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Open manifolds with non-homeomorphic positively curved souls

BY DAVID GONZÁLEZ-ÁLVARO†
Université de Fribourg, Switzerland.
e-mail: david.gonzalezalvaro@unifr.ch

AND MARCUS ZIBROWIUS‡
Heinrich–Heine–Universität Düsseldorf, Germany
e-mail: marcus.zibrowius@cantab.net

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Abstract

We extend two known existence results to simply connected manifolds with positive sectional curvature: we show that there exist pairs of simply connected positively-curved manifolds that are tangentially homotopy equivalent but not homeomorphic, and we deduce that an open manifold may admit a pair of non-homeomorphic simply connected and positively-curved souls. Examples of such pairs are given by explicit pairs of Eschenburg spaces. To deduce the second statement from the first, we extend our earlier work on the stable converse soul question and show that it has a positive answer for a class of spaces that includes all Eschenburg spaces.

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1. *Introduction*

The Soul Theorem [**4**] determines the structure of an open manifold N endowed with a metric g of non-negative sectional curvature: there exists a closed totally convex submanifold S, called the soul, such that N is diffeomorphic to the normal bundle of S. This soul may not be unique, but for a given metric g any two souls are isometric. Our work is motivated then by the following question: if N admits different non-negatively curved metrics g_1, g_2, what can be said about the corresponding souls S_1, S_2? For convenience we will say that S is a soul of N if S is a soul of (N, g) in the usual sense for *some* metric g of non-negative sectional curvature.

Open manifolds with *different* souls can be constructed in the following ways. It is well known that there exist 3-dimensional lens spaces L_1, L_2 that are homotopy equivalent but not homeomorphic, and such that their products with $\mathbb{R}^3$ are diffeomorphic [**28**, section 2]. Thus, the obvious product metrics on $L_1 \times \mathbb{R}^3 \cong L_2 \times \mathbb{R}^3$ have two non-homeomorphic souls. In a similar vein, all of the fourteen exotic 7-dimensional spheres Σ^7 (i. e. manifolds which are homeomorphic but not diffeomorphic to the standard sphere $\mathbb{S}^7$) admit non-negatively curved

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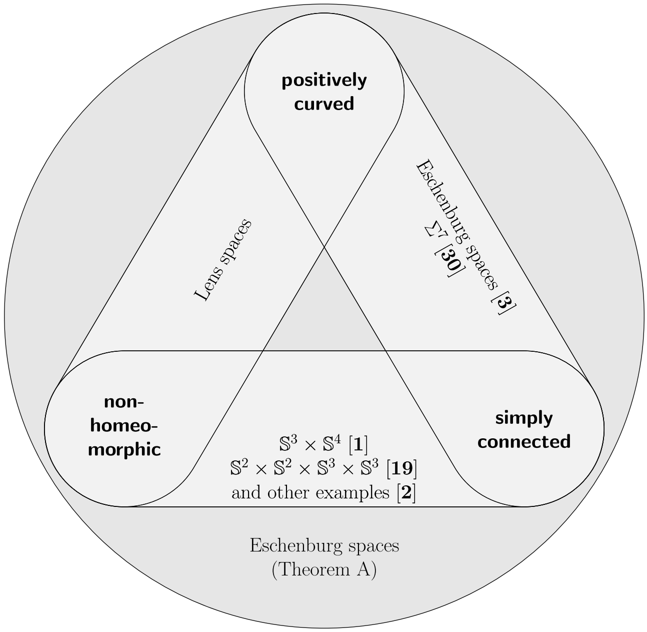

Fig. 1. Existing results on pairs of distinct souls.

metrics (see [15] and the recent preprint [12]), and they all become diffeomorphic after taking the product with $\mathbb{R}^3$. Thus, the obvious product metrics yield fifteen non-diffeomorphic souls of $\mathbb{S}^7 \times \mathbb{R}^3$.

In a more elaborate construction, Belegradek showed that $\mathbb{S}^3 \times \mathbb{S}^4 \times \mathbb{R}^5$ admits infinitely many souls that are pairwise non-homeomorphic [1]. In [19] the same statement was shown over $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{R}^k$ for any $k > 10$, where the souls satisfy certain curvature-diameter properties. Finally, in [2] further examples in the same vein were constructed with the additional property that the souls have codimension four.

Our main interest in this note is the existence of souls with *positive* sectional curvature. For example, the lens spaces described above have metrics with constant positive sectional curvature. Unpublished work by Petersen–Wilhelm [30] announces a positively curved metric on one of the exotic spheres Σ^7; this would yield two non-diffeomorphic souls with positive curvature on $\mathbb{S}^7 \times \mathbb{R}^3$. It also follows from [3] that there exist open manifolds with pairs of non-diffeomorphic homeomorphic souls with positive curvature: see Theorem 16 below for the precise statement and its proof. In all of the above examples, however, the pairs of souls satisfy at most two of the following three properties: they are simply connected, they are non-homeomorphic, they have positive sectional curvature. The situation is summarised in Figure 1. Here, we present open manifolds with pairs of souls that satisfy all three properties simultaneously:

THEOREM A. *There exist simply connected open manifolds with a pair of non-homeomorphic souls of positive sectional curvature.*

In combination with results of [2, 19], Theorem A yields some consequences on the topology of the moduli space of Riemannian metrics with non-negative sectional curvature on the corresponding spaces. This is explained in Section 6.

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Theorem A will be proved in the following more explicit form:

THEOREM A'. *There exist Eschenburg spaces M with the following property: the total space of every real vector bundle over M of rank ≥ 8 admits a pair of non-homeomorphic souls of positive sectional curvature.*

Of course, one of the souls is the given Eschenburg space M; the other soul is a homotopy equivalent but non-homeomorphic Eschenburg space M'. Recall that Eschenburg spaces [**9**] form an infinite family of 7-dimensional quotients of $SU(3)$ under certain circle actions. They inherit non-negatively curved metrics from $SU(3)$ which in many cases have positive sectional curvature (see Section 4 for details). The only known examples of pairs of simply connected manifolds with positive curvature which are homotopy equivalent but non-homeomorphic occur among these Eschenburg spaces [**5**, **32**]. On the other hand, there are only finitely many homeomorphism classes of Eschenburg spaces in each homotopy type [**5**, proposition 1·7], so our strategy behind proving Theorem A' cannot yield infinite families of non-homeomorphic souls.

This strategy is as follows. We use the classical fact that *the total spaces of a vector bundle of high rank and its pull-back under a tangential homotopy equivalence are diffeomorphic*. Here, two manifolds M_1, M_2 of the same dimension are called *tangentially homotopy equivalent* if there exists a homotopy equivalence $f: M_1 \to M_2$ such that the tangent bundle TM_1 and f^*TM_2 are stably isomorphic, i. e. such that $TM_1 \times \mathbb{R}^k$ and $f^*TM_2 \times \mathbb{R}^k$ are isomorphic as bundles over M_1 for some integer $k \geq 0$. Thus, Theorem A' is a consequence of the two following results, in which each Eschenburg space is understood to come equipped with some metric which descends from a circle invariant non-negatively curved metric on $SU(3)$.

THEOREM B. *There exist pairs of positively curved Eschenburg spaces which are tangentially homotopy equivalent but not homeomorphic.*

THEOREM C. *Let M be an Eschenburg space. The total space of every real vector bundle over M of rank ≥ 8 admits a metric with non-negative sectional curvature whose soul is isometric to M.*

Explicit pairs of Eschenburg spaces as in Theorem B are listed in Table I below. They constitute the first known examples of simply connected positively curved non-homeomorphic spaces that are tangentially homotopy equivalent. On the other hand, any two homeomorphic Eschenburg spaces are in particular tangentially homotopy equivalent. (This implication holds for many closed manifolds of dimension at most 7; see Corollary 3.) Pairs of simply connected *non-negatively* curved manifolds that are tangentially homotopy equivalent but not homeomorphic are already known: Crowley exhibited an explicit such pair of $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ [**7**, p. 114], which carry metrics of non-negative sectional curvature by the work of Grove and Ziller [**15**].

Theorem C should be seen in the context of the *converse soul question*: does every vector bundle over a manifold with non-negative sectional curvature itself admit a metric of non-negative sectional curvature? While this is known to be false for general base manifolds, very little is known about this question for simply connected bases. Every vector bundle over a sphere $\mathbb{S}^n$ with $2 \leq n \leq 5$ admits such a metric [**15**], and there exist partial positive

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results over cohomogeneity-one four-manifolds [16]. A stable version of the question is known to have an affirmative answer for all spheres [31], and also for many other families of homogeneous spaces including almost all the positively curved ones [13, 14]. On the other hand, there is not a single known example of a vector bundle over a simply connected non-negatively curved closed manifold whose total space admits no metric of non-negative sectional curvature. Using the same techniques as in the proof of Theorem C, we can further extend the list of examples in which the converse soul question has a positive answer, at least after some form of stabilisation:

THEOREM C'. *Let M be any of the closed manifolds listed below. The total space of every real vector bundle over M of rank $\geq r$ admits a metric of non-negative sectional curvature, where r depends on M as listed*:

(i) *generalised Witten spaces M with $H^4(M)$ of odd order ($r = 8$);*
(ii) *generalised Witten spaces M with $H^4(M)$ of even order ($r = 18$);*
(iii) *products of spheres $\mathbb{S}^2 \times \mathbb{S}^m$ with $m \equiv 3, 5 \mod 8$ ($r = m + 3$);*
(iv) *the total space of the unique non-trivial linear $\mathbb{S}^m$-bundle over $\mathbb{S}^2$ where either $m = 3$ or $m \equiv 5 \mod 8$ (in any case $r = m + 3$).*

The generalised Witten spaces appearing here are a family of manifolds M_{k,l_1,l_2} defined as quotients of $\mathbb{S}^5 \times \mathbb{S}^3$ under the circle action

$$\begin{array}{ccc} S^1 \times \mathbb{S}^5 \times \mathbb{S}^3 & \longrightarrow & \mathbb{S}^5 \times \mathbb{S}^3 \\ (z, (u_1, u_2, u_3), (v_1, v_2)) & \longmapsto & \big((z^k u_1, z^k u_2, z^k u_3), (z^{l_1} v_1, z^{l_2} v_2)\big), \end{array}$$

where $\mathbb{S}^5 \subset \mathbb{C}^3$, $\mathbb{S}^3 \subset \mathbb{C}^2$, and k, l_1, l_2 are nonzero integers such that k, l_j are coprime for $j = 1, 2$; for such a space $H^4(M_{k,l_1,l_2}) = \mathbb{Z}_{l_1 l_2}$. We refer to [11] for details.

The unifying feature of the examples appearing in Theorem C' is that the base manifolds come equipped with a principal S^1-bundle that carries an invariant metric of non-negative sectional curvature, and whose associated complex line bundle generates the Picard group of the base manifold. The idea is then to show that any real vector bundle is stably equivalent to a sum of at most $r/2$ complex line bundles. See Proposition 8 below for a general form of Theorems C and C'.

Note that there are infinitely many manifolds in Theorems C and C' that are not diffeomorphic to homogeneous spaces. Indeed, there are infinitely many spaces among Eschenburg and generalised Witten spaces that are not even homotopy equivalent to any homogeneous space [11, 32].

Outline

The paper is organised as follows. All theorems above follow from a study of stable equivalence classes of real vector bundles over manifolds of dimension at most seven, with which we begin in Section 2. Theorems C and C' are deduced in Section 3. In Section 4, we use the results on stable equivalence classes to refine the homotopy classification of Eschenburg spaces due to Kruggel, Kreck and Stolz to a classification up to tangential homotopy equivalence. A search for pairs as in Theorem B can then easily be implemented as a computer program. The code we use is briefly discussed at the end of Section 4; we have made it freely available [40]. Theorem A is finally proved in Section 5. We close in Section 6 with a brief discussion of implications for moduli spaces.

Notation

We write $H^*(-)$ to denote (singular) cohomology with integral coefficients, i.e. $H^*(X) := H^*(X, \mathbb{Z})$.

2. *Vector bundles over seven-manifolds*

Two real vector bundles F and F' over a common base X are **stably equivalent** if $F \oplus \mathbb{R}^k \cong F' \oplus \mathbb{R}^{k'}$ for certain integers k and k'. The main result of this section is that, over certain classes of 7-manifolds, any real vector bundle is stably equivalent to a sum of complex line bundles. See Proposition 4 for the precise statement and Remark 6 for slight generalisations.

Our calculations will make use of the **Spin characteristic class** q_1 constructed by Thomas [**35**]. Assume for the following brief discussion that our base X is a finite-dimensional connected CW complex. A Spin bundle F over X is a real vector bundle whose first two Stiefel–Whitney classes $w_1 F$ and $w_2 F$ vanish. Equivalently, a real vector bundle F is a Spin bundle if and only if its classifying map $f_F : X \to BO$ lifts to a map $\hat{f}_F : X \to BSpin$. The Spin characteristic class $q_1 F \in H^4(X)$ of such a Spin bundle is defined as the pullback under $\hat{f}_F$ of a distinguished generator of $H^4(BSpin)$. We will make frequent use of the following properties of the Spin characteristic class and its relation to the first Pontryagin class p_1 and the Chern classes c_1 and c_2.

PROPOSITION 1. *Let F and F' be two Spin bundles over X, let E be a complex vector bundle over X, and let rE be the underlying real vector bundle.*

(i) $q_1(F) = 0$ *if F is a trivial vector bundle.*
(ii) $q_1(F \oplus F') = q_1 F + q_1 F'$.
(iii) $2q_1(F) = p_1(F)$ — *"The Spin class is half the Pontryagin class."*
(iv) $p_1(rE) = (c_1 E)^2 - 2c_2 E$.
(v) $q_1(rE) = -c_2 E$ *if $c_1 E = 0$.*

For the last identity, note the rE is a Spin bundle if and only if the mod-2-reduction of $c_1 E$ in $H^2(X, \mathbb{Z}_2)$ vanishes. In particular, the stated stronger condition $c_1 E = 0$ implies that rE is a Spin bundle.

Proof. The first claim is clear from the definition. For (ii) and (iii), see equations 1·10 and 1·5 in theorem 1·2 of [**35**]. Claim (iv) is a direct consequence of the definition of Pontryagin classes. Claim (v) is immediate from (iii) and (iv) when $H^4(X)$ contains no 2-torsion, an assumption we will frequently make below. To see that (v) also holds in general, note that stable equivalence classes of bundles with vanishing first Chern class are classified by BSU. So $q_1 \circ r$ defines a natural transformation $[X, BSU] \to H^4(X)$ and hence corresponds to an element of $H^4(BSU) = \mathbb{Z}c_2$. To see which element it is, we can evaluate, say, on $X = \mathbb{S}^4$ and then use (iv).

PROPOSITION 2. *Suppose X is a connected CW complex of dimension ≤ 7. Then two Spin bundles F, F' over X are stably equivalent if and only if their Spin characteristic classes agree.*

Suppose in addition that $H^4(X)$ contains no 2-torsion. Then two real bundles F, F' over X are stably equivalent if and only if their Stiefel–Whitney classes w_1 and w_2 and their first Pontryagin classes p_1 agree.

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The distinction of cases here is necessary because, in contrast to w_1, w_2 and p_1, the Spin characteristic class q_1 is not defined for arbitrary real vector bundles.

Proof. Let $\widetilde{\mathrm{KO}}(X)$ denote the reduced real K-group of X, i. e. the group of stable equivalence classes of real vector bundles over X. (For background, see for example [18].) Let $\widetilde{\mathrm{KSpin}}(X)$ denote the subgroup of stable equivalence classes of Spin bundles. Points (i) and (ii) of the previous proposition show that q_1 defines a homomorphism $q_1 \colon \widetilde{\mathrm{KSpin}}(X) \to H^4(X)$. By [26, corollary 1], this homomorphism is an isomorphism for X of dimension at most seven, so the claim follows.

In general, $\widetilde{\mathrm{KSpin}}(X)$ and $\widetilde{\mathrm{KO}}(X)$ fit into a short exact sequence as follows [26, remark 2]:

$$0 \longrightarrow \widetilde{\mathrm{KSpin}}(X) \hookrightarrow \widetilde{\mathrm{KO}}(X) \xrightarrow{(w_1,w_2)} H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2) \longrightarrow 0$$

Here, the group structure on $H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2)$ is defined such that the map (w_1, w_2) is a homomorphism. Given two real vector bundles F and F' whose Stiefel–Whitney classes w_1 and w_2 agree, we obtain an element $F - F' \in \widetilde{\mathrm{KO}}(X)$ that lies in the kernel of (w_1, w_2) and hence in $\widetilde{\mathrm{KSpin}}(X)$. If furthermore $p_1(F) = p_1(F')$, we find that $p_1(F - F') = 0$ because the Whitney sum formula holds for Pontryagin classes up to 2-torsion [29, theorem 15·3] and because we have assumed that $H^4(X)$ does not contain any such torsion. Using Proposition 1 (iii) and the same assumption on $H^4(X)$, we deduce that $q_1(F - F') = 0$. As we saw in the first part of the proof, this implies that $F - F' = 0$ in $\widetilde{\mathrm{KSpin}}(X)$. So F and F' are stably equivalent.

As q_1 is a homeomorphism invariant [6, 1·1/remark 2·1], and as Stiefel–Whitney classes are even homotopy invariants, the above proposition implies:

COROLLARY 3. *Any two homeomorphic closed Spin manifolds of dimension ≤ 7 are tangentially homotopy equivalent. Similarly, any two homeomorphic closed manifolds of dimension ≤ 7 for which $H^4(-)$ contains no 2-torsion are tangentially homotopy equivalent.*

We introduce the following notation for a CW complex X with $H^4(X)$ finite:

$$\sigma_4(X) := \begin{cases} 1 & \text{if} \quad H^4(X) = 0, \\ 4 & \text{if} \quad |H^4(X)| \text{ is odd}, \\ 9 & \text{if} \quad |H^4(X)| \text{ is even and non-zero.} \end{cases} \qquad (2{\cdot}1)$$

PROPOSITION 4. *Let X be a connected CW complex of dimension ≤ 7 such that $H^1(X, \mathbb{Z}_2) = 0$, $H^2(X)$ is (non-zero) cyclic, $H^3(X)$ contains no 2-torsion, and $H^4(X)$ is finite cyclic and generated by the square of a generator of $H^2(X)$. Then any real vector bundle over X is stably equivalent to (the underlying real bundle of) a Whitney sum of $\sigma_4(X)$ complex line bundles.*

Proof. Under our assumptions, the Bockstein sequence shows that the reduction map $H^2(X) \to H^2(X, \mathbb{Z}_2)$ is surjective, and that either $H^2(X, \mathbb{Z}_2) = 0$ or $H^2(X, \mathbb{Z}_2) \cong \mathbb{Z}_2$. We identify $H^4(X)$ with $\mathbb{Z}_s$ for some positive integer s. We will not distinguish between integers and their images in any of these residue groups notationally. Given an integer a, we write L_a for the complex line bundle with $c_1(L_a) = a \in H^2(X)$. More generally, a sum of such line bundles will be denoted $L_{a_1,\ldots,a_k} := L_{a_1} \oplus \cdots \oplus L_{a_k}$.

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If $s = 0$, i. e. if $H^4(X)$ vanishes, then by the second half of Proposition 2 the stable equivalence class of a real vector bundle F over X is determined by $w_2(F)$. Thus F is stably equivalent to either $r(L_0)$ or $r(L_1)$.

Next, consider the case that the order s of $H^4(X)$ is odd. Let F be an arbitrary given real vector bundle over X. By the second part of Proposition 2, it suffices to find integers $a_1, \ldots, a_4$ such that

$$w_2(rL_{a_1,\ldots,a_4}) = w_2(F) \quad \text{(i)}$$

$$p_1(rL_{a_1,\ldots,a_4}) = p_1(F). \quad \text{(ii)}$$

If $H^2(X, \mathbb{Z}_2) = 0$, we can ignore the first condition; otherwise, $w_2(rL_{a_1,\ldots,a_4}) = a_1 + a_2 + a_3 + a_4 \mod 2$. For the Pontryagin class, part (iv) of Proposition 1 implies that

$$p_1(rL_{a_1,\ldots,a_4}) = a_1^2 + a_2^2 + a_3^2 + a_4^2 \quad \in H^4(X).$$

So we can find integers a_i satisfying condition (ii) by appealing to Lagrange's Four Square Theorem: any positive integer can be written as a sum of a most four squares. In case these integers do not already satisfy condition (i), we can replace a_1 by $a_1 + s$: as $a_1 + s = a_1 + 1 \mod 2$ and $(a_1 + s)^2 = a_1^2 \mod s$, the new set of integers will then satisfy both conditions.

Finally, for arbitrary s, we can argue as follows. Let F again be some given real vector bundle over X, but assume to begin with that F is a Spin bundle. Then in view of Proposition 2 it suffices to show that there exists a Whitney sum of (at most nine) complex line bundles $L_{a_1,\ldots,a_k}$ such that $rL_{a_1,\ldots,a_k}$ is a Spin bundle with the same Spin characteristic class as F. As the first Chern class of such a sum is given by

$$c_1(L_{a_1,\ldots,a_k}) = a_1 + \cdots + a_k,$$

$rL_{a_1,\ldots,a_k}$ is certainly a Spin bundle whenever $a_1 + \cdots + a_k \equiv 0 \mod 2$. Moreover, part (v) of Proposition 1 applies whenever $a_1 + \cdots + a_k = 0$ in $\mathbb{Z}$. In particular, we find that $q_1(rL_{a,-a}) = a^2$, and more generally that

$$q_1(rL_{a_1,-a_1,a_2,-a_2,a_3,-a_3,a_4,-a_4}) = a_1^2 + a_2^2 + a_3^2 + a_4^2 \quad \in H^4(X).$$

So, again by Lagrange's Four Square Theorem, we can find integers a_1, a_2, a_3, a_4 such that $q_1(rL_{a_1,-a_1,\ldots,a_4,-a_4}) = q_1F$, whatever the given value of q_1F. So our Spin bundle F is stably equivalent to a Whitney sum of eight complex line bundles.

When F is an arbitrary real vector bundle, we can pick a complex line bundle L_b such that $w_2(rL_b) = w_2(F)$. Then $F - rL_b$ is a stable equivalence class in $\widetilde{\text{KSpin}}(X)$, the previous argument shows that $F - rL_b = rL_{a_1,-a_1,\ldots,a_4,-a_4}$ in $\widetilde{\text{KSpin}}$, and hence F is stably equivalent to the Whitney sum of nine complex line bundles $rL_{a_1,-a_1,\ldots,a_4,-a_4,b}$.

COROLLARY 5. *Let X be a connected CW complex satisfying the assumptions of Proposition 4. Any real vector bundle over X of rank $\geq \max\{2\sigma_4(X), \dim(X) + 1\}$ is isomorphic to a Whitney sum of $\sigma_4(X)$ complex line bundles and a trivial bundle.*

Proof. This is immediate from Proposition 4 and the general fact that the notions of stable equivalence and isomorphism agree for bundles of sufficiently high rank: if two real vector bundles of the same rank F and F' over an n-dimensional CW complex are stably equivalent, and if the common rank of these bundles is greater than n, then F and F' are isomorphic (e.g. [**18**, chapter 9, proposition 1·1]).

Remark 6. We have deliberately refrained from stating Propositions 2 and 4 and Corollary 5 with minimal assumptions. In Proposition 2, the condition that X is a connected CW complex of dimension ≤ 7 could easily be replaced with the following weaker assumptions:

(i) X is a connected finite-dimensional CW complex, and
(ii) the inclusion of the seven-skeleton X^7 induces an isomorphism $\widetilde{\mathrm{KO}}(X^7) \cong \widetilde{\mathrm{KO}}(X)$. The Atiyah–Hirzebruch spectral sequence shows that a sufficient criterion for this to be the case is that all non-vanishing integral cohomology groups $H^i(X)$ in degrees $i \geq 5$ are torsion-free and concentrated in degrees i with $(i \mod 8) \in \{3, 5, 6, 7\}$.

The additional assumptions needed in Proposition 4 and Corollary 5 are that $H^1(X, \mathbb{Z}_2)$, $H^2(X)$, $H^3(X)$ and $H^4(X)$ have the properties stated in Proposition 4.

3. *Non-negative curvature*

In this section we review a common construction of non-negatively curved metrics on vector bundles and prove Theorems C and C', which give partial positive answers to the converse soul question for Eschenburg spaces and a few other spaces.

Let G be a Lie group and let $P \to M$ be a principal G-bundle. Given a representation $\rho\colon G \to \mathbb{R}^m$, there exists a natural diagonal action on the product $P \times \mathbb{R}^m$ whose quotient space $E_\rho = P \times_G \mathbb{R}^m$ is the total space of a real vector bundle over M. This construction yields a natural semiring homomorphism:

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(M).$$

Suppose now that P admits a G-invariant metric g_P with non-negative sectional curvature. By the Gray–O'Neill formula for Riemannian submersions, M inherits a metric $\bar{g}_P$ with non-negative sectional curvature. Now suppose that $\rho\colon G \to \mathbb{R}^m$ is an orthogonal representation with respect to the usual Euclidian metric g_0 on $\mathbb{R}^m$. Equip $P \times \mathbb{R}^m$ with the product metric $g_P \times g_0$. Then $P \times \mathbb{R}^m$ also has non-negative sectional curvature and the diagonal G-action on $P \times \mathbb{R}^m$ is by isometries. So, again by the Gray–O'Neill formula, E_ρ inherits a metric with non-negative sectional curvature for which the zero-section $(P \times_G \{0\}, \bar{g}_p) = (M, \bar{g}_P)$ is a soul.

At the present time, this is the only known construction of open manifolds with non-negative sectional curvature, up to a change of metric (see [**39**, section 3·1]). It is natural to ask which vector bundles over M can be constructed in this way, a purely topological question that is discussed at length in [**14**] for the case when $P \to M$ is the canonical G-bundle over a homogeneous space G'/G. Here, we consider circle bundles, i. e. the case $G = S^1$.

PROPOSITION 7. *Let $P \to M$ be a principal circle bundle over a closed manifold M. Assume that P is 2-connected and that it admits an invariant metric g_P of non-negative sectional curvature. Then the total space of any Whitney sum of complex line bundles over M admits a metric of non-negative sectional curvature and with soul isometric to $(M, \bar{g}_P)$, where $\bar{g}_P$ denotes the quotient metric inherited from g_P.*

Proof. As explained in [**3**, section 12], the fact that P is 2-connected implies that $H^2(M) = \mathbb{Z}$ and that the first Chern class of the bundle is a generator of $H^2(M)$. It follows that any

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complex line bundle over M has the form $E_\rho = P \times_{S^1} \mathbb{C}$ for some character ρ of S^1, and more generally that any Whitney sum of complex line bundles has the form $E_\rho = P \times_{S^1} \mathbb{C}^k$ for some direct sum of characters $\rho \in \mathrm{Rep}(S^1)$. So the claim follows immediately from the discussion above.

Conditions for a circle bundle to admit invariant metrics with non-negative sectional curvature are given in [**33**].

Theorems C and C' of the introduction are particular cases of the following more general statement. Recall from equation (2·1) in Section 2 our notation $\sigma_4(M)$ for a space with $H^4(M)$ finite.

PROPOSITION 8. *Let $P \to M$ be a principal circle bundle over a closed manifold M. Assume that:*

(i) *P is 2-connected (so that $H^1(M) = 0$ and $H^2(M) = \mathbb{Z}$) and that it admits an invariant metric g_P of non-negative sectional curvature, and that*
(ii) *$H^3(M)$ contains no 2-torsion, $H^4(M)$ is finite cyclic and generated by the square of a generator of $H^2(M)$, and all non-vanishing integral cohomology groups $H^i(M)$ in degrees $i \geq 5$ are torsion-free and concentrated in degrees i with $(i \bmod 8) \in \{3, 5, 6, 7\}$.*

Then the total space of every real vector bundle of rank $\geq \max\{2\sigma_4(M), \dim(M) + 1\}$ over M admits a metric with non-negative sectional curvature and soul isometric to $(M, \bar{g}_P)$, where $\bar{g}_P$ is the induced quotient metric on M.

Proof. Corollary 5 and Remark 6 show that any real vector bundle F over M of rank $\geq \max\{2\sigma_4(M), \dim(M) + 1\}$ is isomorphic to a Whitney sum of complex line bundles and a trivial vector bundle. The Whitney sum of complex line bundles admits a metric of non-negative sectional curvature by Proposition 7, and thus the product metric of this metric with the flat metric on the trivial summand yields a metric on F with the desired properties.

To prove Theorems C and C', it now suffices to check that the spaces in question satisfy the assumptions of Proposition 8.

Proof of Theorems C and C'. The cohomology of Eschenburg and generalised Witten spaces is well known [**9**, **11**]: they are manifolds of type r (see Definition 10 below). For Eschenburg spaces $|H^4(M)|$ is odd, while for generalized Witten spaces it can be either odd or even so both $\sigma_4(M) = 4$ and $\sigma_4(M) = 9$ occur. The total spaces of the corresponding principal bundles are $SU(3)$ and $\mathbb{S}^3 \times \mathbb{S}^5$, respectively, which clearly satisfy the topological assumptions of Proposition 8. The corresponding metrics on $SU(3)$ were constructed by Eschenburg [**9**], see Section 4 below. As for the generalised Witten spaces, the circle actions are by isometries with respect to the standard product metric on $\mathbb{S}^3 \times \mathbb{S}^5$ (see [**11**]).

The products $\mathbb{S}^2 \times \mathbb{S}^m$ and the unique non-trivial $\mathbb{S}^m$-bundle over $\mathbb{S}^2$ with $m \geq 2$ have the same cohomology ring, which clearly satisfies the topological assumptions when $m \equiv 3, 5 \mod 8$. The products $\mathbb{S}^2 \times \mathbb{S}^m$ are just quotients of $\mathbb{S}^3 \times \mathbb{S}^m$ via the Hopf fibration over the first factor. The unique non-trivial linear $\mathbb{S}^m$-bundle over $\mathbb{S}^2$ with $m = 3$ or $m \equiv 5 \mod 8$ can be described as a circle quotient of $\mathbb{S}^3 \times \mathbb{S}^m$ as well. Moreover, the corresponding action

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is by isometries with respect to the standard product metric on $\mathbb{S}^3 \times \mathbb{S}^m$: see [8] for the case $m = 3$ and [38, item (b) above corollary 4] for the cases $m \equiv 5 \mod 8$.

4. *Eschenburg spaces*

Eschenburg spaces, first introduced and studied in [9], generalise the homogeneous 7-manifolds known as Aloff–Wallach spaces. Each Eschenburg space is a quotient of $SU(3)$ by a free action of S^1 of the following form:

$$S^1 \times SU(3) \longrightarrow SU(3)$$
$$(z, A) \longrightarrow \mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3}) \cdot A \cdot \mathrm{diag}(z^{-l_1}, z^{-l_2}, z^{-l_3})$$

Following [5], we specify the action of S^1 and the resulting Eschenburg space $M = M(k, l)$ by the six-tuple of integer parameters $(k, l) = (k_1, k_2, k_3, l_1, l_2, l_3)$. We refer to this six-tuple as the parameter vector of M. The parameters need to satisfy $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$, as well as some further conditions that ensure that the S^1-action is free, see [5, (1.1)]. The Aloff–Wallach spaces are the Eschenburg spaces $M(k, l)$ with $l_1 = l_2 = l_3 = 0$.

All Aloff–Wallach spaces $M(k, 0)$ with $k_1 k_2 k_3 \neq 0$ admit an invariant metric of positive sectional curvature. The interest in more general Eschenburg spaces arises from the fact that they include some of the very few known examples of *non*-homogeneous manifolds with positive sectional curvature. Any metric on SU(3) invariant under the circle action defined by (k, l) descends to a metric on the Eschenburg space $M(k, l)$. We refer to a metric on an Eschenburg space arising in this way as a *submersion metric*. Every Eschenburg space comes equipped with non-negatively curved submersion metrics. For example, one could consider metrics induced by bi-invariant metrics on SU(3), but there are also lots of other choices. Eschenburg constructed submersion metrics with *positive* sectional curvature on infinitely many Eschenburg spaces [10, Satz 414]. In particular, he did so for all Eschenburg spaces $M(k, l)$ whose parameter vector satisfies the following condition:

$$k_1 \geq k_2 > l_1 \geq l_2 \geq l_3 = 0. \qquad (\dagger)$$

In fact, as explained in [5, lemma 1·4], each of the Eschenburg spaces on which Eschenburg constructed a positively curved submersion metric is diffeomorphic to one of the spaces $M(k, l)$ satisfying (†).

Positively curved Eschenburg spaces display interesting phenomena that are not visible when studying the Aloff–Wallach subfamily alone. The following proposition is one example of this. Part (b) was already stated as Theorem B of the introduction.

PROPOSITION 9. *For Aloff–Wallach spaces, the notions of homotopy equivalence, tangential homotopy equivalence and homeomorphism coincide. In contrast, for general positively curved Eschenburg spaces, these notions differ*:

(a) *there exist pairs of positively curved Eschenburg spaces which are homotopy equivalent to each other but not tangentially homotopy equivalent.*
(b) *there exist pairs of positively curved Eschenburg spaces which are tangentially homotopy equivalent but not homeomorphic.*

Examples of both phenomena are displayed in Table I.

Table I. The "first" six pairs of homotopy equivalent but not tangentially homotopy equivalent pairs of positively curved Eschenburg spaces (top half of table), and the "first" six pairs of tangentially homotopy equivalent but non-homeomorphic pairs of such spaces. "First" means that these are the pairs of spaces satisfying (†) with smallest value of r.

(k_1,	k_2,	k_3,	l_1,	l_2,	l_3)	r	s	Σ	p_1	s_{22}	s_2
Homotopy equivalent but not tangentially homotopy equivalent:											
(8,	7,	−5,	6,	4,	0)	43	−21	1	13	1/6	−59/516
(21,	21,	−2,	20,	20,	0)	43	−21	1	26	1/6	55/516
(12,	10,	−8,	9,	5,	0)	101	−50	−1	21	1/6	565/1212
(50,	50,	−2,	49,	49,	0)	101	−50	−1	55	1/6	−125/1212
(19,	17,	−7,	16,	13,	0)	137	−68	−1	23	1/6	−743/1644
(68,	68,	−2,	67,	67,	0)	137	−68	−1	73	1/6	241/1644
(30,	26,	−6,	25,	25,	0)	181	−26	−1	164	−1/6	−193/2172
(16,	16,	−10,	13,	9,	0)	181	26	1	85	1/6	−443/2172
(15,	14,	−11,	12,	6,	0)	181	−43	0	35	0	−55/181
(45,	43,	−4,	42,	42,	0)	181	−43	0	89	0	36/181
(16,	13,	−11,	12,	6,	0)	183	−91	0	33	−1/6	−991/2196
(91,	91,	−2,	90,	90,	0)	183	−91	0	96	−1/6	413/2196
		⋮									
Tangentially homotopy equivalent but not homeomorphic:											
(58,	54,	−34,	39,	39,	0)	2 197	1 032	0	845	1/2	1147/8788
(45,	41,	−47,	39,	0,	0)	2 197	1 032	0	845	1/2	−3247/8788
(81,	69,	−84,	56,	10,	0)	7 571	74	0	5 352	1/2	−9219/30284
(108,	63,	−69,	56,	46,	0)	7 571	74	0	5 352	1/2	5923/30284
(88,	61,	−107,	30,	12,	0)	10 935	−5 179	0	1 368	−1/6	55529/131220
(77,	77,	−106,	30,	18,	0)	10 935	5 179	0	1 368	1/6	−11789/131220
(79,	58,	−131,	6,	0,	0)	13 365	−1 183	0	72	1/3	−3794/8019
(92,	47,	−127,	6,	6,	0)	13 365	1 183	0	72	−1/3	−1552/8019
(115,	79,	−116,	72,	6,	0)	13 851	1 184	0	9 576	−1/6	−77167/166212
(128,	107,	−97,	72,	66,	0)	13 851	−1 184	0	9 576	1/6	−61343/166212
(1112,	1111,	−13,	1110,	1100,	0)	14 467	2 246	−1	11 744	−1/6	68945/173604
(127,	103,	−106,	88,	36,	0)	14 467	−2 246	1	11 744	1/6	17857/173604
(188,	176,	−82,	145,	137,	0)	16 625	3 341	0	6 608	1/2	−25007/66500
(176,	164,	−94,	163,	83,	0)	16 625	3 341	0	6 608	1/2	8243/66500
		⋮									

For Aloff–Wallach spaces, the equivalence of the notions of homotopy equivalence and homeomorphism is due to Dickinson and Shankar [**32**]. A slightly weaker version of the statements for positively curved Eschenburg spaces, namely the existence of pairs of positively curved Eschenburg spaces which are homotopy equivalent but not homeomorphic, is known by [**5**, **32**]. Also, there are known pairs of positively curved Eschenburg spaces [**5**, table 2] and even of Aloff–Wallach spaces [**21**, corollary on p. 467] which are homeomorphic but not diffeomorphic. The situation is illustrated in Figure 2.

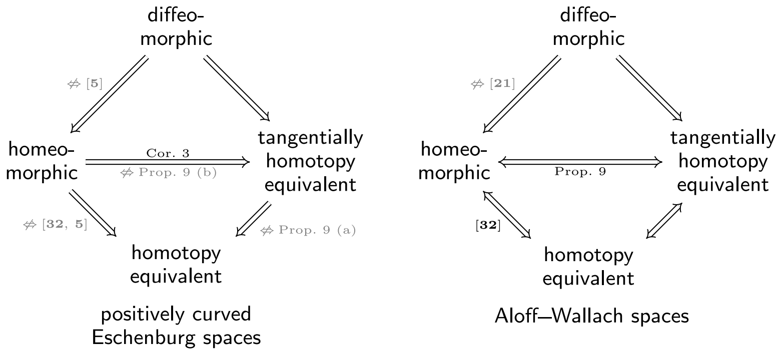

Fig. 2. Implications between different notions of isomorphism for positively curved Eschenburg spaces and for the subfamily of Aloff–Wallach spaces, respectively. All indicated implications (⇒) are strict. The references in gray refer to counterexamples to the inverse implications.

Given the concrete examples in Table I, Proposition 9 can be treated as an application of the classification of Eschenburg spaces. We will first discuss this classification and then say a few words about how the examples were obtained.

Classifications of Eschenburg spaces are known up to various notions of equivalence. Most relevant for us are the classifications up to homotopy and homeomorphism due to Kruggel [23, 24, 25]. The simplest homotopy invariant used in these classifications is obtained via cohomology. Namely, all Eschenburg spaces are type-r-manifolds in the following sense [9, proposition 36]:

Definition 10 ([23]). A **type-r-manifold** is a simply connected closed 7-manifold M whose cohomology has the following structure:

$H^2(M) \cong \mathbb{Z}$, generated by some class u;
$H^4(M) \cong \mathbb{Z}_r$, generated by u^2, for some finite integer $r \geq 1$;
$H^5(M) \cong \mathbb{Z}$, generated by some class v;
$H^7(M) \cong \mathbb{Z}$, generated by uv;
$H^d(M) = 0$ in all other degrees $d > 0$.

In particular, the order r of the fourth cohomology group is a homotopy invariant of Eschenburg spaces. A homeomorphism invariant used in Kruggel's classification is the first Pontryagin class $p_1 \in H^4(M)$. Note that we can *canonically* identify $H^4(M)$ with $\mathbb{Z}_r$ as the generator u^2 does not depend on any (sign) choices. The additional invariants used by Kruggel are the linking number and certain invariants s_i developed by Kreck and Stolz for arbitrary type-r-manifolds [20]. Closed expressions for the Kreck–Stolz invariants of Eschenburg spaces $M(k, l)$ are known only for spaces whose parameter vector (k, l) satisfies a certain numerical "condition (C)" [5, section 2]. However, spaces violating this condition are relatively rare, see Examples 13 below. One last homotopy invariant of positively curved Eschenburg spaces worth mentioning is the value of $\Sigma := k_1 + k_2 + k_3 \mod 3$ [27, 32, proposition 12]. This invariant is not used in Kruggel's classification, but it can still be useful when looking for the kind of phenomena we are studying here.

Table II attempts to give an overview over the different invariants, while Table III summarises the classification results. Note that the displayed classification of Eschenburg spaces

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Table II. Some invariants of an Eschenburg space $M(k, l)$. Our notation mostly follows the notation used in [5]. In the explicit formulae for the invariants, σ_i denotes the ith elementary symmetric polynomial, i. e. $\sigma_1(k) = k_1 + k_2 + k_3$, $\sigma_2(k) = k_1k_2 + k_2k_3 + k_1k_3$ and $\sigma_3(k) = k_1k_2k_3$. The oriented invariants ("or.") change signs under a change of orientation.

	definition		interpretation	invariance
r	$\lvert\sigma_2(k) - \sigma_2(l)\rvert$	$\in \mathbb{Z}$	order of $H^4(M(k, l))$	homotopy
s	$\sigma_3(k) - \sigma_3(l)$	$\in \mathbb{Z}_r^\times$	${}^{-s^{-1}}/_{(\sigma_2(k)-\sigma_2(l))} \in \mathbb{Q}/\mathbb{Z}$ is the linking number	or. homotopy
Σ	$\sigma_1(l)$	$\in \mathbb{Z}_3$	–	or. homotopy
p_1	$2\sigma_1(l)^2 - 6\sigma_2(l)$	$\in \mathbb{Z}_r$	first Pontryagin class	tangential homotopy
s_{22}	$(2rs_2)$	$\in \mathbb{Q}/\mathbb{Z}$	–	or. homotopy
s_2	(non-polynomial)	$\in \mathbb{Q}/\mathbb{Z}$	(Kreck-Stolz invariant)	or. homeomorphism

Table III. Classification of Eschenburg spaces satisfying Kruggel's condition (C), up to various notions of equivalence. For example, the first line says that two such spaces are homotopy equivalent via an orientation preserving equivalence if and only if their invariants r, s, and s_{22} agree. For a more extensive and detailed summary, see [5, theorem 2·3].

invariants ... agree	$\Leftrightarrow$	spaces agree up to ...	references
r, s, s_{22}	$\Leftrightarrow$	oriented homotopy equivalence	[5, 24]
r, s, s_{22}, p_1	$\Leftrightarrow$	oriented tangential homotopy equivalence	Proposition 11
r, s, s_2, p_1	$\Leftrightarrow$	oriented homeomorphism	[5, 25]

up to *tangential* homotopy equivalence is immediate from the classification up to homotopy equivalence:

PROPOSITION 11. *Two Eschenburg spaces are tangentially homotopy equivalent if and only if they are homotopy equivalent and their first Pontryagin classes agree.*

Proof. The invariant r, the order of $H^4(M)$, is odd for any Eschenburg space M [5, above proposition 1·7]. In particular, $H^4(M)$ contains no two-torsion, so that the claim follows directly from the second statement in Corollary 3.

Proof of Proposition 9. The classification results summarised in Table III and the examples in Table I immediately imply the claims concerning general positively curved Eschenburg spaces.

As for the statement concerning Aloff–Wallach spaces, the equivalence of the notions of homeomorphism and homotopy equivalence was proven in [32, proposition A·1]. Finally, the equivalence of the notions of homotopy equivalence and tangential homotopy equivalence follows from Proposition 11 since $p_1 = 0$ for Aloff–Wallach spaces (see Table II).

To find the examples listed in Table I, we followed the basic strategy outlined in [5]. That is, we employed a computer program that first generates all positively curved Eschenburg spaces satisfying (†) with r bounded by some upper bound R, and then looks for families of spaces whose invariants agree. More precisely, given an upper bound $R \in \mathbb{N}$, the main steps of the program are:

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(i) generate all parameter vectors (k, l) satisfying (†) with $r \leq R$;
(ii) among these parameter vectors, find all maximal families of two or more parameter vectors for which the invariants r, s and Σ agree, up to simultaneous sign changes of s and Σ. (This intermediate step is necessary to avoid time-consuming computations of the invariant s_{22} for all generated parameter vectors.)
(iii) within those families, find all maximal (sub)families of two or more parameter vectors for which, in addition, the invariant s_{22} agrees, again up to simultaneous sign changes of s, Σ and s_{22}. This results in a list of families of parameter vectors that describe homotopy equivalent positively curved Eschenburg spaces;
(iv) within the remaining families, find all maximal (sub)families of two or more parameter vectors for which, in addition, the first Pontryagin class agrees. This results in a list of families of parameter vectors that describe tangentially homotopy equivalent positively curved Eschenburg spaces;
(v) within the remaining families, find all maximal (sub)families of two or more parameter vectors for which, in addition, the invariant s_2 agrees (up to simultaneous sign changes of s, Σ, s_{22} and s_2). This results in a list of families of parameter vectors that describe homeomorphic Eschenburg spaces.

The examples in Table I were obtained by comparing the different lists generated by the program. Unfortunately, the C-code referred to in [5] seems to have been lost, so we reimplemented the whole program from scratch and added the additional functionality we needed (in particular steps (iii)–(v)). The new program, written completely in C++, is freely available [40], and we encourage the reader to play around with it. Invariants of individual spaces can alternatively be computed using some Maple code that is still available from Wolfgang Ziller's homepage.

The following empirical data obtained using the program is supplied purely for the reader's amusement.

Statistics 12. Within the range of $r \leq 100\,000$, there are

101 870 124 to 101 872 253 distinct homotopy classes,
103 602 166 distinct tangential homotopy classes, and
103 602 344 distinct homeomorphism classes

of positively curved Eschenburg spaces satisfying (†). We do not know the exact number of distinct homotopy classes due to the failure of Kruggel's condition C in some cases.

Examples 13 (Condition C failures). Examples of positively curved Eschenburg spaces for which Kruggel's condition C fails are discussed in [5]. An example of such a space with minimal value of r among those satisfying (†), taken from [5, section 2], is displayed as space M_0 in Table IV. The spaces (M_1, M_2) in Table IV constitute a pair of positively curved Eschenburg spaces for which the invariants r, s, Σ and p_1 agree, while we cannot compare the Kreck–Stolz invariants due to the failure of condition C for one of the spaces. The value $r = 141\,151$ is minimal among all such pairs of spaces satisfying (†).

Example 14 (Larger exotic families). The literature on Eschenburg spaces only studies *pairs* of exotic structures, for example pairs of homotopy equivalent spaces. However, there also seem to be lots of triples, quadruples, etc. of homotopy equivalent Eschenburg spaces.

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Table IV. Some examples of positively curved Eschenburg spaces

$M($	$k_1,$	$k_2,$	$k_3,$	$l_1,$	$l_2,$	$l_3)$	r	s	Σ	p_1	s_{22}	s_2
$M_0 := M($	35,	21,	−34,	12,	10,	0)	1 289	499	1	248	[condition C fails]	
$M_1 := M($	440,	168,	−320,	159,	129,	0)	141 151	−58 968	0	42 822	0	$-35047/141151$
$M_2 := M($	400,	168,	−352,	165,	51,	0)	141 151	−58 968	0	42 822	[condition C fails]	
$M_3 := M($	410,	259,	−457,	192,	20,	0)	203 383	−79 707	−1	66 848	$-1/6$	$614891/2440596$
$M_4 := M($	548,	497,	−335,	374,	336,	0)	203 383	−79 707	−1	50 833	$-1/6$	$-621835/2440596$
$M_5 := M($	370,	287,	−457,	126,	74,	0)	203 383	−79 707	−1	24 056	$-1/6$	$404657/2440596$
$M_6 := M($	610,	491,	−325,	462,	314,	0)	203 383	−79 707	−1	130 561	$-1/6$	$123017/2440596$
$M_7 := M($	650,	491,	−305,	432,	404,	0)	203 383	−79 707	−1	147 241	$-1/6$	$659411/2440596$
$M_8 := M($	548,	469,	−355,	432,	230,	0)	203 383	−79 707	−1	76 945	$-1/6$	$-947995/2440596$

For example, the spaces M_3, M_4, ..., M_8 in Table IV constitute a six-tuple of homotopy equivalent, positively curved Eschenburg spaces, no two of which are tangentially homotopy equivalent. In contrast, we have not been able to find a single triple of tangentially homotopy equivalent but non-homeomorphic Eschenburg spaces. There appear to be no such triples of spaces satisfying (†) with $r \leq 300\,000$.

5. *Proof of Theorem A*

We are now ready to prove our main result. By Theorem B, there exist pairs of positively curved Eschenburg spaces M_1, M_2 that are tangentially homotopy equivalent but non-homeomorphic. Pick one such pair and a tangential homotopy equivalence $f\colon M_1 \to M_2$. We claim that $M := M_2$ has the property stated in Theorem A'. Indeed, let $E \to M_2$ be an arbitrary real vector bundle of rank ≥ 8. Denote by $f^*E \to M_1$ its pullback along f. The induced map $h\colon f^*E \to E$ is still a tangential homotopy equivalence, see for example the proof of Proposition 1.3 in [14]. Now we need the following well-known corollary of a classical result of Siebenmann; it appears, for example, as [37, theorem 10·1·6], where it is dubbed "Work Horse Theorem":

THEOREM 15 (Siebenmann, Belegradek). *Let $E_1 \to M_1$ and $E_2 \to M_2$ be two vector bundles of the same rank l over two closed manifolds of the same dimension n. Suppose that $l \geq 3$ and $l > n$. Then any tangential homotopy equivalence $h\colon E_1 \to E_2$ is homotopic to a diffeomorphism.*

Proof sketch. Note first that we might as well assume M_1 and M_2 to be connected, as we may argue one component at a time. For $n = 0$ or $n = 1$, the statement can be checked by elementary means. For $n \geq 2$, a proof is outlined in [1] below Proposition 5, as follows: first one observes that the total space E of a vector bundle of rank ≥ 3 over a closed connected manifold M of dimension ≥ 2 satisfies hypothesis (3) in [34, theorem 2·2]: it has one end, π_1 is essentially constant at ∞, and $\pi_1(\infty) \to \pi_1(E)$ is an isomorphism. Thus, if such a total space contains an embedded closed connected manifold S such that the embedding $S \hookrightarrow E$ is a homotopy equivalence, then E admits the structure of a vector bundle over S, with the given embedding as zero section. Slight generalizations of the arguments used in the proof of [34, theorem 2·3] then complete the proof: For $h\colon E_1 \to E_2$ as above and $s_1\colon M_1 \to E_1$ the zero section, the homotopy equivalence $h \circ s_1\colon M_1 \to E_2$ is homotopic to a smooth embedding $g\colon M_1 \to E_2$ by general position arguments [17, chapter 2, theorems 2·6 and 2·13]. It follows that E_2 has the structure of a vector bundle over M_1 and can be identified with the normal bundle N_g of the embedding g. On the other hand, the assumption that h is a *tangential* homotopy equivalence implies that the vector bundles N_g and E_1 over M_1 are stably isomorphic, and since their rank l is greater than n it follows that $N_g \cong M_1$ (see the reference given in the proof of Corollary 5).

Returning to the proof of Theorem A, we find that the total spaces of our bundles $f^*E \to M_1$ and $E \to M_2$ are diffeomorphic. By Theorem C, they admit two metrics with non-negative sectional curvature, one with soul isometric to M_1 and the other with soul isometric to M_2. This completes the proof of Theorem A'/Theorem A.

The pairs of souls we have constructed have codimension ≥ 8. This is probably not optimal. All we know is that any pair of souls as in Theorem A necessarily has codimension at least three: according to [2], any two codimension-two souls of a simply connected open

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manifold are homeomorphic. There is, however, the following result on positively-curved codimension-two souls due to Belegradek, Kwasik and Schultz:

THEOREM 16 ([3]). *There exist Eschenburg spaces M with the following property: the total space of every non-trivial complex line bundle over M admits a pair of non-diffeomorphic, homeomorphic souls of positive sectional curvature.*

Indeed, this is essentially the case $m = 0$ of [3, theorem 1·4]; the exact statement may easily be extracted from the proof of this theorem given there (see page 41). This result does not rely on the "Work Horse Theorem" stated as Theorem 15 above. Rather, the main topological tool that goes into it is [3, theorem 12·1]:

THEOREM. *Let M_1, M_2 be two closed simply connected manifolds of dimension $n \geq 5$ with $n \neq 1 \mod 4$, such that M_1 is the connected sum of M_2 with a homotopy sphere that bounds a parallelisable manifold. Let $L_2 \to M_2$ be a non-trivial line bundle, and let $L_1 \to M_1$ be its pullback via the standard homeomorphism $M_1 \to M_2$. Then the total spaces L_1 and L_2 are diffeomorphic.*

6. *Moduli spaces of Riemannian metrics*

Given a manifold N, denote by $\mathcal{R}(N)$ the space of all (complete) Riemannian metrics on N. We refer to [37, chapter 1] for basic properties of spaces of metrics. They can be topologized in different ways. Following [2], we consider:

(u) the topology of uniform C^∞-convergence;
(c) the topology of uniform C^∞-convergence on compact subsets.

The space of metrics equipped with one of these topologies will be denote $\mathcal{R}^u(N)$ and $\mathcal{R}^c(N)$, respectively. The diffeomorphism group $\mathrm{Diff}(N)$ acts on $\mathcal{R}(N)$ by pulling back metrics. This action is continuous with respect to both topologies. The quotient spaces are called the **moduli spaces of metrics** and will be denoted by $\mathcal{M}^c(N)$ and $\mathcal{M}^u(N)$, respectively. While $\mathcal{M}^c(N)$ is always path-connected, $\mathcal{M}^u(N)$ can have uncountably many connected components if N is non-compact.

For an open manifold N, we are interested in the subspace $\mathcal{R}_{K\geq 0}(N)$ of $\mathcal{R}(N)$ consisting of all metrics with non-negative sectional curvature. Pulling back metrics preserves curvature bounds, so we can consider the corresponding moduli spaces $\mathcal{M}^u_{K\geq 0}(N)$ and $\mathcal{M}^c_{K\geq 0}(N)$. Connectedness properties of these spaces have been the subject of much research; see [36] and [37, chapter 10] for recent surveys on this topic.

Our main result Theorem A suggests to also consider the subspace of those metrics with non-negative sectional curvature $K \geq 0$ whose souls S have positive sectional curvature $K^S > 0$. We will denote this subspace and the corresponding moduli space by $\mathcal{R}_{K\geq 0, K^S>0}(N)$ and $\mathcal{M}_{K\geq 0, K^S>0}(N)$, with the appropriate superscript again indicating the topology. Let us examine how the results above are reflected in the connectedness of these subspaces. We first consider the two topologies separately and then discuss the special case of codimension-one souls, for which both topologies coincide.

Topology of uniform convergence

The following result is an immediate consequence of [2, Theorem 1·5]:

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THEOREM. *Let $g_1, g_2 \in \mathcal{R}^u_{K\geq 0}(N)$ with souls S_1, S_2. If S_1, S_2 are non-diffeomorphic, then the equivalence classes of g_1, g_2 lie in different path components of $\mathcal{M}^u_{K\geq 0}(N)$.*

So $\mathcal{M}^u_{K\geq 0, K^S>0}(N)$ is not path-connected for any N as in Theorem A' or Theorem 16.

Topology of uniform convergence on compact subsets

The following result is an immediate consequence of [**19**, Lemma 6·1]:

THEOREM. *Let $g_1, g_2 \in \mathcal{R}^c_{K\geq 0}(N)$ with souls S_1, S_2. Assume that the normal bundles of S_1 and S_2 in N both have non-trivial rational Euler class. If S_1, S_2 are non-diffeomorphic, then the equivalence classes of g_1, g_2 lie in different path components of $\mathcal{M}^c_{K\geq 0}(N)$.*

For dimensional reasons, the Euler classes of the vector bundles in Theorem A' vanish. On the other hand, the Euler classes of the line bundles in Theorem 16 are non-zero by assumption. Thus, $\mathcal{M}^c_{K\geq 0, K^S>0}(N)$ is not path-connected when N is the total space of a line bundle as in Theorem 16.

Codimension-one souls

In the special case where the souls have codimension one in N both topologies coincide. More precisely, the following result is [**2**, proposition 2·8]:

THEOREM. *If N admits a metric with non-negative curvature and codimension-one soul, then the obvious map $\mathcal{M}^u_{K\geq 0}(N) \to \mathcal{M}^c_{K\geq 0}(N)$ is a homeomorphism. Moreover, the natural map*

$$\textbf{soul: } \mathcal{M}^u_{K\geq 0}(N) \longrightarrow \coprod_i \mathcal{M}_{K\geq 0}(S_i)$$

that assigns to each metric the metric of its soul is a homeomorphism as well, where $\coprod$ denotes disjoint union over are all possible diffeomorphism types S_i of souls of N.

When N is simply-connected all codimension-one souls S are diffeomorphic, so that the map **soul:** $\mathcal{M}^u_{K\geq 0}(N) \to \mathcal{M}_{K\geq 0}(S)$ is a homeomorphism. We can use this result to obtain further open manifolds N such that $\mathcal{M}^u_{K\geq 0, K^S>0}(N)$ is not path-connected: Kreck and Stolz showed in [**22**] that there are Eschenburg spaces M for which the moduli space $\mathcal{M}_{K>0}(M)$ of metrics with *positive* sectional curvature is not path-connected. By considering Riemannian products with the real line we find that $\mathcal{M}^u_{K\geq 0, K^S>0}(M \times \mathbb{R})$ is not path-connected either.

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