Then $F = Fit(G) = \langle c, d, e\rangle \cong \mathbb{Z}^3$ is a characteristic, free abelian subgroup of G with factor group $H = G/F \cong \mathbb{Z}^2$. The group G acts on $F \cong \mathbb{Z}^3$ via the matrices

$$\overline{a} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 0 \end{pmatrix} \quad \text{and} \quad \overline{b} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 4 & 2 \end{pmatrix}.$$

Since the minimal polynomials of these matrices are irreducible of degree 3, we can readily observe that F is $\mathbb{Q}G$-irreducible. This yields that F is semisimple and, since G acts fixed-point freely on F, we obtain that F has an almost complement in G. (But F is not complemented in G.)

Let $K = \langle \overline{a}, \overline{b}\rangle \leq GL(2, \mathbb{Z})$ be the action of G on F and let $A = C_{M_3(\mathbb{Q})}(K)$. Then A is a commutative algebra of dimension 3 and hence F is $\mathbb{R}$-inhomogeneous. Further, $C_H(F) = 1$. Hence $C_{Aut(H)}(H/C_H(F)) = C_{Aut(H)}(H) = 1$ is virtually polycyclic.

Using Theorem 1.2 we can therefore conclude that $Aut(G)$ is polycyclic.

6.1. Groups with finite action. We consider the special case of Theorem 1.2 of a free abelian subgroup with finite action in the following.

THEOREM 6.2. *Let G be virtually polycyclic and let F be a characteristic and free abelian subgroup of G such that $[G : C_G(F)] < \infty$. Denote $H = G/F$ and suppose that F has an almost complement in G. Then $Aut(G)$ is virtually polycyclic if and only if*

- *F is $\mathbb{R}$-inhomogeneous, and*
- *$Aut(H)$ is virtually polycyclic.*

Proof. Since $C_G(F)$ has finite index in G, we have that G acts as a finite group on F and thus F is semisimple as a $\mathbb{Q}G$-module by Maschke's theorem. Hence Theorem 1.2 applies. Further, we note that the factor $Aut(H)/C_{Aut(H)}(H/C_H(F))$ embeds into the

https://doi.org/10.1017/S0017089503001423 Published online by Cambridge University Press

finite group $Aut(G/C_G(F))$. Hence $C_{Aut(H)}(H/C_H(F))$ is virtually polycyclic if and only if $Aut(H)$ is virtually polycyclic. □

As a consequence of Theorem 6.2 we obtain the following theorem which has also been proved in [**3**]. A group G is crystallographic if $Fit(G)$ is free abelian of finite index in G and G does not contain a non-trivial normal torsion subgroup. Further, the group G is a Bieberbach group if G is crystallographic and torsion-free.

COROLLARY 6.3 (Malfait & Szczepanski). *Let G be crystallographic group and let $F = Fit(G)$. Then $Aut(G)$ is virtually polycyclic if and only if F is $\mathbb{R}$-inhomogeneous.*

Proof. Note that F has finite index in G. Thus the trivial subgroup is an almost complement to F in G and $Aut(G/F)$ is finite. □

For a variety of explicit applications of this theorem we refer to [**3**].

6.2. Finitely generated nilpotent groups. Each finitely generated nilpotent group is virtually polycyclic and it has a characteristic central series whose factors are either finite or free abelian. The free abelian factors in such a series are clearly semisimple, but they are not $\mathbb{R}$-inhomogeneous unless they have dimension 1. Hence Theorem 1.1 does not apply to this type of group. The following well-known theorem investigates the structure of the automorphism group of a finitely generated nilpotent group. We refer to [**5**, Chapter 1B], for background.

THEOREM 6.4. *Let G be a finitely generated nilpotent group and let $\psi : Aut(G) \to Aut(G/G')$ be the natural action of $Aut(G)$ on the characteristic abelian factor G/G'. Then $Ker(\psi)$ is nilpotent.*

Hence the action of $Aut(G)$ on the factor G/G' decides whether $Aut(G)$ is virtually polycyclic or contains a non-abelian free subgroup. We give two examples, showing that both cases can occur.

EXAMPLE 6.5. Let $G = \langle a, b, c, d, e \mid [a, b] = c, [a, c] = d, [b, c] = e, (e, d \text{ central})\rangle$ be the free nilpotent group on 2 generators with class 3. Then $G' = \langle c, d, e\rangle$ with $G/G' \cong \mathbb{Z}^2$ and G is generated by a and b. It is straightforward to show that the maps

$$\alpha : G \to G : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \quad \text{and} \quad \beta : G \to G : \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases}$$

induce automorphisms of G. Thus the image of $\psi : Aut(G) \to Aut(G/G')$ and hence also $Aut(G)$ contains a non-abelian free subgroup.

A characteristic semisimple series of G is given by $G > G' > \langle d, e\rangle > 1$. However, its first factor $F = G/G'$ is not $\mathbb{R}$-inhomogeneous, since $F_{\mathbb{Q}}$ is the direct sum of two isomorphic $\mathbb{Q}G$-modules. Hence Theorem 1.1 does not apply.

EXAMPLE 6.6. Let $G = \langle a, b, c, d \mid [a, b] = c, [a, c] = d, [b, c] = 1, (d \text{ central})\rangle$. Then $G' = \langle c, d\rangle \cong \mathbb{Z}^2$ and G is generated by a and b. Let $\alpha \in Aut(G)$ such that α induces an element of $SL(2, \mathbb{Z})$ on G/G'. Then $c^\alpha \equiv c \bmod \langle d\rangle$. Write $b^\alpha = a^e b^f x$ for some $e, f \in \mathbb{Z}$ and $x \in G'$. Then we obtain

$$1 = [b, c]^\alpha = [b^\alpha, c^\alpha] = [a^e b^f x, c] = [a^e, c]^{b^f x}[b^f, c]^x[x, c] = [a^e, c]^{b^f x} = d^e.$$

https://doi.org/10.1017/S0017089503001423 Published online by Cambridge University Press

Hence $e = 1$ and $b^\alpha \equiv b^f \bmod G'$. Therefore $Aut(G)$ induces an upper triangular matrix group on $Aut(G/G')$. Thus $Aut(G/G')$ and also $Aut(G)$ are virtually polycyclic.

A characteristic semisimple series of G is given by $G > G' > \langle d \rangle > 1$ and, as above, its first factor G/G' is not $\mathbb{R}$-inhomogeneous. This shows that the condition of Theorem 1.1 is not necessary to enforce that $Aut(G)$ is virtually polycyclic.

REFERENCES

1. H. Brown, J. Neubüser and H. Zassenhaus, On integral groups. III: Normalizer. *Math. Comp.* **27** (1973), 167–182.

2. J. D. Dixon, *The structure of linear groups* (Van Nostrand Reinhold Company, 1971).

3. W. Malfait and A. Szczepanksi, The structure of the (outer) automorphism group of a Bieberbach group, *Composito Mathematica*, to appear.

4. D. J. Robinson, Applications of cohomology to the theory of groups, in *Groups – St. Andrews 1981*, London Math. Soc. Lecture Note Series, No. 71 (Cambridge University Press, 1981), 46–80.

5. D. Segal, *Polycyclic groups* (Cambridge University Press, 1983).

6. B. A. F. Wehrfritz, *Infinite linear groups* (Springer-Verlag, 1973).

7. H. Zassenhaus, When is the unit group of a Dedekind order solvable? *Comm. Alg.* **6** (1978), 1621–1627.