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Sharp convergence for sequences of Schrödinger means and related generalizations

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For decreasing sequences $\{t_n\}_{n=1}^{\infty}$ converging to zero and initial data $f \in H^s(\mathbb{R}^N)$, $N \geqslant 2$, we consider the almost everywhere convergence problem for sequences of Schrödinger means $\mathrm{e}^{it_n\Delta}f$, which was proposed by Sjölin, and was open until recently. In this paper, we prove that if $\{t_n\}_{n=1}^{\infty}$ belongs to Lorentz space $\ell^{r,\infty}(\mathbb{N})$, then the a.e. convergence results hold for $s > \min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$. Inspired by the work of Lucà-Rogers, we construct a counterexample to show that our a.e. convergence results are sharp (up to endpoints). Our results imply that when $0 < r < \frac{N}{N+1}$, there is a gain over the a.e. convergence result from Du-Guth-Li and Du-Zhang, but not when $r \geqslant \frac{N}{N+1}$, even though we are in the discrete case. Our approach can also be applied to get the a.e. convergence results for the fractional Schrödinger means and nonelliptic Schrödinger means.

Keywords: Schrödinger mean; almost everywhere convergence; maximal functions; pointwise convergence

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1. Introduction

The solution of the Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) - \Delta u(x,t) = 0 \; x \in \mathbb{R}^N, t \in \mathbb{R}^+, \\ u(x,0) = f(x) \end{cases} \tag{1.1}$$

can be formally written as

$$e^{it\Delta}f(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi+it|\xi|^2}\hat{f}(\xi)\,\mathrm{d}\xi, \tag{1.2}$$

where $\hat{f}(\xi)=\int_{\mathbb{R}^N} e^{-ix\cdot\xi}f(x)\,\mathrm{d}x$. $e^{it\Delta}f$ is referred to as the Schrödinger mean of f at time t. The problem of almost everywhere convergence as $t\to 0$ has been studied extensively, under the assumption that f belongs to the Sobolev space $H^s(\mathbb{R}^N)$. The a.e. convergence result holds for $s \geqslant 1/4$ when $N=1$ by Carleson [3], and for $s>\frac{N}{2(N+1)}$ when $N\geqslant 2$ by Du-Guth-Li [8] and Du-Zhang [9]. These results are sharp (except for the endpoints when $N\geqslant 2$) according to Dahlberg–Kenig [6] and Bourgain [2]. It is worth mentioning that a different counterexample was raised by Lucà-Rogers [12] for $N\geqslant 2$.

In this paper, we consider a related problem: to investigate the almost everywhere convergence properties of $e^{it_n\Delta}f$, where t_n belongs to some decreasing sequence $\{t_n\}_{n=1}^{\infty}$ converging to zero. One may expect that less regularity on f is enough to ensure a.e. convergence along some special sequences $\{t_n\}_{n=1}^{\infty}$, such as $t_n=2^{-n}$, $n\in\mathbb{N}$. However, this is not always true for general discrete sequence $\{t_n\}_{n=1}^{\infty}$. For example, when $N=1$ and $t_n=1/n$, $n=1,\,2,\,\cdots$, Carleson [3] proved that the a.e. convergence result holds for $s>1/4$ but fails for $s<\frac{1}{8}$. Indeed, it actually fails for $s<1/4$ by the counterexample in Dahlberg–Kenig [6]; a detailed explanation can be found in Section 3 of Lee–Rogers [11]. Recently, this kind of problem was further considered by Dimou–Seeger [7] when $N=1$, Sjölin [14] and Sjölin–Strömberg [15] in general dimensions. In particular, under the assumption that $\{t_n\}_{n=1}^{\infty}\in\ell^{r,\infty}(\mathbb{N})$, $0<r<\infty$, i.e.,

$$\sup_{b>0} b^r\sharp\Big\{n\in\mathbb{N}: t_n>b\Big\}<\infty, \tag{1.3}$$

it has been shown in [7] that $e^{it_n\Delta}f$ converges almost everywhere to f if $s\geqslant\min\{\frac{r}{2r+1},\,\frac{1}{4}\}$; moreover in [7], this condition is also shown to be necessary under the additional assumption that t_n-t_{n+1} is decreasing. By Theorem 1 in [14], the a.e. convergence results hold if $s>\min\{r,\,\frac{N}{2(N+1)}\}$ for general dimension N. Theorem 3 and Corollary 6 in [15] imply that $s>\min\{\frac{r}{r+1},\,\frac{N}{2(N+)}\}$ suffices for a.e. convergence. In this paper, we obtain essentially sharp results in all dimensions.

1.1. Outline of this paper

We first state the main results on a.e. convergence for sequences of Schrödinger means, which are sharp (up to endpoints). Then, we obtain some generalizations to the fractional Schrödinger means $e^{it\Delta^{\frac{a}{2}}}f$ $(1<a<\infty)$ and nonelliptic Schrödinger means $e^{it_nL}f$, where

$$e^{it_n\Delta^{\frac{a}{2}}}f(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi+it_n|\xi|^a}\hat{f}(\xi)\,\mathrm{d}\xi, \tag{1.4}$$

and

$$e^{it_nL}f(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi+it_n(\xi_1^2-\xi_2^2\pm\ldots\pm\xi_N^2)}\hat{f}(\xi)\,\mathrm{d}\xi. \tag{1.5}$$

Conventions: Throughout this article, we shall use the notation $A \gg B$, which means if there is a sufficiently large constant G, which does not depend on the relevant parameters arising in the context in which the quantities A and B appear, such that $A \geqslant GB$. We write $A \sim B$, and mean that A and B are comparable. By $A \lesssim B$ we mean that $A \leqslant CB$ for some constant C independent of the parameters related to A and B.

1.2. Almost everywhere convergence for sequences of Schrödinger means

THEOREM 1.1. *Let $N \geqslant 2$ and $r \in (0, \infty)$. For any decreasing sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ converging to zero and $\{t_n\}_{n=1}^{\infty} \subset (0, 1)$, we have*

$$\lim_{n\to\infty} e^{it_n\Delta} f(x) = f(x) \ a.e.\ x \in \mathbb{R}^N \tag{1.6}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = \min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$.

By standard arguments, it is sufficient to show a corresponding maximal estimate in $\mathbb{R}^N$.

THEOREM 1.2. *Under the assumptions of theorem 1.1, we have*

$$\left\|\sup_{n\in\mathbb{N}} |e^{it_n\Delta} f|\right\|_{L^2(B(0,1))} \leqslant C\|f\|_{H^s(\mathbb{R}^N)}, \tag{1.7}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = \min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$, where the constant C does not depend on f.

By translation invariance in the x-direction, $B(0, 1)$ in theorem 1.2 can be replaced by any ball of radius 1 in $\mathbb{R}^N$, which implies theorem 1.1. The a.e. convergence result is almost sharp by the Nikisšin–Stein maximal principle and the fact that theorem 1.2 is sharp up to the endpoints.

THEOREM 1.3. *For each $r \in (0, \infty)$, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ which belongs to $\ell^{r,\infty}(\mathbb{N})$, for which the maximal estimate (1.7) fails if $s < s_0 = \min\{\frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$.*

REMARK 1.4. One expects that the sparser the time sequences become, the lower the regularity of almost everywhere convergence requires. Theorems 1.2 and 1.3 reveal a perhaps surprising phenomenon, namely if $0 < r < \frac{N}{N+1}$, there is a gain over the almost everywhere convergence result from [**2**, **8**, **9**, **12**] when time tends continuously to zero, but not when $r \geqslant \frac{N}{N+1}$. In fact this phenomenon already appeared in the one-dimensional case [**7**].

Our counterexample is presented in § 3. The construction is inspired by the work [**12**], which is an alternative proof for Bourgain's counterexample that showed the necessary condition for $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$, a.e. $x \in \mathbb{R}^N$.

Next, we briefly explain how to prove theorem 1.2. Notice that when $\frac{r}{\frac{N+1}{N}r+1} \geqslant \frac{N}{2(N+1)}$, theorem 1.2 follows from the celebrated results by [8] ($N=2$), and [9] ($N \geqslant 3$). Therefore, we only need to consider the case when $\frac{r}{\frac{N+1}{N}r+1} < \frac{N}{2(N+1)}$, so we always assume that $0 < r < \frac{N}{N+1}$ in what follows.

By Littlewood–Paley decomposition and standard arguments, we just concentrate on the case when $\mathrm{supp}\hat{f} \subset \{\xi : |\xi| \sim 2^k\}$, $k \gg 1$. We consider the maximal functions

$$\sup_{n\in\mathbb{N}: t_n \geqslant 2^{-\frac{2k}{(N+1)r/N+1}}} |e^{it_n\Delta} f|$$

and

$$\sup_{n\in\mathbb{N}: t_n < 2^{-\frac{2k}{(N+1)r/N+1}}} |e^{it_n\Delta} f|.$$

We deal with the first term by the assumption that the decreasing sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ and Plancherel's theorem. For the second term, since $k < \frac{2k}{\frac{N+1}{N}r+1} < 2k$, the proof can be completed by the following theorem.

THEOREM 1.5. *Let $j \in \mathbb{R}$ with $k < j < 2k$. For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f| \right\|_{L^2(B(0,1))} \leqslant C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)}+\epsilon k} \|f\|_{L^2(\mathbb{R}^N)}, \tag{1.8}$$

for all f with supp $\hat{f} \subset \{\xi : |\xi| \sim 2^k\}$. The constant C_ϵ does not depend on f, j and k.

In the case $N = 1$, a similar result was proved in [7] by the TT^* argument and stationary phase method. But their method seems not to work well in the higher dimensional case. In order to prove theorem 1.5, we first observe that (1.8) holds true if the spatial variable is restricted to a ball of radius 2^{k-j}. Due to references [8, 9], for any function g with supp $\hat{g} \subset \{\xi : |\xi| \sim 2^{2k-j}\}$, there holds

$$\left\| \sup_{t\in(0,2^{-(2k-j)})} |e^{it\Delta} g| \right\|_{L^2(B(0,1))} \leqslant C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)}+\epsilon k} \|g\|_{L^2(\mathbb{R}^N)}.$$

By scaling, we have

$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} g| \right\|_{L^2(B(0,2^{k-j}))} \leqslant C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)}+\epsilon k} \|g\|_{L^2(\mathbb{R}^N)} \tag{1.9}$$

whenever supp $\hat{g} \subset \{\xi : |\xi| \sim 2^k\}$. Then, we obtain the following lemma by translation invariance in the x-direction.

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LEMMA 1.6. *When $k < j < 2k$, for any $\epsilon > 0$ and $x_0 \in \mathbb{R}^N$, there exists a constant $C_\epsilon > 0$ such that*

$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f| \right\|_{L^2(B(x_0,2^{k-j}))} \leqslant C_\epsilon 2^{(2k-j)\frac{N}{2(N+1)}+\epsilon k} \|f\|_{L^2(\mathbb{R}^N)}, \tag{1.10}$$

whenever supp $\hat{f} \subset \{\xi : |\xi| \sim 2^k\}$. The constant C_ϵ does not depend on x_0 and f.

Then we can obtain theorem 1.5 with the help of lemma 1.6, wave packets decomposition and an orthogonality argument. See § 2 for details. Moreover, we give the following remark on theorem 1.5.

REMARK 1.7. We notice that theorem 1.5 is almost sharp when $j = k$ and $j = 2k$. Indeed, when $j = 2k$, Sobolev's embedding implies

$$\left\| \sup_{t\in(0,2^{-2k})} |e^{it\Delta} f| \right\|_{L^2(B(0,1))} \leqslant C \|f\|_{L^2(\mathbb{R}^N)}. \tag{1.11}$$

By taking $\hat{f}$ as the characteristic function on the set $\{\xi : |\xi| \sim 2^k\}$, it can be observed that the uniform estimate (1.11) is optimal. When $j = k$, it follows from [**8**, **9**] that

$$\left\| \sup_{t\in(0,2^{-k})} |e^{it\Delta} f| \right\|_{L^2(B(0,1))} \leqslant C 2^{\frac{N}{2(N+1)}k+\epsilon k} \|f\|_{L^2(\mathbb{R}^N)}. \tag{1.12}$$

The above inequality (1.12) is sharp up to the endpoints according to the counterexample in [**2**] or [**12**]. However, the presence of $2^{\epsilon k}$ on the right-hand side of inequality (1.8) leads us to lose the endpoint results in theorem 1.2.

1.3. Related generalizations

The method we adopted to prove theorem 1.2 can be generalized to the fractional case and the nonelliptic case. Then, the corresponding a.e. convergence results follow. We omit most of details of the proof because they are very similar with that of theorem 1.2. Moreover, the sharpness of the result for the nonelliptic case will be proved in § 4.

Firstly, for the fractional case, we have the following maximal estimate. When $a = 2$, it coincides with theorem 1.2.

THEOREM 1.8. *Under the conditions of theorem 1.2, for $1 < a < \infty$, we have*

$$\left\| \sup_{n\in\mathbb{N}} |e^{it_n \Delta^{\frac{a}{2}}} f| \right\|_{L^2(B(0,1))} \leqslant C \|f\|_{H^s(\mathbb{R}^N)}, \tag{1.13}$$

whenever $f \in H^s(\mathbb{R}^N)$ and $s > s_0 = \min\{\frac{a}{2} \cdot \frac{r}{\frac{N+1}{N}r+1}, \frac{N}{2(N+1)}\}$, where the constant C does not depend on f.

We now consider maximal estimates for the nonelliptic Schrödinger means, as defined in (1.5). The following result is sharp up to the endpoints, as will be shown in § 4 below.

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Theorem 1.9. *Under the conditions of theorem 1.2, we have*

$$\left\|\sup_{n\in\mathbb{N}}|e^{it_nL}f|\right\|_{L^2(B(0,1))}\leqslant C\|f\|_{H^s(\mathbb{R}^N)},\tag{1.14}$$

whenever $f\in H^s(\mathbb{R}^N)$ *and* $s>s_0=\min\{\frac{r}{r+1},\frac{1}{2}\}$, *where the constant* C *does not depend on* f.

The proof of theorem 1.9 depends heavily on the following theorem.

Theorem 1.10. *If supp* $\hat{f}\subset\{\xi:|\xi|\sim\lambda\}$, $\lambda\geqslant 1$, *then for any interval* I *with* $\lambda^{-2}\leqslant|I|\leqslant\lambda^{-1}$, *we have*

$$\left\|\sup_{t\in I}|e^{itL}f|\right\|_{L^2(B(0,1))}\leqslant C\lambda|I|^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^N)},\tag{1.15}$$

where the constant C *does not depend on* f.

Theorem 1.10 follows directly from Sobolev's embedding. Specially, theorem 1.10 is sharp when $|I|=\lambda^{-1}$ according to the counterexample in Rogers–Vargas–Vega [**13**]. When $|I|=\lambda^{-2}$, the sharpness can be proved by taking $\tilde{f}$ as the characteristic function over the annulus $\{\xi:|\xi|\sim\lambda\}$. We point out that the sharpness of theorem 1.10 enables us to apply the similar decomposition as Proposition 2.3 in [**7**] to get a stronger result than theorem 1.9 when $r\in(0,1)$.

Theorem 1.11. *If* $\{t_n\}_{n=1}^{\infty}\in\ell^{r(s),\infty}(\mathbb{N})$, $r(s)=\frac{s}{1-s}$. *Then for any* $0<s<\frac{1}{2}$, *we have*

$$\left\|\sup_{n\in\mathbb{N}}|e^{it_nL}f|\right\|_{L^2(B(0,1))}\leqslant C\|f\|_{H^s(\mathbb{R}^N)},\tag{1.16}$$

whenever $f\in H^s(\mathbb{R}^N)$, *where the constant* C *does not depend on* f.

Remark 1.12. In table 1, we synthesize our theorems and all results to our best knowledge, and list all almost sharp requirements of regularity on a.e. convergence for different Schrödinger-type operators. We also notice that some results in the table 1 come from theorems 1.1, 1.8, 1.9 and 1.11 in this paper. For the remaining results, readers can refer to the relevant results of the nonelliptic Schrödinger operators in [**13**]; the conclusions about the fractional Schrödinger operators when t continuously tends to 0 can be found in [**4**] ($a>1$) and [**16**] ($0<a<1$); other results were introduced at the beginning of the introduction and will not be repeated here.

Remark 1.13. Shortly after our paper was posted on arXiv.org in July 2022, Cho, Ko, Koh and Lee posted a paper with partially overlapping results, which has now been published in [**5**].

Table 1. *A summary of a.e. convergence for some Schrödinger operators.*

Operators type	Spatial dimensions	Continuous case $t \to 0$	Discrete case $t_n \to 0$
Schrödinger	$N = 1$	$s \geqslant \frac{1}{4}$	$s \geqslant \min\{\frac{1}{4}, \frac{r}{2r+1}\}$
operator	$N \geqslant 2$	$s > \frac{N}{2(N+1)}$	$s > \min\{\frac{N}{2(N+1)}, \frac{r}{\frac{N+1}{N}r+1}\}$
Nonelliptic	$N = 2$	$s \geqslant \frac{1}{2}$	$s \geqslant \min\{\frac{1}{2}, \frac{r}{r+1}\}$
Schrödinger	$N \geqslant 3$	$s > \frac{1}{2}$	$s > \min\{\frac{1}{2}, \frac{r}{r+1}\}$
Fractional	$N = 1$	$s \geqslant \frac{1}{4}$	$s \geqslant \min\{\frac{1}{4}, \frac{a}{2}\frac{r}{2r+1}\}$
$a > 1$	$N \geqslant 2$	$s > \frac{N}{2(N+1)}$	$s > \min\{\frac{N}{2(N+1)}, \frac{a}{2}\frac{r}{\frac{N+1}{N}r+1}\}$
Fractional	$N = 1$	$s > \frac{a}{4}$	$s > \min\{\frac{a}{4}, \frac{a}{2}\frac{r}{2r+1}\}$
$0 < a < 1$	$N \geqslant 2$	Sharp result is open	Sharp result is open

2. Proof of theorems 1.2 and 1.5

Proof of theorem 1.2. Let $s_1 = \frac{r}{\frac{N+1}{N}r+1} + \epsilon$ for some constant $\epsilon > 0$. We decompose f as $f = \sum_{k=0}^{\infty} f_k$, where $\operatorname{supp}\hat{f}_0 \subset B(0, 1)$, $\operatorname{supp}\hat{f}_k \subset \{\xi : |\xi| \sim 2^k\}$, $k \geqslant 1$. Then, we have

$$\left\|\sup_{n\in\mathbb{N}} |e^{it_n\Delta} f|\right\|_{L^2(B(0,1))} \leqslant \sum_{k=0}^{\infty} \left\|\sup_{n\in\mathbb{N}} |e^{it_n\Delta} f_k|\right\|_{L^2(B(0,1))}. \tag{2.1}$$

For $k \lesssim 1$ and arbitrary $x \in B(0, 1)$, $|e^{it_n\Delta} f_k(x)| \lesssim \|f_k\|_{L^2(\mathbb{R}^N)}$, it is obvious that

$$\left\|\sup_{n\in\mathbb{N}} |e^{it_n\Delta} f_k|\right\|_{L^2(B(0,1))} \lesssim \|f\|_{H^{s_1}(\mathbb{R}^N)}. \tag{2.2}$$

For each $k \gg 1$, we decompose $\{t_n\}_{n=1}^{\infty}$ as

$$A_k^1 := \left\{t_n : t_n \geqslant 2^{-\frac{2k}{\frac{N+1}{N}r+1}}\right\}$$

and

$$A_k^2 := \left\{t_n : t_n < 2^{-\frac{2k}{\frac{N+1}{N}r+1}}\right\}.$$

Then, we have

$$\begin{aligned} &\left\|\sup_{n\in\mathbb{N}} |e^{it_n\Delta} f_k|\right\|_{L^2(B(0,1))} \\ &\leqslant \left\|\sup_{n\in\mathbb{N}:t_n\in A_k^1} |e^{it_n\Delta} f_k|\right\|_{L^2(B(0,1))} + \left\|\sup_{n\in\mathbb{N}:t_n\in A_k^2} |e^{it_n\Delta} f_k|\right\|_{L^2(B(0,1))} \\ &:= I + II. \end{aligned} \tag{2.3}$$

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We first estimate I. Since $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$, we have

$$\sharp A_k^1 \leqslant C 2^{\frac{2rk}{\frac{N+1}{N}r+1}}, \tag{2.4}$$

which implies that

$$I \leqslant \left(\sum_{n\in\mathbb{N}:t_n\in A_k^1} \left\| e^{it_n\Delta} f_k \right\|^2_{L^2(B(0,1))} \right)^{1/2} \lesssim 2^{\frac{rk}{\frac{N+1}{N}r+1}} \|f_k\|_{L^2(\mathbb{R}^N)} \lesssim 2^{-\epsilon k} \|f\|_{H^{s_1}(\mathbb{R}^N)}. \tag{2.5}$$

For II, since

$$A_k^2 \subset \left(0, 2^{-\frac{2k}{\frac{N+1}{N}r+1}}\right).$$

By previous discussion, we have $k < \frac{2k}{\frac{N+1}{N}r+1} < 2k$. Then it follows from theorem 1.5 that,

$$II \lesssim 2^{(\frac{r}{\frac{N+1}{N}r+1}+\frac{\epsilon}{2})k} \|f_k\|_{L^2(\mathbb{R}^N)} \leqslant 2^{-\frac{\epsilon}{2}k} \|f\|_{H^{s_1}(\mathbb{R}^N)}. \tag{2.6}$$

Inequalities (2.3), (2.5), and (2.6) yield for $k \gg 1$,

$$\left\| \sup_{n\in\mathbb{N}} |e^{it_n\Delta} f_k| \right\|_{L^2(B(0,1))} \lesssim 2^{-\frac{\epsilon k}{2}} \|f\|_{H^{s_1}(\mathbb{R}^N)}. \tag{2.7}$$

Combining inequalities (2.1), (2.2), and (2.7), inequality (1.7) holds true for s_1. Because $\epsilon > 0$, we have finished the proof of theorem 1.2. It remains to prove theorem 1.5. □

Proof of theorem 1.5. We use a wave packets decomposition and an orthogonality argument to prove theorem 1.5.

• **Wave packets decomposition.**

We first decompose $e^{it\Delta} f$ on $B(0,\,1) \times (0,\,2^{-j})$ in a standard way. For this goal, we decompose the annulus $\{\xi : |\xi| \sim 2^k\}$ into almost disjoint 2^{j-k}-cubes θ with sides parallel to the coordinate axes in $\mathbb{R}^N$. Let 2^{k-j}-cube ν be dual to θ and cover $\mathbb{R}^N$ by almost disjoint cubes ν. Denote the centre of θ by $c(\theta)$ and the centre of ν by $c(\nu)$. We notice that if $\nu \neq \nu'$, then $|c(\nu) - c(\nu')| \geqslant 2^{k-j}$.

Let φ be a Schwartz function defined on $\mathbb{R}^N$ whose fourier transform is non-negative and supported in a small neighbourhood of the origin, and identically equal to 1 in another smaller ball centred at the origin. Let $\widehat{\varphi_\theta}(\xi) = 2^{-\frac{(j-k)N}{2}} \hat{\varphi}(\frac{\xi - c(\theta)}{2^{j-k}})$ and $\widehat{\varphi_{\theta,\nu}}(\xi) = e^{-ic(\nu)\cdot\xi} \widehat{\varphi_\theta}(\xi)$. Then f can be decomposed by

$$f = \sum_\nu \sum_\theta f_{\theta,\nu} = \sum_\nu \sum_\theta \langle f, \varphi_{\theta,\nu} \rangle \varphi_{\theta,\nu},$$

and

$$\|f\|^2_{L^2} \sim \sum_\nu \sum_\theta |\langle f, \varphi_{\theta,\nu} \rangle|^2.$$

When $t \in (0,\, 2^{-j})$, integration by parts implies

$$|e^{it\Delta}\varphi_{\theta,\nu}(x)| \leqslant \frac{C_M 2^{\frac{(j-k)N}{2}}}{(1+2^{j-k}|x-c(\nu)+2tc(\theta)|)^M}.$$

Here, M can be sufficiently large. Therefore, $e^{it\Delta}\varphi_{\theta,\nu}(x)$ is essentially supported in a tube

$$T_{\theta,\nu} := \{(x,t), |x-c(\nu)+2tc(\theta)| \leqslant 2^{(j-k)(-1+\delta)}, 0 \leqslant t \leqslant 2^{-j}\},$$

where $\delta = \epsilon^3$. The direction of $T_{\theta,\nu}$ is parallel to the vector $(-2c(\theta),\, 1)$, and the angle between $(-2c(\theta),\, 1)$ and the x-plane is approximately 2^{-k}.

- **Orthogonality argument.**
  We decompose $B(0,\,1)$ by $B(0,\,1) = \cup_{\nu'} B(c(\nu'),\, 2^{k-j})$ with $|c(\nu')| \lesssim 1$. Then

$$\left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f(x)| \right\|^2_{L^2(B(0,1))} \leqslant \sum_{\nu'} \left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f(x)| \right\|^2_{L^2(B(c(\nu'),2^{k-j}))}. \tag{2.8}$$

We will consider two cases: (i) $j < k + \frac{\epsilon k}{N}$ and (ii) $j \geqslant k + \frac{\epsilon k}{N}$, respectively. In case (i), let $j = k + \epsilon_0 k$, $0 < \epsilon_0 < \frac{\epsilon}{N}$, by lemma 1.6,

$$\begin{aligned} \left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f(x)| \right\|_{L^2(B(0,1))} &\leqslant \left(\sum_{\nu'} \left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f(x)| \right\|^2_{L^2(B(c(\nu'),2^{k-j}))} \right)^{1/2} \\ &\lesssim 2^{(2k-j)\frac{N}{2(N+1)}+\epsilon k} \|f\|_{L^2}. \end{aligned} \tag{2.9}$$

We use an orthogonality argument in the proof of case (ii). Fix $c(\nu')$, we divide f into two terms

$$f_1 = \sum_{\theta} \sum_{\nu:|c(\nu)-c(\nu')|\leqslant 2^{(j-k)(-1+10\delta)}} f_{\theta,\nu},$$

and

$$f_2 = \sum_{\theta} \sum_{\nu:|c(\nu)-c(\nu')|> 2^{(j-k)(-1+10\delta)}} f_{\theta,\nu}.$$

For f_1, by lemma 1.6 and the L^2-orthogonality, we have

$$\begin{aligned} &\sum_{\nu'} \left\| \sup_{t\in(0,2^{-j})} |e^{it\Delta} f_1(x)| \right\|^2_{L^2(B(c(\nu'),2^{k-j}))} \\ &\sim C_\epsilon 2^{(2k-j)\frac{N}{N+1}+\epsilon k} \sum_{\nu'} \sum_{\theta} \sum_{\nu:|c(\nu)-c(\nu')|\leqslant 2^{(j-k)(-1+10\delta)}} \|f_{\theta,\nu}\|^2_{L^2} \\ &\lesssim C_\epsilon 2^{(2k-j)\frac{N}{N+1}+2\epsilon k} \|f\|^2_{L^2}. \end{aligned} \tag{2.10}$$

We will complete the proof by showing that the contribution from $|e^{it\Delta} f_2|$ is negligible when $(x,\, t)$ belongs to $B(c(\nu'),\, 2^{k-j}) \times (0,\, 2^{-j})$.

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Indeed, by the Cauchy–Schwarz inequality and the L^2-orthogonality, there holds

$$\begin{aligned}|e^{it\Delta}f_2| &\leqslant \|f\|_{L^2}\Bigg(\sum_{\theta}\sum_{\nu:|c(\nu)-c(\nu')|>2^{(j-k)(-1+10\delta)}}|e^{it\Delta}\varphi_{\theta,\nu}|^2\Bigg)^{1/2}\\ &\leqslant \|f\|_{L^2}C_M 2^{\frac{(j-k)N}{2}}\\ &\quad\times\Bigg(\sum_{\theta}\sum_{\nu:|c(\nu)-c(\nu')|>2^{(j-k)(-1+10\delta)}}\frac{1}{(1+2^{j-k}|x-c(\nu)+2tc(\theta)|)^{2M}}\Bigg)^{1/2}.\end{aligned}$$

For each θ, $|x-c(\nu)+2tc(\theta)|\geqslant |c(\nu)-c(\nu')|/2$, then we have

$$\begin{aligned}&\sum_{\nu:|c(\nu)-c(\nu')|>2^{(j-k)(-1+10\delta)}}\frac{1}{(1+2^{j-k}|x-c(\nu)+2tc(\theta)|)^{2M}}\\ &\leqslant 2^{2M}\sum_{\substack{l\in\mathbb{N}^+\\ l\geqslant 2^{10\delta\epsilon k/N}}}\sum_{\nu:l2^{k-j}\leqslant|c(\nu)-c(\nu')|<(l+1)2^{k-j}}\frac{1}{(1+2^{j-k}|c(\nu)-c(\nu')|)^{2M}}\\ &\leqslant 2^{2M}\sum_{\substack{l\in\mathbb{N}^+\\ l\geqslant 2^{10\delta\epsilon k/N}}}\frac{C_N l^N}{(1+l)^{2M}}\\ &\leqslant C_{M,N}2^{-M\epsilon^4 k}.\end{aligned}$$

Notice that the number of θ's is dominated by 2^{Nk}. So by choosing M sufficiently large, for each $(x,\, t)\in B(c(\nu'),\, 2^{k-j})\times(0,\, 2^{-j})$, we have

$$|e^{it\Delta}f_2|\leqslant C_N 2^{-1000k}\|f\|_{L^2}.$$

Then, the proof is finished since

$$\sum_{\nu'}\Big\|\sup_{t\in(0,2^{-j})}|e^{it\Delta}f_2(x)|\Big\|^2_{L^2(B(c(\nu'),2^{k-j}))}\leqslant C_N^2 2^{-2000k}\|f\|^2_{L^2}. \qquad \square$$

3. A counterexample: theorem 1.3

We notice that the counterexample for $r=\frac{N}{N+1}$ can be also applied to the case when $r>\frac{N}{N+1}$, since $\ell^{N/(N+1),\infty}(\mathbb{N})\subset\ell^{r,\infty}(\mathbb{N})$ and $\min\{\frac{r}{\frac{N+1}{N}r+1},\, \frac{N}{2(N+1)}\}=\frac{N}{2(N+1)}$ when $r>\frac{N}{N+1}$. Therefore, next we always assume $r\in(0,\, \frac{N}{N+1}]$.

Fix $r\in(0,\, \frac{N}{N+1}]$, we first construct a sequence which belongs to $\ell^{r,\infty}(\mathbb{N})$. Put $\beta=\frac{2}{\frac{N+1}{N}r+1}$. Let $R_1=2$ and for each positive integer n, $R_{n+1}^{-\beta}\leqslant\frac{1}{2}R_n^{-\beta(r+1)}$. Denote the intervals $I_n=[R_n^{-\beta(r+1)},\, R_n^{-\beta})$, $n\in\mathbb{N}^+$. On each I_n, we get an equidistributed subsequence t_{n_j}, $j=1,\, 2,\, \ldots,\, j_n$ such that

$$\{t_{n_j}, 1\leqslant j\leqslant j_n\}=:R_n^{-\beta(r+1)}\mathbb{Z}\cap I_n,$$

and $t_{n_j}-t_{n_{j+1}}=R_n^{-\beta(r+1)}$. We claim that the sequence t_{n_j}, $j=1,\, 2,\, \ldots,\, j_n$, $n=1,\, 2,\, \ldots$ belongs to $\ell^{r,\infty}(\mathbb{N})$.

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Indeed, according to Lemma 3.2 from [7], it suffices to show that

$$\sup_{b>0} b^r \sharp\left\{(n, j) : b < t_{n_j} \leqslant 2b\right\} \lesssim 1. \tag{3.1}$$

Notice that we only need to consider $0 < b < 1$ because $t_{n_j} \in (0, 1)$ for each n and j. Assume that $(b, 2b] \cap I_n \neq \emptyset$ for some n, then we have $b < R_n^{-\beta}$, $2b \geqslant R_n^{-\beta(r+1)}$. Therefore,

$$2b < 2R_n^{-\beta} \leqslant R_{n-1}^{-\beta(r+1)}, \quad b \geqslant \frac{1}{2} R_n^{-\beta(r+1)} \geqslant R_{n+1}^{-\beta}.$$

This yields $(b, 2b] \cap I_{n'} = \emptyset$ for any $n' \neq n$, hence

$$b^r \sharp\left\{(n, j) : b < t_{n_j} \leqslant 2b\right\} \leqslant b^{r+1} R_n^{\beta(r+1)} < 1.$$

Then (3.1) follows by the arbitrariness of b.

Our counterexample comes from the following lemma.

LEMMA 3.1. *Let $R \gg 1$ and $I = [R^{-\beta(r+1)}, R^{-\beta})$. Assume that the sequence $\{t_j : 1 \leqslant j \leqslant j_0\} = R^{-\beta(r+1)}\mathbb{Z} \cap I$ and $t_j - t_{j+1} = R^{-\beta(r+1)}$ for each $1 \leqslant j \leqslant j_0 - 1$. Then there exists a function f with supp $\hat{f} \subset B(0, 2R)$ such that*

$$\left\| \sup_{1 \leqslant j \leqslant j_0} |e^{i\frac{t_j}{2\pi}\Delta} f| \right\|_{L^2(B(0,1))} \gtrsim R^{\frac{1-\beta}{2}} R^{\frac{\beta}{2}} R^{(N-1)(1-\frac{(r+1)\beta}{2})-\epsilon}, \tag{3.2}$$

and

$$\|f\|_{H^s(\mathbb{R}^N)} \lesssim R^s R^{\frac{\beta}{4}} R^{\frac{N-1}{2}(1-\frac{(r+1)\beta}{2})}. \tag{3.3}$$

Here, ϵ is any positive number.

We use lemma 3.1 to show the counterexample here and prove the lemma a moment later. Assume that the maximal estimate

$$\left\| \sup_n \sup_j |e^{i\frac{t_{n_j}}{2\pi}\Delta} f| \right\|_{L^2(B(0,1))} \leqslant C\|f\|_{H^s(\mathbb{R}^N)} \tag{3.4}$$

holds for some $s > 0$ and each $f \in H^s(\mathbb{R}^N)$, then for each $n \in \mathbb{N}^+$, we have

$$\left\| \sup_j |e^{i\frac{t_{n_j}}{2\pi}\Delta} f| \right\|_{L^2(B(0,1))} \leqslant C\|f\|_{H^s(\mathbb{R}^N)} \tag{3.5}$$

whenever $f \in H^s(\mathbb{R}^N)$. Lemma 3.1 and inequality (3.5) yield

$$R_n^{\frac{2-\beta}{4}} R_n^{\frac{N-1}{2}(1-\frac{(r+1)\beta}{2})-\epsilon} \leqslant C R_n^s. \tag{3.6}$$

Then, we have $s \geqslant \frac{r}{\frac{N+1}{N}r+1}$, since R_n can be sufficiently large and ϵ is arbitrarily small. Finally, we obtain a sequence $\frac{t_{n_j}}{2\pi}$, $j = 1, 2, \ldots, j_n$, $n = 1, 2, \ldots \in \ell^{r,\infty}(\mathbb{N})$ such that the maximal estimate (3.4) holds only if $s \geqslant \frac{r}{\frac{N+1}{N}r+1}$.

In the rest of this section, we prove lemma 3.1. Setting

$$\Omega_1 = \left(-\frac{1}{100}R^{\frac{\beta}{2}}, \frac{1}{100}R^{\frac{\beta}{2}}\right),$$
$$\Omega_2 = \left\{\bar{\xi} \in \mathbb{R}^{N-1} : \bar{\xi} \in 2\pi R^{\frac{(r+1)\beta}{2}}\mathbb{Z}^{N-1} \cap B(0, R^{1-\epsilon})\right\} + B\left(0, \frac{1}{1000}\right),$$

then we define $\hat{f}_1(\xi_1) = \hat{h}(\xi_1 + \pi R)$, $\hat{f}_2(\bar{\xi}) = \hat{g}(\bar{\xi} + \pi R\theta)$, where $\hat{h} = \chi_{\Omega_1}$, $\hat{g} = \chi_{\Omega_2}$, and some $\theta \in \mathbb{S}^{N-2}$ (when $N = 2$, $\theta \in (0, 1)$) which will be determined later. Define f by $\hat{f} = \hat{f}_1\hat{f}_2$, it is easy to check that f satisfies (3.3). We are left to prove that inequality (3.2) holds for such f. Notice that

$$|e^{i\frac{t_j}{2\pi}\Delta} f(x_1, \bar{x})| = |e^{i\frac{t_j}{2\pi}\Delta} f_1(x_1)||e^{i\frac{t_j}{2\pi}\Delta} f_2(\bar{x})|. \tag{3.7}$$

We first consider $|e^{i\frac{t_j}{2\pi}\Delta} f_1(x_1)|$. A change of variables implies

$$|e^{i\frac{t_j}{2\pi}\Delta} f_1(x_1)| = |e^{i\frac{t_j}{2\pi}\Delta} h(x_1 - Rt_j)|.$$

It is easy to check that $|e^{i\frac{t_j}{2\pi}\Delta} h(x_1)| \gtrsim |\Omega_1|$ for each j whenever $|x_1| \leqslant R^{-\frac{\beta}{2}}$. Note that for each $x_1 \in (0, R^{1-\beta})$, there exists at least one t_j such that $|x_1 - Rt_j| \leqslant R^{1-\beta(r+1)} \leqslant R^{-\frac{\beta}{2}}$ since $\{t_j\}_{j=1}^{j_0} \subset [R^{-\beta(r+1)}, R^{-\beta})$ and $t_j - t_{j+1} = R^{-\beta(r+1)}$. Hence, we have

$$|e^{i\frac{t_j}{2\pi}\Delta} f_1(x_1)| \gtrsim |\Omega_1|, \tag{3.8}$$

whenever $x_1 \in (0, \frac{1}{2}R^{1-\beta})$ and $Rt_j \in (x_1, x_1 + R^{-\frac{\beta}{2}})$.

For $|e^{i\frac{t_j}{2\pi}\Delta} f_2(\bar{x})|$, we have

$$|e^{i\frac{t_j}{2\pi}\Delta} f_2(\bar{x})| = |e^{i\frac{t_j}{2\pi}\Delta} g(\bar{x} - Rt_j\theta)|.$$

According to Barceló–Bennett–Carbery–Ruiz–Vilela [1], for each j and $\bar{x} \in U_0$,

$$|e^{i\frac{t_j}{2\pi}\Delta} g(\bar{x})| \gtrsim |\Omega_2|, \tag{3.9}$$

here

$$U_0 = \left\{\bar{x} \in \mathbb{R}^{N-1} : \bar{x} \in R^{-\frac{(r+1)\beta}{2}}\mathbb{Z}^{N-1} \cap B(0, 2)\right\} + B\left(0, \frac{1}{1000}R^{-1+\epsilon}\right).$$

We sketch the main idea of the proof of inequality (3.9) for the reader's convenience. Indeed, for each $\bar{\xi} \in \Omega_2$, we write $\bar{\xi} = 2\pi R^{\frac{(r+1)\beta}{2}} l + \bar{\eta}$, $l \in \mathbb{Z}^{N-1}$, $2\pi|l| \leqslant R^{1-\frac{(r+1)\beta}{2}-\epsilon}$, $\bar{\eta} \in B(0, \frac{1}{1000})$. Then for any $\bar{x}_m = R^{-\frac{(r+1)\beta}{2}} m$, $m \in \mathbb{Z}^{N-1}$, $|m| \leqslant$

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$2R^{\frac{(r+1)\beta}{2}}$, $t_j = R^{-(r+1)\beta}(j_0+1-j)$, $1 \leqslant j \leqslant j_0$, we have

$$\begin{aligned} & e^{i\frac{t_j}{2\pi}\Delta} g(\bar{x}_m) \\ &= \frac{1}{(2\pi)^N} \sum_{l\in\mathbb{Z}^{N-1}:2\pi|l|\leqslant R^{1-(r+1)\beta/2-\epsilon}} e^{2\pi i m\cdot l + 2\pi i (j_0+1-j)|l|^2} \\ &\quad \times \int_{B(0,\frac{1}{1000})} e^{i\bar{x}_m\cdot\bar{\eta} + 2i\frac{t_j}{2\pi}2\pi R^{\frac{(r+1)\beta}{2}} l\cdot\bar{\eta} + i\frac{t_j}{2\pi}|\bar{\eta}|^2}\, \mathrm{d}\bar{\eta} \\ &= \frac{1}{(2\pi)^N} \sum_{l\in\mathbb{Z}^{N-1}:2\pi|l|\leqslant R^{1-(r+1)\beta/2-\epsilon}} \int_{B(0,\frac{1}{1000})} e^{i\bar{x}_m\cdot\bar{\eta} + 2i\frac{t_j}{2\pi}2\pi R^{\frac{(r+1)\beta}{2}} l\cdot\bar{\eta} + i\frac{t_j}{2\pi}|\bar{\eta}|^2}\, \mathrm{d}\bar{\eta}. \end{aligned}$$

Noting that $|\bar{x}_m| \leqslant 2$, $|t_j| \leqslant R^{-\beta}$ and $|\bar{\eta}| \leqslant \frac{1}{1000}$ imply

$$\left| \bar{x}_m \cdot \bar{\eta} + 2\frac{t_j}{2\pi} 2\pi R^{\frac{(r+1)\beta}{2}} l\cdot\bar{\eta} + \frac{t_j}{2\pi}|\bar{\eta}|^2 \right| \leqslant \frac{1}{100},$$

then, we have

$$|e^{i\frac{t_j}{2\pi}\Delta} g(\bar{x}_m)| \geqslant \frac{1}{2(2\pi)^N}|\Omega_2|.$$

Moreover, for each $\bar{x} \in U_0$, there exits an $\bar{x}_m$ such that $|\bar{x} - \bar{x}_m| \leqslant \frac{1}{1000}R^{-1+\epsilon}$, by the mean value theorem and the fact that $|\bar{\xi}| \leqslant 2R^{1-\epsilon}$,

$$|e^{i\frac{t_j}{2\pi}\Delta} g(\bar{x}) - e^{i\frac{t_j}{2\pi}\Delta} g(\bar{x}_m)| \leqslant \int_{\mathbb{R}^{N-1}} |\bar{x} - \bar{x}_m||\bar{\xi}|\hat{g}(\bar{\xi})\, \mathrm{d}\bar{\xi} \leqslant \frac{1}{500}|\Omega_2|.$$

Finally, we arrive at inequality (3.9) by the triangle inequality.

Therefore, we have

$$|e^{i\frac{t_j}{2\pi}\Delta} f_2(\bar{x})| \gtrsim |\Omega_2|, \qquad \bar{x} \in U_j = U_0 + Rt_j\theta. \tag{3.10}$$

Set $U_{x_1} = \bigcup_{j:Rt_j\in R^{1-(r+1)\beta}\mathbb{Z}\cap(x_1,\, x_1+R^{-\beta/2})} U_0 + Rt_j\theta$. Then inequalities (3.8) and (3.10) imply that for each $x_1 \in (0,\, \frac{1}{2}R^{1-\beta})$ and $\bar{x} \in U_{x_1}$, there holds

$$\sup_j |e^{i\frac{t_j}{2\pi}\Delta} f(x_1, \bar{x})| \gtrsim |\Omega_1||\Omega_2|. \tag{3.11}$$

Next, we need to select a $\theta \in \mathbb{S}^{N-2}$, such that $|U_{x_1}| \gtrsim 1$ for each $x_1 \in (0,\, \frac{1}{2}R^{1-\beta})$, which follows if we can prove that there exists a $\theta \in \mathbb{S}^{N-2}$ such that $B(0,\, 1/2) \subset U_{x_1}$ for all $x_1 \in (0,\, \frac{1}{2}R^{1-\beta})$. So it remains to prove the claim that there exists a $\theta \in \mathbb{S}^{N-2}$ such that

$$\bigcup_{j:Rt_j\in R^{1-\beta(r+1)}\mathbb{Z}\cap(x_1,x_1+R^{-\beta/2})} \left\{ \bar{x} \in \mathbb{R}^{N-1} : \bar{x} \in R^{-\frac{(r+1)\beta}{2}}\mathbb{Z}^{N-1} \cap B(0,2) \right\} + Rt_j\theta$$

is $\frac{1}{1000}R^{-1+\epsilon}$ dense in the ball $B(0,\, 1/2)$. In order to apply Lemma 2.1 from Lucà–Rogers [12] to get this claim, we first rescale by $R^{\frac{\beta(r+1)}{2}}$, and replace

$R^{1+\frac{\beta(r+1)}{2}}t_j$ by s_j, replace $R^{\frac{\beta r}{2}}$ by R', recall that $\beta = \frac{2}{\frac{N+1}{N}r+1}$, then we are reduced to show

$$\bigcup_{j:s_j\in(R')^{1/N}\mathbb{Z}\cap((R')^{(r+1)/r}x_1,(R')^{(r+1)/r}x_1+R')} \left\{\bar{x} : \bar{x} \in \mathbb{Z}^{N-1} \cap B(0, 2(R')^{(r+1)/r})\right\} + s_j\theta$$

is $\frac{1}{1000}(R')^{-\frac{1}{N}+\frac{(\frac{N+1}{N}r+1)\epsilon}{r}}$ dense in the ball $B(0, \frac{1}{2}(R')^{(r+1)/r})$, which is equivalent to prove that for any $y \in B(0, \frac{1}{2}(R')^{(r+1)/r})$, there exist

$$\bar{x}_y \in \mathbb{Z}^{N-1} \cap B(0, 2(R')^{(r+1)/r}) \quad \text{and}$$
$$s_y \in (R')^{1/N}\mathbb{Z} \cap ((R')^{(r+1)/r}x_1, (R')^{(r+1)/r}x_1 + R'),$$

such that

$$|y - \bar{x}_y - s_y\theta| < \frac{1}{1000}(R')^{-\frac{1}{N}+\frac{\left(\frac{N+1}{N}r+1\right)\epsilon}{r}}, \tag{3.12}$$

for a fixed $\theta \in \mathbb{S}^{N-2}$, which is independent of y and x_1. This is implied by the following Lemma 3.2 from Lucà–Rogers [**12**], but we prefer to omit the proof of (3.12), because a similar but more detailed proof can be found in Corollary 2.2 of [**12**].

LEMMA 3.2 [**12**, Lemma 2.1]. *Let $d \geqslant 2$, $0 < \epsilon, \delta < 1$ and $\kappa > \frac{1}{d+1}$. Then, if $\delta < \kappa$ and $R > 1$ is sufficiently large, there is $\theta \in \mathbb{S}^{d-1}$ for which, given any $[y] \in \mathbb{T}^d$ and $a \in \mathbb{R}$, there is a $t_y \in R^\delta\mathbb{Z} \cap (a, a + R)$ such that*

$$|[y] - [t_y\theta]| \leqslant \epsilon R^{(\kappa-1)/d},$$

where '$[\cdot]$' means taking the quotient $\mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$. Moreover, this remains true with $d = 1$, for some $\theta \in (0, 1)$.

Finally, it follows from (3.7) and (3.11) that

$$\begin{aligned}&\int_{B(0,1)} \sup_j |e^{i\frac{t_j}{2\pi}\Delta}f(x_1,\bar{x})|^2 \,\mathrm{d}\bar{x}\,\mathrm{d}x_1 \\ &\quad \geqslant \int_0^{\frac{R^{1-\beta}}{2}} \int_{U_{x_1}} \sup_j |e^{i\frac{t_j}{2\pi}\Delta}f(x_1,\bar{x})|^2 \,\mathrm{d}\bar{x}\,\mathrm{d}x_1 \gtrsim R^{1-\beta}|\Omega_1|^2|\Omega_2|^2,\end{aligned}$$

which implies inequality (3.2).

4. A counterexample for theorem 1.9

For convenience, we first set $N = 2$. By changing of variables, the nonelliptic Schrödinger operator can be written as

$$e^{it\Box}f(x) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi+it\xi_1\xi_2}\hat{f}(\xi)\,\mathrm{d}\xi. \tag{4.1}$$

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For each $r \in (0,1]$, we will show that there exists $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$, such that the maximal estimate

$$\left\| \sup_{n\in\mathbb{N}} |e^{it_n\Box} f| \right\|_{L^2(B(0,1))} \leqslant C\|f\|_{H^s} \tag{4.2}$$

holds for all $f \in H^s(\mathbb{R}^2)$ only if $s \geqslant \frac{r}{r+1}$.

Indeed, we may choose $t_n = 1/n^{\frac{1}{r}+\epsilon}$, it is clear that $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$ but never belongs to $\ell^{r-\epsilon,\infty}(\mathbb{N})$ for any small $\epsilon > 0$. Moreover, $t_n - t_{n+1}$ is decreasing. According to Lemma 3.2 in [7], we can select $\{b_j\}_{j=1}^{\infty}$ and $\{M_j\}_{j=1}^{\infty}$ satisfying $\lim_{j\to\infty} b_j = 0$, $\lim_{j\to\infty} M_j = \infty$, and

$$M_j b_j^{1-r+\epsilon} \leqslant 1, \tag{4.3}$$

such that

$$\sharp\left\{ n : b_j < t_n \leqslant 2b_j \right\} \geqslant M_j b_j^{-r+\epsilon}. \tag{4.4}$$

By the similar argument as Proposition 3.3 in [7], when $t_n \leqslant b_j$, we have

$$t_n - t_{n+1} \leqslant 2M_j^{-1} b_j^{r-\epsilon+1}. \tag{4.5}$$

For fixed j, choose $\lambda_j = \frac{1}{1000} M_j^{\frac{1}{2}} b_j^{-\frac{r-\epsilon+1}{2}}$ and $\widehat{f_j}(\xi_1, \xi_2) = \frac{1}{\lambda_j} \chi_{[0,\lambda_j]\times[-\lambda_j-1,-\lambda_j]}$ (ξ_1, ξ_2). Therefore,

$$\|f_j\|_{H^{\frac{r-\epsilon}{r-\epsilon+1}}} \leqslant \lambda_j^{\frac{r-\epsilon}{r-\epsilon+1}-\frac{1}{2}}. \tag{4.6}$$

Let $U_j = (0, \frac{\lambda_j b_j}{2}) \times (-\frac{1}{1000}, \frac{1}{1000})$. Notice that $U_j \subset B(0,1)$ due to inequality (4.3). Next, we will show that for each $x \in U_j$,

$$\sup_{n\in\mathbb{N}} |e^{it_n\Box} f_j| > \frac{1}{2(2\pi)^2}. \tag{4.7}$$

After changing variables, we have for each $n \in \mathbb{N}$,

$$|e^{it_n\Box} f_j(x)| = \frac{1}{(2\pi)^2} \left| \int_{-1}^{0} \int_{0}^{1} e^{i\lambda_j(x_1-\lambda_j t_n)\eta_1 + ix_2\eta_2 + it_n\lambda_j\eta_1\eta_2} \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \right|. \tag{4.8}$$

For each $x \in U_j$, there exists a unique $n(x,j)$ such that

$$x_1 \in (\lambda_j t_{n(x,j)+1}, \lambda_j t_{n(x,j)}].$$

It is obvious that $t_{n(x,j)+1} \leqslant \frac{b_j}{2}$, then $t_{n(x,j)} \leqslant b_j$ due to inequality (4.4) and the assumption that $t_n - t_{n+1}$ is decreasing. Then it follows from inequality (4.5) that

$$|\lambda_j(x_1 - \lambda_j t_{n(x,j)})\eta_1| \leqslant 2\lambda_j^2 M_j^{-1} b_j^{r-\epsilon+1} \leqslant \frac{1}{1000}.$$

Also, $|x_2\eta_2| \leqslant \frac{1}{1000}$, and by inequality (4.3), we have $|\lambda_j t_{n(x,j)}\eta_1\eta_2| \leqslant \lambda_j b_j \leqslant \frac{1}{1000}$. Therefore, if we take $n = n(x,j)$ in (4.8), then the phase function will be sufficiently small such that $|e^{it_{n(x,j)}\Box} f_j(x)| > \frac{1}{2(2\pi)^2}$ for each $x \in U_j$, which implies inequality (4.7). Then, it follows from inequalities (4.6) and (4.7) that

$$\frac{\| \sup_{n\in\mathbb{N}} |e^{it_n\Box} f_j| \|_{L^2(B(0,1))}}{\|f_j\|_{H^{\frac{r-\epsilon}{r-\epsilon+1}}}} \geqslant CM_j^{\frac{1}{2(r-\epsilon+1)}}.$$

This implies that the maximal estimate (4.2) can not hold when $s \leqslant \frac{r-\epsilon}{r-\epsilon+1}$, hence when $s < \frac{r}{r+1}$ by the arbitrariness of ϵ.

REMARK 4.1. The original idea we adopted to construct the above counterexample comes from [**13**]. The same idea remains valid in general dimensions. For example, in $\mathbb{R}^3$, by changing variables, we can write

$$e^{itL} f(x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi + it(\xi_1\xi_2 \pm \xi_3^2)} \hat{f}(\xi)\, \mathrm{d}\xi.$$

In order to prove the necessary condition, we only need to take

$$U_j = (0, \frac{\lambda_j b_j}{2}) \times \left(-\frac{1}{1000}, \frac{1}{1000}\right) \times \left(-\frac{1}{1000}, \frac{1}{1000}\right)$$

and

$$\widehat{f_j}(\xi_1, \xi_2, \xi_3) = \frac{1}{\lambda_j} \chi_{[0,\lambda_j]\times[-\lambda_j-1,-\lambda_j]\times(0,1)}(\xi_1, \xi_2, \xi_3).$$

Data

Our manuscript has no associated data.

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Competing interest

None.

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