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Asymptotic spectral gap and Weyl law for Ruelle resonances of open partially expanding maps

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Abstract. We consider a simple model of an open partially expanding map. Its trapped set $\mathcal{K}$ in phase space is a fractal set. We first show that there is a well-defined discrete spectrum of Ruelle resonances which describes the asymptotic of correlation functions for large time and which is parametrized by the Fourier component ν in the neutral direction of the dynamics. We introduce a specific hypothesis on the dynamics that we call 'minimal captivity'. This hypothesis is stable under perturbations and means that the dynamics is univalued in a neighborhood of $\mathcal{K}$. Under this hypothesis we show the existence of an asymptotic spectral gap and a fractal Weyl law for the upper bound of density of Ruelle resonances in the semiclassical limit $\nu \to \infty$. Some numerical computations with the truncated Gauss map and Bowen–Series maps illustrate these results.

Contents

1. *Introduction*

The study of 'Ruelle resonances' was initiated in the 1970s by D. Ruelle and R. Bowen in order to study the decay of correlations in dynamical systems. In a modern approach these Ruelle resonances show up as the discrete spectrum of transfer operators in suitable Banach spaces. While, for analytic expanding maps, such function spaces were already known in the early works of Ruelle [**48**] for hyperbolic systems, they were constructed much later by the work of Kitaev [**34**], Blank, Keller and Liverani [**6**], Baladi and Tsujii [**3**, **4**] and Gouëzel and Liverani [**28**]. In a series of papers by the second author together with Roy and Sjöstrand, it has been shown that semiclassical techniques provide a natural approach for the construction of such suitable function spaces. Up to now this semiclassical approach to the transfer operators has been established for expanding [**22**] and partially expanding maps [**21**], Anosov diffeomorphisms [**23**], and Anosov flows [**24**]. All these systems have in common that they are closed dynamical systems, i.e. systems where the non-wandering set equals the full manifold.

The purpose of this work is to establish the semiclassical approach to 'iterated function schemes' (IFSs) [**19**, **33**]. In these dynamical systems the non-wandering set consists of a fractal subset of the whole system and they can thus be considered as a simple model of an open dynamical system with a trapped set (i.e. non-closed). Beside being a toy-model for such an open system they also appear naturally in various contexts, for example in the reduction of the geodesic flow on convex co-compact hyperbolic surfaces via Bowen–Series maps [**7**, **33**] or in complex dynamics in the analysis of Julia sets [**33**]. We will study the spectral behavior of a certain family of transfer operators that are associated to these IFSs and, using semiclassical techniques, we are able to prove the existence of a discrete spectrum in Sobolev spaces as well as a spectral gap and a fractal Weyl law in a certain semiclassical limit. The concrete form of the transfer operators which we study, as well as the semiclassical limit which we consider, is again motivated from two directions. First of all, these families of transfer operators naturally arise from a decomposition of an open partially expanding map, which has a neutral direction. The existence of a discrete spectrum together with the result on the spectral gap enables us to prove exponential decay of correlations for these systems. Secondly, these transfer operators appear in the dynamical approach for Selberg zeta functions on convex co-compact surfaces and a famous result of Patterson and Perry connects the spectrum of these transfer operators to the resonances of the Laplace operator on these surfaces.

The article is organized as follows. In §2 we will introduce some basic definitions, state the main theorems and discuss their relation to previously known results in the literature. We also show how these transfer operators arise from open partially expanding maps and we obtain a result on the decay of correlations in such systems. Section 3 is dedicated to the semiclassical construction of the Sobolev spaces as well as to the proof of the existence of the discrete spectrum in these spaces. In §4 we provide a detailed study of the dynamics on the cotangent space that appears in our semiclassical approach and we are led to an important assumption on this dynamics which we call minimal captivity. In particular in §4.3 we show that this 'minimally captive assumption' implies the 'non-local integrability assumption' of Dolgopyat [**16**] and Naud [**40**]. Sections 5 and 6 are then dedicated to the proof of the spectral gap estimate and the fractal Weyl law. Finally, in §7 we provide two important examples, show that they fulfill the minimally captive assumption and compare numerical results with the predictions of our theorems.

2. *Basic definitions and statement of the main results*

2.1. *Iterated function scheme.*

The transfer operator studied in this paper is constructed from a simple model of chaotic dynamics called 'an iterated function scheme, IFS' [**20**, Ch. 9]. We give the definition below and refer to §7 where several standard examples are presented.

Definition 2.1. (An iterated function scheme (IFS)) Let $N \in \mathbb{N}$, $N \geq 1$. Let $I_1, \ldots, I_N \subset \mathbb{R}$ be a finite collection of *disjoint bounded and closed* intervals. Let A be aN $N \times N$ matrix, called the adjacency matrix, with $A_{i,j} \in \{0, 1\}$. We will use the notation $i \rightsquigarrow j$

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if $A_{i,j} = 1$. Assume that for each pair $i, j \in \{1, \ldots, N\}$ such that $i \rightsquigarrow j$, we have a smooth invertible map $\phi_{i,j} : I_i \to \phi_{i,j}(I_i) \subset \mathrm{Int}(I_j)$. Assume that the map $\phi_{i,j}$ is a *strict contraction*, i.e. there exists $0 < \theta < 1$ such that, for every $x \in I_i$,

$$|\phi'_{i,j}(x)| \leq \theta. \tag{2.1}$$

We suppose that different images of the maps $\phi_{i,j}$ do not intersect (this is the 'strong separation condition' in [**19**, p. 35]):

$$\phi_{i,j}(I_i) \cap \phi_{k,l}(I_k) \neq \emptyset \Rightarrow i = k \quad \text{and} \quad j = l. \tag{2.2}$$

Note that in general the derivatives $\phi'_{i,j}(x)$ may be negative. Notice also that we assume smoothness of the maps for our results (see Remark 2.7).

As a first illustration we will give the following example of a truncated Gauss map. Further examples will be given in §7.

Example 2.2. The Gauss map is

$$G : \begin{cases}]0, 1] \to]0, 1[, \\ y \to \left\{ \dfrac{1}{y} \right\}, \end{cases} \tag{2.3}$$

where $\{a\} := a - [a] \in [0, 1[$ denotes the fractional part of $a \in \mathbb{R}$. Let $j \in \mathbb{N} \backslash \{0\}$, and $y \in \mathbb{R}$ such that $1/(j+1) < y \leq 1/j$, then $G(y) = G_j(y) := (1/y) - j$. Notice that $dG/dy < 0$. The inverse map is $y = G_j^{-1}(x) = 1/(x+j)$.

Let $N \geq 1$. We will consider only the first N 'branches' $(G_j)_{j=1,\ldots,N}$. In order to have a well-defined IFS according to Definition 2.1, for $1 \leq i \leq N$, let $\alpha_i := G_i^{-1}(1/(N+1))$, $a_i = 1/(1+i)$, b_i such that $\alpha_i < b_i < 1/i$, and intervals $I_i := [a_i, b_i]$. On these intervals $(I_i)_{i=1\ldots N}$, we define the maps

$$\phi_{i,j}(x) = G_j^{-1}(x) = \frac{1}{x+j}, \quad j = 1, \ldots, N. \tag{2.4}$$

See Figure 1. The adjacency matrix is $A = (A_{i,j})_{i,j}$, the full $N \times N$ matrix with all entries $A_{i,j} = 1$. The values of a_i, b_i are somewhat arbitrary but satisfy hypothesis (2.2). We will show below that the spectral results are independent of the intervals $I_i = [a_i, b_i]$ and depend only on the set of branches, here $\{1, \ldots, N\}$, as soon as the intervals I_i are large enough to contain the trapped set K defined below. See Remark 2.7(3). So we call this model the *truncated Gauss map with N branches*.

In order to shorten the notation we write

$$I := \bigcup_{i=1}^{N} I_i \tag{2.5}$$

and introduce the multivalued map

$$\phi : I \to I, \quad \phi = (\phi_{i,j})_{i,j}.$$

The map ϕ can be iterated and generates a multivalued† map $\phi^n : I \to I$ for $n \geq 1$. From hypothesis (2.2) the inverse map

$$\phi^{-1} : \phi(I) \to I$$

† For any $x \in I$, we have $\sharp\{\phi^n(x)\} \leq N^n$.

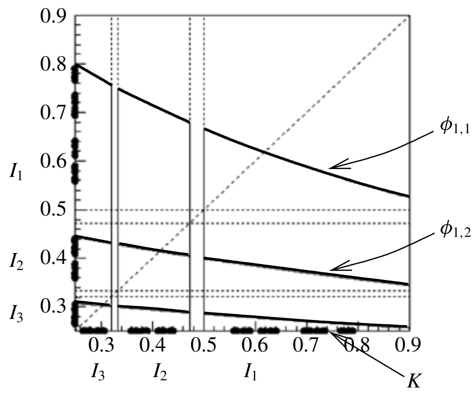

FIGURE 1. The IFS defined from the truncated Gauss map (2.3). Here we have $N = 3$ branches. The maps ϕ: $\phi_{i,j} : I_i \to I_j$, $i, j = 1, \ldots, N$ are contracting and given by $\phi_{i,j}(x) = 1/(x + j)$. The trapped set K defined in (2.7) is an N-adic Cantor set. It is obtained as the limit of the sets $K_0 = (I_1 \cup I_2 \cup \cdots \cup I_N) \supset K_1 = \phi(K_0) \supset K_2 = \phi(K_1) \supset \cdots \supset K$.

is univalued. If we define $K_0 := I$ and

$$K_n := \phi^n(I) \subset I \tag{2.6}$$

for all $n \in \mathbb{N}$ then we have $K_{n+1} \subset K_n$ and we can define the limit set

$$K := \bigcap_{n\in\mathbb{N}} K_n, \tag{2.7}$$

called the *trapped set*. On this set the map

$$\phi^{-1} : K \to K \tag{2.8}$$

is well defined and univalued.

2.2. *Model of dynamics and transfer operators.* From the IFS defined above we first define a dynamical map f that is partially expanding and introduce the transfer operator $\hat{\mathcal{F}}$ associated to it. We first recall the following notation: we denote by $C_0^\infty(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with compact support. If $B \subset \mathbb{R}$ is a compact set, we denote $C_0^\infty(B) \subset C_0^\infty(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with compact support included in B. If not further specified, we will consider complex-valued functions. If we want to specify the values we write, e.g., $C_0^\infty(B; \mathbb{R})$ for real-valued functions.

2.2.1. *Partially expanding maps and transfer operators.* Let ϕ be an iterated function scheme as defined in Definition 2.1. Recall that the map $\phi^{-1} : \phi(I) \to I$ is univalued and expanding. Let $\tau \in C^\infty(\phi(I); \mathbb{R})$ be a smooth, real-valued function called a *roof function*. We define the map

$$f : \begin{cases} \phi(I) \times S^1 \to I \times S^1, \\ (x, y) \to (\phi^{-1}(x), y + \tau(x)), \end{cases} \tag{2.9}$$

with $S^1 := \mathbb{R}/\mathbb{Z}$. Notice that the map f is expanding in the x variable whereas it is neutral in the y variable in the sense that $\partial f/\partial y = 1$. This is called a partially expanding

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map and may serve as a very simple model for the general study of partially hyperbolic dynamics [**44**] such as Axiom A flows. Let $V \in C^{\infty}(\phi(I); \mathbb{C})$, which we call a *potential* function.

Definition 2.3. The transfer operator of the map f with potential V is

$$\hat{\mathcal{F}} : \begin{cases} C_0^{\infty}(I \times S^1) \to C_0^{\infty}(\phi(I) \times S^1), \\ \psi(x, y) \mapsto e^{V(x)} \psi(f(x, y)). \end{cases} \tag{2.10}$$

Notice that $\psi(x, y)$ can be decomposed into Fourier modes in the y direction. For $\nu \in \mathbb{Z}$, a Fourier mode is

$$\psi_{\nu}(x, y) = \varphi(x) e^{i2\pi\nu y}$$

and we have

$$\begin{aligned} (\hat{\mathcal{F}}\psi_{\nu})(x, y) &= e^{V(x)} \psi_{\nu}(f(x, y)) = e^{V(x)} \varphi(\phi^{-1}(x)) e^{i2\pi\nu(y+\tau(x))} \\ &= (\hat{F}_{1/(2\pi\nu)}\varphi)(x) e^{i2\pi\nu y}, \end{aligned}$$

with a *reduced transfer operator* $\hat{F}_{1/(2\pi\nu)} : C_0^{\infty}(I) \to C_0^{\infty}(\phi(I))$ given by

$$(\hat{F}_{1/(2\pi\nu)}\varphi)(x) := e^{V(x)} e^{i2\pi\nu\tau(x)} \varphi(\phi^{-1}(x)). \tag{2.11}$$

So the operator $\hat{\mathcal{F}}$ is the direct sum of operators $\bigoplus_{\nu \in \mathbb{Z}} \hat{F}_{1/(2\pi\nu)}$. In this paper we are interested in the spectral properties of the operators $\hat{F}_{1/(2\pi\nu)}$ in the limit of high Fourier modes $\nu \to \infty$, which corresponds to strong oscillations in the neutral direction y.

2.2.2. *Reduced transfer operators.* Let us consider a direct definition for those reduced transfer operators like (2.11) which does not restrict ν to integers.

Definition 2.4. Let $\tau \in C^{\infty}(\phi(I); \mathbb{R})$ and $V \in C^{\infty}(\phi(I); \mathbb{C})$ be smooth functions called the *roof function* and *potential* function, respectively. Let $\hbar > 0$. We define the *transfer operator*:

$$\hat{F}_{\hbar} : \begin{cases} C_0^{\infty}(I) \to C_0^{\infty}(\phi(I)), \\ \varphi \mapsto \begin{cases} e^{V(x)} e^{i(1/\hbar)\tau(x)} \varphi(\phi^{-1}(x)) & \text{if } x \in \phi(I), \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{2.12}$$

See Figure 2.

Remark 2.5.

- To be more precise, we consider (2.12) as a family of transfer operators depending on the parameter $\hbar > 0$. We will be interested in the spectrum of these operators in the 'semiclassical limit' $\hbar \to 0$.
- For any $\varphi \in C_0^{\infty}(I)$, $n \geq 0$ we have

$$\operatorname{supp}(\hat{F}_{\hbar}^{n}\varphi) \subset K_n, \tag{2.13}$$

  with K_n defined in (2.6).
- In the definition (2.12) we can write $e^{V(x)} e^{i(1/\hbar)\tau(x)} = \exp(i(1/\hbar)\mathcal{V}(x))$ with $\mathcal{V}(x) := \tau(x) + \hbar(-iV(x))$. More generally, we may consider a finite series $\mathcal{V}(x) = \sum_{j=0}^{n} \hbar^j \mathcal{V}_j(x)$ with leading term $\mathcal{V}_0(x) = \tau(x)$ and complex-valued sub-leading terms $\mathcal{V}_j : I \to \mathbb{C}$, $j \geq 1$.

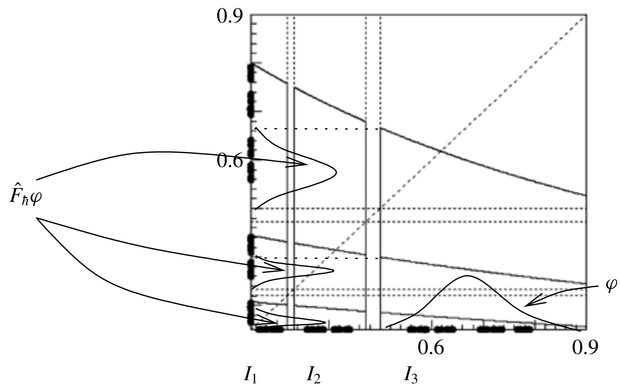

FIGURE 2. Action of the transfer operator $\hat{F}_\hbar$ on a function φ as defined in (2.12). In this schematic figure we have $V = 0$ and $\tau = 0$. In general the factor $e^{V(x)}$ changes the amplitude and $e^{i(1/\hbar)\tau(x)}$ creates some fast oscillations if $\hbar \ll 1$.

2.3. *Discrete spectrum.* The transfer operator $\hat{F}_\hbar$ has been defined on smooth functions $C_0^\infty(I)$ in (2.12). For the proof of the discrete spectrum we will need to extend it to the space of distributions (in §3.1) and it will turn out for technical reasons that we need to compose $\hat{F}_\hbar$ with a cutoff function. We thus introduce a cutoff function $\chi \in C_0^\infty(I)$ such that $\chi(x) = 1$ for every $x \in K_1 = \phi(I)$, i.e. $\chi(\phi_{i,j}(x)) = 1$ for every $x \in I_i$ and j such that $i \rightsquigarrow j$. We denote as $\hat{\chi}$ the multiplication operator by the function χ and define

$$\hat{F}_{\hbar,\chi} := \hat{F}_\hbar \circ \hat{\chi}. \tag{2.14}$$

The first main Theorem 2.6 below states that the transfer operator $\hat{F}_{\hbar,\chi}$ (for any $\hbar$) has a discrete spectrum called 'Ruelle resonances' in ordinary Sobolev spaces with negative order and that the spectrum does not depend on the choice of χ. Recall that, for $m \in \mathbb{R}$, the *Sobolev space* $H^{-m}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ is defined by [**52**, p. 271]

$$H^{-m}(\mathbb{R}) := \langle \hat{\xi} \rangle^m (L^2(\mathbb{R})), \tag{2.15}$$

with the differential operator $\hat{\xi} := -i(d/dx)$ and the notation $\langle x \rangle := (1 + x^2)^{1/2}$. We also recall that a compact operator $\hat{K}$ has a discrete spectrum on $\mathbb{C}\backslash\{0\}$ (i.e. isolated generalized eigenvalues with finite multiplicities) and, if $\hat{R}$ is an operator with norm $\|\hat{R}\| \leq \epsilon$, then $(\hat{K} + \hat{R})$ still has a discrete spectrum on the domain $|z| > \epsilon$, because the essential spectrum is invariant under compact perturbations.

THEOREM 2.6. (Discrete spectrum of resonances) *For any fixed $\hbar$, any $m \in \mathbb{R}$, the transfer operator $\hat{F}_{\hbar,\chi}$ in (2.14) can be extended to a bounded operator on the Sobolev space $H^{-m}(\mathbb{R})$ and can be written as*

$$\hat{F}_{\hbar,\chi} = \hat{K} + \hat{R}, \tag{2.16}$$

where $\hat{K}$ is a compact operator and $\hat{R}$ is such that

$$\|\hat{R}\|_{H^{-m}(\mathbb{R})} \leq r_m \quad \text{with } r_m := c(\theta + \epsilon)^m, \tag{2.17}$$

where $0 < \theta < 1$ is given in (2.1), with any $\epsilon > 0$ (taken so that $\theta + \epsilon < 1$) and c that does not depend on m. This implies that the operator $\hat{F}_{\hbar,\chi}$ has a discrete spectrum

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on the domain $|z| > r_m$ and that $r_m \to 0$ as $m \to +\infty$. These eigenvalues of $\hat{F}_{\hbar,\chi}$ and their eigenspace do not depend on m nor on χ. The support of the eigendistributions is contained in the trapped set K. These discrete eigenvalues are denoted

$$\mathrm{Res}(\hat{F}_\hbar) := \{\lambda_i^\hbar\}_i \subset \mathbb{C}^* \tag{2.18}$$

and are called Ruelle resonances. (See Figure 5 later.)

Remark 2.7.

(1) In this paper we assume for simplicity that the maps $\phi_{i,j}$ are C^∞. This assumption allows us to consider the limit $m \to \infty$ in Theorem 2.6. It may be possible to assume weaker regularity, say C^k. Then Theorem 2.6 would be valid only for $m \le k - 1$.

(2) In the case of an IFS with analytic branches and with analytic potential and roof function, it has even been shown that these transfer operators are trace class in Banach spaces of analytic functions [**33**, **48**]. However, we will prove this result with completely different techniques (microlocal or semiclassical analysis) by the construction of an escape function in the cotangent bundle T^*I. In §7 we will show on different examples that these techniques are also useful for concrete numerical calculations of the spectrum for Ruelle resonances.

(3) The independence of the spectrum of χ implies that the spectral properties of the truncated Gauss map in Example 2.2 do not depend on the explicit choice of boundary points $[a_i, b_i]$.

2.4. *Asymptotic spectral radius.* Next we want to state a result on an asymptotic bound for the spectral radius $r_s(\hat{F}_{\hbar,\chi})$ of the operators $\hat{F}_{\hbar,\chi}$ in the limit $\hbar \to \infty$. A well-known general bound on the spectral radius of transfer operators is given in terms of the topological pressure that we recall now [**49**]. The topological pressure can be defined from the periodic points, which are points $x \in K$ such that $x = \phi^{-n}(x)$, as follows.

Definition 2.8. [**19**, p. 72] The *topological pressure* of a continuous function $\varphi \in C(I)$ is

$$\mathrm{Pr}(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{x=\phi^{-n}(x)} e^{\varphi_n(x)} \right), \tag{2.19}$$

where $\varphi_n(x)$ is the Birkhoff sum of φ along the periodic orbit:

$$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi(\phi^{-k}(x)).$$

It is particularly interesting to consider the topological pressure of the *unstable Jacobian function* which is defined by

$$J(x) := \log\left| \frac{d\phi^{-1}}{dx}(x) \right| \tag{2.20}$$

for $x \in \phi(I)$. From (2.1) we obtain

$$\text{for all } x, \quad J(x) \ge \log \frac{1}{\theta} > 0. \tag{2.21}$$

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One has the general bound, for every $\hbar > 0$ [**49**],

$$\log(r_s(\hat{F}_{\hbar,\chi})) \leq \gamma_{\max} := \Pr(\mathrm{Re}(V) - J). \tag{2.22}$$

In order to give the asymptotic bound (for $\hbar \to 0$) below let us first introduce the so-called *damping function*: $D \in C^\infty(\phi(I))$

$$D := \mathrm{Re}(V) - \tfrac{1}{2}J. \tag{2.23}$$

(This function appears naturally in different models [**25**, **26**].) We can then give the following result on the spectral radius of the operator $\hat{F}_{\hbar,\chi}$ in the limit $\hbar \to \infty$.

THEOREM 2.9. (Asymptotic spectral gap) *If the roof function τ is 'minimally captive' (see Assumption 4.7 for a precise definition) and if m is sufficiently large so that r_m (2.17) fulfills* $\log(r_m) < \gamma_+$, *then the spectral radius $r_s(\hat{F}_{\hbar,\chi})$ of the operators $\hat{F}_{\hbar,\chi} : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ satisfies, in the semiclassical limit $\hbar \to 0$,*

$$\gamma_{\mathrm{asympt}} := \log\left(\limsup_{\hbar\to 0}(r_s(\hat{F}_{\hbar,\chi}))\right) \leq \gamma_+, \tag{2.24}$$

with

$$\gamma_+ := \lim_{n\to\infty}\left(\sup_{x\in\phi^n(I)} \frac{1}{n}\sum_{k=1}^{n} D(\phi^{-k}(x))\right), \tag{2.25}$$

that corresponds to the worst Birkhoff average of the damping function D along the dynamics of the IFS. Moreover, the norm of the resolvent is controlled uniformly with respect to $\hbar$: for any $\rho > e^{\gamma_+}$, there exists $C_\rho > 0$, $\hbar_\rho > 0$ such that, for all $\hbar < \hbar_\rho$, for all $|z| > \rho$ we have

$$\|(z - \hat{F}_\hbar)^{-1}\|_{H^{-m}(\mathbb{R})} \leq C_\rho. \tag{2.26}$$

Remark 2.10.

(1) The limit on the right-hand side of (2.25) exists: the sequence $a_n := \sup_{x\in\phi^n(I)} \sum_{k=1}^{n} D(\phi^{-k}(x))$ is subadditive (i.e. $a_n + a_m \geq a_{n+m}$) and Fekete's lemma guarantees existence of the limit $\gamma_+ = \lim_{n\to\infty} a_n/n$.

(2) From the general bound (2.22) valid every $\hbar$ we have $\gamma_{\mathrm{asympt}} \leq \gamma_{\max} := \Pr(\mathrm{Re}(V) - J)$ and we may define precisely $g_{\mathrm{asympt}} := \gamma_{\max} - \gamma_{\mathrm{asympt}} \geq 0$ to be the *asymptotic spectral gap*. In many cases (but not always), see the concrete examples in §7, we have $\gamma^+ < \gamma_{\max}$. In particular, for closed systems treated in [**21**] and for $V = 0$, one has always $\gamma_{\max} = 0$ and $D = -\frac{1}{2}J < 0$, hence $\gamma_+ < \gamma_{\max}$.

(3) Naud obtained in [**40**] (using techniques of Dolgopyat), an asymptotic bound on the spectral radius under a so-called 'non-local integrability' condition, weaker than the 'minimally captive assumption' and which is discussed below. Translated to our setting he showed the existence of $\epsilon > 0$ such that $\gamma_{\mathrm{asympt}} \leq \gamma_{\max} - \epsilon < \gamma_{\max}$, i.e. that $g_{\mathrm{asympt}} > 0$. For systems where $\gamma^+ < \gamma_{\max}$ the result (2.24) improves this bound as it gives an explicit estimate $g_{\mathrm{asympt.}} > \gamma_{\max} - \gamma_+ > 0$. However, for a general system one may have $\gamma^+ > \gamma_{\max}$ and the result (2.24) gives no asymptotic spectral gap, whereas Naud's result always gives one.

(4) Notice that Theorem 2.9 depends on the roof function τ only implicitly through Assumption 4.7. The value of the upper bound (2.25) does not depend on τ. It is, however, known that such results cannot hold for a general roof function τ (for example it does not hold for roof functions that are cohomologous to a constant, see [**21**, Appendix A]).

(5) Dolgopyat [**16**, **17**] and Naud [**40**] used the so-called 'non-local integrability' (NLI) condition and we will see in §4.3 that our minimally captive assumption implies this non-local integrability condition. The 'minimally captivity' condition which we use arises naturally in the semiclassical approach used in the proof (see §4 for a detailed introduction and definition). It is a similar, but stronger assumption than the condition which appeared in [**21**, **54**] and which was coined 'partially captive' in the latter reference. With only moderate effort Theorem 2.9 could be proven also under the weaker assumption of 'partial captivity' but it will turn out that minimal captivity makes the phase space dynamics on T^*I particularly easy and is essential in the proof of the fractal Weyl law. This is why we decided to put this condition at the center of attention in this article.

(6) In §7 we will illustrate with numerical results on the example of the truncated Gauss map, that the bound (2.24) does not seem to be optimal. Also, other related numerical and physical experiments [**5**] have supported the conjecture that the rigorously known spectral gap estimates are not sharp. The question of finding sharp estimates of asymptotic spectral gaps is an important open question (see e.g. [**41**] for an overview and further references).

2.5. *Expansion of correlations for partially expanding maps.* In this section we present a quite immediate consequence of the existence of an asymptotic spectral radius e^{γ_+} obtained in Theorem 2.9: we obtain a finite expansion for correlation functions $\langle v|\hat{\mathcal{F}}^n u\rangle$ of the extended transfer operator $\hat{\mathcal{F}}$ defined in (2.10).

We first introduce some notation: for a given $\nu \in \mathbb{Z}$, we have seen in Theorem 2.6 that the transfer operator $\hat{F}_{1/(2\pi\nu),\chi}$ has a discrete spectrum of resonances. For $\rho > 0$ such that there is no eigenvalue on the circle $|z| = \rho$, and for any $\nu \in \mathbb{Z}$, we denote by $\Pi_{\rho,\nu}$ the spectral projector of the operator $\hat{F}_{1/(2\pi\nu),\chi}$ on the domain $\{z \in \mathbb{C}, |z| > \rho\}$. These projection operators have obviously finite rank and each commutes with $\hat{F}_{1/(2\pi\nu),\chi}$.

THEOREM 2.11. (Expansion of correlations) *For any $\rho > e^{\gamma_+}$, m large enough (such that $r_m < e^{\gamma_+}$ in Theorem 2.6), there exists $\nu_0 \in \mathbb{N}$ and $C_\rho > 0$ such that for any $u \in H^{-m}(I) \otimes L^2(S^1)$, $v \in H^m(I) \otimes L^2(S^1)$, in the limit $n \to \infty$,*

$$\left| \langle v|\hat{\mathcal{F}}^n u\rangle - \sum_{|\nu|\le\nu_0} \langle v_\nu|(\hat{F}_{1/(2\pi\nu),\chi}\Pi_{\rho,\nu})^n u_\nu\rangle \right| \le C_\rho \rho^n \|u\|^2_{H^{-m}\otimes L^2(S^1)} \|v\|^2_{H^m\otimes L^2(S^1)}. \tag{2.27}$$

Here $u_\nu \in H^{-m}(I)$, $v_\nu \in H^m(I)$ stand for the Fourier components in the S^1 direction of u, v and $\langle v|u\rangle = \int_{I\times S^1} \overline{v}(x)u(x)\,dx$ (extended to distributions).

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Remark 2.12.

- The second term in equation (2.27) is a finite sum and each operator $\hat{F}_{1/(2\pi\nu),\chi}\Pi_{\rho,\nu}$ has finite rank. Using the spectral decomposition of $\hat{F}_\nu$ we get an expansion of the correlation function $\langle v|\hat{\mathcal{F}}^n u\rangle$ with a finite number of terms which involve the leading Ruelle resonances (i.e. those with modulus greater than ρ) and an error term that is $O(\rho^n)$.
- The novelty of this Theorem is that the correlations can be expanded up to this error term $O(\rho^n)$ for any $\rho > e^{\gamma_+}$. As discussed in Remark 2.10, previous results are restricted to error terms $(e^{\Pr(\mathrm{Re}(V)-J)-\varepsilon})^n$ with some non-explicit $\varepsilon > 0$ and more restrictive function spaces for u, v.

Proof. Let $\rho > e^{\gamma_+}$. Recall that $\hbar = 1/2\pi\nu$ and that $|\nu| \to \infty$ corresponds to $\hbar \to 0$. In Theorem 2.6 we have, for $\hbar \to 0$, that $r_s(\hat{F}_{1/(2\pi\nu),\chi}) \le e^{\gamma_+} + o(1)$. Let the value of ν_0 be such that $r_s(\hat{F}_{1/(2\pi\nu),\chi}) < \rho$ for every $|\nu| > \nu_0$. Then, for any $u \in H^{-m}(I) \otimes L^2(S^1)$, $v \in H^m(I) \otimes L^2(S^1)$,

$$\begin{aligned}\langle v|\hat{\mathcal{F}}^n u\rangle = &\sum_{|\nu|\le\nu_0} \langle v_\nu|(\hat{F}_{1/(2\pi\nu),\chi}\Pi_{\rho,\nu})^n u_\nu\rangle \\ &+ \sum_{|\nu|\le\nu_0} \langle v_\nu|(\hat{F}_{1/(2\pi\nu),\chi}(\mathrm{Id} - \Pi_{\rho,\nu}))^n u_\nu\rangle \\ &+ \sum_{|\nu|>\nu_0} \langle v_\nu|\hat{F}^n_{1/(2\pi\nu),\chi} u_\nu\rangle. \end{aligned} \tag{2.28}$$

We have $|\langle v_\nu|\hat{F}^n_{1/(2\pi\nu),\chi} u_\nu\rangle| \le \|v_\nu\|_{H^m}\|u_\nu\|_{H^{-m}}\|\hat{F}^n_{1/(2\pi\nu),\chi}\|_{H^{-m}}$. For $|\nu| \le \nu_0$ we have

$$\|(\hat{F}_{1/(2\pi\nu),\chi}(\mathrm{Id} - \Pi_{\rho,\nu}))^n\| \le C_{\nu_0}\rho^n,$$

where C_{ν_0} depends on ν_0; hence,

$$\left|\sum_{|\nu|\le\nu_0} \langle v_\nu|(\hat{F}_{1/(2\pi\nu),\chi}(\mathrm{Id} - \Pi_{\rho,\nu}))^n u_\nu\rangle\right| \le C_{\nu_0}\rho^n \sum_{|\nu|\le\nu_0} \|v_\nu\|_{H^m}\|u_\nu\|_{H^{-m}}.$$

On the one hand, as a sequence with respect to $\nu \in \mathbb{Z}$, one has $(\|u_\nu\|_{H^m})_\nu, (\|v_\nu\|_{H^{-m}})_\nu \in l^2(\mathbb{Z})$ and $\sum_{\nu\in\mathbb{Z}} \|u_\nu\|^2_{H^m} = \|u\|^2_{H^m\otimes L^2(S^1)}$. Additionally, the resolvent bound (2.26) gives us the existence of a constant C_ρ such that

$$\|(z - \hat{F}_{1/(2\pi\nu)})^{-1}\|_{H^{-m}} \le C_\rho$$

uniformly in $|z| > \rho$ and $|\nu| > \nu_0$. From the Cauchy formula

$$\hat{F}^n_{1/(2\pi\nu),\chi} = \frac{1}{2\pi i}\oint_\gamma z^n(z - \hat{F}_{1/(2\pi\nu),\chi})^{-1}\,dz,$$

where γ is the circle of radius ρ, one deduces that $\|\hat{F}^n_{1/(2\pi\nu),\chi}\|_{H^{-m}} \le C_\rho\rho^n$. So

$$\begin{aligned}\left|\sum_{|\nu|>\nu_0} \langle v_\nu|\hat{F}^n_{1/(2\pi\nu),\chi} u_\nu\rangle\right| &\le C_\rho\rho^n \sum_{|\nu|>\nu_0} \|v_\nu\|_{H^m}\|u_\nu\|_{H^{-m}} \\ &\le C_\rho\rho^n\|u\|^2_{H^m\otimes L^2(S^1)}\|v\|^2_{H^m\otimes L^2(S^1)}.\end{aligned}$$

Then (2.28) gives (2.27). □

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2.6. *Upper bound for the density of resonances (fractal Weyl law).* We will, finally, formulate a fractal Weyl law on the number of Ruelle resonances and therefore introduce the following definition of fractal dimension.

Definition 2.13. ([**38**, p. 76], [**19**, p. 20]) If $B \subset \mathbb{R}^d$ is a non-empty bounded set, its *upper Minkowski dimension* (or box dimension) is

$$\dim_M B := d - \operatorname{codim}_M B, \tag{2.29}$$

with

$$\operatorname{codim}_M B := \sup\left\{ s \in \mathbb{R} \mid \limsup_{\delta \downarrow 0} \delta^{-s} \cdot \operatorname{Leb}(B_\delta) < +\infty \right\}, \tag{2.30}$$

where $B_\delta := \{x \in \mathbb{R}^d, \operatorname{dist}(x, B) \leq \delta\}$ and $\operatorname{Leb}(\cdot)$ is the Lebesgue measure.

Remark 2.14. In general,

$$\limsup_{\delta \downarrow 0} \delta^{-\operatorname{codim}_M B} \cdot \operatorname{Leb}(B_\delta) < +\infty \tag{2.31}$$

does not hold, but if it does, B is said to be of *pure dimension*†. It is known that the trapped set K defined in (2.7) has pure dimension and that the above definition of Minkowski dimension coincides with the more usual *Hausdorff dimension* of K [**19**, p. 68]:

$$\dim_M K = \dim_H K \in [0, 1[. \tag{2.32}$$

Using the following lemma, the topological pressure defined in (2.19) provides an efficient way to calculate the Hausdorff dimension of the trapped set numerically (see Figure 3 for an illustration for the example of the truncated Gauss map).

LEMMA 2.15. [**19**, p. 77] *If $\beta > 0$ is a real parameter and J the unstable Jacobian defined in (2.20), then* $\Pr(-\beta J)$ *is continuous and strictly decreasing as a function of β and its unique zero is given by $\beta = \dim_H K$.*

We can finally formulate the following theorem.

THEOREM 2.16. (Fractal Weyl upper bound) *Suppose that the Assumption 4.7 of minimal captivity holds and that the adjacency matrix A is symmetric. For any $\varepsilon > 0$, any $\eta > 0$, we have, for $\hbar \to 0$,*

$$\sharp\{\lambda_i^\hbar \in \operatorname{Res}(\hat{F}_\hbar) \mid |\lambda_i^\hbar| \geq \varepsilon\} = \mathcal{O}(\hbar^{-\dim_H(K) - \eta}). \tag{2.33}$$

The first result of a fractal Weyl law upper bound has been obtained by Sjöstrand [**51**] for a wide class of semiclassical operators with analytic coefficients. This pioneering work has also triggered theoretical and experimental studies in physics [**35**, **36**, **46**, **50**] as well as an extension of this theorem to various other settings like convex co-compact surfaces [**30**, **56**], manifolds with hyperbolic ends [**14**] and the scattering at several convex bodies [**42**]. To our knowledge there are not yet any rigorous results of a fractal Weyl law upper bound for classical Ruelle resonances; however, in the physics literature the existence of such laws has been observed [**18**]. The minimal captivity assumption, however, allows us to

† See [**51**] for comments and further references.

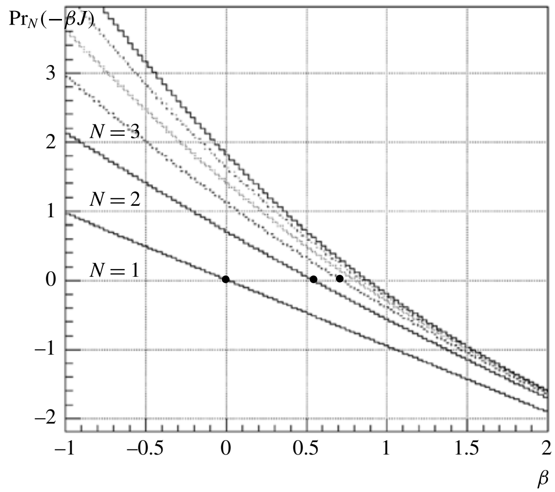

FIGURE 3. $\mathrm{Pr}_N(-\beta J)$ is the topological pressure (2.19) for the truncated Gauss map (Example 2.2), with $N = 1, 2, 3\ldots$ being the number of branches. The black points mark the zero of $\mathrm{Pr}_N(-\beta J) = 0$ giving the fractal dimension of the trapped set K_N for each value of N: $\dim_H K_1 = 0$, $\dim_H K_2 = 0.531\ldots$, $\dim_H K_3 = 0.705\ldots$, and $\dim_H K_N \to_{N\to\infty} 1$.

interpret the transfer operator as a quantization of a bijective map (see [**25**] for discussions about this interpretation) and it should also be possible to obtain this result using the recent work of Nonnenmacher *et al* [**42**]. We will, however, provide an independent proof which is directly based on the semiclassical approach for the transfer operators, by further refining the escape function that appears in the proofs of Theorems 2.6 and 2.9.

The upper bound on the exponent in terms of the Hausdorff dimension is conjectured to be sharp [**36**], meaning that it is also a lower bound (see also [**41**] for an overview and further references). This conjecture has been supported by several numerical experiments, for example for quantum n-disk systems [**36**] or convex co-compact surfaces [**8**]. Also in the case of iterated function schemes, the bound seems to be also a lower bound as suggested by the numerical results shown in Figure 6 later.

3. *Proof of Theorem 2.6 about the discrete spectrum*

For this proof we follow closely the proof† of Theorem 2 in [**21**], which uses semiclassical analysis. However, we have to deal with an additional difficulty associated with the 'openness' of the system. This will be taken into account by the introduction of the cutoff function χ (cf. (2.14)). In this section, §3, the parameter $\hbar$ is fixed.

Here is the strategy. We first show in §3.1 that the transfer operator $\hat{F}_{\hbar,\chi}$ has a well-defined and unique extension to distributions on $\mathbb{R}$. Then in §3.2 we explain that the transfer operator is a Fourier integral operator and compute its associated symplectic map on the cotangent space $\mathfrak{F} : T^*I \to T^*I$. We observe that under this map $\mathfrak{F}$, the trajectory of a point $(x, \xi) \in T^*I$ escape towards infinity if $|\xi| > 0$. In §3.3 we construct an escape function (or Lyapounov function) $A_m(x, \xi)$ that decreases strictly along the trajectories.

† See also Theorem 4 in [**23**] which concerns hyperbolic maps and anisotropic Sobolev spaces.

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We consider the corresponding pseudodifferential operator $\hat{A}_m := \mathrm{Op}(A_m)$ and in §3.4 we show that the conjugated operator $\hat{A}_m \circ \hat{F}_{\hbar,\chi} \circ \hat{A}_m^{-1}$ has a discrete spectrum in $L^2(\mathbb{R})$. Equivalently the transfer operator $\hat{F}_{\hbar,\chi}$ has a discrete spectrum in the Sobolev space $\hat{A}_m^{-1}(L^2(\mathbb{R})) = H^{-m}(\mathbb{R})$.

3.1. *Extension of the transfer operator to distributions on* $\mathbb{R}$. Recall that in (2.14) we introduced the cutoff function $\chi \in C_0^\infty(I)$, with $\chi(x) = 1$ for every $x \in K_1 = \phi(I)$, as well as the truncated transfer operators

$$\hat{F}_{\hbar,\chi} := \hat{F}_\hbar \circ \hat{\chi}.$$

Note that for any $\varphi \in C_0^\infty(K_1)$ we have $\hat{\chi}\varphi = \varphi$, hence $(\hat{F}_\hbar \hat{\chi})\varphi = \hat{F}_\hbar \varphi$. One has $\hat{\chi} : C_0^\infty(\mathbb{R}) \to C_0^\infty(I)$, hence $\hat{F}_{\hbar,\chi}$ is defined on $C_0^\infty(\mathbb{R})$.

The formal adjoint operator $\hat{F}^*_{\hbar,\chi} : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})$ is defined by

$$\langle \varphi | \hat{F}^*_{\hbar,\chi} \psi \rangle = \langle \hat{F}_{\hbar,\chi} \varphi | \psi \rangle \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}),\ \psi \in C_0^\infty(\mathbb{R}), \tag{3.1}$$

with the L^2-scalar product $\langle u|v\rangle := \int \overline{u}(x) v(x)\, dx$. Note that for any test function $\varphi \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\varphi) \cap I = \emptyset$ we have $\hat{F}_{\hbar,\chi}\varphi = 0$, which directly implies that $\hat{F}^*_{\hbar,\chi} : C_0^\infty(\mathbb{R}) \to C_0^\infty(I)$.

LEMMA 3.1. *Let* $\psi \in C_0^\infty(\mathbb{R})$ *and* $y \in I_i$. *Then the adjoint operator* $\hat{F}^*_{\hbar,\chi} : C_0^\infty(\mathbb{R}) \to C^\infty(I)$ *is given by*

$$(\hat{F}^*_{\hbar,\chi}\psi)(y) = \chi(y) \sum_{j \,\mathrm{s.t.}\, i \rightsquigarrow j} |\phi'_{i,j}(y)| e^{\overline{V(\phi_{i,j}(y))}} e^{-(i/\hbar)\tau(\phi_{i,j}(y))} \psi(\phi_{i,j}(y)). \tag{3.2}$$

Proof. Using the definition (3.2), we calculate

$$\begin{aligned}
\langle \varphi | \hat{F}^*_{\hbar,\chi}\psi \rangle &= \int_I \overline{\varphi}(y) (\hat{F}^*_{\hbar,\chi}\psi)(y)\, dy \\
&= \sum_i \int_{I_i} \overline{\varphi}(y) (\hat{F}^*_{\hbar,\chi}\psi)(y)\, dy \\
&= \sum_i \sum_{j \,\mathrm{s.t.}\, i \rightsquigarrow j} \int_{I_i} \overline{\varphi}(y) \chi(y) |\phi'_{i,j}(y)| e^{\overline{V(\phi_{i,j}(y))}} e^{-(i/\hbar)\tau(\phi_{i,j}(y))} \psi(\phi_{i,j}(y))\, dy.
\end{aligned}$$

Now we can perform a change of variables $x = \phi_{i,j}(y)$ in each of the integrals and obtain

$$\begin{aligned}
\langle \varphi | \hat{F}^*_{\hbar,\chi}\psi \rangle &= \sum_i \sum_{j \,\mathrm{s.t.}\, i \rightsquigarrow j} \int_{\phi_{i,j}(I_i)} \overline{e^{V(x)} e^{i(1/h)\tau(x)} \varphi(\phi^{-1}(x)) \chi(\phi^{-1}(x))} \psi_j(x)\, dx \\
&= \int_I \overline{\hat{F}_{\hbar,\chi}\varphi(x)} \psi(x)\, dx \\
&= \langle \hat{F}_{\hbar,\chi}\varphi | \psi \rangle.
\end{aligned}$$

□

PROPOSITION 3.2. *By duality the transfer operator, (2.14) extends to distributions*

$$\begin{aligned}
\hat{F}_{\hbar,\chi} &: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}), \\
\hat{F}^*_{\hbar,\chi} &: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}).
\end{aligned} \tag{3.3}$$

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Similarly to (2.13) we have that, for any $n \geq 1$, any $\alpha \in \mathcal{D}'(\mathbb{R})$,

$$\mathrm{supp}(\hat{F}^n_{\hbar,\chi}\alpha) \subset K_n, \tag{3.4}$$

with K_n defined in (2.6).

Proof. As $\hat{F}^*_{\hbar,\chi}$ is continuous on the space of test functions $C^\infty_0(\mathbb{R})$ the extension can directly be defined by

$$\hat{F}_{\hbar,\chi}(\alpha)(\psi) = \alpha(\overline{\hat{F}^*_{\hbar,\chi}\overline{\psi}}), \quad \alpha \in \mathcal{D}'(\mathbb{R}), \ \psi \in C^\infty_0(\mathbb{R}). \tag{3.5}$$

If $\psi(\phi_{i,j}(y)) = 0$ for all $i \rightsquigarrow j$ and all $y \in I_i$, then (3.2) shows that $\hat{F}^*_{\hbar,\chi}\psi \equiv 0$. More generally, let $\psi \in C^\infty_0(\mathbb{R})$ with $\mathrm{supp}(\psi) \cap K_n = \emptyset$ with $n \geq 1$ and K_n defined in (2.6). Then

$$(\hat{F}^*_{\hbar,\chi})^n\psi \equiv 0. \tag{3.6}$$

For any $\alpha \in \mathcal{D}'(\mathbb{R})$, we deduce that $(\hat{F}^n_{\hbar,\chi}\alpha)(\overline{\psi}) = \alpha(\overline{(\hat{F}^*_{\hbar,\chi})^n\psi}) = 0$. By definition, this means that $\mathrm{supp}(\hat{F}^n_{\hbar,\chi}\alpha) \subset K_n$. □

Remark 3.3.

- Without the cutoff function χ the image of $\hat{F}^*_\hbar$ may not be continuous on the boundary of I and the extension to distribution space in Proposition 3.2 would not have been possible.
- Another more general possibility would have been to consider $\chi \in C^\infty_0(I)$ such that $0 < \chi(x)$ for $x \in \mathrm{Int}(I)$ (without the assumption that $\chi \equiv 1$ on K_1) and define

$$\hat{F}_{\hbar,\chi} := \hat{\chi}^{-1}\hat{F}_\hbar\hat{\chi} : C^\infty_0(\mathbb{R}) \to C^\infty_0(\mathbb{R}), \tag{3.7}$$

  which is well defined since $\mathrm{supp}(\hat{F}_\hbar\hat{\chi}\varphi) \subset \mathrm{Int}(I)$ where χ does not vanish. This more general definition (3.7) may be more useful in some cases, e.g. we use it in numerical computation. We recover the previous definition (2.14) if we make the additional assumption that $\chi \equiv 1$ on K_1.

3.2. *Dynamics on the cotangent space T^*I.* The remark of fundamental importance given in Proposition 3.4 below is that each operator $\hat{F}_{\hbar,\chi}$, although it is a simple composition operator, can be considered as a 'Fourier integral operator' whose associate canonical map $\mathfrak{F}$ is the map $\phi : I \to I$ lifted on the cotangent space T^*I. The definition of a Fourier integral operator and its associated canonical map will be given in a more general context at the beginning of §4 and the following proposition can be considered as a particular case of Lemma 4.2. Then, according to the 'semiclassical approach' we know that in order to study the spectral properties of the transfer operator $\hat{F}_{\hbar,\chi}$ we have first to study the dynamics of its canonical map $\mathfrak{F} : T^*I \to T^*I$. It is not necessary to known what a Fourier integral operator is to read the main part of this paper.

PROPOSITION 3.4. *Considering $\hbar > 0$ fixed, the transfer operator $\hat{F}_{\hbar,\chi}$ restricted to $C^\infty_I(\mathbb{R})$ is a Fourier integral operator (FIO). Its canonical transform is a multivalued symplectic map $\mathfrak{F} : T^*I \to T^*I$ on the cotangent space $T^*I \equiv I \times \mathbb{R}$ given by*

$$\mathfrak{F} : \begin{cases} T^*I \to T^*I, \\ (x,\xi) \mapsto \{\mathfrak{F}_{i,j}(x,\xi) \text{ with } i, j \text{ s.t. } x \in I_i, i \rightsquigarrow j\}, \end{cases}$$

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with

$$\mathfrak{F}_{i,j}:\begin{cases}x'=\phi_{i,j}(x),\\ \xi'=\dfrac{1}{\phi'_{i,j}(x)}\xi.\end{cases} \tag{3.8}$$

Remarks.

- For the proof we refer to the proof of Lemma 4.2 with the following remark. Here $\hbar$ is fixed (it is not a semiclassical parameter), hence the term $e^{i(1/\hbar)\tau(x')}$ in (2.12) contributes to the amplitude and not to the phase function. That explains why the canonical map $\mathfrak{F}$ differs from the canonical map F which will be introduced in (4.5).
- In [**21**, §3.2] we explain the action of the FIO $\hat{F}_{\hbar,\chi}$ in terms of wave packets and the clear relation with the symplectic map $\mathfrak{F}$.
- For short, we can write

$$\mathfrak{F}:\begin{cases}T^*I\to T^*I,\\ (x,\xi)\mapsto\left(\phi(x),\dfrac{1}{\phi'(x)}\xi\right).\end{cases} \tag{3.9}$$

- Observe from (3.9) that the dynamics of the map $\mathfrak{F}$ on T^*I has a quite simple property: the zero section $\{(x,\xi)\in I\times\mathbb{R},\ \xi=0\}$ is globally invariant and any other point (x,ξ) with $\xi\neq 0$ escapes towards infinity ($\xi\to\pm\infty$) in a controlled manner, because $|\phi'_{i,j}(x)|<\theta<1$, with θ given in (2.1), hence

$$|\xi'|\geq\frac{1}{\theta}|\xi|. \tag{3.10}$$

- Due to hypothesis (2.2) the map $\phi_{i,j}^{-1}$ is univalued (when it is defined). Therefore, the map $\mathfrak{F}^{-1}$ is also univalued and one has

$$\mathfrak{F}^{-1}\circ\mathfrak{F}=\mathrm{Id}_{T^*I}. \tag{3.11}$$

3.3. *The escape function.*

Definition 3.5. [**53**, p. 2] For $m\in\mathbb{R}$, the *class of symbols* $S^{-m}(T^*\mathbb{R})$, with *order* m, is the set of functions on the cotangent space $A\in C^\infty(T^*\mathbb{R})$ such that, for any $\alpha,\beta\in\mathbb{N}$, there exists $C_{\alpha,\beta}>0$ such that

$$\text{for all } (x,\xi)\in T^*\mathbb{R},\quad |\partial_x^\alpha\partial_\xi^\beta A(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{-m-|\beta|}\quad\text{with } \langle\xi\rangle=(1+\xi^2)^{1/2}. \tag{3.12}$$

LEMMA 3.6. *Let $m>0$ and let*

$$A_m(x,\xi):=\langle\xi\rangle^{-m}\in S^{-m}(T^*\mathbb{R}).$$

We have

$$\text{for all } R>0, \text{for all } |\xi|>R, \text{for all } i\rightsquigarrow j, \text{for all } x\in I_i,\quad \frac{A_m(\mathfrak{F}_{i,j}(x,\xi))}{A_m(x,\xi)}\leq C^m, \tag{3.13}$$

with $C=\sqrt{(R^2+1)/(R^2/\theta^2+1)}<1$. Equation (3.13) shows that A_m decreases strictly along the trajectories of $\mathfrak{F}$ outside the zero section. We say that A_m is an escape function.

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Proof. From equations (3.8) and (3.10) we have

$$\frac{A_m(\mathfrak{F}_{i,j}(x,\xi))}{A_m(x,\xi)} = \frac{(1+\xi^2)^{m/2}}{(1+(\xi')^2)^{m/2}} \le \frac{(1+\xi^2)^{m/2}}{(1+\xi^2/\theta^2)^{m/2}} \le \left(\frac{1+R^2}{1+R^2/\theta^2}\right)^{m/2} = C^m.$$

The last inequality is because the function decreases with $|\xi|$. □

Using the standard quantization rule [**53**, p. 2] the symbol A_m can be quantized into a pseudodifferential operator (PDO) $\hat{A}_m$, which is self-adjoint and invertible on $C_0^\infty(\mathbb{R})$:

$$(\hat{A}_m\varphi)(x) = \frac{1}{2\pi}\int A_m(x,\xi)e^{i(x-y)\xi}\varphi(y)\,dy\,d\xi. \tag{3.14}$$

Conversely, A_m is called the symbol of the PDO $\hat{A}_m$. In our simple case, this is very explicit: in Fourier space, $\hat{A}_m$ is simply the multiplication by $\langle\xi\rangle^m$. Its inverse $\hat{A}_m^{-1}$ is the multiplication by $\langle\xi\rangle^{-m}$.

3.4. *Use of the Egorov theorem.* Let

$$\hat{Q}_m := \hat{A}_m\hat{F}_{\hbar,\chi}\hat{A}_m^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

which is unitarily equivalent to $\hat{F}_{\hbar,\chi} : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ (from the definition of $H^{-m}(\mathbb{R})$, equation (2.15)). This is expressed by the following commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\hat{Q}_m} & L^2(\mathbb{R}) \\ \big\downarrow \hat{A}_m^{-1} & & \big\downarrow \hat{A}_m^{-1} \\ H^{-m}(\mathbb{R}) & \xrightarrow{\hat{F}_{\hbar,\chi}} & H^{-m}(\mathbb{R}) \end{array} \tag{3.15}$$

We will therefore study the operator $\hat{Q}_m$ on $L^2(\mathbb{R})$. Notice that $\hat{Q}_m$ is defined *a priori* on a dense domain $C_0^\infty(\mathbb{R})$. Define

$$\hat{P} := \hat{Q}_m^*\hat{Q}_m = \hat{A}_m^{-1}(\hat{F}_{\hbar,\chi}^*\hat{A}_m^2\hat{F}_{\hbar,\chi})\hat{A}_m^{-1} = \hat{A}_m^{-1}\hat{B}\hat{A}_m^{-1}, \tag{3.16}$$

with

$$\hat{B} := \hat{F}_{\hbar,\chi}^*\hat{A}_m^2\hat{F}_{\hbar,\chi} = \hat{\chi}\hat{F}_\hbar{}^*\hat{A}_m^2\hat{F}_\hbar\hat{\chi}. \tag{3.17}$$

Now, the crucial step in the proof is to use the Egorov theorem.

LEMMA 3.7. (Egorov theorem) *$\hat{B}$ defined in (3.17) is a pseudodifferential operator with symbol in $S^{-2m}(T^*\mathbb{R})$ given by*

$$B(x,\xi) = \left(\chi^2(x)\sum_{j\,\text{s.t.}\,i\rightsquigarrow j}|\phi'_{i,j}(x)|e^{2\mathrm{Re}(V(\phi_{i,j}(x)))}A_m^2(\mathfrak{F}_{i,j}(x,\xi))\right) + R, \tag{3.18}$$

where $R \in S^{-2m-1}(T^*\mathbb{R})$ has a lower order, $x \in I_i$, $\xi \in \mathbb{R}$.

Proof. $\hat{F}_\hbar$ and $\hat{F}_\hbar^*$ are FIOs whose canonical maps are respectively $\mathfrak{F}$ and $\mathfrak{F}^{-1}$. The PDO $\hat{A}_m$ can also be considered as an FIO whose canonical map is the identity. By composition we deduce that $\hat{B} = \hat{\chi}\hat{F}_\hbar^*\hat{A}_m^2\hat{F}_\hbar\hat{\chi}$ is an FIO whose canonical map is the identity since $\mathfrak{F}^{-1}\circ\mathfrak{F} = I$ from (3.11). Therefore $\hat{B}$ is a PDO. Using the precise expressions for $\hat{F}_\hbar$ (equation (2.12)) and $\hat{F}_\hbar^*$ (equation (3.2)), as well as the behavior of PDOs under a change of variables (see [**29**, Theorem 3.9]), we obtain that the principal symbol of $\hat{B}$ is the first term of (3.18). □

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Remark. Contrary to (3.17), $\hat{F}_\hbar \hat{A}_m \hat{F}_\hbar^*$ is not a PDO, but an FIO whose canonical map $\mathfrak{F} \circ \mathfrak{F}^{-1}$ is multivalued.

Now by the *theorem of composition of PDOs* [**53**, p. 11], equations (3.16) and (3.18) imply that $\hat{P}$ is a PDO with symbol in $S^0(\mathbb{R})$ and for $x \in I_i$, $\xi \in \mathbb{R}$ the principal symbol is given by

$$P(x,\xi) = \frac{B(x,\xi)}{A_m^2(x,\xi)} = \left(\chi^2(x) \sum_{j \text{s.t.} i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\text{Re}(V(\phi_{i,j}(x)))} \frac{A_m^2(\mathfrak{F}_{i,j}(x,\xi))}{A_m^2(x,\xi)}\right). \quad (3.19)$$

The estimate (3.13) gives the following upper bound for any $R > 0$, $x \in I$ and $|\xi| > R$:

$$|P(x,\xi)| \le \chi^2(x) C^{2m} \sum_{j, i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\text{Re}(V(\phi_{i,j}(x)))} \le C^{2m} N \theta e^{2V_{\max}},$$

with $V_{\max} = \max_{x \in I} \text{Re}(V(x))$.

We apply the L^2*-continuity theorem for a PDO* to $\hat{P}$ as given† in [**29**, Theorem 4.5 p. 42]. The result is that, for any $\varepsilon > 0$,

$$\hat{P} = \hat{k}_\varepsilon + \hat{p}_\varepsilon$$

with $\hat{k}_\varepsilon$ a smoothing operator (hence compact) and $\|\hat{p}_\varepsilon\| \le C^{2m} N \theta e^{2V_{\max}} + \varepsilon$.

If $\hat{Q}_m = \hat{U} |\hat{Q}_m|$ is the polar decomposition of $\hat{Q}_m$, with $\hat{U}$ unitary, then from (3.16), $\hat{P} = |\hat{Q}_m|^2$, hence $|\hat{Q}_m| = \sqrt{\hat{P}}$ and the spectral theorem [**53**, p. 75] gives that $|\hat{Q}_m|$ has a similar decomposition

$$|\hat{Q}_m| = \hat{k}'_\varepsilon + \hat{q}_\varepsilon,$$

with $\hat{k}'_\varepsilon$ compact and $\|\hat{q}_\varepsilon\| \le \sqrt{C^{2m} N \theta e^{2V_{\max}}} + \varepsilon$, with any $\varepsilon > 0$. Since $\|\hat{U}\| = 1$ we deduce a similar decomposition for $\hat{Q}_m = \hat{U} |\hat{Q}_m| : L^2(I) \to L^2(I)$, i.e. $\hat{Q}_m = \hat{k}''_\varepsilon + \hat{q}'_\varepsilon$. We also use the fact that $C \to \theta$ as $R \to \infty$ in (3.13) to get that $\|\hat{q}'_\varepsilon\| \le r_m := c(\theta + \epsilon)^m$, with c independent of m and any $\epsilon > 0$. Equivalently, from the diagram (3.15), this gives that $\hat{F}_{\hbar,\chi} : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ can be written $\hat{F}_{\hbar,\chi} = \hat{K} + \hat{R}$, with $\hat{K}$ compact and $\|\hat{R}\| \le r_m$. We have obtained (2.16) and (2.17).

The fact that the eigenvalues λ_i and their generalized eigenspaces do not depend on the choice of space $H^{-m}(\mathbb{R})$ is due to density of $C_0^\infty(\mathbb{R})$ in Sobolev spaces. We refer to the argument given in the proof of corollary 1 in [**23**].

Finally, if φ is an eigendistribution of $\hat{F}_{\hbar,\chi}$, i.e. $\hat{F}_{\hbar,\chi}\varphi = \lambda\varphi$ with $\lambda \ne 0$, we deduce that $\varphi = (1/\lambda^n)\hat{F}^n_{\hbar,\chi}\varphi$ for any $n \ge 1$, and (3.4) implies that $\text{supp}(\varphi) \subset K = \bigcap_{n \in \mathbb{N}} K_n$. On the trapped set we have $\chi = 1$, hence the eigendistribution and eigenvalues of $\hat{F}_{\hbar,\chi}$ do not depend on χ. This finishes the proof of Theorem 2.6.

† Actually, we cannot apply directly the L^2-continuity theorem [**29**, Theorem 4.5, p. 42] for a PDO to $\hat{P}$ because $\hat{P}$ does no t have a compactly supported Schwartz kernel. However, $\hat{B}$ obviously has a compactly supported Schwartz kernel due to the presence of $\hat{\chi}$ in equation (3.17). The trick is to approximate $\hat{A}_m^{-1}$ by a properly supported operator Λ_m as it is done in [**29**, p. 45] and then apply the L^2-continuity theorem to $\hat{\Lambda}_m \hat{B} \hat{\Lambda}_m$.

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4. *Dynamics of the canonical map $F : T^*I \to T^*I$*

In Theorem 2.6 the operator $\hat{F}_{\hbar,\chi}$ is considered for a fixed $\hbar$. On the contrary, in Theorems 2.9 and 2.16 they are considered as a family of $\hbar$-FIOs and we give partial results on the distribution of their Ruelle resonances in the semiclassical limit $\hbar \to 0$. As a consequence, the oscillations of the phase multiplication by $e^{(i/\hbar)\tau(x)}$ are not uniformly bounded anymore and contrary to Proposition 3.4 this multiplication operator is not a pseudodifferential operator anymore but contributes to the phase space dynamics of the canonical map. In this section we will introduce this canonical map and study its dynamics in phase space, which becomes significantly more complicated compared to the map $\mathfrak{F}$ which appeared in Proposition 3.4. In §4.1 we will introduce the trapped set in phase space and will naturally be led to the minimally captive property. Then we will introduce the symbolic dynamics in §4.2, which nicely describes the dynamics in the cotangent space and which is a central technical ingredient in the proofs of Theorems 2.9 and 2.16. Finally, in §4.4 we will see that under the assumption of minimal captivity, the trapped set in phase space also has a fractal structure and that its Hausdorff dimension equals twice the Hausdorff dimension of the trapped set K of the underlying IFS

We first recall that, on $\mathbb{R}^n$, an $\hbar$-FIO is a linear operator $\hat{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ of the form [**57**, Theorem 10.4]

$$(\hat{F}\varphi)(x') = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{(i/\hbar)(\Phi(x',\xi)-x\cdot\xi)} b(x', \xi; \hbar)\varphi(x)\, dx\, d\xi, \tag{4.1}$$

where $\Phi(x', \xi)$ is real-valued and called the 'phase function' and $b(x', \xi; \hbar)$ is the amplitude. The Fourier integral operator $\hat{F}$ has an associated canonical map, which is the symplectic map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $(x', \xi') = F(x, \xi)$ given by [**57**, Lemma 10.5]

$$\xi' = (\partial_{x'}\Phi)(x', \xi), \qquad x = (\partial_\xi \Phi)(x', \xi). \tag{4.2}$$

Remark 4.1. As explained in [**12**] one interpretation of the canonical map is the following. Since we are interested in the situation of high frequencies we write $\xi/\hbar$ for the frequency with $\hbar \ll 1$. In particular, the $\hbar$-Fourier transform of a function u is $(\mathcal{F}\varphi)(\xi) := (1/(2\pi\hbar)^{n/2}) \int_{\mathbb{R}^n} e^{-i(\xi/\hbar)\cdot x}\varphi(x)\, dx$. If an $\hbar$-family of functions $\varphi_\hbar$ is *micro-localized* at point $x \in \mathbb{R}^n$ and its $\hbar$-Fourier transform is *micro-localized* at point $\xi \in T_x^*\mathbb{R}^n$, which means that these functions decay fast outside these points as $\hbar \to 0$, then the operator $\hat{F}$ transforms these functions $\varphi_\hbar$ to functions $\hat{F}\varphi_\hbar$ micro-localized at another point $(x', \xi') = F(x, \xi) \in T^*\mathbb{R}^n$, where F is the associated canonical map.

According to this previous definition, we give now the canonical map F for the family of $\hbar$-FIOs $(\hat{F}_\hbar)_\hbar$ that concern us and that were defined in (2.12).

LEMMA 4.2. *The family of operators $(\hat{F}_\hbar)_\hbar$ restricted to $C_0^\infty(I)$ is an $\hbar$-FIO. Its canonical map is a multivalued symplectic map $F : T^*I \to T^*I$ (with $T^*I \cong I \times \mathbb{R}$) given by*

$$F : \begin{cases} T^*I \to T^*I, \\ (x, \xi) \mapsto \{F_{i,j}(x, \xi) \text{ with } i, j \text{ s.t. } x \in I_i,\ i \rightsquigarrow j\}, \end{cases} \tag{4.3}$$

with

$$F_{i,j} : \begin{cases} x' = \phi_{i,j}(x), \\ \xi' = \dfrac{1}{\phi'_{i,j}(x)}\xi + \dfrac{d\tau}{dx}(x'). \end{cases} \tag{4.4}$$

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Proof. From (2.12),

$$(\hat{F}_{i,j}\varphi)(x') = e^{V(x')}e^{i(1/\hbar)\tau(x')}\varphi(\phi_{i,j}^{-1}(x')) = (\hat{F}_2 \circ \hat{F}_1\varphi)(x')$$

that we have decomposed into a first operator (we will use $\delta(x) = (1/(2\pi\hbar)) \int_{\mathbb{R}} e^{(i/\hbar)x\cdot\xi}\, d\xi$)

$$\begin{aligned}(\hat{F}_1\varphi)(x') := \varphi(\phi_{i,j}^{-1}(x')) &= \int_{\mathbb{R}} \delta(\phi_{i,j}^{-1}(x') - x)\varphi(x)\, dx \\ &= \frac{1}{(2\pi\hbar)} \int_{\mathbb{R}^2} e^{(i/\hbar)(\phi_{i,j}^{-1}(x')\cdot\xi - x\cdot\xi)}\varphi(x)\, dx\, d\xi,\end{aligned}$$

which shows from (4.1) that $\hat{F}_1$ is an FIO with amplitude $b = 1$ and phase function $\Phi(x', \xi) = \phi_{i,j}^{-1}(x') \cdot \xi$. Its canonical map is then $(x', \xi') = F_1(x, \xi)$, given from (4.2) by

$$\xi' = (\partial_{x'}\Phi)(x', \xi) = \frac{1}{\phi'_{i,j}(x)}\xi, \quad x = (\partial_\xi\Phi)(x', \xi) = \phi_{i,j}^{-1}(x').$$

Similarly, for the second operator we write

$$\begin{aligned}(\hat{F}_2\varphi)(x') := e^{V(x')}e^{i\tau(x')/\hbar}\varphi(x') &= \int_{\mathbb{R}} e^{V(x')}e^{i\tau(x')/\hbar}\delta(x' - x)\varphi(x)\, dx \\ &= \frac{1}{(2\pi\hbar)} \int_{\mathbb{R}^2} e^{(i/\hbar)(x'\cdot\xi + \tau(x') - x\cdot\xi)}e^{V(x')}\varphi(x)\, dx\, d\xi,\end{aligned}$$

which shows from (4.1) that $\hat{F}_2$ is an FIO with amplitude $b(x', \xi; \hbar) = e^{V(x')}$ and phase function $\Phi(x', \xi) = x' \cdot \xi + \tau(x')$. Its canonical map is then $(x', \xi') = F_2(x, \xi)$, given from (4.2) by

$$\xi' = (\partial_{x'}\Phi)(x', \xi) = \xi + \tau'(x'), \quad x = (\partial_\xi\Phi)(x', \xi) = x'.$$

By composition [**29**, Theorem 11.12] we deduce that $\hat{F}_{i,j} = \hat{F}_2 \circ \hat{F}_1$ is an FIO with canonical map $F_{i,j} = F_2 \circ F_1$ given by (4.4). □

Remark 4.3. For short, we can write

$$F : \begin{cases} T^*I \to T^*I, \\ (x, \xi) \to \left(\phi(x), \dfrac{1}{\phi'(x)}\xi + \tau'(\phi(x))\right). \end{cases} \tag{4.5}$$

We will study the dynamics of F in detail in later sections, but we can already make some remarks. The term $(d\tau/dx)(x')$ in the expression of ξ', equation (4.4), complicates significantly the dynamics near the zero section $\xi = 0$. However, the next lemma shows that a trajectory from an initial point (x, ξ), with $|\xi|$ large enough, escapes towards infinity.

LEMMA 4.4. *For any $1 < \kappa < 1/\theta$, with θ defined in (2.1), there exists $R \geq 0$ such that for any (x, ξ), with $|\xi| > R$, and any $i \rightsquigarrow j$, with $x \in I_i$, we have*

$$|\xi'| > \kappa|\xi|, \tag{4.6}$$

where $(x', \xi') = F_{i,j}(x, \xi)$.

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Proof. From (4.4), one has $\xi' = (1/\phi'_{i,j}(x))\xi + \tau'(x')$. Also $|1/\phi'_{i,j}(x)| \geq \theta^{-1}$, hence

$$|\xi'| - \kappa|\xi| = \left|\frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x')\right| - \kappa|\xi| \geq \left|\frac{1}{\phi'_{i,j}(x)}\xi\right| - |\tau'(x')| - \kappa|\xi|$$
$$\geq \left(\frac{1}{\theta} - \kappa\right)|\xi| - \max_x |\tau'(x)| > 0.$$

The last inequality holds true if $|\xi| > R := ((1/\theta) - \kappa)^{-1} \max_x |\tau'|$. □

4.1. *The trapped set $\mathcal{K}$ in phase space.*

Definition 4.5. The *trapped set* in phase space T^*I is defined as

$$\mathcal{K} = \{(x, \xi) \in T^*I, \exists C \Subset T^*I \text{ compact}, \forall n \in \mathbb{Z}, F^n(x, \xi) \cap C \neq \emptyset\}. \tag{4.7}$$

Remark 4.6. Since the map $F : T^*I \to T^*I$ is a lift of the map ϕ, we have $\mathcal{K} \subset (K \times \mathbb{R})$. We can make this more precise: for any R given from Lemma 4.4,

$$\mathcal{K} \subset (K \times [-R, R]).$$

For $\varepsilon > 0$, let $\mathcal{K}_\varepsilon$ denote an ε-neighborhood of the trapped set $\mathcal{K}$, namely

$$\mathcal{K}_\varepsilon := \{(x, \xi) \in T^*I, \exists (x_0, \xi_0) \in \mathcal{K}, \max(|x - x_0|, |\xi - \xi_0|) \leq \varepsilon\}.$$

Recall that the canonical map F is multivalued. The definition of the trapped set requires that at least one of the future trajectories of points in $\mathcal{K}$ stays bounded. The following assumption on the map basically demands that exactly one trajectory stays bounded.

Assumption 4.7. We assume the following property called *minimal captivity*:

$$\text{there exists } \varepsilon > 0, \text{ for all } (x, \xi) \in \mathcal{K}_\varepsilon, \quad \sharp\left\{F(x, \xi) \bigcap \mathcal{K}_\varepsilon\right\} \leq 1. \tag{4.8}$$

This means that the dynamics of F is univalued on the trapped set $\mathcal{K}$.

Remarks.

- In the paper [**21**] the second author introduced the property of *partial captivity* which is weaker than *minimal captivity*: partial captivity roughly states that most trajectories escape from the trapped set $\mathcal{K}$, whereas minimal captivity states that every trajectory, except one, escapes from the trapped set $\mathcal{K}$.
- Note that the complexity of the dynamics of the map F in (4.5) is due to the term $\tau'(\phi_{i,j}(x))$, so the minimally captive property can also be considered as a condition on the behavior of the roof function along the trajectories of the IFS. In particular, for trivial (i.e. constant) roof functions the condition cannot be fulfilled. In this case, the canonical map F equals the simpler map $\mathfrak{F}$ of Proposition 3.4. Then the trapped set is given by $\mathcal{K} = K \times \{0\}$ and all trajectories in the trapped set stay on the trapped set. The same holds for all roof functions, that are cohomologous to a constant (cf. [**21**, Appendix A]).

We will give now a more precise description of the trapped set $\mathcal{K}$. Recall that the inverse maps ϕ^{-1} and F^{-1} are univalued. For any integer $m \geq 0$, let

$$\tilde{K}_m := F^{-m}(K_m \times [-R, R]),$$

where $K_m = \phi^m(I)$ has been defined in (2.6) and R is given by Lemma 4.4. In particular $\tilde{K}_0 = I \times [-R, R]$. Let $\pi : (x, \xi) \in T^*I \to x \in I$ be the projection map. From (4.5) we have

$$\pi(\tilde{K}_m) = I,$$

and a short computation† gives

$$\tilde{K}_{m+1} \subset \tilde{K}_m. \tag{4.9}$$

Let us define

$$\tilde{K} := \bigcap_m \tilde{K}_m. \tag{4.10}$$

Now we combine the sets K_n defined in (2.6) with the sets $\tilde{K}_m$ and define, for any integers $a, b \geq 0$,

$$\mathcal{K}_{a,b} := \pi^{-1}(K_a) \bigcap \tilde{K}_b. \tag{4.11}$$

We have

$$\mathcal{K}_{a+1,b} \subset \mathcal{K}_{a,b}, \quad \mathcal{K}_{a,b+1} \subset \mathcal{K}_{a,b} \tag{4.12}$$

and

$$F^{-1}(\mathcal{K}_{a,b}) = \mathcal{K}_{a-1,b+1}. \tag{4.13}$$

Remark 4.8. We can interpret the trapped set $K \subset I$ with respect to the lifted map $F : T^*I \to T^*I$, as follows. The trapped set $\pi^{-1}(K) \subset T^*I$ is characterized by

$$\pi^{-1}(K) = \{(x, \xi) \in T^*I, \exists \text{ compact } C \Subset T^*I, \forall n \geq 0,\ F^{-n}(x, \xi) \in C\},$$

i.e. $\pi^{-1}(K)$ can be considered as the 'trapped set of the map F in the past'. Similarly, $\tilde{K} \subset T^*I$ can be interpreted as the 'trapped set of the map F in the future' and $\mathcal{K} \subset T^*I$ as the full trapped set (past and future) since they are characterized by

$$\begin{aligned}
\tilde{K} &= \{(x, \xi) \in T^*I, \exists \text{ compact } C \Subset T^*I, \forall n \geq 0,\ F^n(x, \xi) \cap C \neq \emptyset\},\\
\mathcal{K} &= \{(x, \xi) \in T^*I, \exists \text{ compact } C \Subset T^*I, \forall n \in \mathbb{Z},\ F^n(x, \xi) \cap C \neq \emptyset\} \qquad (4.14)\\
&= \pi^{-1}(K) \cap \tilde{K}.
\end{aligned}$$

From this previous remark, we have the following expression for the trapped set equivalent to (4.7).

PROPOSITION 4.9. *The trapped set $\mathcal{K} \subset T^*I$ of the map F is*

$$\mathcal{K} = \bigcap_{a=0}^{\infty} \mathcal{K}_{a,a}. \tag{4.15}$$

† From Lemma 4.4 we have

$$(K_{m+1} \times [-R, R]) \subset F(K_m \times [-R, R]),$$

hence

$$\tilde{K}_{m+1} = F^{-m}(F^{-1}(K_{m+1} \times [-R, R])) \subset F^{-m}(K_m \times [-R, R]) = \tilde{K}_m.$$

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The hypothesis of minimal captivity has been defined in Assumption 4.7. The following proposition gives an equivalent and a slightly weaker definition of minimal captivity that will be used in §5.2.

PROPOSITION 4.10.

(1) *The map F is minimally captive (i.e. equation (4.8) holds true) if and only if the map F satisfies*

$$\text{there exists } a, \text{for all } (x,\xi)\in\mathcal{K}_{a,a}, \quad \sharp\{F(x,\xi)\cap\mathcal{K}_{a,a}\}\le 1. \tag{4.16}$$

(2) *If map F is minimally captive then*

$$\text{there exists } a, \text{ there exists } C, \text{ such that for all } (x,\xi)\in\mathcal{K}_{a,0}, \text{for all } n,\\ \sharp\{F^n(x,\xi)\cap\mathcal{K}_{a,0}\}\le C, \tag{4.17}$$

where $\mathcal{K}_{a,0} := (\pi^{-1}(K_a)\cap[-R, R])$ has been defined in (4.11).

Proof. The fact that (4.16) is equivalent to (4.8) is because

$$\text{for all } \varepsilon>0, \text{ there exists } a, \text{ such that } \mathcal{K}_{a,a}\subset\mathcal{K}_\varepsilon,\\ \text{for all } a, \text{ there exists } \varepsilon>0, \text{ such that } \mathcal{K}_\varepsilon\subset\mathcal{K}_{a,a}.$$

Now we prove (4.17). Let $a>0$ be an even integer. Let $(x,\xi)\in\mathcal{K}_{a,0}$ and $n>a/2$. We write $F^n(x,\xi)=F^{a/2}(F^{n-a/2}(x,\xi))$. Let $(x',\xi')\in F^{n-a/2}(x,\xi)$. From (4.13), we have $F^{-a/2}(\mathcal{K}_{a,0})=\mathcal{K}_{a/2,a/2}$, hence if $(x',\xi')\notin\mathcal{K}_{a/2,a/2}$ then $F^{a/2}(x',\xi')\notin\mathcal{K}_{a,0}$. On the contrary, for $(x',\xi')\in\mathcal{K}_{a/2,a/2}$, then the set $F^{a/2}(x',\xi')$ has cardinal less than $N^{a/2}$, so we obtain

$$\sharp\{F^n(x,\xi)\cap\mathcal{K}_{a,0}\}\le N^{a/2}\cdot\sharp\{F^{n-a/2}(x,\xi)\cap\mathcal{K}_{a/2,a/2}\}.$$

So if $\sharp\{F^{n-a/2}(x,\xi)\cap\mathcal{K}_{a/2,a/2}\}=\emptyset$ we have finished the proof. Suppose, on the contrary, that $\sharp\{F^{n-a/2}(x,\xi)\cap\mathcal{K}_{a/2,a/2}\}\neq\emptyset$. From (4.13) we have $(x,\xi)\in\mathcal{K}_{0,n}\cap\mathcal{K}_{a,0}\subset\mathcal{K}_{a/2,a/2}$. Finally, we suppose that assumption that (4.16) is fulfilled for $a/2$. This gives that

$$\sharp\{F^{n-a/2}(x,\xi)\cap\mathcal{K}_{a/2,a/2}\}\le 1$$

and $\sharp\{F^n(x,\xi)\cap\mathcal{K}_{a,0}\}\le N^{a/2}$. We have obtained (4.17) with the bound $C=N^{a/2}$. □

4.2. *Symbolic dynamics.* The purpose of this section is to describe precisely the dynamics of ϕ and F using 'symbolic dynamics'. This is very standard for expanding maps [**11**]. This description refines the structure of the sets $\mathcal{K}_{a,b}$ introduced before. We would like to emphasize that the use of symbolic dynamics in this paper is related to the fact that the initial IFS model in Definition 2.1 is a multivalued map ϕ defined on a union of intervals $(I_i)_{i=1,\ldots,N}$. This is not a 'discontinuous Markov partition of a continuous dynamics' [**11**, p. 134].

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4.2.1. *Symbolic dynamics on the trapped set $K \subset I$.* Let

$$\mathcal{W}_- := \{(\dots, w_{-2}, w_{-1}, w_0) \in \{1, \dots, N\}^{-\mathbb{N}}, w_{l-1} \rightsquigarrow w_l, \forall l \leq 0\} \tag{4.18}$$

be the set of *admissible left semi-infinite sequences*. For $w \in \mathcal{W}_-$ and $i < j$ we write $w_{i,j} := (w_i, w_{i+1}, \dots, w_j)$ for an extracted sequence. For simplicity, we will use the notation

$$\phi_{w_{i,j}} := \phi_{w_{j-1},w_j} \circ \cdots \circ \phi_{w_i,w_{i+1}} : I_{w_i} \to I_{w_j} \tag{4.19}$$

for the composition of maps. For $n \geq 0$, let

$$I_{w_{-n,0}} := \phi_{w_{-n,0}}(I_{w_{-n}}) \subset I_{w_0}. \tag{4.20}$$

For any $0 < m < n$ we have the strict inclusions

$$I_{w_{-n,0}} \subset I_{w_{-m,0}} \subset I_{w_0}.$$

From (2.1), the size of $I_{w_{-n,0}}$ is bounded by

$$|I_{w_{-n,0}}| \leq \theta^n |I_{w_0}|,$$

hence the sequence of sets $(I_{w_{-n,0}})_{n\geq 1}$ is a sequence of non-empty and decreasing closed intervals and $\bigcap_{n=1}^{\infty} I_{w_{-n,0}}$ is a point in K. We define the following.

Definition 4.11. The *symbolic coding map* is

$$S : \begin{cases} \mathcal{W}_- \to K, \\ w \mapsto S(w) := \displaystyle\bigcap_{n=1}^{\infty} I_{w_{-n,0}}. \end{cases} \tag{4.21}$$

In some sense we have decomposed the sets K_n, equation (2.6), into individual components:

$$K_n = \bigcup_{w_{-n,0} \in \mathcal{W}_-} I_{w_{-n,0}}, \tag{4.22}$$

$$K = \bigcup_{w \in \mathcal{W}_-} S(w).$$

Let us introduce the *left shift,* a multivalued map, defined by

$$L : \begin{cases} \mathcal{W}_- \to \mathcal{W}_-, \\ (\dots, w_{-2}, w_{-1}, w_0) \mapsto (\dots, w_{-2}, w_{-1}, w_0, w_1), \end{cases}$$

with $w_1 \in \{1, \dots, N\}$ such that $w_0 \rightsquigarrow w_1$. Let the *right shift* be the univalued map defined by

$$R : \begin{cases} \mathcal{W}_- \to \mathcal{W}_-, \\ (\dots, w_{-2}, w_{-1}, w_0) \mapsto (\dots, w_{-2}, w_{-1}). \end{cases}$$

PROPOSITION 4.12. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{W}_- & \xrightarrow{S} & K \\ R \uparrow\downarrow L & & \phi^{-1} \uparrow\downarrow \phi \\ \mathcal{W}_- & \xrightarrow{S} & K \end{array} \tag{4.23}$$

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and the map $S : \mathcal{W}_- \to K$ is one to one. This means that the dynamics of points on the trapped set K under the maps ϕ^{-1}, ϕ is equivalent to the symbolic dynamics of the shift maps R, L on the set of admissible words $\mathcal{W}_-$. Notice that the maps R and ϕ^{-1} are univalued, whereas the maps L and ϕ are (in general) multivalued.

Proof. From the definition of S we have

$$\phi_{w_0 w_1}(S(\ldots, w_{-2}, w_{-1}, w_0)) = S(\ldots, w_{-2}, w_{-1}, w_0, w_1) \tag{4.24}$$

and

$$\phi^{-1}_{w_{-1} w_0}(S(\ldots, w_{-2}, w_{-1}, w_0)) = S(\ldots, w_{-2}, w_{-1}), \tag{4.25}$$

which gives the diagram (4.23). The map $S : \mathcal{W}_- \to K$ is surjective by construction. Let us show that the hypothesis (2.2) implies that it is also injective. Let $w, w' \in \mathcal{W}_-$ and suppose that $w \neq w'$, i.e. there exists $k \geq 0$ such that $w_{-k} \neq w'_{-k}$. From (2.2) we have $\phi_{w_{-k}, w_{-k+1}}(I_{w_{-k}}) \cap \phi_{w'_{-k}, w'_{-k+1}}(I_{w'_{-k}}) = \emptyset$. We deduce recursively that $\phi_{w_{-k,0}}(I_{w_{-k}}) \cap \phi_{w'_{-k,0}}(I_{w'_{-k}}) = \emptyset$. Since $S(w) \in \phi_{w_{-k,0}}(I_{w_{-k}})$ and $S(w') \in \phi_{w'_{-k,0}}(I_{w'_{-k}})$ we deduce that $S(w) \neq S(w')$. Hence S is one to one. □

4.2.2. *The 'future trapped set' $\tilde{K}$ in phase space T^*I.* Let

$$\mathcal{W}_+ := \{(w_0, w_1, w_2 \ldots) \in \{1, \ldots, N\}^{\mathbb{N}}, w_l \rightsquigarrow w_{l+1}, \forall l \geq 0\}$$

be the set of admissible right semi-infinite sequences. We still use the notation $w_{i,j} := (w_i, w_{i+1}, \ldots, w_j)$ for an extracted sequence. For any $n \geq 0$ let

$$\tilde{I}_{w_{0,n}} := F^{-n}(I_{w_{0,n}} \times [-R, R]) \tag{4.26}$$

be the image of the rectangle under the univalued map F^{-n}. Notice that $\pi(\tilde{I}_{w_{0,n}}) = I_{w_0}$, where $\pi(x, \xi) = x$ is the canonical projection map. The map F^{-1} contracts strictly in the ξ-variable by the factor $\theta < 1$, thus $(\tilde{I}_{w_{0,n}})_{n \in \mathbb{N}}$ is a sequence of decreasing sets: $\tilde{I}_{w_{0,n+1}} \subset \tilde{I}_{w_{0,n}}$, and we can define the limit

$$\tilde{S} : w \in \mathcal{W}_+ \to \tilde{S}(w) := \bigcap_{n \geq 0} \tilde{I}_{w_{0,n}} \subset \tilde{K}. \tag{4.27}$$

PROPOSITION 4.13. *For every $w \in \mathcal{W}_+$, the set $\tilde{S}(w)$ is a smooth curve given by*

$$\tilde{S}(w) = \{(x, \zeta_w(x)), x \in I_{w_0}, w \in \mathcal{W}_+\},$$

with

$$\zeta_w(x) = -\sum_{k \geq 1} \phi'_{w_{0,k}}(x) \cdot \tau'(\phi_{w_{0,k}}(x)). \tag{4.28}$$

We have an estimate of regularity, uniform in w: for all $\alpha \in \mathbb{N}$, there exists $C_\alpha > 0$ such that, for all $w \in \mathcal{W}_+$, for all $x \in I_{w_0}$,

$$|(\partial_x^\alpha \zeta_w)(x)| \leq C_\alpha. \tag{4.29}$$

Moreover, with the Assumption 4.7 of minimal captivity there exists $a \geq 1$ such that these branches do not intersect on $\pi^{-1}(K_a)$,

$$\text{for all } w, w' \in \mathcal{W}_+, \quad w \neq w' \Rightarrow \pi^{-1}(K_a) \cap \tilde{S}(w) \cap \tilde{S}(w') = \emptyset. \tag{4.30}$$

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The set (4.10) can be expressed as

$$\tilde{K} = \bigcup_{w \in \mathcal{W}_+} \tilde{S}(w).$$

Proof. From (4.4) we get

$$F^{-1}(\phi_{i,j}(x), \xi) = (x, \phi'_{i,j}(x)(\xi - \tau'(\phi_{i,j}(x)))). \tag{4.31}$$

Iterating this equation we get that

$$\zeta_{w,n}(x) := -\sum_{k=1}^{n} \phi'_{w_{0,k}}(x) \cdot \tau'(\phi_{w_{0,k}}(x))$$

fulfills

$$(x, \zeta_{w,n}(x)) = F^{-n}(\phi_{w_{0,n}}(x), 0),$$

thus $(x, \zeta_{w,n}(x)) \in \tilde{I}_{w_{0,n}}$, for all $n \in \mathbb{N}$, and we get (4.28).

In order to prove (4.29) we can check, that the series of $\zeta_{w,n}(x)$ and $\partial_x^\alpha \zeta_{w,n}(x)$ converge with uniform bounds in w which follows after some calculations from (2.1) and the fact that $|\phi'_{w_{0,k}}(x)| \leq \theta^k$ independent of w.

In order to see (4.30) let $w, w' \in \mathcal{W}_+$, with $w \neq w'$ and $n \in \mathbb{N}$ such that $w_{0,n} \neq w'_{0,n}$, and suppose that there exist $x \in K_a$ and $\xi \in \mathbb{R}$ such that $(x, \xi) \in \tilde{S}(w) \cap \tilde{S}(w')$. Then by the definition of $\tilde{S}$, $(x, \xi) \in \tilde{I}_{w_{0,n+a}} \cap \tilde{I}_{w'_{0,n+a}} \subset \tilde{K}_{n+a}$. Consequently, there are $(x_1, \xi_1) \in I_{w_{0,n+a}} \times [-R, R]$ and $(x_2, \xi_2) \in I_{w'_{0,n+a}} \times [-R, R]$ with $(x, \xi) = F^{-n-a}(x_1, \xi_1) = F^{-n-a}(x_2, \xi_2)$ and we have

$$F^{-a}(x_1, \xi_1), F^{-a}(x_2, \xi_2) F^n \in (x, \xi).$$

But as $F^{-a}(x_1, \xi_1) \in \pi^{-1}(I_{w_{0,n}})$ and $F^{-a}(x_2, \xi_2) \in \pi^{-1}(I_{w'_{0,n}})$ we clearly have $F^{-a}(x_1, \xi_1) \neq F^{-a}(x_2, \xi_2)$ because $w_{0,n} \neq w'_{0,n}$. And additionally we have chosen $(x, \xi) \in \mathcal{K}_{a,n+a}$ and from the definition of (x_1, ξ_1) and (x_2, ξ_2) we get $F^{-a}(x_1, \xi_1), F^{-a}(x_2, \xi_2) \in \mathcal{K}_{n+a,a}$. Thus we have found $(x, \xi) \in \mathcal{K}_{a,a}$ with $\sharp\{F^n(x, \xi) \cap \mathcal{K}_{a,a}\} \geq 2$, which contradicts Assumption 4.7. □

4.2.3. *Symbolic dynamics on the trapped set $\mathcal{K}$ in phase space T^*I.* Recall from (4.14) that $\mathcal{K} = \pi^{-1}(K) \cap \tilde{K}$. Let

$$\mathcal{W} := \{(\ldots w_{-2}, w_{-1}, w_0, w_1, \ldots) \in \{1, \ldots, N\}^{\mathbb{Z}}, w_l \rightsquigarrow w_{l+1}, \forall l \in \mathbb{Z}\}$$

be the set of bi-infinite admissible sequences. For a given $w \in \mathcal{W}$ and $a, b \in \mathbb{N}$, let

$$\mathcal{I}_{w_{-a,0}, w_{0,b}} := (\pi^{-1}(I_{w_{-a,0}}) \cap \tilde{I}_{w_{0,b}}) \subset \mathcal{K}_{a,b},$$

where $\mathcal{K}_{a,b}$ has been defined in (4.11).

Definition 4.14. The *symbolic coding map* is

$$\mathcal{S} : \begin{cases} \mathcal{W} \to \mathcal{K}, \\ w \mapsto \mathcal{S}(w) := \bigcap_{n=1}^{\infty} \mathcal{I}_{w_{-n,0}, w_{0,n}} = (\pi^{-1}(S(w_-)) \cap \tilde{S}(w_+)), \end{cases} \tag{4.32}$$

with $w_- = (\ldots w_{-1}, w_0) \in \mathcal{W}_-$, $w_+ = (w_0, w_1, \ldots) \in \mathcal{W}_+$.

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More precisely, we can express the point $\mathcal{S}(w) \in \mathcal{K}$ as

$$\mathcal{S}(w) = (x_{w_-}, \xi_w), \quad x_{w_-} = S(w_-), \quad \xi_w = \zeta_{w_+}(S(w_-)), \tag{4.33}$$

with ζ_{w_+} given in (4.28). We also have

$$\mathcal{K}_{a,b} = \bigcup_{w \in \mathcal{W}} \mathcal{I}_{w_{-a,0}, w_{0,b}}.$$

PROPOSITION 4.15. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\mathcal{S}} & \mathcal{K} \\ R \uparrow\downarrow L & & F^{-1} \uparrow\downarrow F \\ \mathcal{W} & \xrightarrow{\mathcal{S}} & \mathcal{K} \end{array} \tag{4.34}$$

If Assumption 4.7 of minimal captivity holds true then the map $\mathcal{S} : \mathcal{W} \to \mathcal{K}$ is one to one. This means that the univalued dynamics of points on the trapped set $\mathcal{K}$ under the maps F^{-1}, F is equivalent to the symbolic dynamics of the full shift maps R, L on the set of words $\mathcal{W}$.

Proof. Commutativity of the diagram comes from the construction of $\mathcal{S}$. Also, $\mathcal{S}$ is surjective. Let us show that $\mathcal{S}$ is injective. Let $w, w' \in \mathcal{W}$, with $w \neq w'$. There exists $n \geq 0$ such that $(L^n(w))_- \neq (L^n(w'))_-$. So $S((L^n(w))_-) \neq S((L^n(w'))_-)$ because $S : \mathcal{W}_- \to K$ is one to one from Lemma 4.12. Hence $\mathcal{S}(L^n(w)) \neq \mathcal{S}(L^n(w'))$ and $F^n(\mathcal{S}(w)) \neq F^n(\mathcal{S}(w'))$ from commutativity of the diagram. We apply F^{-n} and deduce that $\mathcal{S}(w) \neq \mathcal{S}(w')$ because F^{-1} and F^{-n} are injective on $\mathcal{K}$ from Assumption 4.7. □

4.3. *Relation to the non-local integrability condition of Dolgopyat.* We can now discuss the relation between the minimally captive assumption and the non-local integrability (NLI) condition used by Naud and Dolgopyat [**16**, **40**] in order to obtain exponential decay of correlation. For the discussion we use the version of the NLI condition introduced in [**40**] where Naud first introduces, for a symbolic sequence $w \in \mathcal{W}_+$ and $u, v \in I_{w_0}$, the quantity

$$\Delta_w(u, v) := \sum_{k=1}^{\infty} \tau(\phi_{w_{0,k}}(u)) - \tau(\phi_{w_{0,k}}(v)), \tag{4.35}$$

as well as the temporal distance function for $w, w' \in \mathcal{W}_+$ with $w_0 = w'_0$:

$$\varphi_{w,w'}(u, v) := \Delta_w(u, v) - \Delta_{w'}(u, v).$$

According to [**40**, Definition 2.1] the roof function τ fulfills the NLI condition if there exists w, w' with $w_0 = w'_0$ and $u_0, v_0 \in I_{w_0} \cap K$ such that

$$\frac{\partial \varphi_{w,w'}}{\partial u}(u_0, v_0) \neq 0.$$

Note that (4.35) implies that

$$\frac{\partial \Delta_w}{\partial u}(u_0, v_0) = -\zeta_w(u_0),$$

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where ζ_w are exactly the functions defined in (4.28). Thus, translated into the language of our article, the NLI condition is the existence of two words $w, w' \in \mathcal{W}_+$, with $w_0 = w'_0$, and a point in the trapped set $u_0 \in I_{w_0} \cap K$ such that $\zeta_w(u_0) \neq \zeta_{w'}(u_0)$ (i.e. the two branches $\zeta_w(u_0)$, $\zeta_{w'}(u_0)$ above the point $u_0 \in K$ are disjoint). In Proposition 4.15 we have, however, shown that, under the condition of minimal captivity, this is true for all $u_0 \in K$ and for all $w \neq w'$ we have $\zeta_w(u_0) \neq \zeta_{w'}(u_0)$. The minimal captivity assumption thus implies NLI and is much stronger. It also has, however, stronger implications, for example for the fractal structure of the trapped set $\mathcal{K}$ as shown in the following section.

4.4. *Dimension of the trapped set $\mathcal{K}$.* We will now show that the assumption of minimal captivity allows us to characterize the fractal structure of the trapped set $\mathcal{K}$.

PROPOSITION 4.16. *If Assumption 4.7 holds true and if the adjacency matrix A is symmetric, then*

$$\dim_M \mathcal{K} = 2 \dim_M K, \tag{4.36}$$

where $\dim_M B$ *stands for the Minkowski dimension of a set B as defined in equation (2.29).*

Recall from (2.32) that $\dim_H K = \dim_M K$.

For $w = (w_k)_{k\in\mathbb{Z}} \in \mathcal{W}$, we note that $w_- = (\ldots, w_{-2}, w_{-1}, w_0) \in \mathcal{W}_-$ and $w_+ = (w_0, w_1, \ldots) \in \mathcal{W}_+$. Let

$$\mathrm{Inv}(w_+) := (\ldots, w_2, w_1, w_0)$$

be the reversed word. Since the adjacency matrix A is supposed to be symmetric, we have that $\mathrm{Inv}(w_+) \in \mathcal{W}_-$. Then, let us consider the following one to one map:

$$D : \begin{cases} \mathcal{W} \to (\mathcal{W}_- \times \mathcal{W}_-)_l, \\ w \to (w_-, \mathrm{Inv}(w_+)), \end{cases}$$

where

$$(\mathcal{W}_- \times \mathcal{W}_-)_l := \{(w, w') \in \mathcal{W}_- \times \mathcal{W}_-, w_0 = w'_0\} \tag{4.37}$$

is a subset of $\mathcal{W}_- \times \mathcal{W}_-$. The index l stands for 'linked'. Let

$$\Phi := (S \otimes S) \circ D \circ \mathcal{S}^{-1} : \mathcal{K} \to K \times K,$$

where $\mathcal{S} : \mathcal{W} \to \mathcal{K}$ has been defined in (4.32) and is shown in Proposition 4.15 to be one to one under Assumption 4.7. The map $S : \mathcal{W}_+ \to K$ has been defined in (4.21) and is also one to one. Consider

$$(K \times K)_l := (S \otimes S)((\mathcal{W}_- \times \mathcal{W}_-)_l) \subset K \times K \tag{4.38}$$

the image of (4.37) under the map $S \otimes S$. From the previous remarks, the map $\Phi : \mathcal{K} \to (K \times K)_l$ is one to one.

LEMMA 4.17. *The map $\Phi : \mathcal{K} \to (K \times K)_l$ is bi-Lipschitz.*

This lemma is illustrated in Figure 4 which shows clearly that the trapped set $\mathcal{K}$ has a product structure. Before proving Lemma 4.17, let us show how to deduce

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Proposition 4.16 from it. Since the Hausdorff and Minkowski dimension are invariant under bi-Lipschitz maps [**19**, p. 24], we deduce from this lemma that

$$\dim_M(\mathcal{K}) = \dim_M(K \times K)_l. \tag{4.39}$$

Let us temporarily write $K_i := K \cap I_i$. From (4.38) we have that

$$(K \times K)_l = \bigcup_i K_i \times K_i,$$

hence

$$\dim_M(K \times K)_l = \max_{1 \le i \le N} (2 \dim_M K_i) = 2 \dim_M K. \tag{4.40}$$

Equations (4.39) and (4.40) give Proposition 4.16.

Proof of Lemma 4.17. Let $w \in \mathcal{W}$. We write $w = (w_-, w_+)$ as before and $x_{w_-} := S(w_-) \in K$, $\rho = (x_{w_-}, \xi_w) = \mathcal{S}(w) \in \mathcal{K}$. Similarly, for another $w' \in \mathcal{W}$ we get another point $\rho' = (x_{w'_-}, \xi_{w'}) \in \mathcal{K}$. We have that

$$\Phi(\rho) = (S(w_-), S(\mathrm{Inv}(w_+))) = (x_{w_-}, x_{\mathrm{Inv}(w_+)}) \in K \times K.$$

That the map Φ is bi-Lipschitz means that

$$|\Phi(\rho) - \Phi(\rho')| \asymp |\rho - \rho'|$$

uniformly† over ρ, ρ'. Equivalently, this is

$$|x_{w_-} - x_{w'_-}| + |x_{\mathrm{Inv}(w_+)} - x_{\mathrm{Inv}(w'_+)}| \asymp |x_{w_-} - x_{w'_-}| + |\xi_w - \xi_{w'}| \tag{4.41}$$

uniformly over $w, w' \in \mathcal{W}$. Let us show (4.41). Let $w, w' \in \mathcal{W}$, and let $n \ge 0$ be the integer such that $(w_+)_j = (w'_+)_j$ for $0 \le j \le n$ but $(w_+)_{n+1} \ne (w'_+)_{n+1}$. From the definition (4.20) of the intervals $I_{w_{-n,0}}$, we see that the two points $x_{\mathrm{Inv}(w_+)}, x_{\mathrm{Inv}(w'_+)}$ both belong to the interval $I_{(\mathrm{Inv}(w_+))_{-n,0}}$, but inside it they belong to the disjoint sub-intervals $I_{(\mathrm{Inv}(w_+))_{-n-1,0}}$ and $I_{(\mathrm{Inv}(w'_+))_{-n-1,0}}$, respectively. Hence

$$|x_{\mathrm{Inv}(w_+)} - x_{\mathrm{Inv}(w'_+)}| \asymp |I_{(\mathrm{Inv}(w_+))_{-n,0}}|$$

uniformly over $w, w' \in \mathcal{W}$, where $|I|$ is the length of the interval I. From the definition (4.26) of the sets $\tilde{I}_{w_{0,n}}$ we observe that the points $\rho = (x_{w_-}, \xi_w)$ and $\rho' = (x_{w'_-}, \xi_{w'})$ belong, respectively, to the sets $\tilde{I}_{w_{0,n}}$ and $\tilde{I}_{w'_{0,n}}$. Let $\tilde{w}' := (w'_-, w_+)$. We have

$$\begin{aligned} |\rho - \rho'| &= |(x_{w_-}, \xi_w) - (x_{w'_-}, \xi_{w'})| & (4.42)\\ &\asymp |(x_{w_-}, \xi_w) - (x_{w_-}, \xi_{\tilde{w}'})| + |(x_{w_-}, \xi_{\tilde{w}'}) - (x_{w'_-}, \xi_{w'})| & (4.43)\\ &\asymp |x_{w_-} - x_{w'_-}| + |\xi_w - \xi_{\tilde{w}'}|. \end{aligned}$$

The points $\xi_w, \xi_{\tilde{w}'}$ belong to the same set $\tilde{I}_{w_{0,n}}$. However, if the assumption of 'minimal captivity' holds, they belong to disjoint sub-sets $\tilde{I}_{w_{0,n+1}}$ and $\tilde{I}_{w'_{0,n+1}}$, respectively. Hence

$$|\xi_w - \xi_{\tilde{w}'}| \asymp |J_{w,n}|, \tag{4.44}$$

† The notation $|\Phi(\rho) - \Phi(\rho')| \asymp |\rho - \rho'|$ means precisely that there exists $C > 0$ such that, for every ρ, ρ', $C^{-1}|\rho - \rho'| \le |\Phi(\rho) - \Phi(\rho')| \le C|\rho - \rho'|$.

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with the interval $J_{w,n} := \tilde{I}_{w_{0,n}} \cap \pi^{-1}(x_{w_-})$. From the bounded distortion principle [**19**] we have that

$$\text{for all } x,\, y \in I_{w_{-n,0}}, \quad |(D\phi_{w_{-n,0}})(x)| \asymp |(D\phi_{w_{-n,0}})(y)| \asymp |I_{w_{-n,0}}|$$

uniformly with respect to w, n, x, y. From the expression of the canonical map F in (4.4) and the bounded distortion principle, we have that

$$|J_{w,n}| \asymp |(D\phi_{w_{-n,0}})(x)| \quad \text{for all } x \in I_{w_0},$$

uniformly with respect to w, n, x. Using the previous results we get

$$\begin{aligned} |x_{w_-} - x_{w'_-}| + |\xi_w - \xi_{w'}| &\asymp |x_{w_-} - x_{w'_-}| + |\xi_w - \xi_{\tilde{w}'}| \\ &\asymp |x_{w_-} - x_{w'_-}| + |J_{w,n}| \\ &\asymp |x_{w_-} - x_{w'_-}| + (D\phi_{w_{0,n}})(x) \quad \text{for all } x \in I_{w_0} \\ &\asymp |x_{w_-} - x_{w'_-}| + |I_{\mathrm{Inv}(w_{0,n})}| \\ &\asymp |x_{w_-} - x_{w'_-}| + |x_{\mathrm{Inv}(w_+)} - x_{\mathrm{Inv}(w'_+)}|. \end{aligned}$$

We have obtained (4.41) and finished the proof of Lemma 4.17 and Proposition 4.16.

5. *Proof of Theorem 2.9 for the spectral gap in the semiclassical limit*

For the proof of Theorem 2.9, we will follow step by step the same analysis as in §3 (and also follow closely the proof of Theorem 2 in [**21**]). The main difference now is that $\hbar \ll 1$ is a semiclassical parameter (not fixed anymore). In other words, we just perform a linear rescaling in cotangent space: $\xi_h := \hbar\xi$. Our quantization rule for a symbol $A(x, \xi_h) \in S^{-m}(\mathbb{R})$, equation (3.14) is now written (see [**37**] p. 22), for $\varphi \in \mathcal{S}(\mathbb{R})$,

$$(\hat{A}\varphi)(x) := \frac{1}{2\pi\hbar} \int A(x, \xi_h) e^{i(x-y)\xi_h/\hbar} \varphi(y)\, dy\, d\xi_h. \tag{5.1}$$

For simplicity we will still write ξ instead of ξ_h below.

5.1. *The escape function.*

Let $1 < \kappa < 1/\theta$ and $R > 0$ given in Lemma 4.4. Let $m > 0$, $\eta > 0$ (small) and consider a C^∞ function $A_m(x, \xi)$ on $T^*\mathbb{R}$ so that

$$A_m(x, \xi) := \begin{cases} \langle \xi \rangle^{-m} & \text{for } |\xi| > R + \eta, \\ 1 & \text{for } \xi \le R, \end{cases}$$

where $\langle \xi \rangle := (1 + \xi^2)^{1/2}$. A_m belongs to the symbol class $S^{-m}(\mathbb{R})$ defined in (3.12).

From equation (4.6) we can deduce, similarly to equation (3.13) and if η is small enough, that

$$\text{for all } x \in I, \text{ for all } |\xi| > R, \text{ for all } i \rightsquigarrow j,$$
$$\frac{A_m(F_{i,j}(x, \xi))}{A_m(x, \xi)} \le C^m < 1 \quad \text{with } C = \sqrt{\frac{R^2 + 1}{\kappa^2 R^2 + 1}} < 1. \tag{5.2}$$

This means that the function A_m is an *escape function* since it decreases strictly along the trajectories of F outside the zone $\mathcal{Z}_0 := I \times [-R, R]$. For any point $(x, \xi) \in T^*I$ we have the more general bound

$$\text{for all } x \in I, \text{ for all } \xi \in \mathbb{R}, \text{ for all } i \rightsquigarrow j, \quad \frac{A_m(F_{i,j}(x, \xi))}{A_m(x, \xi)} \le 1. \tag{5.3}$$

Let $\hbar > 0$. Using the quantization rule (5.1), the symbol A_m can be quantized, giving a $\hbar$-pseudodifferential operator $\hat{A}_m$ which is self-adjoint and invertible on $C^\infty(I)$. In our case $\hat{A}_m$ is simply a multiplication operator by $A_m(\xi)$ in $\hbar$-Fourier space.

5.2. *Using the Egorov theorem.* Let us consider the Sobolev space

$$H^{-m}(\mathbb{R}) := \hat{A}_m^{-1}(L^2(\mathbb{R})),$$

which is the usual Sobolev space, as a linear space, except for the norm which depends on $\hbar$. Then $\hat{F} : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ is unitary equivalent to

$$\hat{Q} := \hat{A}_m \hat{F}_{\hbar,\chi} \hat{A}_m^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Let $n \in \mathbb{N}^*$ be a fixed time, which will be made large at the end of the proof, and define

$$\hat{P}^{(n)} := \hat{Q}^{*n} \hat{Q}^n = \hat{A}_m^{-1} \hat{F}_{\hbar,\chi}^{*n} \hat{A}_m^2 \hat{F}_{\hbar,\chi}^n \hat{A}_m^{-1}. \tag{5.4}$$

From the Egorov theorem, as in Lemma 3.7, we have that $\hat{B} := \hat{F}_{\hbar,\chi}^* \hat{A}_m^2 \hat{F}_{\hbar,\chi}$ is a PDO with principal symbol

$$\begin{aligned} B(x, \xi) &= \chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\mathrm{Re}(V(\phi_{i,j}(x)))} A_m^2(F_{i,j}(x, \xi)), \quad (x, \xi) \in T^*I, \\ &= \chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} e^{2D((\phi_{i,j}(x)))} A_m^2(F_{i,j}(x, \xi)), \end{aligned}$$

where we have used the 'damping function' $D(x) := \mathrm{Re}(V(x)) - \frac{1}{2}\log(|(\phi^{-1})'(x)|)$ already defined in (2.23). Iteratively for every $n \geq 1$, Egorov's theorem gives that $(\hat{F}_{\hbar,\chi}^*)^n \hat{A}_m^2 \hat{F}_{\hbar,\chi}^n$ is a PDO with principal symbol

$$B_n(x, \xi) = \chi^2(x) \sum_{w_{-n,0} \in \mathcal{W}_-} e^{2D_{w_{-n,0}}(x)} A_m^2(F_{w_{-n,0}}(x, \xi)),$$

where $\mathcal{W}_+$ is the set of admissible sequences defined in (4.18), with the Birkhoff sum $D_{w_{-n,0}}(x) := \sum_{k=1}^n D(\phi_{w_{-n,-k}}(x))$ and

$$F_{w_{-n,0}} := F_{w_{-1},w_0} \circ \cdots \circ F_{w_{-n},w_{-n+1}}.$$

With the theorem of composition of a PDO [**57**, Ch. 4] we obtain that $\hat{P}^{(n)}$ is a PDO of order 0 with principal symbol given by

$$P^{(n)}(x, \xi) = \left(\chi^2(x) \sum_{w_{-n,0} \in \mathcal{W}_-} e^{2D_{w_{-n,0}}(x)} \frac{A_m^2(F_{w_{-n,0}}(x, \xi))}{A_m^2(x, \xi)}\right). \tag{5.5}$$

We define

$$\gamma_{(n)} := \sup_{x \in I, w_{-n,0} \in \mathcal{W}_-} \frac{1}{n} D_{w_{-n,0}}(x),$$

hence $e^{2D_{w_{-n,0}}(x)} \leq e^{2n\gamma_{(n)}}$.

From Theorem 2.6, the spectrum of $\hat{F}_{\hbar,\chi}$ does not depend on the choice of χ. Here we take $a \geq 0$ as given in Assumption 4.7 and we choose χ such that $\chi \equiv 1$ on K_{a+1}, $\chi \equiv 0$ on $\mathbb{R} \backslash K_a$. We have $P(x, \xi) = 0$ if $x \in \mathbb{R} \backslash K_a$.

Now we will bound the positive symbol $P^{(n)}(x, \xi)$ from above, considering $x \in K_a$ and the following different possibilities for the trajectory $F_{w_{-n,0}}(x, \xi)$.

(1) If $|\xi| > R$, equation (5.2) gives

$$\frac{A_m^2(F_{w_{-n,0}}(x,\xi))}{A_m^2(x,\xi)} = \frac{A_m^2(F_{w_{-n,0}}(x,\xi))}{A_m^2(F_{w_{-n,-1}}(x,\xi))}\frac{A_m^2(F_{w_{-n,-1}}(x,\xi))}{A_m^2(F_{w_{-n,-2}}(x,\xi))}\cdots\frac{A^2(F_{w_{-n,-n+1}}(x,\xi))}{A^2(x,\xi)} \tag{5.6}$$

$$\leq (C^{2m})^n, \tag{5.7}$$

therefore

$$P^{(n)}(x,\xi) \leq (\sharp\mathcal{W}_n)e^{2n\gamma_{(n)}}(C^{2m})^n \leq (Ne^{2\gamma_{(n)}}C^{2m})^n.$$

We have used that $\sharp\mathcal{W}_n \leq N^n$. Notice that C^{2m} can be made arbitrarily small if m is large.

(2) If $|\xi| \leq R$, we have from the hypothesis of minimal captivity (Assumption 4.7) and Proposition 4.10 that, at time $(n-1)$, every point (x',ξ') of the set $F^{n-1}(x,\xi)$, except finitely many points, satisfies $|\xi'| > R$. Using (5.3) and (5.2), for all these points one has $A_m^2(F_{w_{-n,0}}(x,\xi))/A_m^2(x,\xi) \leq C^{2m}$ and for the exceptional point one can only write $A_m^2(F_{w_{-n,0}}(x,\xi))/A_m^2(x,\xi) \leq 1$. This gives

$$P^{(n)}(x,\xi) \leq e^{2n\gamma_{(n)}}((\sharp\mathcal{W}_n - 1)C^{2m} + C') \leq \mathcal{B},$$

with the bound

$$\mathcal{B} := e^{2n\gamma_{(n)}}(N^nC^{2m} + C'). \tag{5.8}$$

With the L^2-*continuity theorem* for pseudodifferential operators [**15**, **37**] this implies that, in the limit $\hbar \to 0$,

$$\|\hat{P}^{(n)}\|_{L^2} \leq \mathcal{B} + \mathcal{O}_n(\hbar). \tag{5.9}$$

Polar decomposition of $\hat{Q}^n$ gives

$$\|\hat{Q}^n\|_{L^2} \leq \||\hat{Q}^n|\|_{L^2} = \sqrt{\|\hat{P}^{(n)}\|_{L^2}} \leq (\mathcal{B} + \mathcal{O}_n(\hbar))^{1/2}. \tag{5.10}$$

Let $\gamma_+ = \limsup_{n\to\infty}\gamma_{(n)}$. If we let $\hbar \to 0$ first, and $m \to +\infty$ giving $C^{2m} \to 0$, and $n \to \infty$, we obtain $(\mathcal{B} + \mathcal{O}_n(\hbar))^{1/(2n)} \to e^{\gamma_+}$. Therefore for any $\rho > e^{\gamma_+}$, there exists $n_0 \in \mathbb{N}$, $\hbar_0 > 0$, $m_0 > 0$ such that, for any $\hbar \leq \hbar_0$, $m > m_0$,

$$\|\hat{F}_{\hbar,\chi}^{n_0}\|_{H^{-m}} = \|\hat{Q}^{n_0}\|_{L^2} \leq \rho^{n_0}. \tag{5.11}$$

Also, there exists $c > 0$ independent of $\hbar \leq \hbar_0$ such that, for any r such that $0 \leq r < n_0$, we have $\|\hat{Q}^r\|_{L^2} < c\rho^r$. As a consequence, for any $n \in \mathbb{N}$ we write $n = kn_0 + r$ with $0 \leq r < n_0$ and

$$\|\hat{F}_{\hbar,\chi}^n\|_{H^{-m}} = \|\hat{Q}^n\|_{L^2} \leq \|\hat{Q}^{n_0}\|_{L^2}^k\|\hat{Q}^r\|_{L^2} \leq \rho^n\frac{\|\hat{Q}^r\|_{L^2}}{\rho^r} \leq c\rho^n.$$

This estimate implies as well as the bound on the spectral radius (2.24) the bound on the resolvent (2.26), as follows.

For any n, the spectral radius of $\hat{Q}$ satisfies [**47**, p. 192]

$$r_s(\hat{Q}) \leq \|\hat{Q}^n\|^{1/n} \leq c^{1/n}\rho.$$

So we get that, for $\hbar \to 0$,

$$r_s(\hat{F}_{\hbar,\chi}) = r_s(\hat{Q}) \leq e^{\gamma_+} + o(1). \tag{5.12}$$

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In order to obtain the bound (2.26) on the resolvent, let $|z| > \rho_2 > \rho$. The relation $(z - \hat{F}_{\hbar,\chi})^{-1} = z^{-1} \sum_{n\geq 0} (\hat{F}_{h,\chi}/z)^n$ gives that

$$\|(z - \hat{F}_{\hbar,\chi})^{-1}\|_{H^{-m}} \leq |z|^{-1} \sum_{n\geq 0} \frac{\|\hat{F}^n_{\hbar,\chi}\|_{H^{-m}}}{|z|^n} \leq |z|^{-1} c_{\rho_1} \sum_{n\geq 0} \frac{\rho_1^n}{|z|^n}$$
$$= \frac{c_{\rho_1}}{|z| - \rho_1} \leq \frac{c_{\rho_1}}{\rho_2 - \rho_1} =: C_{\rho_2}$$

which finishes the proof of Theorem 2.9.

6. *Proof of Theorem 2.16 about the fractal Weyl law*

We will prove this result once more by conjugating the transfer operator by an escape function as in the previous section, §5. However, we need first to improve the properties of the escape function. The fractal Weyl estimate will then follow from general trace estimates of PDOs and general lemmas on singular values of compact operators, which we recall in the Appendices.

6.1. *A refined escape function.*

6.1.1. *Distance function.* The escape function A will be constructed from a distance function δ. For $x \in I$, let

$$\tilde{K}(x) := \tilde{K} \cap (\{x\} \times \mathbb{R}), \tag{6.1}$$

where $\tilde{K}$ has been defined in (4.10). With this notation we can define the following distance function.

Definition 6.1. Let $x \in I_{w_0}$ and $\xi \in \mathbb{R}$. We define the distance of (x, ξ) to the set $\tilde{K}$ given in (4.10) by

$$\delta(x, \xi) := \operatorname{dist}(\xi, \tilde{K}(x)) = \min_{w\in\mathcal{W}_+} |\xi - \zeta_w(x)|. \tag{6.2}$$

We will show that the distance function $\delta(x, \xi)$ decreases along the trajectories of F. First, the next lemma shows how the branches ζ_w are transformed under the canonical map F. This formula follows from straightforward calculations.

LEMMA 6.2. *For every $w_+ = (w_0, w_1, \ldots) \in \mathcal{W}_+$, $x \in I_{w_0}$ we have*

$$F_{w_0,w_1}(x, \zeta_{w_+}(x)) = (x', \zeta_{L(w_+)}(x')), \tag{6.3}$$

with $L(w_+) := (w_1, w_2, \ldots)$ and $x' = \phi_{w_0,w_1}(x)$.

LEMMA 6.3. *For all $i, j, i \rightsquigarrow j$, for all $x \in I_i$, for all $\xi \in \mathbb{R}$,*

$$\delta(F_{i,j}(x, \xi)) \geq \frac{1}{\theta}\delta(x, \xi),$$

where $\theta < 1$ is given by (2.1).

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Proof. Let $i = w_0 \rightsquigarrow j = w_1$, $x \in I_{w_0}$. Let $(x', \xi') := F_{w_0,w_1}(x, \xi)$ with $x' \in I_{w_1}$. We use (6.3) and also that F_{w_0,w_1} is expansive in ξ by a factor larger than $\theta^{-1} > 1$ (equation (4.4)), and get

$$|\xi' - \zeta_{L(w_+)}(x')| = |(F_{w_0,w_1}(x, \xi) - F_{w_0,w_1}(x, \zeta_{w_+}(x)))_\xi| \geq \frac{1}{\theta}|\xi - \zeta_{w_+}(x)|.$$

Thus

$$\begin{aligned}\delta(F_{w_0,w_1}(x, \xi)) &= \min_{w \in \mathcal{W}_+} |\xi' - \zeta_{w_+}(x')| = \min_{w_+ \in \mathcal{W}_+} |\xi' - \zeta_{L(w_+)}(x')| \\ &\geq \frac{1}{\theta} \min_{w_+ \in \mathcal{W}_+} |\xi - \zeta_{w_+}(x)| = \frac{1}{\theta}\delta(x, \xi).\end{aligned}$$ □

6.1.2. *Escape function.* The aim of this section is to prove the existence of an escape function with the following properties.

PROPOSITION 6.4. *For all $1 < \kappa < \theta^{-1}$, there exists $\mathsf{C}_0 > 0$, such that for all μ, $0 \leq \mu < \frac{1}{2}$, for all $m > 0$, there exists an $\hbar$-dependent order function $A_{m,\mu} \in \mathcal{OF}^{m\mu}(\langle\xi\rangle^{-m})$ (as defined in Definition B.3) which fulfills the following 'decay condition':*

for all i, j, such that $i \rightsquigarrow j$ and for all $(x, \xi) \in I_i \times \mathbb{R}$ such that $\delta(x, \xi) > \mathsf{C}_0\hbar^\mu$ the following estimate holds:

$$\left(\frac{A_{m,\mu} \circ F_{i,j}}{A_{m,\mu}}\right)(x, \xi) \leq \kappa^{-m}. \tag{6.4}$$

In order to prove the above proposition we first remark that the distance function (6.2) is not differentiable; however, it is Lipschitz.

LEMMA 6.5. *Let $C_1 := \sup_{x \in I, \omega \in \mathcal{W}_+} |(\partial_x \zeta_\omega)(x)|$. Then $\delta : T^*I \to \mathbb{R}^+$ is a Lipschitz function with constant $C_1 + 1$.*

Proof. Let $x, y \in I_i$, then from the fact that $|(\partial_x \zeta_\omega)(x)|$ is uniformly bounded by C_1 we have

$$|\delta(x, \xi) - \delta(y, \xi)| \leq C_1|x - y|.$$

On the other hand, clearly

$$|\delta(y, \xi) - \delta(y, \zeta)| \leq |\xi - \zeta|,$$

thus

$$|\delta(x, \xi) - \delta(y, \zeta)| \leq C_1|x - y| + |\xi - \zeta| \leq (C_1 + 1)\mathrm{dist}((x, \xi), (y, \zeta)).$$ □

Next we choose $0 \leq \mu < 1/2$ and regularize the function δ at the scale $\hbar^\mu$. For this we choose $\chi \in C_0^\infty(\mathbb{R}^2)$ with support in the unit ball $B_1(0)$ of $\mathbb{R}^2$ and $\chi(x, \xi) > 0$ for $\|(x, \xi)\| < 1$. This function can be rescaled to

$$\chi_{\hbar^\mu}(x, \xi) := \frac{1}{\hbar^{2\mu}\|\chi\|_{L^1}}\chi\left(\frac{x}{\hbar^\mu}, \frac{\xi}{\hbar^\mu}\right)$$

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such that $\mathrm{supp}\chi_{\hbar^\mu} \subset B_{\hbar^\mu}(0)$ and $\int \chi_{\hbar^\mu}(x)\,dx = 1$. Now we can define the regularized distance function by

$$\tilde{\delta}(x,\xi) := \int_{T^*I} \delta(x',\xi')\chi_{\hbar^\mu}(x-x',\xi-\xi')\,dx'\,d\xi'.$$

This smoothed distance function $\tilde{\delta}$ differs only at order $\hbar^\mu$ from the original one because

$$\begin{aligned}|\tilde{\delta}(x,\xi)-\delta(x,\xi)| &= \left|\int_{\mathbb{R}^2}(\delta(x,\xi)-\delta(x-x',\xi-\xi'))\chi_{\hbar^\mu}(x',\xi')\,dx'\,d\xi'\right| \\ &\leq \sup_{(x',\xi')\in B_{\hbar^\mu}(0)} |(\delta(x,\xi)-\delta(x-x',\xi-\xi')| \\ &\leq (C_1+1)\hbar^\mu. \end{aligned} \tag{6.5}$$

Furthermore, we get the following estimates for its derivatives.

LEMMA 6.6. *For all $\alpha,\beta\in\mathbb{N}$ the estimate*

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{\delta}(x,\xi)| \leq C_{\alpha,\beta}\hbar^{-\mu(\alpha+\beta)}(\delta(x,\xi)+C\hbar^\mu)$$

holds.

Proof. From the definition of $\chi_{\hbar^\mu}$ we have $\|\partial_x^\alpha\partial_\xi^\beta\chi_{\hbar^\mu}\|_\infty \leq C_{\alpha,\beta}\hbar^{-2-(\alpha+\beta)\mu}$ and thus

$$\begin{aligned}|(\partial_x^\alpha\partial_\xi^\beta\tilde{\delta}(x,\xi))| &= \int_{T^*I}\delta(x',\xi')\partial_x^\alpha\partial_\xi^\beta\chi_{\hbar^\mu}(x-x',\xi-\xi')\,dx'\,d\xi' \\ &\leq \pi\hbar^{2\mu}\|\delta\|_{\infty,B_{\hbar^\mu(x,\xi)}}C_{\alpha,\beta}\hbar^{-(2+\alpha+\beta)\mu} \\ &\leq \pi C_{\alpha,\beta}\hbar^{-(\alpha+\beta)\mu}(\delta(x,\xi)+(C_1+1)\hbar^\mu),\end{aligned}$$

where we have used the Lipschitz property of δ (Lemma 6.5) in the last inequality. □

As $|\delta(x,\xi)|\leq|\xi|+C$, the above lemma gives us directly that $\tilde{\delta}\in S^1_\mu(T^*I)$. Now we define the escape function as

$$A_{m,\mu}(x,\xi) := \hbar^{m\mu}(\hbar^{2\mu}+(\tilde{\delta}(x,\xi))^2)^{-m/2}. \tag{6.6}$$

This is obviously a smooth function and it fulfills the conditions of an $\hbar$-dependent order function (cf. Definition B.3)

LEMMA 6.7. *The function $A_{m,\mu}$ defined in (6.6) is an $\hbar$-dependent order function $A_{m,\mu}\in \mathcal{OF}^{m\mu}(\langle\xi\rangle^{-m})$ as defined in Definition B.3.*

Proof. As $\tilde{K}\subset I\times[-R,R]$, we obtain $\min(0,|\xi|-R)\leq\tilde{\delta}(x,\xi)\leq|\xi|+R$. This implies that $A_{m,\mu}(x,\xi)\leq\tilde{C}\langle\xi\rangle^{-m}$ and that $A_{m,\mu}(x,\xi)\geq C'\hbar^{m\mu}\langle\xi\rangle^{-m}$. It remains thus to show that, for arbitrary $\alpha,\beta\in\mathbb{N}$, one has

$$|\partial_x^\alpha\partial_\xi^\beta A_{m,\mu}(x,\xi)|\leq C_{\alpha,\beta}\hbar^{-\mu(\alpha+\beta)}A_{m,\mu}(x,\xi), \tag{6.7}$$

where $C_{\alpha,\beta}$ depends only on α and β. First, consider the case $\alpha=1$, $\beta=0$:

$$|\partial_x A_{m,\mu}(x,\xi)| = \left|\hbar^{m\mu}m\frac{(\partial_x\tilde{\delta}(x,\xi))\tilde{\delta}(x,\xi)}{(\hbar^{2\mu}+(\tilde{\delta}(x,\xi))^2)^{(m+2)/2}}\right| \leq C\hbar^{-\mu}A_{m,\mu}(x,\xi),$$

where we have used $\tilde{\delta} \le \sqrt{\hbar^{2\mu} + \tilde{\delta}^2}$ and $|\partial_x \tilde{\delta}| \le C\hbar^{-\mu}\sqrt{\hbar^{2\mu} + \tilde{\delta}^2}$, which follows from Lemma 6.6 together with (6.5). Inductively, one obtains the estimate for arbitrary $\alpha, \beta \in \mathbb{N}$ by repeated use of Lemma 6.6 and (6.5). □

Finally, it remains to show the decay estimates for $(A_{m,\mu} \circ F_{i,j}/A_{m,\mu})(x,\xi)$. Combining (6.5) with Lemma 6.3 we then get

$$\begin{aligned}\tilde{\delta}(F_{i,j}(x,\xi)) &\ge \delta(F_{i,j}(x,\xi)) - (C_1+1)\hbar^{\mu} \ge \frac{1}{\theta}\delta(x,\xi) - (C_1+1)\hbar^{\mu}\\ &\ge \frac{1}{\theta}\tilde{\delta}(x,\xi) - \left(\frac{1}{\theta}+1\right)(C_1+1)\hbar^{\mu}\end{aligned}$$

and thus

$$\frac{A_{m,\mu}(F_{i,j}(x,\xi))}{A_{m,\mu}(x,\xi)} \le \left(\frac{1 + ((1/\theta)\cdot(\tilde{\delta}(x,\xi)/\hbar^{\mu}) - \tilde{C})^2}{1 + (\tilde{\delta}(x,\xi)/\hbar^{\mu})^2}\right)^{m/2}, \tag{6.8}$$

where $\tilde{C} = ((1/\theta)+1)(C_1+1)$. Clearly the right side of (6.8) converges to $(1/\theta)^{-m}$ for $\tilde{\delta}(x,\xi)/\hbar^{\mu} \to \infty$, which proves the existence of a desired C_0 and finishes the proof of Proposition 6.4.

6.1.3. *Truncation in x.* Here we choose a similar truncation operator $\hat{\chi}$ as in equation (2.14) but in a finer vicinity of the trapped set K. First, notice that $K_{\hbar^{\mu}} \Subset \phi^{-1}(K_{\hbar^{\mu}})$ where $K_{\hbar^{\mu}}$ has been defined in Definition 2.13. For $\hbar$ small enough we have $\phi^{-1}(K_{\hbar^{\mu}}) \Subset I$. Let $\chi \in C^{\infty}_{\phi^{-1}(K_{\hbar^{\mu}})}$ such that $\chi(x) = 1$ for $x \in K_{\hbar^{\mu}}$. χ can be considered as a function $\chi(x,\xi) := \chi(x)$ (independent of ξ) and we have that $\chi_{\mu} \in S^0_{\mu}(T^*\mathbb{R})$. As in equation (2.14) we define $\hat{\chi} := \mathrm{Op}^w_{\hbar}(\chi)$, which is the multiplication operator by χ and

$$F_{\hbar,\chi} := \hat{F}_{\hbar}\hat{\chi}.$$

6.2. *Weyl law.* The Weyl law will give an upper bound on the number of eigenvalues of $\hat{F}_{\hbar,\chi}$ in the Sobolev spaces H^m. These estimates will be obtained by conjugating $\hat{F}_{\hbar,\chi}$ with $Op^w_{\hbar}(A_{m,\mu})$ in the same way as for the discrete spectrum or the spectral gap. Note that we use the Weyl quantization (see Definition B.2) in this section, because we want to obtain self-adjoint operators. In order to be able to conjugate we have to show that $Op^w_{\hbar}(A_{m,\mu}) : H^{-m} \to L^2$ is an isomorphism. We already know that $Op^w_{\hbar}(\langle\xi\rangle^m) : L^2 \to H^{-m}$ is an isomorphism, thus it suffices to show that $\hat{B} := Op^w_{\hbar}(A_{m,\mu})Op^w_{\hbar}(\langle\xi\rangle^m) : L^2 \to L^2$ is invertible. From the $\hbar$-local symbol calculus (Theorem B.7) it follows that $\hat{B}$ is an elliptic operator in the $\hbar$-local symbol class $S_{\mu}(A_{m,\mu}\langle\xi\rangle^m)$ and thus the invertibility follows from Proposition B.10. Note that it is necessary to work in the $\hbar$-local symbol classes as $\hat{B}$ would not be an elliptic operator in $S_{\mu}(1)$. Proposition B.10 also gives us the leading order of our inverse $\hat{B}^{-1}$, which is $A^{-1}_{m,\mu}\langle\xi\rangle^{-m}$. So the inverse of $Op^w_{\hbar}(A_{m,\mu})$ is again a PDO with leading symbol $A^{-1}_{m,\mu}$.

With the isomorphism $Op^w_{\hbar}(A_{m,\mu}) : H^{-m} \to L^2$ we can thus define a different scalar product on the Sobolev spaces which turns $Op^w_{\hbar}(A_{m,\mu})$ into a unitary operator. The Sobolev space equipped with this scalar product will be denoted by $\mathcal{H}^{-m}_{\hbar,\mu}$ and the study of $\hat{F}_{\hbar}$ is thus unitary equivalent to the study of $\hat{Q}_m$ defined by the following commutative

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diagram (where we have noted $\hat{A}_{m,\mu} := Op_{\hbar}^{w}(A_{m,\mu})$):

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\hat{Q}_m} & L^2(\mathbb{R}) \\ \downarrow \hat{A}_{m,\mu}^{-1} & & \downarrow \hat{A}_{m,\mu}^{-1} \\ \mathcal{H}_{\hbar,\mu}^{-m} & \xrightarrow{\hat{F}_{\hbar,\chi}} & \mathcal{H}_{\hbar,\mu}^{-m} \end{array} \tag{6.9}$$

In the next lemma, C_0 and κ are as in Lemma 6.4.

LEMMA 6.8. *Let C_0 be as in Lemma 6.4. Then for every $\epsilon > 0$ and $0 \le \mu < \frac{1}{2}$ there exist $m_0 > 0$ and $\tilde{C}_1, \tilde{C}_2 > 0$ such that, for all $m > m_0$ and in the limit $\hbar \to 0$, we have*

$$\sharp\{\lambda_i^{\hbar} \in \sigma(\hat{F}_{\hbar,\chi}|_{\mathcal{H}_{\hbar,\mu}^{m}}) \mid |\lambda_i^{\hbar}| \ge \epsilon\} \le \frac{1}{2\pi\hbar}(\tilde{C}_1 \mathrm{Leb}\{\mathcal{K}_{\mathsf{C}_0\hbar^{\mu}}\} + \tilde{C}_2\hbar). \tag{6.10}$$

Before proving Lemma 6.8, let us show that it implies Theorem 2.16. From Theorem 2.6, the discrete spectrum of $\hat{F}_{\hbar,\chi}|_{\mathcal{H}_{\hbar,\mu}^{m}}$ is the Ruelle spectrum of resonances $\mathrm{Res}(\hat{F}_{\hbar})$, independent of μ and m. With Assumption 4.7 we can use equation (4.36) and that $\mathcal{K}$ has pure dimension, thus equation (2.31) gives $\mathrm{Leb}\{\mathcal{K}_{C_1\hbar^{\mu}}\} = \mathcal{O}((\hbar^{\mu})^{\mathrm{codim}_M(\mathcal{K})})$. As $\mathrm{codim}_M(\mathcal{K}) < 2$ and $\mu < \frac{1}{2}$, equation (6.10) gives

$$\begin{aligned} \sharp\{\lambda_i^{\hbar} \in \mathrm{Res}(\hat{F}_{\hbar}) \mid \lambda_i^{\hbar}| \ge \epsilon\} &= \mathcal{O}(\hbar^{-1}(\hbar^{\mu})^{\mathrm{codim}_M(\mathcal{K})}) \\ &= \mathcal{O}(\hbar^{-1}(\hbar^{\mu})^{2-2\dim_H(K)}) = \mathcal{O}(\hbar^{2\mu-1-2\mu\dim_H(K)}) \end{aligned}$$

for any fixed $0 \le \mu < 1/2$. This gives Theorem 2.16 with $\eta = (1-2\mu)(1-\dim_{\mathrm{H}}(K))$.

Proof of Lemma 6.8. From (6.9), $\hat{F}_{\hbar,\chi} : \mathcal{H}_{\hbar,\mu}^{-m} \to \mathcal{H}_{\hbar,\mu}^{-m}$ is unitary equivalent to

$$\hat{Q}_{m,\mu} := \mathrm{Op}_{\hbar}^{w}(A_{m,\mu})\hat{F}_{\hbar}\hat{\chi}\mathrm{Op}_{\hbar}^{w}(A_{m,\mu})^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Consider

$$\hat{P}_{\mu} := \hat{Q}_{m,\mu}^{*}\hat{Q}_{m,\mu} = \mathrm{Op}_{\hbar}^{w}(A_{m,\mu})^{-1}\hat{\chi}\hat{F}_{\hbar}^{*}\mathrm{Op}_{\hbar}^{w}(A_{m,\mu})^{2}\hat{F}_{\hbar}\hat{\chi}\mathrm{Op}_{\hbar}^{w}(A_{m,\mu})^{-1}.$$

By the composition Theorem B.7 and the Egorov Theorem B.11 for $\hbar$-local symbols, $\hat{P}_{\mu}$ is a PDO with leading symbol $P_{\mu}(x,\xi) \in S_{\mu}^{0}$. For $x \in I_i$, $\xi \in \mathbb{R}$, the leading symbol is given by the same expression as in (3.19)†:

$$P_{\mu}(x,\xi) = \chi^2(x)\sum_{j \text{s.t.} i \rightsquigarrow j} |\phi_{i,j}'(x)| e^{2\mathrm{Re}(V(\phi_{i,j}(x)))} \frac{A_{m,\mu}^{2}(F_{i,j}(x,\xi))}{A_{m,\mu}^{2}(x,\xi)} \bmod \hbar^{1-2\mu} S_{\mu}^{-1}(T^*\mathbb{R}). \tag{6.11}$$

Now using the definition of χ, equation (6.4) and Lemma 6.4, the operator $\hat{P}_{\mu}$ can be decomposed into self-adjoint operators:

$$\hat{P}_{\mu} = \hat{k}_{\mu} + \hat{r}_{\mu},$$

† Also for this calculation it is crucial to work with the $\hbar$-local calculus in order to obtain sufficient remainder estimates.

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where $\hat{k}_\mu$ is a PDO with symbol $k_\mu \in S_\mu^{-\infty}$ supported on $\mathcal{K}_{\mathsf{C}_0\hbar^\mu}$ for C_0 being the constant from Lemma 6.4. Hence $\hat{k}_\mu$ is a trace-class operator. The operator $\hat{r}_\mu$ is a PDO with symbol $r_\mu \in S_\mu^0$ such that

$$\|r_\mu\|_\infty \leq \theta e^{2\|\mathrm{Re}(V)\|_\infty} \kappa^{-2m} + \mathcal{O}(\hbar^{1-2\mu}),$$

hence $\|\hat{r}_\mu\| \leq C\kappa^{-2m} + \mathcal{O}(\hbar^{1-2\mu})$. Here $\kappa < 1$ is the constant from Lemma 6.4.

Using Lemma C.1 in Appendix C we have that, for every $\epsilon > 0$, in the limit $\hbar \to 0$,

$$\sharp\{\mu_i^\hbar \in \sigma(\hat{k}_\mu) \mid |\mu_i^\hbar| \geq \epsilon\} \leq (2\pi\hbar)^{-1}(\tilde{C}_1 \mathrm{Leb}\{\mathcal{K}_{\mathsf{C}_0\hbar^\mu}\} + \tilde{C}_2\hbar). \tag{6.12}$$

By a standard perturbation argument the same estimate holds for the operator $\hat{P}_\mu$ (for m sufficiently large): for every $\epsilon > 0$, in the limit $\hbar \to 0$,

$$\sharp\{\mu_i^\hbar \in \sigma(\hat{P}_\mu) \mid |\mu_i^\hbar| \geq \epsilon + \|\hat{r}_\mu\|\} \leq (2\pi\hbar)^{-1}(\tilde{C}_1 \mathrm{Leb}\{\mathcal{K}_{\mathsf{C}_0\hbar^\mu}\} + \tilde{C}_2\hbar). \tag{6.13}$$

From the definition $\hat{P}_\mu := \hat{Q}_{m,\mu}^* \hat{Q}_{m,\mu}$, the $\sqrt{\mu_i^\hbar}$ are singular values of $\hat{Q}_{m,\mu}$. Then Corollary A.2 from Appendix A shows that the same estimate holds true for the eigenvalues of $\hat{Q}_{m,\mu}$, hence of $\hat{F}_{\hbar,\chi}$, yielding the result (6.10).

7. *Numerical results for the truncated Gauss map and Bowen–Series maps*

In this section we will present numerical results for two important classes of IFS: the truncated Gauss map and the Bowen–Series maps for convex co-compact hyperbolic surfaces. We will show that both examples satisfy the partially captive property. We will then give some numerical illustrations of the main theorems presented in this paper and finally discuss the connection between the spectrum of these transfer operators and the resonance spectrum of the Laplacian on hyperbolic surfaces.

7.1. *The truncated Gauss map.*

In this section we consider the IFS defined from the truncated Gauss map with N intervals presented in Example 2.2. We choose the roof function τ and the potential function V, which enter in the definition of the transfer operator (2.12), to be

$$\tau(x) = -J(x), \qquad V(x) = (1-a)J(x), \qquad a \in \mathbb{R}, \tag{7.1}$$

where $J(x) = \log(|(\phi^{-1})'(x)|) = \log(|G'(x)|) = -2\log(x)$ has been defined in (2.20). Let us write

$$s = a + ib \in \mathbb{C}, \qquad b = \frac{1}{\hbar} > 0.$$

Then, for every $s \in \mathbb{C}$, the transfer operator $\hat{F}$ given in (2.12) will be written $\hat{L}_s = \hat{F}$ and is given by

$$\hat{L}_s\varphi = \hat{F}\varphi = e^{V(x)} e^{i(1/\hbar)\tau(x)} \varphi \circ \phi^{-1} = e^{(1-s)J} \varphi \circ \phi^{-1}. \tag{7.2}$$

As explained in §7.1.1 below, this choice is interesting due its relation with the dynamics on the modular surface. The (adjoint of the) transfer operator $\hat{F}$ constructed in this way is usually called the *Gauss–Kuzmin–Wirsing transfer operator* or 'Dieter–Mayer transfer operator' for the truncated Gauss map [**39**, **55**].

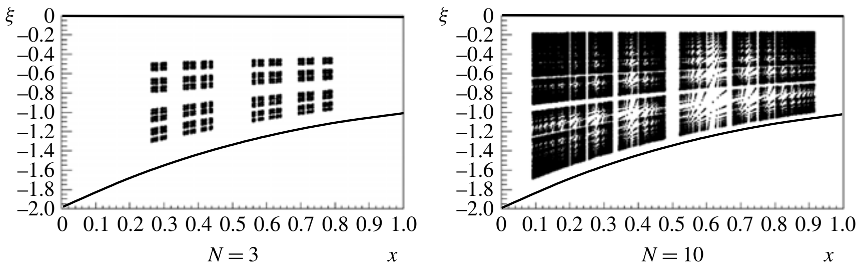

FIGURE 4. The trapped set $\mathcal{K}_N := \mathcal{K}$ for the truncated Gauss map with functions (7.1), for the cases of $N = 3$ and $N = 10$ branches. This corresponds to the Gauss–Kuzmin–Wirsing transfer operator (7.2). We have $\mathcal{K}_N \subset \mathcal{K}_{N+1}$ and, for $N \to \infty$, the limit trapped set $\mathcal{K}_\infty = \bigcup_{N\geq 0} \mathcal{K}_N = \{(x, \xi), x \in]0, 1[, -2/(1+x) < \xi < 0\}$ is the band between the marked black lines. (More precisely, we have represented the periodic points with period $n = 6$. That explains the sparse aspect of the trapped set.)

PROPOSITION 7.1. *For every $N \geq 1$, the minimal captivity Assumption 4.7 holds true for the truncated Gauss transfer operator defined by (7.2).*

The proof is given in §7.3 below. In this proof we explain the structure of the trapped set $\mathcal{K}$ with more details.

Consequently, we can apply Theorem 2.9 and deduce that there is an asymptotic spectral gap. In Figure 5 we present the numerical Ruelle resonances of the truncated Gauss map with three branches for different values of $\hbar$ and compare them with the prediction of the spectral gap. For the numerical calculation we directly use the conjugated transfer operator $\hat{Q}_m$ that appears in the proofs of the main theorems and develop it in a Fourier basis (see [**23**, §7] for more details on the numerical calculation of Ruelle resonances via the semiclassical approach). One observes on Figure 5 that the asymptotic spectral gap given by γ_+ is smaller than the general topological pressure bound $\mathrm{Pr}(-J)$, equation (2.22). The numerical results indicate, however, that this gap γ_+ is still not optimal.

We can also apply Theorem 2.16 and deduce a fractal Weyl upper bound for the density of resonances. In Figure 6 we determine the behavior of the counting function in relation to the semiclassical parameter in a double logarithmic plot. In these numerical results we do indeed observe an algebraic dependence and the exponent agrees very well with the upper bound of the Hausdorff dimension from Theorem 2.16. Note that this is particularly interesting because it is an important open conjecture that the fractal Weyl upper bounds are sharp (see e.g. [**41**, §6] for a review and further references). This conjecture has been supported by numerical experiments in different contexts, like quantum n-disk systems [**36**] or convex co-compact hyperbolic surfaces [**8**]. The data presented in Figure 6 provides more support for the general validity of the fractal Weyl law.

7.1.1. *Relation with the zeros of the Selberg zeta function.* For the geodesic flow on the modular surface $\mathrm{SL}_2\mathbb{Z}\backslash\mathrm{SL}_2\mathbb{R}$ it is possible to define the Selberg zeta function (see §7.2.1 below for more comments and references):

$$\zeta_{\mathrm{Selberg}}(s) = \prod_{\gamma} \prod_{m\geq 0} (1 - e^{-(s+m)|\gamma|}), \qquad s \in \mathbb{C},$$

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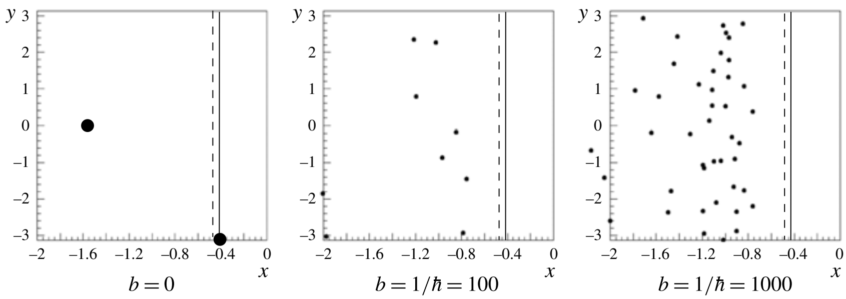

FIGURE 5. The discrete spectrum of Ruelle resonances λ_j (in log scale, writing $\log\lambda = x + iy$) for the truncated Gauss–Kuzmin–Wirsing transfer operator (7.2) associated to the Gauss map, for $N = 3$ branches and parameters $a = 1$, $b = 1/\hbar = 0$, 100, 1000. The full vertical line is at $x = \Pr(-J) \simeq -0.4$. For $b = 0$ there is the eigenvalue $\lambda = e^{\Pr(-J)}$ plotted at $(x, y) = (\Pr(-J), -\pi)$, corresponding to the 'equilibrium measure'. The dashed vertical line is at $x = \gamma_+$, which is shown in (2.24) to be an asymptotic upper bound for $b = 1/\hbar \to \infty$. In this example it seems to be not optimal.

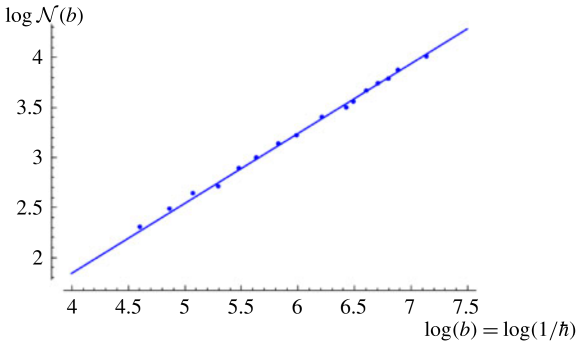

FIGURE 6. This is the Weyl law for the model of Gauss map with $N = 3$ branches. The points represent the number of resonances $\mathcal{N}(b) = \sharp\{\lambda_j \in \mathrm{Res}(\hat{L}_s), \log|\lambda_j| > -3.5\}$ computed numerically, as a function of the semiclassical parameter $b = 1/\hbar$ in log scale. The linear fit gives $\log\mathcal{N}(b) = -0.70 \cdot \log b - 0.96$, which has to be compared to the fractal Weyl law (2.33) giving $\log\mathcal{N}(b) \leq -\dim_H(K)\cdot\log b + \mathrm{cste}$. From (3) we have $\dim_H K_3 = 0.705$, giving an excellent agreement with the numerical results and suggesting that the upper bound is in fact optimal.

where the product is over the primitive periodic orbits γ of the geodesic flow and $|\gamma|$ denotes the length of the orbit. This zeta function is absolutely convergent for $\mathrm{Re}(s) > 1$. Using the Gauss map and continued fractions, Bowen and Series [**10**]† have shown that a periodic orbit γ is in one to one correspondence with a periodic sequence $(w_j)_{j\in\mathbb{Z}} \in (\mathbb{N}\backslash\{0\})^{\mathbb{Z}}$ where $w_j \in \mathbb{N}\backslash\{0\}$ is the index of the branch of the Gauss map $G_{w_j}^{-1}$ in (2.4). Given $N \geq 1$, we can restrict the product $\prod_\gamma$ over periodic orbits above to orbits for which $w_j \leq N$, for all $j \in \mathbb{Z}$, and define a truncated Selberg zeta function as follows:

$$\zeta_{\mathrm{Selberg},N}(s) = \prod_{\gamma, w_j \leq N, \text{for all } j} \prod_{m\geq 0} (1 - e^{-(s+m)|\gamma|}), \qquad s \in \mathbb{C}.$$

† For the special case of the modular surface and the Gauss map such a correspondence has indeed been known for a long time, see e.g. [**1**].

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On the other hand, for fixed $s \in \mathbb{C}$, we have from Theorem 2.6 that the operator $\hat{L}_s$ has a discrete spectrum of Ruelle resonances. It is possible to define the dynamical determinant of $\hat{L}_s$ by

$$d(z, s) := \mathrm{Det}(1 - z\hat{L}_s) := \exp\bigg(-\sum_{n \geq 1} \frac{z^n}{n} \mathrm{Tr}^\flat(\hat{L}_s^n)\bigg), \quad z \in \mathbb{C},$$

where $\mathrm{Tr}^\flat(\hat{L}_s^n)$ stands for the flat trace of Atiyah–Bott. The sum is convergent for $|z|$ small enough. It is known that, for fixed s, the zeros of $d(z, s)$ (as a function of z) coincide with multiplicities with the Ruelle resonances of $\hat{L}_s$ [**2**]. In the case $z = 1$, we also have that $d(1, s)$ coincides with the truncated Selberg zeta function [**7**, **45**]:

$$\mathrm{Det}(1 - \hat{L}_s) = \zeta_{\mathrm{Selberg},N}(s), \tag{7.3}$$

which means that the zeros of $\zeta_{\mathrm{Selberg},N}(s)$ are given (with multiplicity) by the event that 1 is a Ruelle resonance of the transfer operator $\hat{L}_s$. This also shows that $\zeta_{\mathrm{Selberg},N}(s)$ has a holomorphic extension to the complex plane $s \in \mathbb{C}$.

Remark 7.2. In [**45**], [**7**, p. 306] they consider the adjoint operator $\hat{L}_s^*$, called the Perron–Frobenius operator.

7.2. *Bowen–Series maps for Schottky surfaces.* The second class of examples that we consider in this section are Bowen–Series maps for Schottky surfaces [**10**]. We will follow the notation of Borthwick's book [**7**, Ch. 15] and recall the definition of a Schottky group given there. Recall that an element $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$ and $\overline{\mathbb{R}} = \partial\mathbb{H}^2$ by $S(x) := (ax + b)/(cx + d)$.

Definition 7.3. Let $D_1, \ldots, D_{2r}$ be disjoint closed half discs in the Poincare half plane $\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$ with center in $\mathbb{R} = \partial\mathbb{H}^2 \backslash \{\infty\}$. There exist elements $S_i \in \mathrm{SL}_2\mathbb{R}$, $i = 1, \ldots, r$, such that $S_i(\partial D_i) = \partial D_{i+r}$ and $S_i(\mathrm{Int}(D_i)) = \mathbb{C} \setminus D_{i+r}$. The group generated by the S_i is called a *Schottky group* $\Gamma = \langle S_1, \ldots, S_r\rangle$.

Remark 7.4. For convenience we will use a cyclic notation for the indices $i = 1, \ldots, 2r$. Then one can also define S_i for $i = r + 1, \ldots, 2r$ as in the definition above and obtain $S_{i+r} = S_i^{-1}$.

Let $I_i := D_i \cap \partial\mathbb{H}$. Then $(I_i)_{i=1,\ldots,2r}$ are $N = 2r$ disjoint closed intervals. One has $S_j(\mathrm{Int}(I_j)) = \partial\mathbb{H} \backslash I_{j+r}$ and we assume that S_j is expanding on I_j (this can always be obtained by taking iterations if necessary and localizing further to the trapped set, see [**7**, Proposition 15.4]). The maps S_j are usually called the Bowen–Series maps. Considering the inverse maps, one obtains an IFS according to Definition 2.1 associated to this Schottky group in the following way. For any $j = 1, \ldots, N$ and $i \neq j + r$ let

$$\phi_{i,j} := S_j^{-1} = S_{j+r} : I_i \to S_j^{-1}(I_i) \subset \mathrm{Int}(I_j).$$

The adjacency matrix $A_{i,j}$ has all entries equal to one except $A_{i,i+r} = 0$ (see Figure 7).

As in (7.1) we make the following choice for the potential and the roof function for $x \in I_j$:

$$\tau(x) = -J(x), \quad V(x) = (1 - a)J(x), \quad a \in \mathbb{R}, \tag{7.4}$$

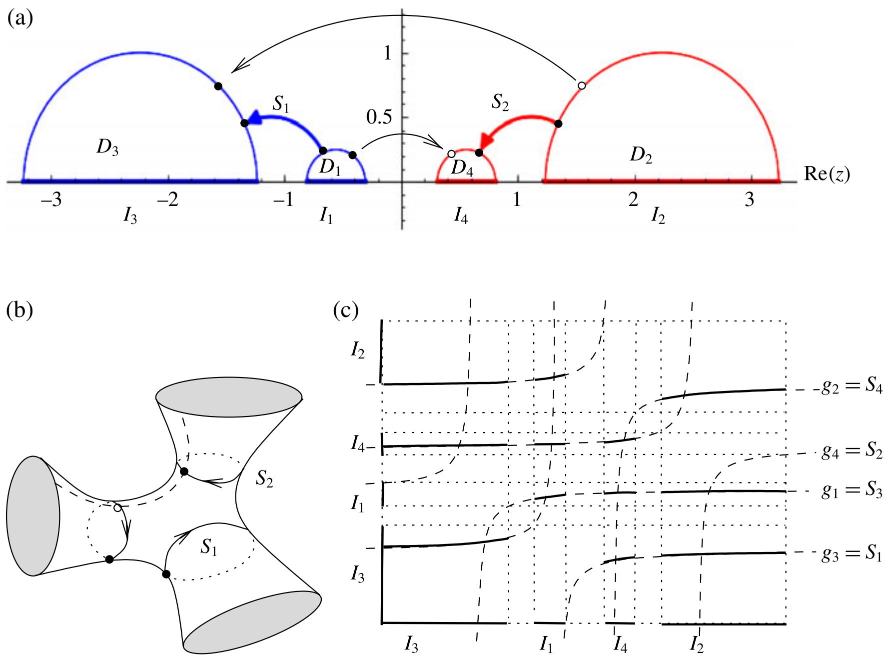

FIGURE 7. In this (arbitrary) example, we have $r = 2$ hyperbolic matrices of $\mathrm{SL}_2\mathbb{R}$: $S_1 = \begin{pmatrix} 4 & \sqrt{5} \\ -\sqrt{5} & -1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} -1 & \sqrt{5} \\ -\sqrt{5} & 4 \end{pmatrix}$ that generate a Schottky group $\Gamma = \langle S_1, S_2 \rangle$. (a) The Dirichlet fundamental domain $\mathbb{H}^2 \backslash (D_1 \cup D_2 \cup D_3 \cup D_4)$ with the intervals I_i, $i = 1, 2, 3, 4$, on which the IFS is defined. (b) The resulting Schottky surface $\Gamma \backslash \mathbb{H}^2$. It has three funnels. (c) The graph of the generating functions $\phi_{i,j} = g_j = S_{j+r} : I_i \to I_j$ of the associated iterated function system.

where $J(x) = \log(|(\phi_{i,j}^{-1})'(x)|) = \log(|S_j'(x)|)$ has been defined in (2.20). Let us write

$$s = a + ib \in \mathbb{C}, \quad b = \frac{1}{\hbar} > 0.$$

Then, for every $s \in \mathbb{C}$, the transfer operator $\hat{F}$ given in (2.12) will be written $\hat{L}_s = \hat{F}$ and is given by

$$\hat{L}_s \varphi = \hat{F}\varphi = e^{V(x)} e^{i(1/\hbar)\tau(x)} \varphi \circ \phi^{-1} = e^{(1-s)J} \varphi \circ \phi^{-1}. \tag{7.5}$$

The adjoint of our transfer operator $L_s^* = \hat{F}^*$ is exactly the Ruelle transfer operator defined in [**7**, p. 304] and, as we will discuss below, its spectrum is closely connected to the spectrum of the Laplace operator on the Schottky surface.

PROPOSITION 7.5. *The minimal captivity Assumption 4.7 holds true for the Bowen–Series transfer operator defined by (7.5).*

The proof is given in §7.3 below. Consequently, we can apply Theorem 2.9 and deduce that there is an asymptotic spectral gap. We can also apply Theorem 2.16 and deduce a fractal Weyl upper bound for the density of resonances.

Remark 7.6. We recall from §4.3 that minimal captivity implies the NLI condition. Naud in [**40**] has already shown that this weaker NLI condition holds true for Schottky surfaces.

7.2.1. *Selberg zeta function and resonances of the Laplacian.* For the geodesic flow on a hyperbolic surface it is possible to define the Selberg zeta function:

$$\zeta_{\mathrm{Selberg}}(s) = \prod_{\gamma} \prod_{m \geq 0} (1 - e^{-(s+m)|\gamma|}),$$

where the product is over primitive periodic orbits γ of the geodesic flow and $|\gamma|$ denotes the length of the orbit. This zeta function is absolutely convergent for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the whole complex plane. This continuation is particularly interesting as its zeros are either 'topological zeros' located on the real axis or resonances of the Laplace operator Δ on the corresponding hyperbolic surface $\Gamma\backslash\mathbb{H}^2$. These resonances $s \in \mathrm{Res}(\Delta)$ are defined as the poles of the meromorphic extension of the resolvent [**7**]:

$$R(s) := (\Delta - s(1-s))^{-1}, \quad s \in \mathbb{C}. \tag{7.6}$$

This correspondence follows from the Selberg trace formula for finite-area surfaces, and has been shown by Patterson and Perry [**43**] for infinite volume surfaces without cusps and Borthwick *et al* [**9**] for infinite volume surfaces with cusps (see also [**7**] for an overview).

For the transfer operators as defined above, one can define a dynamical zeta function by [**7**, p. 305]

$$d(z, s) := \mathrm{Det}(1 - z\hat{L}_s).$$

The dynamical and the Selberg zeta functions are equal, $\zeta_{\mathrm{Selberg}}(s) = d(1, s)$ (see [**7**, Theorem 15.8]). This implies immediately that if $s \in \mathbb{C}$ is a resonance of the Laplacian on the Schottky surface, then 1 has to be an eigenvalue of $\hat{L}_s$:

$$s \in \mathrm{Res}(\Delta) \Leftrightarrow 1 \in \mathrm{Spec}(\hat{L}_s). \tag{7.7}$$

Remark 7.7. For the full Gauss map (i.e. with infinitely many branches) the same correspondence between the resonances of the Laplacian on the modular surface $\mathrm{SL}_2\mathbb{Z}\backslash\mathbb{H}^2$ and the Dieter–Mayer transfer operator $\hat{L}_s$ is true and has been developed by Dieter Mayer [**39**]. For the truncated Gauss map considered in §7.1, to our knowledge, no such corresponding surfaces are known.

Using the relation (7.7) between the Ruelle spectrum of the transfer operator $\hat{L}_s$ and the resonances of the Laplacian, it is possible to deduce from Theorem 2.9 some estimate of the 'asymptotic spectral gap' for the resonances of the Laplacian as follows.

Definition 7.8. The *asymptotic spectral gap* of resonances of the Laplacian Δ is defined by

$$a_{\mathrm{asymp}} := \limsup_{b \to \infty} \{\mathrm{Re}(s) \text{ s.t. } s \in \mathrm{Res}(\Delta), |\mathrm{Im}(s)| > b\}.$$

The setting (7.4) gives $D(x) = V - \frac{1}{2}J = (\frac{1}{2} - a)J(x)$, hence our estimate (2.25) gives that $a_{\mathrm{asympt}} \leq \frac{1}{2}$. However, this result concerning the resonances of the hyperbolic Laplacian is not new: from the self-adjoint properties of the Laplacian Δ in L^2 space we have that $\mathrm{Im}(s(1-s)) \leq 0$ and this gives that

$$a_{\mathrm{asympt}} \leq \tfrac{1}{2}. \tag{7.8}$$

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Remark 7.9. If δ denotes the dimension of the limit set (equal to the dimension of the trapped set K) a result from Naud gives [**40**]: there exists $\varepsilon > 0$ such that

$$a_{\text{asymp}} \leq (1-\varepsilon)\delta$$

which improves (7.8) if $\delta \leq 1/2$.

7.3. *Proof of minimal captivity for both models.* We give now the proof of Propositions 7.1 and 7.5. Note first that in both models the contracting maps are Möebius maps, i.e. of the form $x'_j = \phi_{i,j}(x) = (a_j x + b_j)/(c_j x + d_j) = g_j(x)$ with 2×2 matrices $g_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ with $D_j := \det g_j = \pm 1$. For the truncated Gauss map these matrices are

$$g_j = \begin{pmatrix} 0 & 1 \\ 1 & j \end{pmatrix} = G_j^{-1}, \tag{7.9}$$

with $j = 1, \ldots, N$ and $D := D_j = -1$. For the Bowen–Series maps we have

$$g_j = S_j^{-1} \in \mathrm{SL}_2\mathbb{R}, \tag{7.10}$$

with $j = 1, \ldots, 2r$ and $D := D_j = +1$.

The following proposition shows that there exists coordinates (x, η) on phase space such that the canonical map $F = (F_j)_{j=1,\ldots,N}$ is decoupled in a product of identical maps.

LEMMA 7.10. *The canonical map F defined in (4.3) is the union of the following maps F_j, with $j = 1, \ldots, N$:*

$$\begin{aligned} (x'_j, \xi'_j) = F_j(x, \xi) &= (g_j(x), (g_j^{-1})'(x'_j)\xi + \tau'(x'_j)) & (7.11)\\ &= (g_j(x), D_j \cdot (c_j x + d_j)^2 \xi - 2c_j(c_j x + d_j)). & (7.12)\end{aligned}$$

Using the change of variables $(x, \eta) = \Phi(x, \xi) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and

$$\eta := x - \frac{2D}{\xi}, \tag{7.13}$$

the map F_j gets the simpler 'decoupled expression'

$$(x'_j, \eta'_j) = (\Phi \circ F_j \circ \Phi^{-1})(x, \eta) = (g_j(x), g_j(\eta)). \tag{7.14}$$

Remark 7.11. Geometrically, these new variables (x, η) can be interpreted as the limit points $(x, \eta) \in \partial\mathbb{H}$ of a geodesic. The map $(x', \eta') = (\Phi \circ F \circ \Phi^{-1})(x, \eta)$ is simply the Poincare map of the geodesic flow [**13**].

Proof. One has

$$g_j^{-1} = D_j \cdot \begin{pmatrix} d_j & -b_j \\ -c_j & a_j \end{pmatrix}, \; (g_j^{-1})(y) = \frac{d_j y - b_j}{-c_j y + a_j}$$

and

$$(g_j^{-1})'(y) = D_j \cdot (a_j - c_j y)^{-2} = D_j \cdot (c_j x + d_j)^2 \quad \text{if } y = g_j(x).$$

The roof function is given by (7.1):

$$\begin{aligned} \tau(y) = -J(y) &= -\log(|(\phi_{i,j}^{-1})'(y)|) = -\log(|(g_j^{-1})'(y)|) \\ &= 2\log(a_j - c_j y). \end{aligned}$$

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So $\tau'(y) = -2c_j(a_j - c_j y)^{-1} = -2c_j(c_j x + d_j)$ and

$$(x'_j, \xi'_j) = F_j(x, \xi) \underset{(7.11)}{=} (g_j(x), D \cdot (c_j x + d_j)^2 \xi - 2c_j(c_j x + d_j)), \tag{7.15}$$

giving (7.12). Now we use the change of variable

$$\xi = \frac{2D}{x - \eta}. \tag{7.16}$$

So

$$\begin{aligned} \xi'_j &= D \cdot (c_j x + d_j)^2 \xi - 2c_j(c_j x + d_j) \\ &= D \cdot (c_j x + d_j)^2 \frac{2D}{(x - \eta)} - 2c_j(c_j x + d_j) \\ &= \frac{2(c_j x + d_j)}{(x - \eta)}(c_j \eta + d_j). \end{aligned}$$

Then

$$\begin{aligned} \eta'_j = x'_j - \frac{2D}{\xi'_j} &= \frac{a_j x + b_j}{c_j x + d_j} - \frac{D(x - \eta)}{(c_j x + d_j)(c_j \eta + d_j)} \\ &= \frac{(a_j x + b_j)(c_j \eta + d_j) - (a_j d_j - b_j c_j)(x - \eta)}{(c_j x + d_j)(c_j \eta + d_j)} = \frac{a_j \eta + b_j}{c_j \eta + d_j} = g_j(\eta). \end{aligned}$$ □

Recall that the multivalued map $\phi = (\phi_{i,j} = g_j)_j$ has a trapped set K defined in (2.7) as $K = \bigcap_{n \geq 1} \phi^n(I)$. The basin of K on $\overline{\mathbb{R}}$ is $\mathcal{B}(K) := \{x \in \overline{\mathbb{R}}, \exists n \geq 0, \phi^n(x) \in I\} \subset \overline{\mathbb{R}}$.

LEMMA 7.12. *The trapped set in phase space $\mathcal{K}$ defined in (4.7) is contained in the following set:*

$$\mathcal{K} \subset \{(x, \xi), x \in I, \eta \notin \mathcal{B}(K) \text{ with } (x, \eta) = \Phi(x, \xi)\}, \tag{7.17}$$

Proof. Let $(x, \xi) \in I \times \mathbb{R}$, which does not belong to the set defined on the right-hand side of (7.17). Then $\eta \in \mathcal{B}(K)$. Hence, for every admissible word $w \in \mathcal{W}$, we have that $|\phi_{w_{0,n}}(x) - \phi_{w_{0,n}}(\eta)| \leq C \cdot \theta^n \to_{n \to +\infty} 0$. From the change of variable (7.16) and the expression (7.14) with the new variables, this gives that $(x_n, \xi_n) := F_{w_{0,n}}(x, \xi)$ satisfies

$$|\xi_n| = \frac{2}{|\phi_{w_{0,n}}(x) - \phi_{w_{0,n}}(\eta)|} \geq C' \cdot \theta^{-n} \to +\infty,$$

hence $(x, \xi) \notin \mathcal{K}$. We deduce (7.17). □

Finally, we show minimal captivity of the canonical map F. According to (4.8), we have to show that there exists a neighborhood B of $\mathcal{K}$ such that, for all $(x, \xi) \in B$, $\sharp\{F(x, \xi) \cap B\} \leq 1$. This is true if $B_j := F_j^{-1}(B)$, $j = 1, \ldots, N$ are disjoint sets. Using the coordinates (x, η) which decouple the map F_j, in (7.14), it is equivalent to show that there exists a neighborhood $\mathcal{B}$ of K in $\overline{\mathbb{R}}$ such that $\mathcal{B}_j := g_j^{-1}(\mathcal{B}) \subset \overline{\mathbb{R}}$, $j = 1, \ldots, N$ are disjoint sets. For this we consider both cases as follows.

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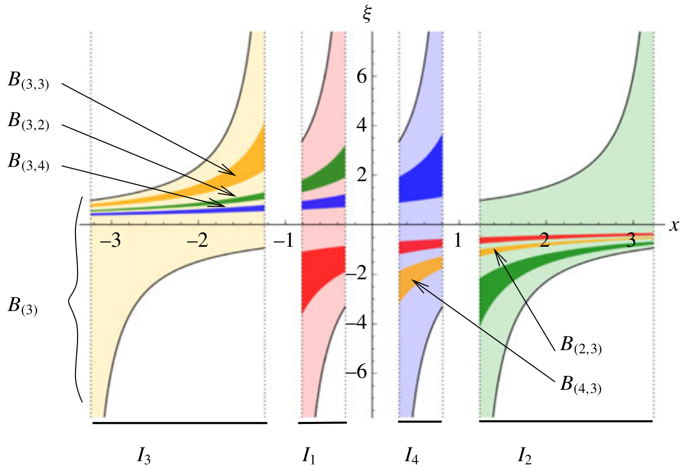

FIGURE 8. This figure illustrates the choice of the bounding functions in the proof of the minimally captive property for the example of a Schottky surface shown in Figure 7. The light shaded regions indicate the set $B_{(j)} := B \cap (I_j \times \mathbb{R})$ while the darker shaded regions indicate the different pre-images $B_{(i,j)} := F_{ij}^{-1}(B \cap (I_j \times \mathbb{R})) \subset B_{(i)}$, $i \neq j+2 \bmod 4$. For example, the dark shaded regions $B_{(3,3)}$, $B_{(4,3)}$, $B_{(2,3)}$ show the three pre-images of the light region $B_{(3)}$. The trapped set $\mathcal{K}$ is contained in the union of these $B_{(i,j)}$.

Minimal captivity of the truncated Gauss map. For this map, let $\mathcal{B} :=]-\infty, -1[$. Then the sets $g_j^{-1}(]-\infty, -1[) =]-j-1, j[$, with $j = 1, \dots, N$, are mutually disjoint. From the argument above this implies that the truncated Gauss map is minimally captive, i.e. Proposition 7.1 holds. Notice that, from (7.16), in variables $(x, \xi) \in T^*[0, 1]$ we have

$$B = \{x \in [0, 1], \eta \in]-\infty, -1[\} = \left\{(x, \xi), x \in [0, 1], \frac{-2}{x+1} < \xi < 0\right\}.$$

This set B contains the trapped set $\mathcal{K}_N$ and is depicted in Figure 4.

Minimal captivity of the Bowen–Series map. For this case, let $\mathcal{B} := I = \bigcup_{j=1}^{2r} I_j$. Then $\mathcal{B}_j = g_j^{-1}(\mathcal{B}) = g_{j+r}(I) \subset I_{j+r}$. Since the sets I_{j+r} are mutually disjoint, the sets $\mathcal{B}_j$ are also disjoint. From the argument above this implies that the Bowen–Series map on phase space is minimally captive, i.e. Proposition 7.5 holds.

Figure 8 shows the sets $B_j = F_j^{-1}(B)$ with $B := \{x \in I, \eta \in \mathcal{B}\}$ and

$$B_j = \{x \in I, \eta \in \mathcal{B}_j\}$$

that we have used in the proof of minimal captivity.

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A. *Appendix. General lemmas on singular values of compact operators*

Let $(P_n)_{n\in\mathbb{N}}$ be a family of compact operators on some Hilbert space. For every $n \in \mathbb{N}$, let $(\lambda_{j,n})_{j\in\mathbb{N}} \in \mathbb{C}$ be the sequence of eigenvalues of P_n counted with multiplicity (i.e. repeated

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according to the dimension of the eigenspace) and ordered decreasingly:

$$|\lambda_{0,n}| \geq |\lambda_{1,n}| \geq \cdots.$$

In the same manner, define $(\mu_{j,n})_{j\in\mathbb{N}} \in \mathbb{R}^+$, the decreasing sequence of singular values of P_n, i.e. the eigenvalues of $\sqrt{P_n^* P_n}$.

LEMMA A.1. *Suppose there exits a map $N : \mathbb{N} \to \mathbb{N}$ such that $N(n) \to_{n\to\infty} \infty$ and $\mu_{N(n),n} \to_{n\to\infty} 0$, then, for all $C > 1$, $|\lambda_{[C\cdot N(n)],n}| \to_{n\to\infty} 0$ where $[\cdot]$ stands for the integer part.*

COROLLARY A.2. *Suppose there exits a map $N : \mathbb{N} \to \mathbb{N}$ such that for all $\varepsilon > 0$, there exists $A_\varepsilon \geq 0$ such that for all $n \geq A_\varepsilon$,*

$$\#\{j \in \mathbb{N} \text{ s.t. } \mu_{j,n} > \varepsilon\} < N(n),$$

then, for all $C > 1$, for all $\varepsilon > 0$, there exists $B_{C,\varepsilon} \geq 0$ such that for all $n \geq B_{C,\varepsilon}$,

$$\#\{j \in \mathbb{N} \text{ s.t. } |\lambda_{j,n}| > \varepsilon\} \leq C \cdot N(n). \tag{A.1}$$

Proof of Corollary A.2. Suppose that for any $\varepsilon > 0$, there exists A_ε such that for all $n \geq A_\varepsilon$, $\#\{j \in \mathbb{N} \text{ s.t. } \mu_{j,n} > \varepsilon\} < N(n)$. Then $\mu_{N(n),n} \to_{n\to\infty} 0$ and from Lemma A.1, for all $C > 1$, $|\lambda_{[C\cdot N(n)],n}| \to_{n\to\infty} 0$, which can be directly restated as (A.1).

Proof of Lemma A.1. Let $m_{j,n} := -\log \mu_{j,n}$ and $l_{j,n} := -\log |\lambda_{j,n}|$, $M_{k,n} := \sum_{j=0}^{k} m_{j,n}$ and $L_{k,n} := \sum_{j=0}^{k} l_{j,n}$. Weyl inequalities relate singular values and eigenvalues by (see [**27**, p. 50] for a proof)

$$\prod_{j=1}^{k} \mu_{j,n} \geq \prod_{j=1}^{k} |\lambda_{j,n}| \quad \text{for all } k \geq 1. \tag{A.2}$$

This can be rewrittin as

$$M_{k,n} \leq L_{k,n} \quad \text{for all } k,\, n. \tag{A.3}$$

The sequence $(l_{j,n})_{j\geq 0}$ is increasing in j so, for all n, for all k, we have

$$k \cdot l_{k,n} \geq L_{k,n}. \tag{A.4}$$

Suppose that $\mu_{N(n),n} \to 0$ as $n \to \infty$, hence

$$m_{N(n),n} \underset{n\to\infty}{\to} \infty. \tag{A.5}$$

Let $C > 1$. The sequence $(m_{j,n})_{j\geq 0}$ is increasing in j, hence

$$M_{[C\cdot N(n)],n} \geq ([C \cdot N(n)] - N(n)) \cdot m_{N(n),n}, \tag{A.6}$$

hence

$$\begin{aligned} l_{[C\cdot N(n)],n} &\underset{(A.4)}{\geq} \frac{1}{[CN(n)]} \cdot L_{[C\cdot N(n)],n} \underset{(A.3)}{\geq} \frac{1}{[C \cdot N(n)]} M_{[C\cdot N(n)],n} \\ &\underset{(A.6)}{\geq} \frac{[C \cdot N(n)] - N(n)}{[C \cdot N(n)]} \cdot m_{N(n),n} \underset{(A.5)}{\longrightarrow} \infty. \end{aligned}$$

Thus $l_{[C\cdot N(n)],n} \to_{n\to\infty} \infty$ and $|\lambda_{[C\cdot N(n)],n}| \to_{n\to\infty} 0$. □

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B. *Appendix. Symbol classes of local $\hbar$-order*

In this appendix we will first repeat the definitions of the standard symbol classes which are used in this article, as well as their well-known quantization rules. Then we will introduce a new symbol class which allows $\hbar$-dependent order functions and will prove some of the classical results which are known in the usual case for these new symbol classes.

B.1. *Standard semiclassical symbol classes and their quantization.* The standard symbol classes (see e.g. [**57**, Ch. 4] or [**15**, Ch. 7]) of $\hbar$-PDOs are defined with respect to an order function $f(x, \xi)$. This order function is required to be a smooth positive-valued function on $\mathbb{R}^{2n}$ such that there are constants C_0 and N_0 fulfilling

$$f(x,\xi) \le C_0 \langle (x,\xi) - (x',\xi') \rangle^{N_0} f(x',\xi'), \tag{B.1}$$

where we have used the notation $\langle y \rangle := \sqrt{1+|y|^2}$ for $y \in \mathbb{R}^k$. An important example of such an order function is given by $f(x,\xi) = \langle \xi \rangle^m$ with $m \in \mathbb{R}$.

Definition B.1. For $0 \le \mu \le \frac{1}{2}$ and $k \in \mathbb{R}$, the symbol classes $\hbar^k S_\mu(f)$ contain all families of functions $a_\hbar(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ parametrized by a parameter $\hbar \in]0, \hbar_0]$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_\hbar(x,\xi)| \le C \hbar^{k-\mu(|\alpha|+|\beta|)} \langle \xi \rangle^m,$$

where C depends only on $\alpha, \beta \in \mathbb{N}^n$.

Unless we want to emphasize the dependence of the symbol $a_\hbar$ on $\hbar$ we will drop the index in the following. For the special case of the order function $f(x,\xi) = \langle \xi \rangle^m$ we also write $S_\mu^m = S_\mu(\langle \xi \rangle^m)$; if $\mu = 0$ we write $S(f) := S_0(f)$.

As quantization we use two different quantization rules in this article, which are called respectively standard quantization and Weyl quantization.

Definition B.2. Let $a_\hbar \in S_\mu(f)$. The Weyl quantization is a family of operators $\mathrm{Op}_\hbar^w(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, defined by

$$(\mathrm{Op}_\hbar^w(a_\hbar)\varphi)(x) = (2\pi\hbar)^{-n} \int e^{(i/\hbar)\xi(x-y)} a_\hbar\left(\frac{x+y}{2}, \xi\right) \varphi(y)\, dy\, d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{B.2}$$

while the standard quantization $\mathrm{Op}_\hbar(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is given by

$$(\mathrm{Op}_\hbar(a_\hbar)\varphi)(x) = (2\pi\hbar)^{-n} \int e^{(i/\hbar)\xi(x-y)} a_\hbar(x,\xi) \varphi(y)\, dy\, d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{B.3}$$

Both quantizations extend continuously to operators on $\mathcal{S}'(\mathbb{R}^n)$. While the standard quantization is slightly easier to define, the Weyl quantization has the advantage that real symbols are mapped to formally self-adjoint operators.

B.2. *Definition of the symbol classes $S_\mu(A_\hbar)$.* In the standard $\hbar$-PDO calculus the symbols are ordered by their asymptotic behavior for $\hbar \to 0$. If we take, for example, a symbol $a \in \hbar^k S_\mu(f)$ then $a(x,\xi)$ is of order $\hbar^k$ for all $(x,\xi) \in \mathbb{R}^{2n}$. The symbol classes that we will now introduce will also allow an $\hbar$-dependent order function, which will allow control of the $\hbar$-order of a symbol locally, i.e. dependent on (x,ξ). First we define these $\hbar$-dependent order functions as follows.

Definition B.3. Let f be an order function on $\mathbb{R}^{2n}$ and $0 \le \mu \le \frac{1}{2}$. Let $A_\hbar \in S_\mu(f)$ be a (possibly $\hbar$-dependent) positive symbol such that, for some $c \ge 0$, there is a constant C that fulfills

$$A_\hbar(x, \xi) \ge C\hbar^c f(x, \xi) \tag{B.4}$$

and that for all multi-indices $\alpha, \beta \in \mathbb{N}^n$,

$$|\partial_x^\alpha \partial_\xi^\beta A_\hbar(x, \xi)| \le C_{\alpha,\beta} \hbar^{-\mu(|\alpha|+|\beta|)} A_\hbar(x, \xi) \tag{B.5}$$

holds. Then we call $A_\hbar$ an $\hbar$-dependent order function and say $A_\hbar \in \mathcal{OF}^c(f)$.

Definition B.4. Let $0 \le \mu \le \frac{1}{2}$ and $A_\hbar$ be an $\hbar$-dependent order function. The symbol class $S_\mu(A_\hbar)$ is then defined to be the space of smooth functions $a_\hbar(x, \xi)$ defined on $\mathbb{R}^{2n}$ and parametrized by a parameter $\hbar \in]0, \hbar_0]$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_\hbar(x, \xi)| \le C_{\alpha,\beta} \hbar^{-\mu(|\alpha|+|\beta|)} A_\hbar(x, \xi). \tag{B.6}$$

As usual, we will denote by $\hbar^k S_\mu(A_\hbar)$ the symbols $a_\hbar$ for which $\hbar^{-k} a_\hbar \in S_\mu(A_\hbar)$.

The set $S_\mu(A_\hbar)$ only depends on the $\hbar$-dependent order function $A_\hbar$ and the real parameter $0 \le \mu \le \frac{1}{2}$. From the Definition B.3 of $\hbar$-dependent order function we conclude, however, that there is an order function f such that $A_\hbar \in \mathcal{OF}^c(f)$.

As $A_h(x, \xi) \le C_0 f(x, \xi)$, and from (B.5), it is obvious that

$$S_\mu(A_\hbar) \subset S_\mu(f) \tag{B.7}$$

and via this inclusion for $a_\hbar \in S_\mu(A_\hbar)$ the standard quantization $Op_\hbar(a)$ and the Weyl quantization $Op_\hbar^w(a_\hbar)$ are well defined and give continuous operators on $\mathcal{S}(\mathbb{R}^n)$, respectively on $\mathcal{S}'(\mathbb{R}^n)$ (see e.g. [**57**, Theorem 4.16]). Furthermore, equation (B.4) gives us a second inclusion

$$S_\mu(f) \subset \hbar^{-c} S_\mu(A_\hbar); \tag{B.8}$$

thus, combining these two inclusions we have

$$\hbar^c S_\mu(f) \subset S_\mu(A_\hbar) \subset S_\mu(f).$$

As for standard $\hbar$-PDO symbols we can define asymptotic expansions as follows.

Definition B.5. Let $a_j \in S_\mu(A_\hbar)$ for $j = 0, 1, \ldots$ then we call $\sum_j \hbar^j a_j$ an asymptotic expansion of $a \in S_\mu(A_\hbar)$ (writing $a \sim \sum_j \hbar^j a_j$) if and only if, for all $k \in \mathbb{N}$,

$$a - \sum_{j<k} \hbar^j a_j \in \hbar^k S_\mu(A_\hbar).$$

As for the standard $\hbar$-PDOs we have some sort of Borel's theorem for symbols in $S_\mu(A_\hbar)$ also.

PROPOSITION B.6. *Let $a_j \in S_\mu(A_\hbar)$ for $j = 0, 1, \ldots$, then there is a symbol $a \in S_\mu(A_\hbar)$ such that, for all $k \in \mathbb{N}$,*

$$a - \sum_{j<k} \hbar^j a_j \in \hbar^k S_\mu(A_\hbar). \tag{B.9}$$

Proof. Once more we can use the inclusion (B.7) into the standard $\hbar$-PDO classes and obtain the existence of a symbol $a \in S_\mu(f)$ such that (see [**57**, Theorem 4.15])

$$a - \sum_{j<k} \hbar^j a_j \in \hbar^k S_\mu(f), \tag{B.10}$$

and we will show that this symbol belongs to $S_\mu(A_\hbar)$ and that (B.9) holds. For the first statement we write

$$a = \underbrace{a - \sum_{j<c} \hbar^j a_j}_{\in \hbar^c S_\mu(f)} + \underbrace{\sum_{j<c} \hbar^j a_j}_{\in S_\mu(A_\hbar)}$$

and use the inverse inclusion (B.8).

In order to prove (B.9) we write

$$a - \sum_{j<k} \hbar^j a_j = \underbrace{a - \sum_{j<k+c} \hbar^j a_j}_{\in \hbar^{c+k} S_\mu(f)} + \underbrace{\sum_{j=k}^{k+c-1} \hbar^j a_j}_{\in \hbar^k S_\mu(A_\hbar)}$$

and use once more (B.8). □

The advantage of this new symbol class is that the order function $A_\hbar(x, \xi)$ itself can depend on $\hbar$ and thus the control in $\hbar$ can be localized. A simple example for such an order function would be $A_\hbar = \hbar^{m\mu} \langle \xi/\hbar^\mu \rangle^m \in \mathcal{OF}^{m\mu}(\langle \xi \rangle^m)$. For $\xi \neq 0$ this function is of order $\hbar^0$, whereas for $\xi = 0$ it is of order $\hbar^{m\mu}$. Thus also all symbols in $S_\mu(A_\hbar)$ have to show this behavior.

B.3. *Composition of symbols.* By using the inclusion (B.7) we will show a result for the composition of symbols absolutely analogous to the one in the standard case [**57**, Theorem 4.18]. We first note that, for $A_\hbar \in \mathcal{OF}^{c_A}(f_A)$ and $B_\hbar \in \mathcal{OF}^{c_B}(f_B)$, the product formula for the derivative yields that $A_\hbar B_\hbar \in \mathcal{OF}^{c_A+c_B}(f_A f_B)$ and can now formulate the following theorem.

THEOREM B.7. *Let $A_\hbar \in \mathcal{OF}^{c_A}(f_A)$ and $B_\hbar \in \mathcal{OF}^{c_B}(f_B)$ be two $\hbar$-dependent order functions and $a \in S_\mu(A_\hbar)$ and $b \in S_\mu(B_\hbar)$ two $\hbar$-local symbols. Then there is a symbol*

$$a\#b \in S_\mu(A_\hbar B_\hbar)$$

such that

$$Op_\hbar^w(a) Op_\hbar^w(b) = Op_\hbar^w(a\#b) \tag{B.11}$$

as operators on $\mathcal{S}$, and at first order we have

$$a\#b - ab \in \hbar^{1-2\mu} S_\mu(A_\hbar B_\hbar). \tag{B.12}$$

Proof. The standard theorem of composition of $\hbar$-PDOs (see e.g. Theorem 4.18 in [**57**]) together with the inclusion of symbol classes (B.7) provides us with a symbol

$a\#b \subset S_\mu(f_A \cdot f_B)$ that fulfills equation (B.11). Furthermore, it provides us with a complete asymptotic expansion for $a\#b$:

$$a\#b - \sum_{k=0}^{N-1}\left(\frac{1}{k!}\left[\frac{i\hbar(\langle D_x, D_\eta\rangle - \langle D_y, D_\xi\rangle)}{2}\right]^k a(x,\xi)b(y,\eta)\right)_{|y=x,\eta=\xi} \in \hbar^{N(1-2\mu)}S_\mu(f_A\cdot f_B). \tag{B.13}$$

In order to prove our theorem it thus is only left to show that $a\#b \in S_\mu(A_\hbar B_\hbar)$ and that equation (B.12) holds. We start with the second one. First, let $N\in\mathbb{N}$ be such that $(N-1)(1-2\mu)\geq c_A + c_B$, then equation (B.13) and inclusion (B.8) ensure that the remainder term in (B.13) is in $\hbar^{1-2\mu}S_\mu(A_\hbar B_\hbar)$. For $0\leq k\leq N-1$, each term in (B.13) can be written as a sum of finitely many terms of the form

$$\frac{(i\hbar)^k}{2^k k!}(D_x^\alpha D_\xi^\beta a(x,\xi))\cdot(D_x^\gamma D_\xi^\delta b(x,\xi)),$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{N}^n$ are multi-indices fulfilling $|\alpha|+|\beta|+|\gamma|+|\delta| = 2k$. Via the product formula one easily checks that these terms are all in $\hbar^{k(1-2\mu)}S_\mu(A_\hbar B_\hbar)$, which proves that $a\#b\in S_\mu(A_\hbar B_\hbar)$. □

B.4. *Ellipticity and inverses.* In this section we will define ellipticity for our new symbol classes and will prove a result on L^2-invertibility.

Definition B.8. We call a symbol $a\in S_\mu(A_\hbar)$ elliptic if there is a constant C such that

$$|a(x,\xi)|\geq CA_\hbar(x,\xi). \tag{B.14}$$

For an $\hbar$-dependent order function $A_\hbar\in\mathcal{OF}^c(f)$, from (B.5) and (B.4) it follows that $\hbar^c A_\hbar^{-1}\in\mathcal{OF}^c(f^{-1})$ is again an $\hbar$-dependent order function and we can formulate the following proposition.

PROPOSITION B.9. *If $a\in S_\mu(A_\hbar)$ is elliptic then $a^{-1}\in\hbar^{-c}S_\mu(\hbar^c A_\hbar^{-1})$.*

Proof. We have to show that $|\partial_x^\alpha\partial_\xi^\beta a^{-1}(x,\xi)|\leq C\hbar^{-\mu(|\alpha|+|\beta|)}A_\hbar^{-1}(x,\xi)$ uniformly in $\hbar$, x and ξ. For some first derivative (i.e. for $\alpha\in\mathbb{N}^{2n}$, $|\alpha|=1$) we have

$$|\partial_{x,\xi}^\alpha a^{-1}| = \frac{|\partial_{x,\xi}^\alpha a|}{|a^2|}\leq C\frac{\hbar^{-\mu}A_\hbar}{A_\hbar^2} = C\hbar^{-\mu}A_\hbar^{-1},$$

where the inequality is obtained by (B.5) and (B.14). The estimates of higher order derivatives can be obtained by induction. □

As for standard $\hbar$-PDOs this notion of ellipticity implies that the corresponding operators are invertible for sufficiently small $\hbar$.

PROPOSITION B.10. *Let $A_\hbar\in\mathcal{OF}^c(1)$ and $a\in S_\mu(A_\hbar)$ be an elliptic symbol, then $Op_\hbar^w(a): L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is a bounded operator. Furthermore, there exists $\hbar_0>0$ such that $Op_\hbar^w(a)$ is invertible for all $\hbar\in\,]0,\hbar_0]$. Its inverse is again bounded and a pseudodifferential operator $Op_\hbar^w(b)$ with symbol $b\in\hbar^{-c}S_\mu(\hbar^c A_\hbar^{-1})$. At leading order its symbol is given by*

$$b - a^{-1}\in\hbar^{1-2\mu-c}S_\mu(\hbar^c A_\hbar^{-1}).$$

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Proof. As $a \in S_\mu(A_\hbar) \subset S_\mu(1)$ the boundedness of $Op_\hbar^w(a)$ follows from [**57**, Theorem 4.23]. By Theorem B.7 we calculate

$$Op_\hbar^w(a) Op_\hbar^w(a^{-1}) = \mathrm{Id} + R,$$

where $R = Op_\hbar^w(r)$ is a PDO with symbol $r \in \hbar^{1-2\mu} S_\mu(1)$. Again from [**57**, Theorem 4.23] we obtain $\|R\|_{L^2} \leq C\hbar^{1-2\mu}$, thus there exists $\hbar_0$ such that $\|R\|_{L^2} < 1$ for $\hbar \in]0, \hbar_0]$. According to [**57**, Theorem C.3] we can conclude that $Op_\hbar^w(a)$ is invertible and that the inverse is given by $Op_\hbar^w(a^{-1})(\mathrm{Id} + R)^{-1}$. The semiclassical version of Beal's theorem allows us to conclude that $(\mathrm{Id} + R)^{-1} = \sum_{k=0}^\infty (-R)^k$ is a PDO with symbol in $S_\mu(1)$ (cf. Theorem 8.3 and the following remarks in [**57**]). The representation of $(\mathrm{Id} - R)^{-1}$ as a series finally gives us the symbol of the inverse operator at leading order. □

B.5. *Egorov's theorem for diffeomorphisms.* In this section we will study the behavior of symbols $a \in S_\mu(A_\hbar)$ under variable changes. Let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that equals identity outside some bounded set, then the pullback with this coordinate change acts as a continuous operator on $\mathcal{S}(\mathbb{R}^n)$ by

$$(\gamma^* u)(x) := u(\gamma(x)),$$

which can be extended by its adjoint to a continuous operator $\gamma^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. By a variable change of an operator we understand its conjugation by γ and we are interested in for which $a \in S_\mu(A_\hbar)$ the conjugated operator $(\gamma^*)^{-1} Op_\hbar(a)\gamma^*$ is again an $\hbar$-PDO with symbol a_γ. At leading order this symbol will be the composition of the original symbol with the so-called canonical transformation

$$T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, (x, \xi) \mapsto (\gamma^{-1}(x), (\partial\gamma(\gamma^{-1}(x)))^T \xi)$$

and the symbol class of a_γ will be $S_\mu(A_\hbar \circ T)$. For the $A_\hbar \in \mathcal{OF}^c(f)$ defined in Definition B.3 the composition $A_\hbar \circ T$ will in general, however, not be an $\hbar$-dependent order function itself because the derivatives in x create a supplementary ξ factor which has to be compensated (cf. discussion in [**57**, Ch. 9.3]). We therefore demand in this section that our order function $A_\hbar$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta A_\hbar(x, \xi)| \leq C_{\alpha,\beta} \hbar^{\mu(|\alpha|+|\beta|)} \langle\xi\rangle^{-|\beta|} A_\hbar(x, \xi). \tag{B.15}$$

A straightforward calculation shows then that $A_\hbar \circ T \in \mathcal{OF}^c(f \circ T)$ is again an $\hbar$-dependent order function. The same condition has to be fulfilled by the symbol of the conjugated operator:

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq \hbar^{-\mu(|\alpha|+|\beta|)} \langle\xi\rangle^{-|\beta|} A_\hbar(x, \xi). \tag{B.16}$$

THEOREM B.11. *Let $A_\hbar$ be an $\hbar$-dependent order function that fulfills (B.15). Let $a \in S_\mu(A_\hbar)$ be a symbol which fulfills (B.16) and has compact support in x (i.e. $\overline{\{x \in \mathbb{R}^n | \exists \xi \in \mathbb{R}^n : a(x, \xi) \neq 0\}}$ is compact) and let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then there is a symbol $a_\gamma \in S_\mu(A_\hbar \circ T)$ such that*

$$(Op_\hbar(a_\gamma)u)(\gamma(x)) = (Op_\hbar(a)(u \circ \gamma))(x) \tag{B.17}$$

for all $u \in \mathcal{S}'(\mathbb{R}^n)$. *Furthermore,* a_γ *has the following asymptotic expansion:*

$$a_\gamma(\gamma(x), \eta) \sim \sum_{n=0}^{k-n} \frac{1}{\nu!} \left\langle i \frac{\hbar}{\langle \eta \rangle} D_y, D_\xi \right\rangle^\nu e^{(i/\hbar)\langle \rho_x(y), \eta \rangle} a(x, \xi)_{|y=0, \xi=(\partial\gamma(x))^T\eta}, \tag{B.18}$$

where $\rho_x(y) = \gamma(y + x) - \gamma(x) - \gamma'(x)y$. *The terms of the series are in* $\hbar^{\nu(1-2\mu)/2} S_\mu(\langle \eta \rangle^{\nu/2} A_\hbar \circ T(\gamma(x), \eta))$.

We will prove this theorem similarly to [**31**, Theorem 18.1.17] by using a parameter dependent stationary phase approximation [**32**, Theorem 7.7.7] as well as the following proposition, which forms the analog to [**31**, Proposition 18.1.4] for our symbol classes and which we will prove first.

PROPOSITION B.12. *Let* $a(x, \xi; \hbar) \in C^\infty(\mathbb{R}^{2n})$ *be a family of smooth functions that fulfills*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C\hbar^{-l} \langle \xi \rangle^l f(x, \xi), \tag{B.19}$$

where C *and* l *may depend on* α *and* β. *Let* $a_j \in S_\mu(A_\hbar)$, $j = 0, 1, \ldots,$ *be a sequence of symbols such that*

$$\left| a(x, \xi) - \sum_{j<k} \hbar^j a_j(x, \xi) \right| \leq C\hbar^{\tau k} \langle \xi \rangle^{-\tau k} f(x, \xi), \tag{B.20}$$

where $\tau > 0$. *Then* $a \in S_\mu(A_\hbar)$ *and* $a \sim \sum \hbar^j a_j$.

Proof. We have to show that, for all $k \geq 0$ and $g_k(x, \xi) := a(x, \xi) - \sum_{j<k} \hbar^j a_j(x, \xi)$, we have $|\partial_x^\alpha \partial_\xi^\beta g_k| \leq C\hbar^{k-\mu(|\alpha|+|\beta|)} A_\hbar$. This result can be obtained by iterating the following argument for the first derivative in x_1.

Let $e_1 \in \mathbb{R}^n$ be the first eigenvector and $0 < \varepsilon < 1$. For arbitrary $j \in \mathbb{N}$ we can write, by Taylor's formula,

$$|g_j(x + \varepsilon e_1, \xi) - g_j(x, \xi) - \partial_{x_1} g_j(x, \xi)\varepsilon)| \leq C\varepsilon^2 \sup_{t \in [0, \varepsilon]} |\partial_{x_1}^2 g_j(x + te_1, \xi)|.$$

From (B.19) and the property that all a_j are in $S_\mu(A_\hbar)$ we get

$$\sup_{t \in [0, \varepsilon]} |\partial_{x_1}^2 g_j(x + te_1, \xi)| \leq C\hbar^{-l} \langle \xi \rangle^l f(x, \xi)$$

for some $l \in \mathbb{R}$, and get

$$|\partial_{x_1} g_j(x, \xi)| \leq C\varepsilon\hbar^{-l} \langle \xi \rangle^l m(x, \xi) + \frac{|g_j(x + \varepsilon e_1, \xi) - g_j(x, \xi)|}{\varepsilon},$$

which turns, for $j > (2k + 2c + l)/\tau$ and $\varepsilon = \hbar^{k+l+c} \langle \xi \rangle^{-(k+l+c)}$, into

$$|\partial_{x_1} g_j(x, \xi)| \leq C\hbar^{c+k} \langle \xi \rangle^{-(c+k)} f(x, \xi) \leq C\hbar^k A_\hbar(x, \xi),$$

where we have used (B.8) in the second equation. Thus

$$|\partial_{x_1} g_k(x, \xi)| \leq C\hbar^k A_\hbar(x, \xi) + \left| \sum_{i=k}^{j} \hbar^i \partial_{x_1} a_i(x, \xi) \right| \leq C\hbar^{k-\mu} A_\hbar(x, \xi),$$

which finishes the proof. □

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After having proven this proposition we can start with the proof of Theorem B.11.

Proof. If we define

$$a_\gamma(\gamma(x), \eta) := e^{-(i/\hbar)\gamma(x)\eta} Op_\hbar(a) e^{(i/\hbar)\gamma(\cdot)\eta} \tag{B.21}$$

then equation (B.17) holds for all $e^{(i/\hbar)x\eta}$ which form a dense subset of $\mathcal{S}'(\mathbb{R}^n)$. We thus have to show that a_γ defined in (B.21) is in $S_\mu(A_\hbar)$ and that (B.18) holds.

We will first write a_γ as an oscillating integral in order to apply the stationary phase theorem. By definition of $Op_\hbar(a)$ one obtains

$$a_\gamma(\gamma(x), \eta) = \frac{1}{(2\pi\hbar)^n} \iint a(x, \tilde{\xi}) e^{(i/\hbar)((x-\tilde{y})\tilde{\xi}+(\gamma(\tilde{y})-\gamma(x))\eta)}\, d\tilde{y}\, d\tilde{\xi},$$

which we can transform by a variable transformation $\tilde{\xi} = \langle\eta\rangle\xi$ and $\tilde{y} = y + x$ into

$$a_\gamma(\gamma(x), \eta) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint a(x, \langle\eta\rangle\xi) e^{(i/\tilde{\hbar})(-y\xi+(\gamma(y+x)-\gamma(x))(\eta/\langle\eta\rangle))}\, dy\, d\xi,$$

where $\tilde{\hbar} = (\hbar/\langle\eta\rangle)$.

The critical points of the phase function are given by

$$y = 0 \quad \text{and} \quad \xi = (\partial\gamma(x))^T \frac{\eta}{\langle\eta\rangle}.$$

Let $\chi \in C_c^\infty([-2, 2]^n)$ such that $\chi = 1$ on $[-1, 1]^n$, then we can write

$$a_\gamma(\gamma(x), \eta) = I_1(\tilde{\hbar}) + I_2(\tilde{\hbar}),$$

with

$$I_1(\tilde{\hbar}) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint \chi(y)\chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle\eta\rangle}\right) \times a(x, \langle\eta\rangle\xi) e^{(i/\tilde{\hbar})(-y\xi+(\gamma(y+x)-\gamma(x))(\eta/\langle\eta\rangle))}\, dy\, d\xi$$

and

$$I_2(\tilde{\hbar}) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint \left(1 - \chi(y)\chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle\eta\rangle}\right)\right) \times a(x, \langle\eta\rangle\xi) e^{(i/\tilde{\hbar})(-y\xi+(\gamma(y+x)-\gamma(x))(\eta/\langle\eta\rangle))}\, dy\, d\xi.$$

While $I_1(\hbar)$ still contains critical points, for $I_2(\hbar)$ there are no critical points in the support of the integrand anymore.

I_1 is of the form studied in [**32**, Theorem 7.7.7]. Here the role of x and y is interchanged and there is an additional parameter $\eta/\langle\eta\rangle$. We thus get from this stationary phase theorem

$$\left| I_1(\tilde{\hbar}) - \sum_{\nu=0}^{k-n} \frac{1}{\nu!} \langle i\tilde{\hbar}D_y, D_\xi\rangle^\nu e^{(i/\tilde{\hbar})\langle\rho_x(y),(\eta/\langle\eta\rangle)\rangle} u(x, \xi, y, \eta)_{|y=0,\xi=(\partial\gamma(x))^T(\eta/\langle\eta\rangle)} \right| \leq C\tilde{\hbar}^{(k+n)/2} \sum_{|\alpha|\leq 2k} \sup_{y,\xi} |D^\alpha_{y,\xi} u(x, \xi, y, \eta)|, \tag{B.22}$$

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where $u(x, \xi, y, \eta) = \chi(y)\chi(\xi - (\partial\gamma(x))^T(\eta/\langle\eta\rangle))a(x, \langle\eta\rangle\xi)$. Because of (B.16) and (B.1) we can estimate

$$\sup_{y,\xi} |D^\alpha_{y,\xi} u(x, \xi, y, \eta)| \leq C\hbar^{-\mu|\alpha|} f(x, (\partial\gamma(x))^T\eta) = C\hbar^{-\mu|\alpha|} f \circ T(\gamma(x), \eta).$$

Thus, transforming the expansion (B.22) back to an expansion in $\hbar$ we get

$$\left| I_1(\hbar) - \sum_{\nu=0}^{k-n} \frac{1}{\nu!}\left\langle i\frac{\hbar}{\langle\eta\rangle} D_y, D_\xi\right\rangle^\nu e^{(i/\hbar)\langle\rho_x(y),\eta\rangle} u(x, \xi, y, \eta)_{|y=0,\xi=(\partial\gamma(x))^T(\eta/\langle\eta\rangle)}\right| \\ \leq C\hbar^{(k(1-2\mu)+n)/2}\langle\eta\rangle^{-(k+n)/2} f \circ T(\gamma(x), \eta).$$

As the stationary points for I_2 are not contained in the support of the integrand, we get by the non-stationary phase theorem

$$|I_2(\hbar)| \leq C\left(\frac{\hbar}{\langle\eta\rangle}\right)^N f \circ T(\gamma(x), \eta)$$

for all $N \in \mathbb{N}$. Thus we finally get

$$\left| a_\gamma(\gamma(x), \eta) - \sum_{\nu=0}^{k-n} \frac{1}{\nu!}\left\langle i\frac{\hbar}{\langle\eta\rangle} D_y, D_\xi\right\rangle^\nu e^{(i/\hbar)\langle\rho_x(y),\eta\rangle} u(x, \xi, y, \eta)_{|y=0,\xi=(\partial\gamma(x))^T(\eta/\langle\eta\rangle)}\right| \\ \leq C\hbar^{(k(1-2\mu)+n)/2}\langle\eta\rangle^{-(k+n)/2} f \circ T(\gamma(x), \eta). \tag{B.23}$$

If we show that the elements of the series are in $\hbar^{\nu(1-2\mu)/2} S_\mu(\langle\eta\rangle^{\nu/2} A_\hbar \circ T(\gamma(x), \eta))$ then this equation is of the form (B.20). The terms of order ν in the series are of the form

$$\left(\frac{i\hbar}{\langle\eta\rangle}\right)^\nu \partial^\alpha_y e^{(i/\hbar)\langle\rho_x(y),\eta\rangle} (\partial^\alpha_\xi a)(x, (\partial\gamma(x))^T\eta)\langle\eta\rangle^\nu_{|y=0},$$

where $\alpha \in \mathbb{N}^n$ with $|\alpha| = \nu$. The second factor $(\partial^\alpha_\xi a)(x, (\partial\gamma(x))^T\eta)\langle\eta\rangle^\nu$ is in $\hbar^{-\mu\nu} S_\mu(A_\hbar \circ T(\gamma(x), \eta))$ as we demanded the condition (B.16) on our symbol a. Thus, it remains to show that the other factor is of order $(\hbar/\langle\eta\rangle)^{\nu/2}$ on the support of a. This is the case because $\rho_x(y)$ vanishes at second order in $y = 0$. Each derivative of $e^{(i/\hbar)\langle\rho_x(y),\eta\rangle}$ produces a factor $(i/\hbar)\langle\partial_{y_i}\rho_x(0), \eta\rangle$. But as $\partial_{y_i}\rho_x(0)$ vanishes we need a second derivative, now acting on $\partial_{y_i}\rho_x(y)$, in order to get a contribution. Thus, in the worst case $\partial^\alpha_y e^{(i/\hbar)\langle\rho_x(y),\eta\rangle}$ is of order $(\hbar/\langle\eta\rangle)^{-\nu/2}$. Thus, we have shown that (B.23) is of the form (B.20).

The last thing that we have to show is, thus, that a_γ fulfills (B.19). If we consider the definition (B.21) of a_γ we see that $\partial^\alpha_x\partial^\beta_\xi a_\gamma(\gamma(x), \eta)$ can be written as a sum of terms of the form $(P(\eta)/\hbar^k)e^{-(i/\hbar)\gamma(x)\eta} Op_\hbar(b)e^{(i/\hbar)\gamma(\cdot)\eta}$, where $b \in S_\mu(A_\hbar\langle\xi\rangle^j)$ and $P(\eta)$ is a polynomial in η. The constants j, k and the degree of $P(\eta)$ depend on α and β. Thus writing these terms as oscillating integrals and applying the same arguments as above one gets (B.19).

We have thus shown that all the conditions for Proposition B.12 are fulfilled and can conclude that a_γ belongs to $S_\mu(A_\hbar)$ and that (B.23) is also an asymptotic expansion with respect to the order function $A_\hbar$. □

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C. *Appendix. Adapted Weyl type estimates*

LEMMA C.1. *Let $a_\hbar \in S_\mu(\langle x\rangle^{-2}\langle \xi\rangle^{-2})$ with $0 \le \mu < \frac{1}{2}$ be a real compactly supported symbol as Definition B.4. For all $\hbar > 0$, $\hat{A} := \mathrm{Op}^w_\hbar(a_\hbar)$ is self-adjoint and trace class on $L^2(\mathbb{R})$ and, for any $\epsilon > 0$, as $\hbar \to 0$,*

$$(2\pi\hbar)\sharp\{\lambda_i^\hbar \in \sigma(\hat{A}) \mid |\lambda_i^\hbar| \ge \epsilon\} \le C_1 \mathrm{Leb}\{(x,\xi);\ |a_\hbar(x,\xi)| > 0\} + C_2\hbar, \tag{C.1}$$

where C_1 and C_2 are independent of $\hbar$.

Proof. As $a_\hbar$ is compactly supported $\hat{A}$ is trace class for every $\hbar$ (see [**57**, Theorem C.17]). Consequently, $(1/\epsilon^2)\hat{A}^2$ is also trace class and its trace is given by Lidskii's theorem by $\mathrm{Tr}((1/\epsilon^2)\hat{A}^2) = \sum_i (\lambda_i^\hbar/\epsilon)^2$. As $\hat{A}$ is self-adjoint all $\lambda_i^\hbar$ are real and one clearly has

$$\sharp\{\lambda_i^\hbar \in \sigma(\hat{A}) \mid |\lambda_i^\hbar| \ge \epsilon\} \le \mathrm{Tr}\bigg(\frac{1}{\epsilon^2}\hat{A}^2\bigg).$$

If we denote by $b_\hbar(x,\xi)$ the complete symbol of $\hat{A^2}$ we can calculate the trace by the following exact formula:

$$\mathrm{Tr}(\hat{A}^2) = \frac{1}{2\pi\hbar}\int b_\hbar(x,\xi)\,dx\,d\xi.$$

For any $\mu < \frac{1}{2}$ let $N_\mu \in \mathbb{N}$ be such that $N_\mu(1-2\mu) \ge 1$. Then using the asymptotic expansion (B.13) for composition of PDOs up to order N_μ, $b_\hbar$ can be written as $b_\hbar = b_\hbar^{(1)} + \hbar b_\hbar^{(2)}$, where $\mathrm{supp} b_\hbar^{(1)} = \mathrm{supp} a_\hbar$ and $b_\hbar^{(2)} \in S_\mu(\langle x\rangle^{-4}\langle\xi\rangle^{-4})$. Note that this decomposition depends on μ via the choice of the order N_μ. Thus

$$\begin{aligned}\frac{1}{\epsilon^2}Tr(\hat{A}^2) &= \frac{1}{2\pi\hbar\epsilon^2}\bigg(\int b_\hbar^{(1)}(x,\xi)\,dx\,d\xi + \hbar\int b_\hbar^{(2)}(x,\xi)\,dx\,d\xi\bigg)\\ &\le \frac{1}{2\pi\hbar}(C_1\mathrm{Leb}(\mathrm{supp}(a_\hbar)) + C_2\hbar).\end{aligned}$$

The estimate of the first term is obtained because $b_\hbar^{(1)} \in S_\mu(\langle x\rangle^{-4}\langle\xi\rangle^{-4})$ implies that $b_\hbar^{(1)}$ is bounded uniformly in $\hbar$. Furthermore, as stated above, $b_\hbar^{(1)}$ is compactly supported in $\mathrm{supp}(a_\hbar)$. The estimate of the second term follows from the integrable upper bound $|b_\hbar^{(2)}| \le C\langle x\rangle^{-4}\langle\xi\rangle^{-4}$. Finally, note that the ϵ dependence can be absorbed in the constants C_1 and C_2. □

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