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From Symmetries of the Modular Tower of Genus Zero Real Stable Curves to a Euler Class for the Dyadic Circle

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Abstract. We construct actions of the spheromorphism group of Neretin (containing Thompson's group V) on towers of moduli spaces of genus zero real stable curves. The latter consist of inductive limits of spaces which are the real parts of the Grothendieck–Knudsen compactification of the moduli spaces of punctured Riemann spheres. By a result of M. Davis, T. Januszkiewicz and R. Scott, these spaces are aspherical cubical complexes whose fundamental groups, the 'pure quasi-braid groups', can be viewed as analogues of the Artin pure braid groups. By lifting the actions of the Thompson and Neretin groups to the universal covers of the towers, we obtain extensions of both groups by an infinite pure quasi-braid group, and construct an 'Euler class' for the Neretin group. We justify this terminology by constructing a corresponding cocycle.

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Introduction

The starting point of this work is to relate geometrically some discrete groups, namely Thompson's group V (acting on the Cantor set) and the spheromorphism group of Neretin N (which is a dyadic analogue of the diffeomorphism group of the circle, and acts on the boundary of the regular dyadic tree, cf. [10–12, 17], to the moduli spaces of genus zero curves. More explicitly, let $\mathcal{M}_{0,n+1}(\mathbb{C})$ denote the moduli space $(\mathbb{C}P^1)^{n+1}\setminus\Delta/PGL(2,\mathbb{C})$, where $\mathbb{C}P^1$ is the complex projective line, Δ the thick diagonal. The Grothendieck–Knudsen compactification $\overline{\mathcal{M}}_{0,n+1}(\mathbb{C})$ has a concrete realization as an iterated blow-up along the irreducible components of a hyperplane arrangement in $\mathbb{C}P^{n-2}$. While the interest of the real part $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ was revealed in [14], its topology has recently been studied in [5–7]. From a naive point of view, the relevance of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ with respect to Thompson's or Neretin's groups relies on the common role played by planar trees: $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is a stratified space, whose strata are labelled by planar trees, while dyadic planar trees appear in symbols defining elements of Thompson's and Neretin's groups.

In [9], P. Greenberg builds a classifying space for Thompson's group F, the smallest of Thompson's groups, with total space an inductive limit of 'bracelet' spaces, combinatorially isomorphic to the Stasheff associahedra K_n's. However, this space is too small to support an action of Thompson's group V. The idea to proceed to an analogous construction for V originated in the observation that a two-sheeted covering space $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ was tiled by $n!$ copies of the associahedron K_n.

The first step of our construction consists in building two spaces, $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$ and $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R})$, the towers of moduli spaces of genus zero real curves, defined as inductive limits of spaces $\widetilde{\mathcal{M}}_{0,n}(\mathbb{R})$ and $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$, respectively (the second of these towers being a real version of a stabilized moduli space considered by Kapranov in [15]). The next step is to make not only Thompson's group V, but also the much larger spheromorphism group N, act cellularly on the towers:

THEOREM 1. *Neretin's group of spheromorphisms N – and so Thompson's group V – acts cellularly on the towers $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$ and $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R})$.*

By lifting the action of N and V to the universal covers of the towers, we deduce the existence of nontrivial extensions of V and N by infinite 'pure quasi-braid groups' PJ_{2^∞} or $Q_{3.2^\infty}$. These are inductive limits of the fundamental groups of the spaces $\widetilde{\mathcal{M}}_{0,n}(\mathbb{R})$ or $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$. Thus defined, they are analogues of the pure Artin braid groups P_n (cf. [6, 7]). We prove:

THEOREM 2. *The group PJ_{2^∞} surjects onto $\mathbb{Z}/2\mathbb{Z}$. By dividing the extension of Neretin's group $1 \to PJ_{2^\infty} \longrightarrow \mathcal{A}_N \longrightarrow N \to 1$ by the kernel of the previous surjection, one obtains a nontrivial central extension of N with kernel $\mathbb{Z}/2\mathbb{Z}$: $0 \to \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{N} \longrightarrow N \to 1$.*

We view the resulting cohomology class as the analogue for N of the Euler class of the homeomorphism group of the circle $\mathrm{Homeo}^+(S^1)$, whose corresponding (real) cocycle is induced by the coboundary of a $\mathbb{Z}$-equivariant function on the universal cover $\widetilde{\mathrm{Homeo}}^+(S^1)$:

THEOREM 3. *There is a ring extension $\mathcal{R}$ of $\mathbb{Z}/2\mathbb{Z}$ and an $\mathcal{R}$-valued Euler-type cocycle on N (i.e. induced by the coboundary of a $\mathbb{Z}/2\mathbb{Z}$-equivariant function on $\widetilde{N}$) associated with the previous cohomology class on N.*

PLAN OF THE ARTICLE

The first section is devoted to the description of the real moduli spaces $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, and of the combinatorics of their stratifications. Section 2 introduces the stabilized

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moduli spaces $\overline{\mathcal{M}}_{0,3.2^{\infty}}(\mathbb{R})$ and $\widetilde{\mathcal{M}}_{0,2^{\infty}}(\mathbb{R})$ (or 'towers'), and the actions of Neretin's and Thompson's groups are defined. In Section 3, we describe the quasi-braid extension of Neretin's group, and construct its Euler class.

1. The Compactified Real Moduli Space $\overline{\mathcal{M}}_{\mathbf{0,n+1}}(\mathbb{R})$

1.1. THE MODULI SPACE OF GENUS ZERO $(n+1)$-STABLE CURVES

DEFINITION 1.1.1 ([13], [14]). Let k be the field $\mathbb{C}$ or $\mathbb{R}$. A genus zero $(n+1)$-stable curve is an algebraic curve $\mathcal{C}$ over the field k with $(n+1)$ smooth marked points $x_0, \ldots, x_n$ such that

(1) each irreducible component of $\mathcal{C}$ is isomorphic to a projective line $\mathbb{P}^1_k$, and each double point of $\mathcal{C}$ is ordinary;
(2) the graph of $\mathcal{C}$ is a tree;
(3) each component of $\mathcal{C}$ has at least three points, double or marked.

The graph of a curve $(\mathcal{C}, x_0, \ldots, x_n)$ is defined as follows: the leaves (or 1-valent vertices) $A_0, \ldots, A_n$ are in correspondence with the marked points of the curve, $x_0, \ldots, x_n$, and the internal vertices $v_1, \ldots, v_k$ with the irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_k$. There is an internal edge $[v_iv_j]$ if $\mathcal{C}_i$ and $\mathcal{C}_j$ intersect (in a double point). Terminal edges $[A_iv_j]$ correspond to pairs $(x_i, \mathcal{C}_j)$ such that $x_i \in \mathcal{C}_j$. The terminal edge with leaf A_0 will be distinguished as the *output edge*. Figure 1 illustrates the construction of the graph of a stable curve; the components are represented by circles.

1.1.2. *Terminology*

By a *rooted planar n-tree* we shall mean a planar tree with $(n+1)$ terminal edges, one of them being distinguished as the *output edge*, the others as the *input edges*. The terminal vertices of the input edges are called *leaves*. The *root* of the tree is the internal vertex of the output edge. When $k = \mathbb{R}$, the graph of a stable curve should be thought of as a planar rooted tree.

1.2 EXPLICIT CONSTRUCTION OF $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, AS AN ITERATED BLOW-UP OF $P^{n-2}_{\mathbb{R}}$

(1) By definition, $\mathcal{M}_{0,n+1}(\mathbb{R}) = ((\mathbb{R}P^1)^{n+1} \setminus \Delta)/(PGL(2, \mathbb{R}))$, where Δ is the thick diagonal of $(\mathbb{R}P^1)^{n+1}$.

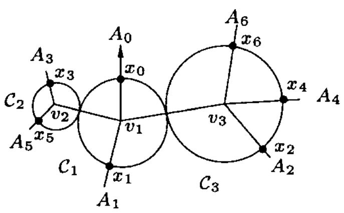

Figure 1.

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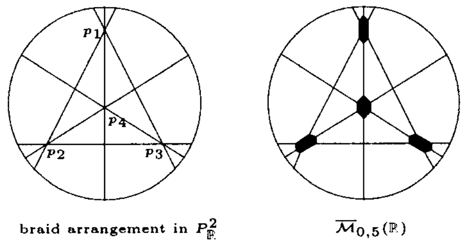

Figure 2.

LEMMA 1.2.1. *$\mathcal{M}_{0,n+1}(\mathbb{R})$ embeds in the $(n-2)$-dimensional real projective space as the complement of a hyperplane arrangement, the braid arrangement.*

Proof. Clearly, $\mathcal{M}_{0,n+1}(\mathbb{R}) \cong (\mathbb{R}^n \setminus \tilde{\Delta})/(A\mathrm{ff}(1,\mathbb{R}))$, where $\tilde{\Delta}$ is the thick diagonal of $\mathbb{R}^n$, and $A\mathrm{ff}(1,\mathbb{R})$ the affine group of $\mathbb{R}$. There is an embedding

$$\frac{\mathbb{R}^n \setminus \tilde{\Delta}}{A\mathrm{ff}(1,\mathbb{R})} \hookrightarrow P^{n-2}_{\mathbb{R}} := P\left\{(a_1,\dots,a_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = 0\right\}$$

in the $(n-2)$-dimensional real projective space, induced by the map sending $(x_1,\dots,x_n) \in \mathbb{R}^n \setminus \tilde{\Delta}$ to $(a_1,\dots,a_n)$, with $a_i = x_i - (1/n)\sum_{j=1}^{n} x_j$.

The image of $\mathcal{M}_{0,n+1}(\mathbb{R})$ in $P^{n-2}_{\mathbb{R}}$ is the complement of the union of hyperplanes $H_{i,j}$: $a_i = a_j$. This hyperplane arrangement is called the *braid arrangement*. So, $\mathcal{M}_{0,n+1}(\mathbb{R})$ has $n!/2$ connected components* which are open simplices, called the *projective Weyl chambers* W^σ: $a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(n)}$, where σ belongs to the symmetric group Σ_n. The projectivisation introduces an identification of the chambers W^σ and $W^{\sigma\omega}$, where ω is the permutation $\binom{1\ \ 2\ \ \cdots\ \ n}{n\ \ n-1\ \ \cdots\ \ 1}$. □

EXAMPLE 1.2.2. $\mathcal{M}_{0,3}(\mathbb{R})$ is a point, $\mathcal{M}_{0,4}(\mathbb{R}) = \mathbb{R}P^1 \setminus \{0,1,\infty\}$ (3 chambers), $\mathcal{M}_{0,5}(\mathbb{R})$ is represented on Figure 2 (12 chambers).

PROPOSITION 1.2.3 ([13, 14]). *There is a projective smooth variety $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ and an iterated blow-down map $p\colon \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \longrightarrow P^{n-2}_{\mathbb{R}}$ which desingularizes the braid arrangement.*

Proof. We content ourselves to recall the construction, as performed in [3]. Denote by $\mathfrak{B}_0$ the set of n points p_i: $a_1 = \cdots = \widehat{a_i} = \cdots = a_n$ ($\widehat{\ }$ being the symbol of omission), $\mathfrak{B}_1$ the set of lines (p_ip_j): $a_1 = \cdots = \widehat{a_i} = \cdots = \widehat{a_j} = \cdots = a_n$, and more generally, by $\mathfrak{B}_k$ the set of k-planes $a_1 = \cdots = \widehat{a_{i_1}} = \cdots = \widehat{a_{i_{k+1}}} = \cdots = a_n$, $i_1 < \cdots < i_{k+1}$, for $k = 0,\dots,n-3$ (they are the irreducible components of the braid arrangement, in the sense of Coxeter groups: this will become clear in Section 1.3). Along components of $\mathfrak{B}_k$, hyperplanes $H_{i,j}$ do not meet transversely. So, the process of desingularization of the nonnormal crossing divisor $\bigcup_{i,j} H_{i,j}$ is the following:

*Note the difference with the complex case: $\mathcal{M}_{0,n+1}(\mathbb{C})$ is connected.

Points of $\mathfrak{B}_0$ are first blown-up, and we obtain the blown-up space X_1 as well as the blow-down map $X_1 \xrightarrow{\pi_1} P_{\mathbb{R}}^{n-2}$. The proper transforms of the lines (p_ip_j) (i.e. the closures of $\pi_1^{-1}((p_ip_j)\backslash\{p_i, p_j\})$) become transverse in X_1, consequently they can be blown-up in any order to produce $X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} P_{\mathbb{R}}^{n-2}$; again the proper transforms of the planes $(p_ip_jp_k)$ become transverse in X_2, and are blown-up in any order. Finally we get the composite of blow-down maps

$$\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) := X_{n-2} \xrightarrow{\pi_{n-2}} \dots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} P_{\mathbb{R}}^{n-2}.$$

We shall denote by $p\colon \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \longrightarrow P_{\mathbb{R}}^{n-2}$ the composition of the iterated blow-ups.

(2) Each blow-up along a smooth algebraic subvariety produces a smooth exceptional divisor in the new variety, which is isomorphic to the projective normal bundle over the subvariety. We denote by $\widehat{\mathfrak{B}}_k$ the set of proper transforms in $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ of the exceptional divisors produced by blowing-up the (proper transforms in X_k of the) components of $\mathfrak{B}_k$: they are smooth irreducible hypersurfaces of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, and meet transversely. So the union $\bigcup_{k=0}^{n-3} \widehat{\mathfrak{B}}_k$ is the set of irreducible components of a *normal crossing divisor* $\hat{D}$.

EXAMPLE 1.2.4. $\overline{\mathcal{M}}_{0,3}(\mathbb{R})$ is a point, $\overline{\mathcal{M}}_{0,4}(\mathbb{R})$ is isomorphic to $\mathbb{R}P^1$ (decomposed into three one-dimensional cells). Contrarily to $\overline{\mathcal{M}}_{0,4}(\mathbb{R})$, all $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ with $n \geqslant 5$ are nonoriented.

In Figure 2, we illustrate how $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is obtained by blowing-up the four points $p_1, \dots, p_4$ of the braid arrangement. Each exceptional divisor produced by a blow-up process is isomorphic to a real projective line, and is represented by a hexagon, whose opposite sides must be identified by the antipodal map. Note that the exceptional divisors truncate the neighboring simplices, so that $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is tiled by 12 pentagons. Topologically, $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is the connected sum of 5 projective real planes.

(3) The real algebraic variety $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is both a stratified space and a cellular complex: the codimension k strata are the nonempty intersections of k irreducible components of the divisor $\hat{D}$, while the (closed) cells are obtained by intersecting the strata with one of the $n!/2$ closures of the preimages $p^{-1}(W^\sigma)$. The open cells are then defined as the complements of all closed strict subcells in a given closed cell. Since the divisor $\hat{D}$ is normal, the strata are smooth closed subvarieties of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$.

Remark 1.2.5. The construction of the complex algebraic variety $\overline{\mathcal{M}}_{0,n+1}(\mathbb{C})$ is similar. The strata of complex codimension k are the nonempty intersections of k irreducible components of the divisor $\hat{D}$. However, the construction does not equip the strata with a natural cell decomposition.

(4) Each codimension k cell $\mathcal{M}$ is coded by a *planar tree* that we now define: If $\overline{\mathcal{M}} = \overline{p^{-1}(W^\sigma)} \cap D_1 \cap \cdots \cap D_k$ is nonempty, then all the components D_α, $\alpha = 1, \dots, k$ are produced by a blow-up along a set which *must be* of the form $a_{\sigma(i_\alpha)} = a_{\sigma(i_\alpha+1)} = \cdots = a_{\sigma(j_\alpha)}$, for some $1 \leqslant i_\alpha < j_\alpha \leqslant n$.

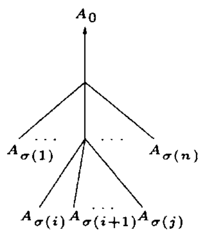

Figure 3.

Let $T(\sigma,(i,j))$ denote the planar rooted n-tree with one output edge and n input edges, labelled from the left to the right by $\sigma(1),\dots,\sigma(n)$, with a unique internal edge, as drawn in Figure 3.

Define a *contraction* of a tree as the operation consisting of collapsing an internal edge on a single vertex. Introduce the partial order on the set of n-planar trees: $T \leqslant T'$ if T' is obtained from T through a sequence of contractions.

DEFINITION 1.2.6. The tree attached to the (closed) cell $\overline{\mathcal{M}} = \overline{p^{-1}(W^\sigma)} \cap D_1 \cap \cdots \cap D_k$ (or to the open cell $\mathcal{M}$) is defined as $T(\sigma) = \inf_{\alpha=1,\dots,k} T(\sigma,(i_\alpha,j_\alpha))$, where (i_α,j_α) is associated with the component D_α as explained in (4). We denote $\overline{\mathcal{M}}$ (resp. $\mathcal{M}$) by $\overline{\mathcal{M}}(T,\sigma)$ (resp. $\mathcal{M}(T,\sigma)$).

NOTATION 1.2.7. $T(\sigma)$, with $\sigma \in \Sigma_n$, will denote a rooted planar n-tree with leaves labelled from $\sigma(1)$ to $\sigma(n)$, leftmost first, and reading from left to right, whereas T will refer to the same tree, with the canonical labelling of the leaves, from 1 (on the left) to n (on the right).

We list without proof:

FACT 1. *The cell* $\overline{\mathcal{M}} = \overline{p^{-1}(W^\sigma)} \cap D_1 \cap \cdots \cap D_k$ *is nonempty if and only if the collection of sets* $S_\alpha = \{i_\alpha, i_\alpha + 1, \dots, j_\alpha\}$, *attached to each* D_α, *for* $\alpha = 1,\dots,k$, *is nested in the following sense:*

$$\forall \alpha,\ \beta,\ \textit{either}\ S_\alpha \cap S_\beta = \emptyset,\ \text{or}\ S_\alpha \subset S_\beta\ \text{or}\ S_\beta \subset S_\alpha.$$

Note that the tree $T(\sigma) = \inf_{\alpha=1,\dots,k} T(\sigma,(i_\alpha,j_\alpha))$ *exists if and only if the collection* $(S_\alpha)_{\alpha=1,\dots,k}$ *is nested.*

EXAMPLE 1.2.8 (Figure 4).

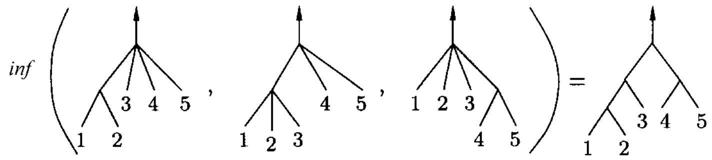

Figure 4.

FACT 2. *The codimension of a cell $\mathcal{M}(T,\sigma)$ is equal to the number of internal edges of the tree T. Indeed, the set of internal edges of T is in one-to-one correspondence with the codimension* 1 *components $D_1,\ldots,D_k$ containing $\overline{\mathcal{M}}(T,\sigma)$. In particular, dyadic trees label the* 0*-cells of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, and trees without internal edges label its maximal cells.*

Remark 1.2.9. In view of Section 1.1, we may think of each cell $\mathcal{M}(T,\sigma)$ as the set of stable curves whose associated planar tree is $T(\sigma)$. Then Fact 1 becomes completely clear.

1.3. COXETER GROUP FORMULATION, FOLLOWING DAVIS, JANUSZKIEWICZ AND SCOTT

Following [6], we now formulate the condition guaranteeing a collection is nested in group theoretic terms, namely in terms of the symmetric group Σ_n.

Denote by $S = \{\sigma_1,\ldots,\sigma_{n-1}\}$ the set of canonical Coxeter generators of Σ_n: σ_i is the transposition $(i, i+1)$. Let T be a rooted planar n-tree (with the canonical labelling of its n leaves). To each vertex v except the root of T, a proper subset $T_v = \{\sigma_i, \sigma_{i+1},\ldots,\sigma_{j-1}\}$ of S is associated, corresponding to a connected subgraph G_{T_v} of the Coxeter graph of Σ_n: $i, i+1,\ldots,j$ are the labels of the leaves of T which are the descendants of v.

Denote by $\mathrm{Vert}^*(T)$ the set of vertices of T, distinct from the root. The collection $\mathcal{T} = \{T_v,\ v\in \mathrm{Vert}^*(T)\}$ is a *nested collection* in the following sense:

DEFINITION 1.3.1 ([6]). A collection $\mathcal{T}$ of proper subsets of S will be called nested if the following conditions are satisfied:

(1) The Coxeter subgraph G_T is connected for all $T\in\mathcal{T}$.
(2) For any $T, T'\in\mathcal{T}$, either $T\subset T'$, $T'\subset T$, or $G_{T\cup T'}$ is not connected.

It is clear that there is a bijection $T\leftrightarrow\mathcal{T}$ between the set of rooted planar n-trees with at least one internal edge and the set of nested collections of the symmetric group Σ_n.

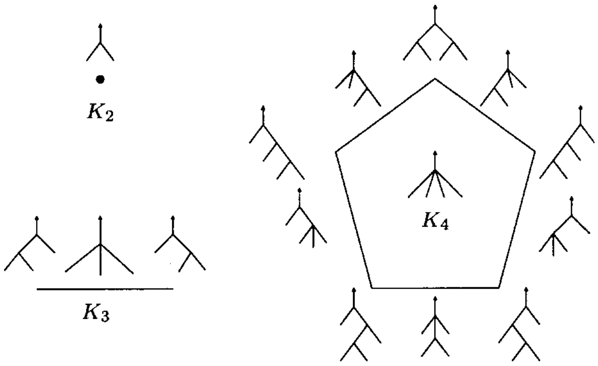

Figure 5.

1.4. COMBINATORICS OF THE CELLULATION OF $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ AND OF ITS TWO-SHEETED COVER $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$

1.4.1. *Associahedra*

LEMMA 1.4.1. *Each* $K^\sigma = \overline{p^{-1}(W^\sigma)}$ *is combinatorially isomorphic to the Stasheff associahedron* K_n (*cf.* [18]).

Proof. This is so because the cell decompositions of both objects are the same. Recall that for each $n \geqslant 2$, K_n is a convex $(n-2)$-dimensional polytope, whose faces are labelled by rooted planar n-trees. If $f(T)$ and $f(T')$ are faces (or cells) of K_n, then $f(T) \subset f(T')$ if and only if $T \leqslant T'$ in the sense of Section 1.2 (4). Following the notation of Definition 1.2.6, the correspondence $f(T) \to \overline{\mathcal{M}}(T, \sigma)$ establishes a combinatorial isomorphism between K_n and $\overline{p^{-1}(W^\sigma)}$. □

EXAMPLE 1.4.2. K_2 is a point, K_3 a segment, K_4 a pentagon (Figure 5).

1.4.2. *The Two-sheeted Cover* $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$

Consider the two-sheeted cover $S^{n-2} \longrightarrow P_{\mathbb{R}}^{n-2}$, where S^{n-2} is the $(n-2)$-dimensional unit sphere of $\{(a_1, \ldots, a_n) \in \mathbb{R}^n \colon \sum_{i=1}^n a_i = 0\}$.

Apply now the process of iterated blow-ups described in Section 1.2 to S^{n-2} with its lifted braid arrangement, and denote by $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ the resulting space. This yields a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{0,n+1} & \longrightarrow & \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \\ \downarrow & & \downarrow \\ S^{n-2} & \longrightarrow & P_{\mathbb{R}}^{n-2} \end{array}$$

where the horizontal arrows are the obvious two-sheeted covering maps, and the vertical ones are the blow-down maps. Since S^{n-2} is tiled by $n!$ Weyl chambers, the covering space $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ will be tiled by $n!$ copies of the associahedron K_n. We denote

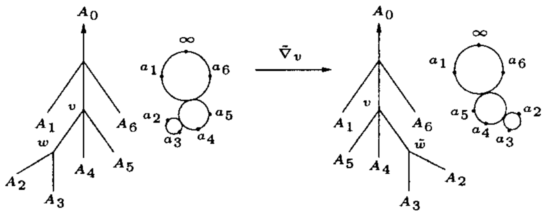

Figure 6.

again by $\mathcal{M}(T,\sigma)$ a cell of $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$. If ω is the permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$, the cells $\mathcal{M}(T,\sigma)$ and $\mathcal{M}(T,\sigma\circ\omega)$ of $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ cover the same cell of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$.

1.4.3 *Combinatorics of the Cellulation of* $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ *and* $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$

Let $T(\sigma)$ be a rooted planar n-tree, and v an internal vertex. The tree $T(\sigma)$ splits in v into two subtrees. Apply a reflection to the subtree which does not contain the output edge – this results in inversing the labelling of its input edges – before gluing both pieces back together to form a new planar n-tree, denoted $\tilde{\nabla}_v T(\sigma)$ (cf. Figure 6, for an example). Let T' and σ' be respectively the tree and the permutation such that $T'(\sigma') = \tilde{\nabla}_v T(\sigma)$.

PROPOSITION 1.4.3 (adapted from [7]). *Two cells* $\mathcal{M}(T,\sigma)$ *and* $\mathcal{M}(T',\sigma')$ *in the double cover* $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ *coincide if and only if* $T(\sigma) \sim T'(\sigma')$*, where* $\sim$ *is the equivalence relation generated by* $T(\sigma) \sim \tilde{\nabla}_v T(\sigma)$ *for any vertex* v *distinct from the root. In other words,* $T(\sigma) \sim T'(\sigma')$ *if and only if there exist* $v_1 \in \mathrm{Vert}^*(T)$*,* $v_i \in \mathrm{Vert}^*(\tilde{\nabla}_{v_{i-1}} \dots \tilde{\nabla}_{v_1} T(\sigma))$*,* $i = 2,\dots,r$*, such that* $T'(\sigma') = \tilde{\nabla}_{v_r} \dots \tilde{\nabla}_{v_1} T(\sigma)$.

Remark 1.4.4. If, moreover, one allows v to be the root in the identification $T(\sigma) \sim \tilde{\nabla}_v T(\sigma)$, then one obtains the combinatorial structure of the cellulation of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$. But the latter admits a more symmetric description if one uses unrooted $(n+1)$-trees, labelled by permutations in Σ_{n+1}. In [7], this combinatorics is expressed in the language of polygons; it can be translated in terms of trees, dual to the polygons.

Proof. The rule $T(\sigma) \sim \tilde{\nabla}_v T(\sigma)$ comes from the projectivisation of the normal bundle over the blown-up components. Indeed, suppose we blow-up the component $a_{\sigma(i)} = a_{\sigma(i+1)} = \dots = a_{\sigma(j)}$; the equations of the blow-up are:

$$\lambda_k(a_{\sigma(l)} - a_{\sigma(i)}) = \lambda_l(a_{\sigma(k)} - a_{\sigma(i)}), \quad k,l = i+1,\dots,j, \; [\lambda_{i+1} : \dots : \lambda_j] \in \mathbb{R}P^{j-i-1},$$

$(a_1,\dots,a_n) \in S^{n-2}$. Since $[\lambda_{i+1} : \dots : \lambda_j] = [-\lambda_{i+1} : \dots : -\lambda_j]$, it follows that a point of the exceptional divisor close to the cell $a_{\sigma(1)} < a_{\sigma(2)} < \dots < a_{\sigma(i)} < \dots < a_{\sigma(j)} < \dots < a_{\sigma(n)}$ is close also to the cell $a_{\sigma(1)} < a_{\sigma(2)} < \dots < a_{\sigma(j)} < \dots <$

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$a_{\sigma(i)} < \cdots < a_{\sigma(n)}$. This gives the identification rules for the codimension 1 cells, and since the other cells are intersections of codimension 1 cells, the complete rule follows easily. □

EXAMPLE 1.4.5 (Figure 6).

1.5. TRANSLATION OF THE IDENTIFICATION RULES IN TERMS OF NESTED COLLECTIONS AND COXETER GROUPS

Let T be a planar rooted n-tree, v a vertex of T distinct from the root, $T_v = \{\sigma_i, \sigma_{i+1}, \dots, \sigma_{j-1}\}$ the corresponding subset of generators of Σ_n (cf. Section 1.3). Denote by ω_{T_v} or ω_v the involution $\begin{pmatrix} i & i+1 & \dots & j \\ j & j-1 & \dots & i \end{pmatrix}$. It is the longest element of Σ_{T_v} (the symmetric group generated by T_v) for the word metric.

If $\mathcal{T}$ is a nested collection (corresponding to an n-planar tree T), then for each T_v, T_w in $\mathcal{T}$, define $j_{T_v}T_w$ the connected subset of $S = \{\sigma_1, \dots, \sigma_{n-1}\}$ by:

$$j_{T_v}T_w = \begin{cases} \omega_v T_w \omega_v, & \text{if } T_w \subset T_v, \text{or equivalently, } w \text{ is a descendant of } v, \\ T_w, & \text{if not.} \end{cases}$$

PROPOSITION 1.5.1. *Under the correspondence $T \leftrightarrow \mathcal{T}$, $T' \leftrightarrow \mathcal{T}'$ between rooted n-trees and nested collections (cf. Section 1.3), $T'(\sigma') = \tilde{\nabla}_v T(\sigma)$ if and only if $\mathcal{T}' = j_{T_v}\mathcal{T}$ and $\sigma' = \sigma\omega_{T_v}$, where $j_{T_v}\mathcal{T}$ is the nested collection $\{j_{T_v}T_w, T_w \in \mathcal{T}\}$.*

EXAMPLE 1.5.2. With the example of Figure 6,

$$\sigma = \mathrm{id}, \qquad \sigma' = \omega_{T_v} = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 \end{pmatrix}, \qquad T_v = \{\sigma_2, \sigma_3, \sigma_4\}, \qquad T_w = \{\sigma_2\}.$$

The only subset changed under j_{T_v} is T_w: $j_{T_v}T_w = \omega_{T_v}T_w\omega_{T_v} = \{\sigma_4\}$.

A reformulation of the results of Section 1.4.3 is

THEOREM A (Davis–Januszkiewicz–Scott [6]). *The two-sheeted covering space $\widetilde{\mathcal{M}}_{0,n+1}$ is Σ_n-equivariantly homeomorphic to the geometric realization $|\Sigma_n\mathcal{N}|$ of the following poset $\Sigma_n\mathcal{N}$:*

1. *Its elements are the equivalence classes of pairs $(\mathcal{T}, \sigma)$, with $\mathcal{T}$ a nested collection, and σ in Σ_n, for the equivalence relation defined by $(\mathcal{T}, \sigma) \sim (\mathcal{T}', \sigma')$ if and only if there exists a subset $\mathcal{T}'' \subset \mathcal{T}$ such that $\sigma' = \sigma\omega_{\mathcal{T}''}$ and $\mathcal{T}' = j_{\mathcal{T}''}\mathcal{T}$. Here $\omega_{\mathcal{T}''} = \omega_{T_1} \dots \omega_{T_r}$ if $\mathcal{T}'' = \{T_1, \dots, T_r\}$ (it can be supposed $i \leqslant j$ if $T_i \subset T_j$), and $j_{\mathcal{T}''} = j_{T_r} \dots j_{T_1}$.*
2. *The partial order is defined by $[\mathcal{T}, \sigma] \leqslant [\mathcal{T}', \sigma']$ if and only if there exists some $\mathcal{T}'' \subset \mathcal{T}$ such that $\sigma' = \sigma\omega_{\mathcal{T}''}$ and $\mathcal{T}' \subset j_{\mathcal{T}''}\mathcal{T}$.*

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Moreover, the free involution $\tilde{a}\colon \widetilde{\mathcal{M}}_{0,n+1} \to \widetilde{\mathcal{M}}_{0,n+1}$ *(lifted from the antipodal involution of* S^{n-2}*), is combinatorial, and given on the poset* $\Sigma_n\mathcal{N}$ *by* $[\mathcal{T}, \sigma] \mapsto [j_S\mathcal{T}, \sigma\omega_S]$*, with* $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$*. Since* $\overline{\mathcal{M}}_{0,n+1} = \widetilde{\mathcal{M}}_{0,n+1}/\tilde{a}$*,* $\overline{\mathcal{M}}_{0,n+1}$ *inherits a natural cell decomposition with poset* $\Sigma_n\mathcal{N}/\tilde{a}$*.*

2. Towers of Genus Zero Real Stable Curves and Action of Thompson's and Neretin's Groups

2.1. CONSTRUCTION OF THE TOWERS $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R})$ AND $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$

We shall use below Kapranov's interpretation of $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ as the moduli space of collections $\mathcal{C}(x_0, x_1, \ldots, x_n, \omega)$, where $\mathcal{C}(x_0, x_1, \ldots, x_n)$ is a stable $(n+1)$-pointed real curve of genus zero, and ω is an orientation of the component of $\mathcal{C}$ which contains x_0 (cf. [14]).

CONSTRUCTION 2.1.1. (1) Embedding $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \hookrightarrow \overline{\mathcal{M}}_{0,2(n+1)}(\mathbb{R})$: if $\mathcal{C}(x_0, x_1 \ldots, x_n) \in \overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is a stable curve, then graft a new circle at any marked point x_i, $i \geqslant 0$, with two marked points on it, y_{2i-1} and y_{2i} for $i \neq 0$, and y_0 and y_{2n+1} for $i = 0$. We obtain a new stable $2(n+1)$-curve, and it is uniquely defined.

(2) Embedding $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}(\mathbb{R})$: if $\mathcal{C}(x_0, x_1, \ldots, x_n, \omega)$ is in the double cover $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$, we may expand unambiguously all the points x_i except x_0, to obtain a curve $\mathcal{C}(x_0, y_1, y_2, \ldots, y_{2n-1}, y_{2n}, \omega)$ in $\widetilde{\mathcal{M}}_{0,2n+1}(\mathbb{R})$.

Remark 2.1.2. The map $\mathcal{C}(x_0, x_1, \ldots, x_n) \mapsto \mathcal{C}(x_0, x_1, \ldots, x_{i-1}, y_i, y_{i+1}, x_{i+1}, \ldots, x_n)$ is a section of a forgetful map $\overline{\mathcal{M}}_{0,n+2}(\mathbb{R}) \to \overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, and is called 'stabilization' by Knudsen (cf. [16]). It is a smooth map.

PROPOSITION 2.1.3. *The embedding* $\overline{\exp}_n\colon \overline{\mathcal{M}}_{0,n}(\mathbb{R}) \hookrightarrow \overline{\mathcal{M}}_{0,2n}(\mathbb{R})$ *is a morphism of cellular complexes. The inductive limit* $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R}) = \varinjlim_n \overline{\mathcal{M}}_{0,3.2^n}(\mathbb{R})$ *inherits a (locally nonfinite) CW-complex structure. The same is true when the moduli spaces are replaced by their two-sheeted covering spaces* $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$*, with embeddings* $\widetilde{\exp}_n$*:* $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}(\mathbb{R})$*, defining a tower* $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) = \varinjlim_n \widetilde{\mathcal{M}}_{0,2^n+1}(\mathbb{R})$*.*

Remark 2.1.4. We may view both towers as pointed spaces, with base-point represented by the unique point of $\overline{\mathcal{M}}_{0,3}(\mathbb{R}) = \widetilde{\mathcal{M}}_{0,2+1}(\mathbb{R})$.

Proof. Both embeddings being composed of stabilization maps, they are smooth embeddings. We give a proof for the covering spaces $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$. If $\mathcal{M}(T, \sigma)$ is a cell of $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$, then $\exp_n(\mathcal{M}(T, \sigma))$ is the cell $\mathcal{M}(\exp_n(T), \exp_n(\sigma))$, where $\exp_n(T)$ is the planar rooted tree obtained from T by expanding each of its leaf with two new edges, and $\exp_n(\sigma)$ is the permutation $\tau \in \Sigma_{2n}$ defined by $\tau(2i-1) = 2\sigma(i) - 1$, $\tau(2i) = 2\sigma(i)$, $i = 1, \ldots, n$. Since $\widetilde{\exp}_n\colon \widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R}) \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}(\mathbb{R})$ is smooth and maps cells onto cells, the proof is done. $\square$

Remark 2.1.5. (1) In the proof above we have introduced a group homomorphism $\exp_n : \Sigma_n \longrightarrow \Sigma_{2n} : \sigma \mapsto \tau = \exp_n(\sigma)$, called the *expansion morphism*.

(2) More generally, if $1 \leqslant i \leqslant n$, there is an *expansion map* $\exp_{n,i} : \Sigma_n \to \Sigma_{n+1}$, such that if σ belongs to Σ_n, then $\exp_{n,i}(\sigma)$ is the natural extension of σ to $\{1, \ldots, n+1\}$ after replacing $\{i\}$ at the source by $\{i, i+1\}$, and $\{\sigma(i)\}$ at the target by $\{\sigma(i), \sigma(i)+1\}$, and imposing $\exp_{n,i}(\sigma)(i) = \sigma(i)$, $\exp_{n,i}(\sigma)(i+1) = \sigma(i)+1$.

(3) If T is a planar rooted n-tree, then $\exp_{n,i}(T)$, for $1 \leqslant i \leqslant n$, will denote the $(n+1)$-tree resulting from T by expanding its ith leaf with two new edges.

(4) When we use an iterated expansion map, we denote it by $\exp : \Sigma_n \to \Sigma_{n+*}$. Similarly, we denote by $\exp(T)$ an iterated expansion of a tree T.

2.2. ACTION OF THE NERETIN GROUP ON THE MODULI TOWERS

2.2.1 *The Groups N and V*

Let T_2 be the dyadic complete planar rooted tree (all its vertices except the leaf of the output edge are 3-valent). Let α be a rooted planar dyadic n-tree, viewed as a finite subtree of T_2. Its internal vertices are 3-valent, and its leaves are canonically labelled from 1 to n, leftmost first, and reading from left to right. For each $i = 1, \ldots, n$, view the ith leaf of α as the root of a dyadic complete planar tree T_i^α, so that $\alpha \cup T_1^\alpha \cup \cdots \cup T_n^\alpha$ is the dyadic complete planar rooted tree T_2.

A *symbol* is a triple $(\alpha_1, \alpha_0, q_\sigma)$, where α_0, α_1 are rooted dyadic n-trees for some $n \geqslant 2$, $\sigma \in \Sigma_n$, and q_σ is a collection of tree isomorphisms $q_i : T_i^{\alpha_0} \to T_{\sigma(i)}^{\alpha_1}$, $i = 1, \ldots, n$. Equivalently, q_σ is a family $(\sigma_k)_{k \in \mathbb{N}}$ in the product $\prod_{k \in \mathbb{N}} \Sigma_{2^k.n}$ such that $\sigma_0 = \sigma$ and σ_{k+1} may differ from $\exp_{2^k.n}(\sigma_k)$ by a product of elementary transpositions of the form $(2i-1, 2i)$. If $\sigma_{k+1} = \exp_{2^k.n}(\sigma_k)$ for all $k \in \mathbb{N}$, we say that the symbol *locally preserves the orientation* of the tree T_2, and we denote it by $(\alpha_1, \alpha_0, \sigma)$.

DEFINITION 2.2.1 (Groups N and V, cf. [4,11,17]). Denote the boundary at infinity of the tree T_2 by ∂T_2, endowed with its natural topology.

(1) Let $(\alpha_1, \alpha_0, q_\sigma)$ be a symbol. Since $\{\partial T_i^{\alpha_0}\}_{i=1,\ldots,n}$ and $\{\partial T_i^{\alpha_1}\}_{i=1,\ldots,n}$ form two partitions of ∂T_2, the collection $(q_i)_{i=1,\ldots,n}$ induces a homeomorphism of ∂T_2, called a spheromorphism. The set N of all spheromorphisms is a subgroup of $\mathrm{Homeo}(\partial T_2)$, namely the *Spheromorphism group of Neretin*.

(2) One says that two symbols are equivalent if they define the same spheromorphism. One denotes by $[\alpha_1, \alpha_0, q_\sigma]$ the spheromorphism associated with the symbol $(\alpha_1, \alpha_0, q_\sigma)$.

(3) One says that the symbol $(\alpha_1', \alpha_0', q_{\sigma'}')$ is an expansion of $(\alpha_1, \alpha_0, q_\sigma)$ if both symbols are equivalent and α_0 (equivalently, α_1) is a subtree of α_0' (equivalently, of α_1').

(4) The set of spheromorphisms induced by locally orientation-preserving symbols is a countable subgroup of N, namely *Thompson's group V*.

Remark 2.2.2. (1) Given two spheromorphisms, one can always find symbols of the form $(\alpha_1, \alpha_0, q_\sigma)$ and $(\alpha_2, \alpha_1, r_\tau)$ which represent them. It follows that the composite $[\alpha_2, \alpha_1, r_\tau][\alpha_1, \alpha_0, q_\sigma]$ of spheromorphisms is equal to $[\alpha_2, \alpha_0, s_{\tau\sigma}]$, where $s_{\tau\sigma}$ is the collection $(s_i = r_{\sigma(i)}q_i \colon T_i^{\alpha_0} \to T_{\tau\circ\sigma(i)}^{\alpha_2})_{i=1,\dots,n}$.

(2) Using the material introduced in Remark 2.1.5, V may be described as the set of equivalence symbols $(\alpha_1, \alpha_0, \sigma)$, for the relation generated by $(\alpha_1, \alpha_0, \sigma) \sim (\exp_{n,\sigma(i)}(\alpha_1), \exp_{n,i}(\alpha_0), \exp_{n,i}(\sigma))$, for $1 \leqslant i \leqslant n$.

(3) With more intricate notations, one could describe similarly the equivalence of symbols defining spheromorphisms.

2.2.2. *Thompson's and Neretin's Groups Acting on the Towers*

Let $g = [\alpha_1, \alpha_0, q_\sigma]$ be in N, $[\mathcal{M}(T, \tau)]$ be a cell of $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$, represented in some $\widetilde{\mathcal{M}}_{0,2^n+1}(\mathbb{R})$.

Represent the cell by a symbol (T_{2^n}, T, τ), where T_{2^n} is the dyadic tree with 2^n leaves, which labels the base-point of the tower, viewed in $\widetilde{\mathcal{M}}_{0,2^n+1}(\mathbb{R})$. After making an expansion of the symbol defining g if necessary, it can be supposed that the trees α_i, $i = 0, 1$, are planar 2^n-trees, with $\alpha_0 = T_{2^n}$. Compose both symbols in the following way:

$$(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)(T_{2^n}, T, \tau) = (\alpha_1, T, \sigma \circ \tau).$$

Interpretation of the symbol $(\alpha_1, T, \sigma \circ \tau)$ *as a label of a cell in an appropriate moduli space*: after making expansions (as in the case of Thompson's group symbols, cf. Remark 2.2.2, 2), though T is not necessarily a dyadic tree), replace $(\alpha_1, T, \sigma \circ \tau)$ by a symbol of the form $(\exp(\alpha_1) = T_{2^m}, \exp(T), \exp(\sigma \circ \tau))$, for some $m \geqslant n \in \mathbb{N}$ (exp denoting the appropriate iterated expansion map, cf. Remark 2.1.5, 4), so that $\exp(\sigma \circ \tau)$ belongs to Σ_{2^m}. Thus, $\overline{\mathcal{M}}(\exp(T), \exp(\sigma \circ \tau))$ is a cell of $\widetilde{\mathcal{M}}_{0,2^m+1}(\mathbb{R})$.

EXAMPLE 2.2.3 (Figure 7).

DEFINITION 2.2.4 (Action of N on the cells of the tower). With the previous notations, the action of $g \in N$ on the set of cells of $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$ is defined by

$$g[\,\overline{\mathcal{M}}(T, \tau)] := [\overline{\mathcal{M}}(\exp(T), \exp(\sigma \circ \tau))].$$

THEOREM 2.2.5. *Denoting by* $\mathrm{Cell}(\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}))$ *the set of cells of the tower* $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$*, the map*

$$N \times \mathrm{Cell}(\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})) \longrightarrow \mathrm{Cell}(\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}))$$

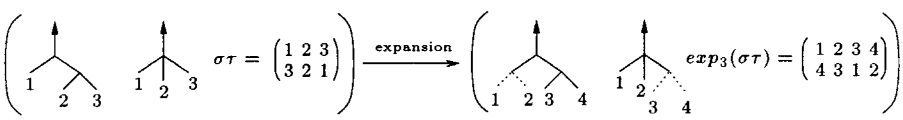

Figure 7.

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introduced above is well-defined, and induces a cellular left action of Neretin's group N on the tower $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$: $\tilde{\gamma}\colon N \longrightarrow \mathrm{Homeo}_{\mathrm{cell}}(\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}))$.

Remark 2.2.6. There is an equivalent definition of the group N acting on the boundary of the regular dyadic *unrooted* tree $\mathcal{T}_2$. Thus defined, N naturally contains $\mathrm{Aut}(\mathcal{T}_2)$, the automorphism group of the tree $\mathcal{T}_2$. The cells of $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R})$ being labelled also by unrooted trees, one defines similarly a cellular action $\bar{\gamma}\colon N \longrightarrow \mathrm{Homeo}_{\mathrm{cell}}(\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R}))$. In [15], Kapranov has noticed that the $\mathrm{Aut}(\mathcal{T}_2)$ acts on the stratified complex tower $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{C})$. In fact, we could prove that this action extends to N, and restricts on the real tower to the action of Theorem 2.2.5.

Proof. We have to prove the independence of the definition of $g[\overline{\mathcal{M}}(T,\tau)]$ (with the notations used in Definition 2.2.4) with respect to the label of the cell and the symbol representing g.

(1) We first check $\mathcal{M}(T,\tau) = \mathcal{M}(T',\tau') \Longrightarrow \mathcal{M}(\exp(T), \exp(\sigma\circ\tau)) = \mathcal{M}(\exp(T'), \exp(\sigma\circ\tau'))$. It is clear that we can restrict ourselves to the case where $\mathcal{T}' = j_{T_v}\mathcal{T}$ and $\tau' = \tau\omega_v$, for some $v \in \mathrm{Vert}^*(T)$ (we use the correspondence $T \leftrightarrow \mathcal{T}$, $T' \leftrightarrow \mathcal{T}'$ between planar trees and nested families, cf. Section 1.3). To simplify the notations, we may suppose that ω_v is of the form

$$\omega_v = \begin{pmatrix} 1 & 2 & \dots & k \\ k & k-1 & \dots & 1 \end{pmatrix} = \omega_{(1,\dots,k)}$$

Using the iterated expansion homomorphism $\exp\colon \Sigma_{2^n} \to \Sigma_{2^m}$, we may write $\exp(\sigma\circ\tau') = \exp(\sigma\circ\tau)\exp(\omega_v)$. By induction it is enough to consider the case $m = n+1$; then with the canonical labelling of $\exp(T)$,

$$\exp(\omega_v) = \begin{pmatrix} 1 & 2 & \dots & 2l-1 & 2l & \dots & 2k-1 & 2k \\ 2k-1 & 2k & \dots & 2k-2l+1 & 2k-2l+2 & \dots & 1 & 2 \end{pmatrix}.$$

Denoting by $\tilde{v}$ the vertex v seen in the expanded tree $\exp(T)$, $\exp(\omega_v)$ differs from

$$\omega_{\tilde{v}} = \begin{pmatrix} 1 & 2 & \dots & 2l-1 & 2l & \dots & 2k-1 & 2k \\ 2k & 2k-1 & \dots & 2k-2l+2 & 2k-2l+1 & \dots & 2 & 1 \end{pmatrix}$$
$$= \omega_{(1,\dots,2k)}$$

by the product of the k transpositions $\omega_{\tilde{v}_1}\dots\omega_{\tilde{v}_k}$, where $\omega_{\tilde{v}_i}$ is simply $\omega_i = (2i-1, 2i)$, and $\tilde{v}_i$ is the ith leaf of T seen in $\exp(T)$ (two terminal edges emanate from each $\tilde{v}_i$ in $\exp(T)$). It follows that

$$\exp(\mathcal{T}') = j_{T_{\tilde{v}_1}}\dots j_{T_{\tilde{v}_k}} j_{T_{\tilde{v}}}\exp(\mathcal{T}),$$

$$\exp(\sigma\circ\tau') = \exp(\sigma\circ\tau)\omega_{\tilde{v}}\omega_{\tilde{v}_k}\dots\omega_{\tilde{v}_1},$$

where in fact the operations $j_{T_{\tilde{v}_i}}$ have no effect. By Theorem A (see 1.5), the expected implication is proved.

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(2) *Independence with respect to the choice of the symbol defining* g: it suffices to replace the symbol $(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)$ by its expansion $\exp_{2^n}(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)$, and check that the result of the product $\exp_{2^n}(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)\exp_{2^n}(T_{2^n}, T, \tau)$ defines the same cell as the symbol $(\alpha_1, T, \sigma \circ \tau)$. Now $\exp_{2^n}(\alpha_1, T_{2^n}, q_\sigma) = (\exp_{2^n}(\alpha_1), T_{2^{n+1}}, \tilde{q}_{\tilde{\sigma}})$, where $\tilde{\sigma}$ may differ from $\exp_{2^n}(\sigma)$ (because g lies in N, not necessarily in V) by a product of transpositions $\omega_i = (2i-1, 2i)$: $\tilde{\sigma} = \exp_{2^n}(\sigma)\omega_{i_1} \ldots \omega_{i_r}$. So,

$$\begin{aligned}(*) :&= \exp_{2^n}(\alpha_1, T_{2^n}, q_\sigma)\exp_{2^n}(T_{2^n}, T, \tau)\\ &= (\exp_{2^n}(\alpha_1), \exp_{2^n}(T), \exp_{2^n}(\sigma)\omega_{i_1} \ldots \omega_{i_r}\exp_{2^n}(\tau)).\end{aligned}$$

But $\exp_{2^n}(\tau)\omega_i = \omega_{\tau(i)}\exp_{2^n}(\tau)$, so that

$$\begin{aligned}(*) &= (\exp_{2^n}(\alpha_1), \exp_{2^n}(T), \exp_{2^n}(\sigma)\exp_{2^n}(\tau)\omega_{\tau^{-1}(i_1)} \ldots \omega_{\tau^{-1}(i_r)})\\ &= (\exp_{2^n}(\alpha_1), \exp_{2^n}(T), \exp_{2^n}(\sigma \circ \tau)\omega_{\tau^{-1}(i_1)} \ldots \omega_{\tau^{-1}(i_r)}).\end{aligned}$$

Since $(\exp_{2^n}(\alpha_1), \exp_{2^n}(T), \exp_{2^n}(\sigma \circ \tau)\ \omega_{\tau^{-1}(i_1)} \ldots \omega_{\tau^{-1}(i_r)})$ labels the same cell as the symbol $(\exp_{2^n}(\alpha_1), \exp_{2^n}(T),\ \exp_{2^n}(\sigma \circ \tau))$ (in the appropriate finite moduli space), we see that the cell defined by $\exp_{2^n}(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)\exp_{2^n}(T_{2^n}, T, \tau)$ coincides indeed in the inductive limit with the cell defined by $(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)(T_{2^n}, T, \tau)$.

(3) Once the action is proved to be well-defined, it is straightforward to check that it is cellular. □

2.3. EXTENSIONS OF THOMPSON'S AND NERETIN'S GROUPS BY AN INFINITE PURE QUASI-BRAID GROUP

It is shown in [5] that $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ (or $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$) is an aspherical space: its universal cover $\widehat{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is contractible. The universal cover of $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$ is the inductive limit of the covers $\widehat{\mathcal{M}}_{0,2^n+1}(\mathbb{R})$, and will be denoted $\widehat{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$.

NOTATION 2.3.1. Denote by PJ_n (resp. Q_n) the fundamental group of $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ (resp. $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$). Since $\widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is a double-cover of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, there is a (non-split) extension $1 \to PJ_n \to Q_{n+1} \to \mathbb{Z}/2\mathbb{Z} \to 0$. Define the infinite pure quasi-braid group to be $PJ_{2^\infty} = \pi_1(\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})) = \varinjlim_n PJ_{2^n}$.

Each transformation $\tilde{\gamma}(g)\colon \widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) \longrightarrow \widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$, with $g \in N$, can be lifted to the universal cover $\widehat{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$.

DEFINITION-PROPOSITION 2.3.2. *The set $\mathcal{A}_N$ of lifted transformations $\tilde{\gamma}(g)$, $g \in N$, is a subgroup of the cellular homeomorphism group of $\widehat{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$.*

The kernel of the natural epimorphism $\mathcal{A}_N \longrightarrow N$ is the automorphism group of the universal covering map $\widehat{\mathcal{M}}_{0,2^\infty}(\mathbb{R}) \longrightarrow \widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$, PJ_{2^∞}. We define the quasi-braid extension of N to be the short exact sequence $1 \to PJ_{2^\infty} \longrightarrow \mathcal{A}_N \longrightarrow N \to 1$. By restriction, one obtains a similar extension $1 \to PJ_{2^\infty} \longrightarrow \mathcal{A}_V \longrightarrow V \to 1$.

Remark 2.3.3. Considering Neretin's group acting on the tower $\overline{\mathcal{M}}_{0,3.2^\infty}(\mathbb{R})$ (cf. Remark 2.2.6), one would obtain a similar extension by the group $Q_{3.2^\infty} = \varinjlim_n Q_{3.2^n}$. However, the expansion morphism $Q_n \hookrightarrow Q_{2n}$ maps Q_n into PJ_{2n-1}, so that $Q_{3.2^\infty}$ is an inductive limit of groups $PJ_{3.2^n-1}$. Thus, the extension should be isomorphic to the previous one.

3. An Analogue of the Euler Class for Neretin's Group of Spheromorphisms

Recall that the elements of V and N are described by symbols involving trees and permutations. In this section, we shall give a similar description of the groups $\mathcal{A}_V$ and $\mathcal{A}_N$ (Section 3.2), the role of the permutation groups Σ_n now being played by the *quasi-braid groups* J_n (described in Section 3.1). The precise description of $\mathcal{A}_N$ will enable us to define a nontrivial central extension for N–the 'Euler class' of N.

3.1. QUASI-BRAID GROUPS J_n

3.1.1 Let J_n be the group defined in [6] by generators and relations, with generators α_T for each strict subset T of $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ such that the corresponding graph G_T is connected, and relations:

- $\alpha_T^2 = 1$ for each T
- $\alpha_T \alpha_{T'} = \alpha_{j_T T'} \alpha_T$ if $T' \subset T$
- $\alpha_T \alpha_{T'} = \alpha_{T'} \alpha_T$ if $G_{T \cup T'}$ is not connected.

These relations are those verified by the involutions ω_T. So, there is a well-defined homomorphism $\phi\colon J_n \to \Sigma_n$, $\alpha_T \mapsto \omega_T$, which is surjective since for $T_i = \{\sigma_i = (i, i+1)\}$, $\phi(\alpha_{T_i}) = \sigma_i$.

THEOREM B (Davis–Januszkiewicz–Scott, [6]). *The universal cover $\widehat{\mathcal{M}}_{0,n+1}$ of the two-sheeted cover $\widetilde{\mathcal{M}}_{0,n+1}$ is J_n-equivariantly homeomorphic to the geometric realization $|J_n \mathcal{N}|$ of the poset $J_n \mathcal{N}$:*

(1) *Its elements are the equivalence classes of pairs $(\mathcal{T}, \alpha)$, with $\mathcal{T}$ a nested collection, and α in J_n, for the equivalence relation defined by $(\mathcal{T}, \alpha) \sim (\mathcal{T}', \alpha')$ if and only if there exists a subset $\mathcal{T}'' \subset \mathcal{T}$ such that $\alpha' = \alpha\alpha_{\mathcal{T}''}$ and $\mathcal{T}' = j_{\mathcal{T}''}\mathcal{T}$. Here $\alpha_{\mathcal{T}''} = \alpha_{T_1} \ldots \alpha_{T_r}$ if $\mathcal{T}'' = \{T_1, \ldots, T_r\}$ (it can be supposed $i \leqslant j$ if $T_i \subset T_j$), and $j_{\mathcal{T}''} = j_{T_r} \ldots j_{T_1}$.*

(2) *The partial order is defined by $[\mathcal{T}, \alpha] \leqslant [\mathcal{T}', \alpha']$ if and only if there exists some $\mathcal{T}'' \subset \mathcal{T}$ such that $\alpha' = \alpha\alpha_{\mathcal{T}''}$ and $\mathcal{T}' \subset j_{\mathcal{T}''}\mathcal{T}$.*

Moreover, there is a natural J_n-left-equivariant map $J_n\mathcal{N} \to \Sigma_n\mathcal{N}$ given by $[\mathcal{T}, \alpha] \to [\mathcal{T}, \phi(\alpha)]$, the J_n-action on $\Sigma_n\mathcal{N}$ being defined by $\alpha\colon [\mathcal{T}, \sigma] \mapsto [\mathcal{T}, \phi(\alpha)\sigma]$.

The kernel $PJ_n := \mathrm{Ker}\phi$ is the fundamental group of $\widetilde{\mathcal{M}}_{0,n+1}$, and there is a short exact sequence $1 \to PJ_n \longrightarrow J_n \longrightarrow \Sigma_n \to 1$.

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3.1.2 *Describing the Morphism $PJ_n \to PJ_{2n}$ and Defining $J_n \to J_{2n}$*

The formalism contained in Theorem B enables us to describe the morphism $PJ_n \to PJ_{2n}$ induced by the embedding $\widetilde{\exp}_n \colon \widetilde{\mathcal{M}}_{0,n+1} \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}$. Each $\alpha = \alpha_{T_1} \ldots \alpha_{T_r}$ in PJ_n projects onto $\omega_{T_1} \ldots \omega_{T_r} = 1$ in Σ_n. We interpret α as the homotopy class of the edge loop

$$\gamma = (\mathrm{id}_n \to \omega_{T_1} \to \omega_{T_1}\omega_{T_2} \to \cdots \to \omega_{T_1}\omega_{T_2} \ldots \omega_{T_r} = \mathrm{id}_n)$$

in the dual cell complex of $|\Sigma_n \mathcal{N}| \cong \widetilde{\mathcal{M}}_{0,n+1}$.

By definition of the embedding $\widetilde{\exp}_n \colon \widetilde{\mathcal{M}}_{0,n+1} \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}$, the loop γ is mapped onto the loop

$$\begin{aligned}\widetilde{\exp}_n(\gamma) = &(\mathrm{id}_{2n} \to \exp(\omega_{T_1}) \to \exp(\omega_{T_1})\exp(\omega_{T_2}) \to \cdots \to \exp(\omega_{T_1}) \\ &\exp(\omega_{T_2}) \ldots \exp(\omega_{T_r}) = \mathrm{id}_{2n}).\end{aligned}$$

We need, however, to make it precise what a path of the form $\mathrm{id}_{2n} \to \exp(\omega_T)$ is: Suppose for simplicity that ω_T is of the form

$$\omega_T = \begin{pmatrix} 1 & 2 & \ldots & k \\ k & k-1 & \ldots & 1 \end{pmatrix},$$

so that

$$\exp(\omega_T) = \begin{pmatrix} 1 & 2 & \ldots & 2k-1 & 2k \\ 2k-1 & 2k & \ldots & 1 & 2 \end{pmatrix}$$

is the product $\omega_{\exp(T)}\omega_{(1,2)} \ldots \omega_{(2k-1,2k)}$, with $\omega_{(2i-1,2i)}$ the transposition σ_{2i-1}. The path $\mathrm{id}_n \to \omega_T$ once embedded in $|\Sigma_{2n}\mathcal{N}^0|$, and after a suitable translation to make its extremities coincide with the barycenters of the cells id_{2n} and $\exp(\omega_T)$ (which are adjacent because exp is cellular, and meet along a codimension $n+1$ cell), becomes the straight line joining the barycenters. We claim this line is homotopic to the edge path

$$\mathrm{id}_{2n} \to \omega_{\exp(T)} \to \omega_{\exp(T)}\omega_{(1,2)} \to \cdots \to \omega_{\exp(T)}\omega_{(1,2)} \cdots \omega_{(2k-1,2k)}.$$

Indeed, the path above passes through cells which all share a same codimension $n+1$ cell, and the line $\mathrm{id}_{2n} \to \exp(\omega_T)$ crosses the same cell.

Now α_T may be lifted in J_{2n} to $\exp(\alpha_T) := \alpha_{\exp(T)}\alpha_{(1,2)} \ldots \alpha_{(2k-1,2k)}$, where $\alpha_{(2i-1,2i)} := \alpha_{T_i}$, with $T_i = \{\sigma_{2i-1}\}$. Finally define $\exp(\alpha)$ as the product $\exp(\alpha_{T_1}) \ldots \exp(\alpha_{T_r})$. We now claim:

PROPOSITION 3.1.1. (1). *The map $J_n \to J_{2n}$: $\alpha \mapsto \exp(\alpha)$, is a well-defined group homomorphism. More generally, each expansion map* $\exp : \Sigma_n \to \Sigma_{n+*}$ *has a canonical lift $J_n \to J_{n+*}$.*

(2) *Its restriction to PJ_n is the morphism $(\widetilde{\exp}_n)_* \colon PJ_n \to PJ_{2n}$ induced at the fundamental group level by the embedding $\widetilde{\exp}_n \colon \widetilde{\mathcal{M}}_{0,n+1} \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}$.*

(3) *The morphisms $(\widetilde{\exp}_n)_*$ are injective for all $n \geqslant 2$.*

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Proof. (1) We must check that exp preserves the relations of the group J_n: For simplicity, suppose $G_T = (1, \dots, j)$ and compute $\exp(\alpha_T)^2$:

$$\exp(\alpha_T)^2 = \alpha_{\exp(T)}\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}\alpha_{\exp(T)}\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}.$$

Now observe that in J_{2n}, for all $i \leqslant j$,

$$\alpha_{\exp(T)}\alpha_{(2i-1,2i)} = \alpha_{2(j-i+1)-1,2(j-i+1)}\alpha_{\exp(T)}.$$

This fact joint to the commutation property of $\alpha_{(1,2)}, \dots, \alpha_{(2j-1,2j)}$ among each other allows to write $\exp(\alpha_T) = \alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}\alpha_{\exp(T)}$, and it comes easily $\exp(\alpha_T)^2 = 1$. Then let $T' \subset T$ be such that $G_{T'} = (1, \dots, i)$, $i \leqslant j$, and check the relation $\alpha_T\alpha_{T'}\alpha_T = \alpha_{j_T T'}$ is preserved by exp.

Compute

$$\begin{aligned}
&\exp(\alpha_T)\exp(\alpha_{T'})\exp(\alpha_T)\\
&\quad = [\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}\alpha_{\exp(T)}].[\alpha_{\exp(T')}\alpha_{(1,2)} \dots \alpha_{(2i-1,2i)}].[\alpha_{\exp(T)}\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}]\\
&\quad = [\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}].\big(\alpha_{\exp(T)}\alpha_{\exp(T')}\alpha_{\exp(T)}\big).[\alpha_{(2(j-i+1)-1,2(j-i+1))} \dots \alpha_{(2j-1,2j)}]\times\\
&\qquad \times [\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}]\\
&\quad = [\alpha_{(1,2)} \dots \alpha_{(2j-1,2j)}].\alpha_{j_{\exp(T)}\exp(T')}.[\alpha_{(1,2)} \dots \alpha_{(2(j-i)-1,2(j-i))}] = (*).
\end{aligned}$$

But

$$j_{\exp(T)}\exp(T') = (2(j-i+1)-1, \dots, 2j),$$

so that

$$\begin{aligned}
&\alpha_{j_{\exp(T)}\exp(T')}.[\alpha_{(1,2)} \dots \alpha_{(2(j-i)-1,2(j-i))}]\\
&\quad = [\alpha_{(1,2)} \dots \alpha_{(2(j-i)-1,2(j-i))}].\alpha_{j_{\exp(T)}\exp(T')},
\end{aligned}$$

and finally,

$$(*) = \alpha_{(2(j-i+1)-1,2(j-i+1))} \dots \alpha_{(2j-1,2j)}\alpha_{j_{\exp(T)}\exp(T')} = \exp(\alpha_{j_{\exp(T)}\exp(T')}),$$

which ends the proof of the first assertion of (1).

If now one performs, say, one simple expansion from the ith label, corresponding to the expansion map $\exp_{n,i}\colon \Sigma_n \to \Sigma_{n+1}$, then there exists a lift $J_n \to J_{n+1}$: if $\alpha = \alpha_{T_r} \dots \alpha_{T_2}\alpha_{T_1} \in J_n$, then (supposing $\alpha_{T_1} = \alpha_{(1,\dots,j)}$ to simplify the notations), define first $\exp(\alpha_{T_1}) = \alpha_{(1,\dots,j,j+1)}\alpha_{(i,i+1)}$ if i belongs to the support of T_1 (if not, don't modify α_{T_1}), next define similarly $\exp(\alpha_{T_2})$ by expanding the $\omega_{T_1}(i)th$ label, and so on. Finally one obtains $\exp(\alpha) := \exp(\alpha_{T_r}) \dots \exp(\alpha_{T_1}) \in J_{n+1}$, which projects onto Σ_{n+1} on the expansion (from the ith label) of the permutation $\omega = \omega_{T_r} \dots \omega_{T_2}\omega_{T_1} \in \Sigma_n$. Again, it can be checked that the relations in the groups J_n and J_{n+1} are preserved by this expansion map, which proves it is well-defined.

(2) Let $\alpha = \alpha_{T_1}\alpha_{T_2} \dots \alpha_{T_r} \in PJ_n = \operatorname{Ker}\phi$, γ the combinatorial loop attached to α, based at id_n. We claim that loop $\widetilde{\exp}_n(\gamma)$ lifts to the path $(1 \to \exp(\alpha_{T_1}) \to \dots \to \exp(\alpha_{T_1}) \dots \exp(\alpha_{T_r}))$, where $1 \to \exp(\alpha_T)$ is defined to be

$$1 \to \alpha_{\exp(T)} \to \alpha_{\exp(T)}\alpha_{(1,2)} \to \cdots \to \alpha_{\exp(T)}\alpha_{(1,2)}\cdots\alpha_{(2k-1,2k)}.$$

Indeed, applying ϕ to this path gives precisely the loop $\widetilde{\exp}_n(\gamma)$, as described in the preliminary of Proposition 3.1.1, for the embedding $\widetilde{\exp}_n : \widetilde{\mathcal{M}}_{0,n+1} \hookrightarrow \widetilde{\mathcal{M}}_{0,2n+1}$. It ends at $\exp(\alpha_{T_1})\ldots\exp(\alpha_{T_r}) = \exp(\alpha)$, so $(\widetilde{\exp}_n)_*(\alpha) = \exp(\alpha)$.

(3) We use the fact that the embedding $\widetilde{\exp}_n$ has a retraction

$$r_n : \widetilde{\mathcal{M}}_{0,2n+1}(\mathbb{R}) \to \widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R}),$$

which is the composite of the forgetful maps $\widetilde{\mathcal{M}}_{0,2n+1}(\mathbb{R}) \to \widetilde{\mathcal{M}}_{0,2n}(\mathbb{R}) \to \cdots \to \widetilde{\mathcal{M}}_{0,n+1}(\mathbb{R})$ (cf. [16]: the map $\overline{\mathcal{M}}_{0,n+1}(\mathbb{C}) \to \overline{\mathcal{M}}_{0,n}(\mathbb{C})$ is a universal family of n-pointed stable curves). □

3.2. DESCRIPTION OF THE EXTENDED GROUP $\mathcal{A}_N$

The group $\mathcal{A}_N$ has a description similar to the group N, by replacing the symmetric group Σ_n and the expansion maps $\exp: \Sigma_n \to \Sigma_{n+*}$ by the quasi-braid groups J_n and the maps $\exp: J_n \to J_{n+*}$ (cf. 3.1.1) respectively.

DEFINITION-PROPOSITION 3.2.1. (1) *By a quasi-braided symbol we mean a triple* $(\alpha_1, \alpha_0, q_\sigma)$, *where* α_0, α_1 *are finite rooted dyadic n-trees for some* $n \geqslant 2$, σ *belongs to some* J_n, *and* q_σ *is a family* $(\sigma_k)_{k\in\mathbb{N}}$ *in the product* $\prod_{k\in\mathbb{N}} J_{2^k.n}$, *such that* $\sigma_0 = \sigma$ *and* σ_{k+1} *may differ from* $\exp_{2^k.n}(\sigma_k)$ *by a product of quasi-braid transpositions of the form* $\alpha_{(2i-1,2i)}$.

(2) *Two such symbols are said to be equivalent if they have a common expansion, this notion being defined as in Remark* 2.2.2(3)*,*

(3) *The group* $\mathcal{A}_N$ *is isomorphic to the set of equivalence classes of quasi-braided symbols endowed with the product induced by the obvious composition of symbols.*

Proof. Easy. □

Remark. The elements of $\mathcal{A}_V \subset \mathcal{A}_N$ will be represented by symbols $(\alpha_1, \alpha_0, \sigma)$, where α_0, α_1 are rooted dyadic n-trees for some $n \geqslant 2$, an σ belongs to J_n (compare with Section 2.2.1).

3.3. A STABLE LENGTH, AND A CENTRAL EXTENSION FOR N

Let $\alpha = \alpha_{T_1}\ldots\alpha_{T_r}$ be in the free monoid freely generated by the generators of J_n. Define its length to be $\ell_n(\alpha) = r + |T_1| + \cdots + |T_r|$, where $|T_i|$ is the length of the graph G_{T_i}.

PROPOSITION 3.3.1 (stable length). *The length* ℓ_n *induces a well-defined group homomorphism* $\ell_n : J_n \to \mathbb{Z}/2\mathbb{Z}$, $\ell_n(\alpha) = r + |T_1| + \ldots + |T_r| \mod 2$. *Moreover, the collection* $\{\ell_n, n \geqslant 1\}$ *is compatible with the direct system* $\{J_n, \exp_n\}$, *and induces a stable length* $\ell_\infty : J_{2^\infty} \to \mathbb{Z}/2\mathbb{Z}$. *More generally, the length is compatible with the dyadic expansion maps* $J_n \to J_{n+*}$ (cf. Proposition 3.1.1, 1.).

The restriction of ℓ_∞ *to the infinite pure braid group* PJ_{2^∞} *is still nontrivial. Finally, the stable length* ℓ_∞ *can be extended to* $\mathcal{A}_V$*, but not to the whole group* $\mathcal{A}_N$.

Proof. The last two relations in the presentation of J_n preserve the length ℓ_n. The first one ($\alpha_T^2 = 1$) preserves the length mod 2 only. So $\ell_n \colon J_n \to \mathbb{Z}/2\mathbb{Z}$ is a well-defined group homomorphism.

On the other hand, if $\alpha_T = \alpha_{(1,\ldots,k)}$ is in J_n and one performs a simple expansion from the first leaf (to simplify the notations), then

$$\exp(\alpha_T) = \alpha_{(1,\ldots,k+1)}\alpha_{(1,2)} \in J_{n+1},$$

and

$$\ell_n(\alpha_T) = 1 + k \bmod 2, \ell_{n+1}(\exp(\alpha_T)) = k + 1 + 1 + 2 + 1 = \ell_n(\alpha_T) \bmod 2.$$

Further observe that the pure braid $p = \alpha_{(1,2)}\alpha_{(2,3)}\alpha_{(1,2)}\alpha_{(1,2,3)} \in PJ_4$ has a stable length equal to $1 \bmod 2$.

Let now $g = [\alpha_1, \alpha_0 = T_{2^n}, \sigma]$ be in $\mathcal{A}_V$, with $\sigma \in J_{2^n}$. Then $\ell_\infty(g) := \ell_\infty(\sigma)$ is well-defined, since expanding the symbol defining g would replace σ by some expansion of it.

The impossibility to extend ℓ_∞ to $\mathcal{A}_N$ is essentially equivalent to Theorem 3.3.2 below. □

Let now $\mathrm{Ker}\,\ell_\infty$ be the kernel of the restriction of ℓ_∞ to PJ_{2^∞}.

THEOREM 3.3.2 (Analogue of the Euler class for N). *The quasi-braid extension* $1 \to PJ_{2^\infty} \to \mathcal{A}_N \to N \to 1$ *induces a nontrivial central extension*

$$1 \to \mathbb{Z}/2\mathbb{Z} \cong PJ_{2^\infty}/\mathrm{Ker}\,\ell_\infty \longrightarrow \widetilde{N} := \mathcal{A}_N/\mathrm{Ker}\,\ell_\infty \longrightarrow N \to 1,$$

which defines a nontrivial cohomology class $\mathrm{Eu} \in H^2(N, \mathbb{Z}/2\mathbb{Z})$.

Proof. Let $g = [\alpha_1, \alpha_0 = T_{2^n}, q_\sigma]$ be in $\mathcal{A}_N$ ($\sigma \in J_{2^n}$), and $p \in PJ_{2^\infty}$, represented by $[\alpha_1, \alpha_1, p_1]$, with $p_1 \in PJ_{2^n}$. It follows that $g^{-1}pg$ is represented in $PJ_{2^n} \subset PJ_{2^\infty}$ by $\sigma^{-1}p_1\sigma$, and

$$\ell_\infty(g^{-1}pg) = \ell_\infty(\sigma^{-1}p_1\sigma) = -\ell_\infty(\sigma) + \ell_\infty(p_1) + \ell_\infty(\sigma) = \ell_\infty(p),$$

thanks to Proposition 3.3.1. This proves that $[\mathcal{A}_N, PJ_{2^\infty}] \subset \mathrm{Ker}\,\ell_\infty$: so, $\mathrm{Ker}\,\ell_\infty$ is normal in $\mathcal{A}_N$, and the extension is central.

Suppose the extension is trivial: the embedding i: $\mathbb{Z}/2\mathbb{Z} \to \mathcal{A}_N$ would admit a retraction r. We prove it is impossible, by writing the generator of the kernel $\mathbb{Z}/2\mathbb{Z} \cong PJ_{2^\infty}/\mathrm{Ker}\,\ell_\infty$ as a product of commutators in $\mathcal{A}_N/\mathrm{Ker}\,\ell_\infty$, i.e. finding a pure quasi-braid with length $1 \bmod 2$ which is a product of commutators in $\mathcal{A}_N$.

Conventions: In the proof below, we shall simplify the representation of a symbol where both trees are the same by a single tree-symbol, and figure out the permutation by arrows indicating its action on the leaves of the tree; moreover, when the permutation occuring in a symbol representing an element of V is the identity, it will be omitted in the representation of the symbol.

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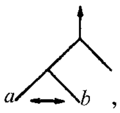
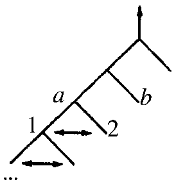
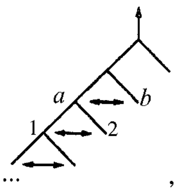
Let $\tau \in V \subset N$ be the transposition defined by the single tree-symbol



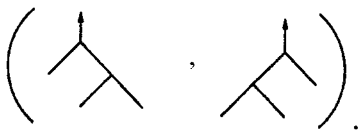
the leaves a and b being permuted. Let $\alpha \in \mathrm{Aut}(T_2) \subset N$ be defined by the single tree-symbol



(the permutations of the leaves indicated by the arrows must be composed *from bottom to top,* see also the definition of $\tilde{\alpha}$ below). Set

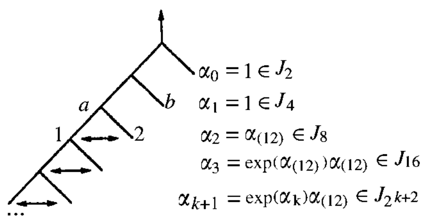
$\gamma := \tau\alpha = \dots$ 

and it appears that γ and α are conjugated by the 'translation'

$\delta =$ 

Precisely, we have $\gamma = \delta\alpha\delta^{-1}$, or equivalently, $\tau = [\delta, \alpha]$.

We now lift τ, δ and α to $\mathcal{A}_N$ in an obvious way: τ is lifted in $\tilde{\tau}$ (same symbol coupled with $\alpha_{(12)} \in J_3$), δ in $\tilde{\delta}$ (same symbol coupled with $1 \in J_3$), and α lifted in $\tilde{\alpha}$ (same symbol coupled with the sequence $\alpha_0 = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_{(12)}$,

$\alpha_{k+1} = \exp(\alpha_k)\alpha_{(12)}, \ k \in \mathbb{N}$): 

Clearly, the same relation as in N holds in $\mathcal{A}_N$: $\tilde{\tau} = [\tilde{\delta}, \tilde{\alpha}] \in \mathcal{A}_N$.

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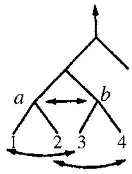
On the other hand,



$\tau =$

may also be written as the product $\tau = \tau_1\tau_2$, where τ_1 exchanges the leaves 1 and 3 (keeping 2 and 4 fixed) and τ_2 exchanges the leaves 2 and 4 (keeping 1 and 3 fixed). We note abusively $\tau_1 = (13)$, $\tau_2 = (24)$. Introducing $\sigma \in V$ defined by $\sigma = (12)(34)$, we have $\tau_2 = \sigma\tau_1\sigma$, and $\tau = [\tau_1, \sigma]$.

We then lift τ_1 and σ to $\mathcal{A}_N$ by $\tilde{\tau_1} = \alpha_{(123)}$, $\tilde{\sigma} = \alpha_{(12)}\alpha_{(34)}$. Now $[\tilde{\tau_1}, \tilde{\sigma}]$ differs from $\tilde{\tau} = [\tilde{\delta}, \tilde{\alpha}] = [\tilde{\alpha}, \tilde{\delta}]$ by a pure quasi-braid

$$p = [\tilde{\tau_1}, \tilde{\sigma}][\tilde{\delta}, \tilde{\alpha}] = [\alpha_{(123)}, \alpha_{(12)}\alpha_{(34)}]\exp(\alpha_{(12)})$$

$$= [\alpha_{(123)}, \alpha_{(12)}\alpha_{(34)}]\alpha_{(12)}\alpha_{(34)}\alpha_{(1234)} = \alpha_{(123)}\alpha_{(12)}\alpha_{(34)}\alpha_{(123)}\alpha_{(1234)}.$$

The miracle is that $\ell_\infty(p) = 1 \bmod 2$ as desired. □

COROLLARY 3.3.3. *The* 2-*cycle* ω *defined by the relation* $[\tau_1, \sigma][\alpha, \delta] = 1 \in N$ *is nontrivial and verifies* $(\mathrm{Eu}, [\omega]) = 1$, *where* $\mathrm{Eu} \in H^2(N, \mathbb{Z}/2\mathbb{Z})$ *is the cohomology class of the extension of* N.

Proof. This is an immediate application of the following standard lemma in homological algebra:

LEMMA 3.3.4. *Let* G *be a perfect group,* $A \to \hat{G} \to G$ *a central extension of* G *with kernel an Abelian group* A, $c \in H^2(G, A)$ *the associated cohomology class. If* ω *is a* 2-*cycle of* G *associated with a relation* $1 = \prod_i [g_i, h_i]$ *in* G, *then* $(c, [\omega]) = a \in A$, *where* a *is computed as* $a = \prod_i [\hat{g}_i, \hat{h}_i]$, *for any choices of lifts* $\hat{g}_i$, $\hat{h}_i$ *of* g_i, h_i.

So, in our case, $(Eu, [\omega]) = \ell_\infty(p) = 1 \bmod 2$. □

3.4. EULER-TYPE COCYCLE

Let $\mathcal{R}$ be the ring of $\mathbb{Z}/2\mathbb{Z}$-valued sequences, divided by the ideal of almost zero sequences: $\mathcal{R} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}/(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$. Denote by $1_{\mathcal{R}}$ its unit.

For each f in $\mathcal{A}_N$ defined by a symbol of the form $(\alpha_1, \alpha_0 = T_{2^n}, q_\sigma)$ (cf. Section 3.2), there is a family $(\sigma_k)_{k \geqslant n}$, $\sigma_k \in J_{2^k}$, with $\sigma_n = \sigma$ and σ_{k+1} differing from $\exp_{2^k}(\sigma_k)$ by a product of quasi-braid transpositions. So there is a well-defined function $\ell\colon \mathcal{A}_N \to \mathcal{R}$, $f \mapsto \ell(f)$, where $\ell(f)$ is represented by the sequence $\ell(f)_k = \ell_\infty(\sigma_k) \in \mathbb{Z}/2\mathbb{Z}$ for $k \geqslant n$ and, say, $\ell(f)_k = 0$ for $k = 0, \ldots, n-1$.

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Denote by

$$j\colon \mathbb{Z}/2\mathbb{Z} = PJ_{2^\infty}/\mathrm{Ker}\,\ell_\infty \hookrightarrow \mathcal{R} \text{ and } i\colon \mathbb{Z}/2\mathbb{Z} = PJ_{2^\infty}/\mathrm{Ker}\,\ell_\infty \hookrightarrow \widetilde{N}$$

the natural embeddings: $1_{\mathcal{R}} = j(1_{\mathbb{Z}/2\mathbb{Z}})$, $\mathbf{1} := i(1_{\mathbb{Z}/2\mathbb{Z}})$.

LEMMA 3.4.1. *For all f in $\mathcal{A}_N$ and $p \in PJ_{2^\infty}$, $\ell(pf) = \ell(p) + \ell(f)$, and ℓ induces a function $\tilde{\ell}\colon \widetilde{N} = \mathcal{A}_N/\mathrm{Ker}\,\ell_\infty \to \mathcal{R}$ such that for all $\tilde{f} \in \widetilde{N}$, $\tilde{\ell}(\mathbf{1}.\tilde{f}) = 1_{\mathcal{R}} + \tilde{\ell}(\tilde{f})$.*

Proof. Choose n such that f is represented by a symbol (T_1, T_{2^n}, q_σ), and p by (T_1, T_1, α), where α belongs to PJ_{2^n}. Thus, $pf = [T_1, T_{2^n}, q_\tau]$ with $\tau = \tau_n = \alpha\sigma$, $\tau_k = \exp^{(k-n+1)}(\alpha)\sigma_k$, $k \geqslant n$ (where $\exp^{(k-n+1)}$ denotes the $(k-n+1)$ times iterated expansion morphism). Since $\ell_\infty(\exp^{(k-n+1)}(\alpha)) = \ell_\infty(\alpha)$ $\forall k \geqslant n$, the proof is done. Note that $\ell(p) = j(\ell_\infty(\alpha))$. □

Since N is perfect, the injection $j\colon \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathcal{R}$ induces an injective morphism $j_*\colon H^2(N, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(N, \mathcal{R})$.

THEOREM 3.4.2. *The image by j_* of the Euler class* Eu $\in H^2(N, \mathbb{Z}/2\mathbb{Z})$ *is the cohomology class of the well-defined cocycle $c\colon N \times N \to \mathcal{R}$ defined by $c(f, g) = \tilde{\ell}(\tilde{f}\tilde{g}) - \tilde{\ell}(\tilde{f}) - \tilde{\ell}(\tilde{g})$, where $\tilde{f}$ and $\tilde{g}$ are any lifts in $\widetilde{N}$ of f and g in N, respectively.*

Proof. First the fact that the cocycle c is well-defined follows from the equivariant relation of Lemma 3.4.1.

Let ω be a 2-cycle of N. It is associated with a relation $\prod_{i=1}^p [f_i, g_i] = 1 \in N$, and may be written

$$\omega = \sum_{i=1}^{p} (f_i, g_i) - (g_i, f_i) - (g_i f_i, (g_i f_i)^{-1}) + (f_i g_i, (g_i f_i)^{-1}) + \\ + \sum_{i=1}^{p-1} ([f_1, g_1] \ldots [f_i, g_i], [f_{i+1}, g_{i+1}]).$$

It follows that $([c], [\omega]) = \tilde{\ell}(\prod_{i=1}^p [\widetilde{f_i}, \widetilde{g_i}])$, for any lifts $\widetilde{f_i}$, $\widetilde{g_i}$ in $\widetilde{N}$ of f_i, g_i. Now $\prod_{i=1}^p [\widetilde{f_i}, \widetilde{g_i}] = \alpha \bmod \mathrm{Ker}\,\ell_\infty$, for some $\alpha \in PJ_{2^\infty}$, and $\tilde{\ell}(\prod_{i=1}^p [\widetilde{f_i}, \widetilde{g_i}]) = \ell(\alpha) = j(\ell_\infty(\alpha))$. But by Lemma 3.3.4, $\ell_\infty(\alpha) = (\mathrm{Eu}, [\omega])$, so that $([c], [\omega]) = j((\mathrm{Eu}, [\omega])) = (j_* Eu, [\omega])$. Since $H^2(N, \mathcal{R}) = \mathrm{Hom}(H_2(N), \mathcal{R})$, this proves indeed $[c] = j_*\mathrm{Eu}$. □

3.5. THE ANALOGY WITH THE EULER CLASS OF HOMEOMORPHISM GROUPS OF THE CIRCLE

(1) Thompson's group T (acting continuously on the circle) has an Euler class (cf. [8]), which is the restriction to T of the Euler class of the group $\mathrm{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of the circle. The latter is the class of the central extension $0 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathrm{Homeo}^+(S^1) \to 1$ obtained by lifting to $\mathbb{R}$ (the universal covering space of S^1) the homeomorphisms of the circle. The boundary ∂T_2 of the dyadic infinite tree is the dyadic analogue of the circle. Since it is totally

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disconnected, it is not possible to go further in the analogy between the Euler class of T and the so-called Euler class of N. However, both are of topological nature, the latter being related to the nontriviality of the homotopy type of $\widetilde{\mathcal{M}}_{0,2^\infty}(\mathbb{R})$.

(2) A deeper analogy relies on the relation with the Euler class of $\mathrm{Homeo}^+(S^1)$ in *bounded cohomology*.

Indeed, recall from [2] that the embeddding of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ maps the integral Euler class of $\mathrm{Homeo}^+(S^1)$ to the class of the real cocycle $eu(f,g) = \tau(\tilde{f} \circ \tilde{g}) - \tau(\tilde{f}) - \tau(\tilde{g})$, where $\tilde{f}, \tilde{g}$ are lifts of f, g, respectively, in $\widetilde{\mathrm{Homeo}}^+(S^1)$, and $\tau(\tilde{f}) = \lim_{n \to +\infty} \tilde{f}^n(0)/n$ is the *translation number of Poincaré*. The cocycle eu is induced by the boundary of the unbounded function τ on $\widetilde{\mathrm{Homeo}}^+(S^1)$, and the class of eu stands in the bounded cohomology group $H_b^2(\mathrm{Homeo}^+(S^1), \mathbb{R})$. We believe the analogy with our class Eu is suggestive, when replacing $\mathbb{Z}$, $\mathbb{R}$ and τ by $\mathbb{Z}/2\mathbb{Z}$, $\mathcal{R}$ and $\tilde{\ell}$ respectively.

(3) In [12] we have introduced an analogue of the Virasoro extension of $\mathrm{Diff}^+(S^1)$, the orientation-preserving diffeomorphism group of the circle, for the discrete group N, and called the associated cohomology class the *combinatorial analogue of the Godbillon–Vey class*. We believe it is different from the Euler class Eu we have just defined. Both classes Eu and Gv are the analogues of classes existing in Thompson's group T: indeed, $H^2(T, \mathbb{Z}) = \mathbb{Z}\overline{gv} \oplus \mathbb{Z}\overline{eu}$, where $\overline{gv}$ is the discrete Godbillon–Vey class, and $\overline{eu}$ the Euler class of T, cf. [8]. This analogy is mysterious, since the embedding $T \hookrightarrow N$ factors through Thompson's group V, which has no cohomology in degree 2.

4. Concluding Remarks

(1) We believe that the restriction of our Euler class is trivial on $PGL(2, \mathbb{Q}_2)$ (this results from tedious computations). In particular, our Euler class is not related with the Euler cocycle of J. Barge constructed on $PSL(2, k)$, for every field k, with values in the Witt group $W(k)$ (cf. [1]).

(2) A central question concerns the relative natures of the three groups concerned or evoked in the paper: the diffeomorphism group of the circle $\mathrm{Diff}^+(S^1)$, Thompson's group T, and Neretin's group of spheromorphisms N, which possess deep and mysterious cohomological analogies. We would like to find a unified way to understand this triangle of groups.

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References

1. Barge, J.: Cocycle d'Euler et K_2, *K-Theory* **7**(1) (1993), 9–16.
2. Barge, J. and Ghys, E.: Cocycles d'Euler et de Maslov, *Math. Ann.* **294** (1992), 235–265.
3. Beilinson, A. and Ginzburg, V.: Infinitesimal structure of moduli spaces of G-bundles, *Internat. Math. Res. Notices* No. 4 (1992), 63–74.
4. Cannon, J., Floyd, W. J. and Parry, W. R.: Introductory notes on Richard Thompson's groups, *Enseign. Math. (2)* **42**(3–4) (1996), 215–256.
5. Davis, M., Januszkiewicz, T. and Scott, R.: Nonpositive curvature of blow-ups, *Selecta Math.* (*N.S.*) **4**(4) (1998), 491–547.
6. Davis, M., Januszkiewicz, T. and Scott, R.: Fundamental groups of minimal blow-ups (Preliminary version), Final version to appear in *Adv. in Math.*
7. Devadoss, S.: Tessellations of moduli spaces and the mosaic operad, In: *Homotopy Invariant Algebraic Structures* (*Baltimore, MD, 1998*), Contemp. Math. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 91–114.
8. Ghys, E. and Sergiescu, V.: Sur un groupe remarquable de difféomorphismes du cercle, *Comment. Math. Helv.* **62**(2) (1987), 185–239.
9. Greenberg, P.: Les espaces de bracelets, les complexes de Stasheff et le groupe de Thompson. Papers in honor of José Adem (Spanish), *Bol. Soc. Mat. Mexicana* (*2*) **37**(1–2) (1992), 189–201.
10. Kapoudjian, C.: Sur des analogues p-adiques du groupe des difféomorphismes du cercle, Thèse de doctorat, Université LYON-I, décembre 1998.
11. Kapoudjian, C.: Simplicity of Neretin's group of spheromorphisms, *Ann. Inst. Fourier* (*Grenoble*) **49**(4) (1999), 1225–1240.
12. Kapoudjian, C.: Virasoro-type extensions for the Higman–Thompson and Neretin groups, *Quart. J. Math. Oxford* **53**(3) (2002), 295–317.
13. Kapranov, M.: Chow quotients of Grassmannians. I, In: *I. M. Gel'fand Seminar*, 29–110, Adv. Soviet Math. 16(2) Amer. Math. Soc., Providence, RI, 1993, pp.
14. Kapranov, M.: The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation, *J. Pure Appl. Algebra* **85**(2) (1993), 119–142.
15. Kapranov, M.: Stabilization of moduli spaces of rational curves (private communication to V. Sergiescu).
16. Knudsen, F.: The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.* **52**(2) (1983), 161–199.
17. Neretin, Y.: Combinatorial analogues of the group of diffeomorphisms of the circle. (Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* **56**(5) (1992), 1072–1085; translation in *Russian Acad. Sci. Izv. Math.* **41**(2) (1993), 337–349.
18. Stasheff, J.: The prehistory of operads, In: J.-L. Loday, J. D. Stasheff, A. A. Voronov (eds.), *Operads: Proceedings of Renaissance Conferences*, Contemp. Math. 202, Amer. Math. Soc., Providence, 1997.