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Orlicz–Besov Extension and Imbedding

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Abstract. We establish criteria for Orlicz–Besov extension/imbedding domains via (global) n-regular domains that generalize the known criteria for Besov extension/imbedding domains.

1 Introduction

Due to important applications in harmonic analysis, geometry analysis, and partial differential equations, the extension and imbedding properties of function spaces (including Sobolev, fractional Sobolev, Hajlasz–Sobolev, Besov, Triebel–Lizorkin space, and Q-spaces) have attracted much attention and have been widely studied in the literature [2, 5, 7–11, 15, 16, 18, 20–23].

In this paper we are interested in the Orlicz–Besov spaces as motivated by [13, 14, 19]. Let ϕ be a Young function, that is, $\phi \in C([0,\infty))$ is convex and satisfies $\phi(0) = 0$, $\phi(t) > 0$ for all $t > 0$. For any $\alpha \in \mathbb{R}$ and domain $\Omega \subset \mathbb{R}^n$, define the *homogeneous Orlicz–Besov space* $\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ as the space of all measurable functions u in Ω with the semi-norm

$$\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} := \inf\Big\{\lambda > 0 : \int_\Omega \int_\Omega \phi\Big(\frac{|u(x) - u(y)|}{\lambda |x-y|^\alpha}\Big)\,\frac{dx\,dy}{|x-y|^{2n}} \leq 1\Big\} < \infty.$$

Define the *inhomogeneous Orlicz–Besov space* $\mathbf{B}^{\alpha,\phi}(\Omega) := L^\phi(\Omega) \cap \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$, equipped with norm $\|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)} := \|u\|_{L^\phi(\Omega)} + \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}$. Here $L^\phi(\Omega)$ is the *Orlicz space*, that is, the set of all functions u with

$$\|u\|_{L^\phi(\Omega)} := \inf\Big\{\lambda > 0 : \int_\Omega \phi\Big(\frac{|u(x)|}{\lambda}\Big)\,dx \leq 1\Big\} < \infty.$$

The Orlicz–Besov spaces generalize the Besov (or fractional Sobolev) spaces; indeed, if $\phi(t) = t^p$ with $p \geq 1$ and $\alpha + n/p > 0$, then $\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ is exactly the homogenous Besov space $\dot{\mathbf{B}}^{\alpha+n/p}_{pp}(\Omega)$ and $\mathbf{B}^{\alpha,\phi}(\Omega)$ is the inhomogenous Besov space $\mathbf{B}^{\alpha+n/p}_{pp}(\Omega)$.

The main purpose of this paper is to establish the following criteria for Orlicz–Besov extension/imbedding when $\alpha \neq 0$, which generalizes the known criteria for Besov extension/imbedding established in [9, 16, 24]. Recall that the case $\alpha = 0$ has already been considered by Liang and Zhou [13].

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Theorem 1.1 *Let $\alpha \in (0,1)$ and let ϕ be a Young function satisfying*

$$\underline{\Lambda}_{\phi}(\alpha) := \sup_{x>0} \int_0^1 \frac{\phi(t^{1-\alpha}x)}{\phi(x)} \frac{dt}{t^{n+1}} < \infty, \tag{1.1}$$

or let $\alpha \in (-n,0)$ and let ϕ be a Young function satisfying (1.1) and

$$\overline{\Lambda}_{\phi}(\alpha) := \sup_{x>0} \int_1^\infty \frac{\phi(t^{-\alpha}x)}{\phi(x)} \frac{dt}{t^{n+1}} < \infty. \tag{1.2}$$

For any domain $\Omega \subset \mathbb{R}^n$, the following statements are equivalent.

(i) *Ω is a global n-regular domain (respectively n-regular domain); that is, there exists a constant $\theta > 0$ such that $|B(x,r) \cap \Omega| \geq \theta r^n$ for all $x \in \Omega$ and $0 < r \leq 2\operatorname{diam}\Omega$ (respectively, $0 < r \leq 1$).*

(ii) *Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-extension domain (respectively, $\mathbf{B}^{\alpha,\phi}$-extension domain).*

(iii) *Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domain (respectively, $\mathbf{B}^{\alpha,\phi}$-imbedding domain).*

Note that a bounded domain Ω is n-regular if and only if it is global n-regular. An unbounded global n-regular Ω must be n-regular, but the converse is not necessarily correct; indeed, the domain $(-1,1) \times \mathbb{R}^{n-1}$ is n-regular, but not global n-regular. Motivated by Besov $\dot{\mathbf{B}}^{\alpha+n/p}_{pp}$- and $\mathbf{B}^{\alpha+n/p}_{pp}$-extension/imbedding domains in [9,16,24], the definitions of $\dot{\mathbf{B}}^{\alpha,\phi}$- and $\mathbf{B}^{\alpha,\phi}$-extension/imbedding domains are as below.

Definition 1.2 (i) For $X = \dot{\mathbf{B}}^{\alpha,\phi}$ or $\mathbf{B}^{\alpha,\phi}$, a domain $\Omega \subset \mathbb{R}^n$ is called an *X-extension domain* if any function $u \in X(\Omega)$ can be extended to be a function $\widetilde{u} \in X(\mathbb{R}^n)$ in a continuous and linear way. In other words, there exists a linear bounded operator $E: X(\Omega) \to X(\mathbb{R}^n)$ such that $Eu|_\Omega = u$ whenever $u \in X(\Omega)$.

(ii) A domain $\Omega \subset \mathbb{R}^n$ is called a *$\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domain* (respectively, $\mathbf{B}^{\alpha,\phi}$-*imbedding domain*) if the following hold.

(a) When $\alpha \in (-n,0)$, there exists a constant $C \geq 1$ such that $\inf_{c\in\mathbb{R}} \|u-c\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}$ (respectively, $n/|\alpha|(\Omega) \leq C\|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)}$) for any $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ (respectively, $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$).

(b) When $\alpha \in (0,1)$, there exists a constant $C \geq 1$ such that for any $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ (respectively, $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$), we can find $\hat{u} \in C(\Omega)$ satisfying $\hat{u} = u$ almost surely (a.s.) and $|\hat{u}(x) - \hat{u}(y)| \leq C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}|x-y|^\alpha$ (respectively, $|\hat{u}(x) - \hat{u}(y)| \leq C\|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)}|x-y|^\alpha$)) for all $x, y \in \Omega$.

Below we give some reasons for assumptions (1.1) and (1.2) on α and ϕ, and also for the restriction on the range of α.

Remark 1.3 (i) When $\alpha \in (0,1)$, (1.2) holds trivially; indeed, since ϕ is increasing, we always have $\phi(t^{-\alpha}x) \leq \phi(x)$ for all $t \geq 1$ and $x > 0$, and hence $\int_1^\infty t^{-(n+1)}\,dt < \infty$ implies (1.2).

When $\alpha \in (-n,0)$, assumptions (1.1) and (1.2) guarantee that both $\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ and $\mathbf{B}^{\alpha,\phi}(\Omega)$ contain smooth functions with compact supports, and hence are nontrivial; see Lemma 2.2 and [19, Lemma 2.3]. Moreover, (1.1) and (1.2) are optimal in the following sense.

Assumption (1.1) is optimal in the sense that both $\dot{\mathbf{B}}^{n/p+\alpha}_{p,p}(\Omega)$ and $\mathbf{B}^{n/p+\alpha}_{p,p}(\Omega)$ (that is, $\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ and $\mathbf{B}^{\alpha,\phi}(\Omega)$ with $\phi(t) = t^p$) are nontrivial when $n/p + \alpha \in (0,1)$, but are trivial when $n/p + \alpha \geq 1$; see [4]. Since $\phi(t) = t^p$ satisfies (1.1) if and only if $n/p + \alpha < 1$, we know that (1.1) is optimal for guaranteeing their non-triviality. When Ω is an unbounded domain, (1.2) is optimal to guarantee $C^1_c(\Omega) \subset \mathbf{B}^{\alpha,\phi}(\Omega)$ and hence $C^1_c(\Omega) \subset \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$; see [19, Remark 2.4].

(ii) Note that (1.1) implies $\alpha < 1$ and (1.2) implies $\alpha > -n$. Indeed, if $\alpha \geq 1$, then we have $\underline{\Lambda}_\phi(\alpha) \geq \int_0^1 t^{-n-1}\,dt = \infty$, and hence (i) fails. If $\alpha \leq -n$, by the convexity of ϕ and $\phi > 0$ in $(0,\infty)$, there exists a constant $c > 0$ such that $\phi(t) - \phi(1) \geq c(t-1)$ for all $t \geq 0$. Thus

$$\overline{\Lambda}_\phi(\alpha) \geq \sup_{x>0} \int_1^\infty \frac{\phi(1) + c(t^{-\alpha}x - 1)}{\phi(x)}\,\frac{dt}{t^{n+1}} = \sup_{x>0}\Big[\frac{\phi(1) - c}{n\phi(x)} + \frac{cx}{\phi(x)}\int_1^\infty \frac{dt}{t^{n+\alpha+1}}\Big] = \infty,$$

and hence (ii) fails.

Now we turn to the proof of Theorem 1.1. Recall that when Ω is a bounded domain or $\Omega = \mathbb{R}^n$, the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1.1 was already proved in [19] by a direct approach (without using extension). But when $\Omega \subsetneq \mathbb{R}^n$ is an unbounded domain, the direct approach in [19] does not work. This is also the partial motivation for us to study the Orlicz–Besov extension.

To prove Theorem 1.1, we first recall the Whitney cubes and their reflected quasi-cubes considered by Shvartsman [16]. By using these quasi-cubes, we get an extension operator E. In Section 4 we prove when Ω is a global n-regular domain, the extension operator E is bounded from $\dot{\mathbf{B}}^{\alpha,\phi}(\Omega) \to \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$, and hence Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-extension domain; see Theorem 4.1. A similar result for $\mathbf{B}^{\alpha,\phi}$ was also proved; see Theorem 4.2. This proves (i) $\Rightarrow$ (ii) in Theorem 1.1. In Section 5 the Orlicz–Besov extension domains are proved to be Orlicz–Besov imbedding domains, that is, (ii) $\Rightarrow$ (iii) in Theorem 1.1; see Theorems 5.1 and 5.3. In Section 6, by the estimate of Orlicz–Besov norms of some test functions given in Section 2 and using some ideas from [5,6], we show that Orlicz–Besov imbedding domains are (global) n-regular domains, that is, (iii) $\Rightarrow$ (i) in Theorem 1.1; see Theorems 6.1 and 6.2.

Finally, we use the following conventions and notations in this paper. Throughout the paper, C will be a positive constant depending only on n, α, ϕ, and Ω, whose value can change from line to line. Its value can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by $\gamma_0 = \gamma_0(n)$. We write $A \lesssim B$ (resp., $A \gtrsim B$) if there exist a constant $C > 0$ such that $A \leq CB$ (resp., $A \geq CB$). The notations f_B or $\fint_B f(x)\,dx$ denote the average value of f on the set B with $0 < |B| < \infty$, *i.e.*, $\frac{1}{|B|}\int_B f(x)\,dx$.

2 Some Basic Properties of Young Functions and Orlicz–Besov Spaces

Note that if ϕ is a Young function, then ϕ is increasing, and $\phi(t) \to \infty$ as $t \to \infty$. Moreover, we have the following properties for Young functions.

Lemma 2.1 *Let ϕ be a Young function.*

(i) *If $\alpha \in (-n,1)$ and ϕ satisfies (1.1), then $\lim_{s\to\infty}\phi(xs^{1-\alpha})s^{-n} = 0$ for all $x > 0$.*
(ii) *If $\alpha \in (-n,0)$ and ϕ satisfies (1.2), then $\lim_{s\to\infty}\phi(xs^{-\alpha})s^{-n} = 0$ for all $x > 0$ and*

$$\phi(xs^{-\alpha}) \le 2^{3n}\overline{\Lambda}_\phi(\alpha)\phi(x)s^n \quad \text{for all } s \ge 1, x > 0. \tag{2.1}$$

Proof When $\alpha \in (-n,0]$, Lemma 2.1 was established in [13, 19]. When $\alpha \in (0,1)$, using a similar argument as in [19], we get (i). ∎

Lemma 2.2 *Let α and ϕ be as in Theorem 1.1. Then*

$$C_c^1(\Omega) \subset \mathbf{B}^{\alpha,\phi}(\Omega) \subset \dot{\mathbf{B}}^{\alpha,\phi}(\Omega) \subset L^1(\Omega).$$

Proof If $\alpha \le 0$, this was established in [13, 19]. If $\alpha \in (0,1)$, this can be proved similarly. We omit the details. ∎

The following ϕ-Poincaré inequality holds.

Lemma 2.3 *Let α and ϕ be as in Theorem 1.1. Then there exists a constant $C \ge 1$ such that $⨍_B |u(x) - u_B|\, dx \le Cr^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(B)}$, where $B := B(z,r) \subset \mathbb{R}^n$ and $u \in \dot{\mathbf{B}}^{\alpha,\phi}(B)$.*

Proof Let $\gamma = |\alpha|$. Then we have

$$\begin{aligned} 0 < \int_B\int_B r^{\gamma-2n}\frac{dx\,dy}{|x-y|^\gamma} &\le \int_B\int_{B(x,2r)} r^{\gamma-2n}\frac{dx\,dy}{|x-y|^\gamma} \\ &\le \omega_n \int_B\int_0^{2r} r^{\gamma-2n}s^{n-\gamma-1}\,dsdx < \infty. \end{aligned}$$

Set $K_\gamma := \int_B\int_B r^{\gamma-2n}\frac{dxdy}{|x-y|^\gamma}$. Notice that

$$r^{-\alpha}⨍_B |u(x) - u_B|\,dx \le \frac{2^{\alpha+\gamma}}{\omega_n^2}r^{\gamma-2n}\int_B\int_B \frac{|u(x)-u(y)|}{|x-y|^\alpha}\frac{dxdy}{|x-y|^\gamma}.$$

Let $u \in \dot{\mathbf{B}}^{\alpha,\phi}(B)$. For $\lambda > \frac{2^{\alpha+\gamma}}{\omega_n^2}K_\gamma\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(B)}$, applying Jensen's inequality, we have

$$\begin{aligned} \phi\left(\frac{r^{-\alpha}⨍_B|u(x)-u_B|\,dx}{\lambda}\right) &\le r^{\gamma-2n}\int_B\int_B \phi\left(\frac{\frac{2^{\alpha+\gamma}}{\omega_n^2}K_\gamma|u(x)-u(y)|}{\lambda|x-y|^\alpha}\right)\frac{dxdy}{K_\gamma|x-y|^\gamma} \\ &\le \frac{2^{2n-\gamma}}{K_\gamma}\int_B\int_B \phi\left(\frac{\frac{2^{\alpha+\gamma}}{\omega_n^2}K_\gamma|u(x)-u(y)|}{\lambda|x-y|^\alpha}\right)\frac{dydx}{|x-y|^{2n}} \\ &\le \frac{2^{2n-\gamma}}{K_\gamma}. \end{aligned}$$

Thus, letting $\lambda \to \frac{2^{\alpha+\gamma}}{\omega_n^2}K_\gamma\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(B)}$, one has

$$⨍_B |u(x)-u_B|\,dx \le \phi^{-1}\left(\frac{2^{2n-\gamma}}{K_\gamma}\right)\frac{2^{\alpha+\gamma}}{\omega_n^2}K_\gamma r^\alpha\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(B)}$$

as desired. This completes the proof of Lemma 2.3. ∎

For $x \in \Omega$ and $0 < r < t < \operatorname{diam} \Omega$, let $B_\Omega(x,t) := \Omega \cap B(x,t)$ and $B_\Omega(x,r) := \Omega \cap B(x,r)$, and set

$$u_{x,r,t}(z) := \begin{cases} 1 & z \in B_\Omega(x,r), \\ \dfrac{t-|x-z|}{t-r} & z \in B_\Omega(x,t) \setminus B_\Omega(x,r), \\ 0 & z \in \Omega \setminus B_\Omega(x,t). \end{cases} \tag{2.2}$$

Then we have the following estimates.

Lemma 2.4 *Let α and ϕ be as in Theorem 1.1. Then there exists a constant C such that for any domain $\Omega \subset \mathbb{R}^n$, $u_{x,r,t} \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ with*

$$\|u_{x,r,t}\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} \le C(t-r)^{-\alpha}\Big[\phi^{-1}\Big(\frac{(t-r)^n}{|B_\Omega(x,t)|}\Big)\Big]^{-1}.$$

Proof If $\alpha \le 0$, this was established in [13, 19]. If $\alpha \in (0,1)$, this can be proved similarly. We omit the details. ∎

3 Whitney Cubes and Reflected Quasi-cubes for (Global) n-regular Domains

Let $\Omega \subset \mathbb{R}^n$ be a domain, and write $U := \mathbb{R}^n \setminus \overline{\Omega}$. Then U admits a Whitney decomposition [16, 20].

Lemma 3.1 *There exists a family $\mathscr{W} = \{Q_i\}_{i\in\mathbb{N}}$ of countable closed cubes satisfying the following.*

- (i) *$U = \bigcup_{i\in\mathbb{N}} Q_i$, and $Q_k^\circ \cap Q_i^\circ = \varnothing$, for all $i,k \in \mathbb{N}$ with $i \ne k$.*
- (ii) *For every $Q \in \mathscr{W}$, $l_Q \le \operatorname{dist}(Q,\partial\Omega) \le 4\sqrt{n}\,l_Q$.*
- (iii) *If $K, Q \in \mathscr{W}$, then $\frac{1}{4}l_Q \le l_K \le 4 l_Q$, whenever $Q \cap K \ne \varnothing$.*

Associated with $\mathscr{W}$, there is a partition of unity [20].

Lemma 3.2 *There exists a family $\{\varphi_Q : Q \in \mathscr{W}\}$ of functions such that the following hold.*

- (i) *For each $Q \in \mathscr{W}$, $0 \le \varphi_Q \in C_0^\infty(\frac{17}{16}Q)$.*
- (ii) *For each $Q \in \mathscr{W}$, $|\nabla \varphi_Q| \le L/l_Q$ for some constant $L > 0$.*
- (iii) *$\sum_{Q\in\mathscr{W}} \varphi_Q = \chi_U$.*

By Lemmas 3.1 and 3.2, we have the following properties of $\mathscr{W}$ and partition of unity.

Lemma 3.3 *For any $Q \in \mathscr{W}$, let $N(Q) = \{P \in \mathscr{W}, P \cap Q \ne \varnothing\}$. Then we have the following.*

- (i) *$P \in N(Q) \Leftrightarrow Q \in N(P) \Leftrightarrow \frac{9}{8}Q \cap P \ne \varnothing \Leftrightarrow \frac{9}{8}P \cap Q \ne \varnothing \Leftrightarrow \frac{9}{8}P \cap \frac{9}{8}Q \ne \varnothing$.*

(ii) *There exists a constant $\gamma_0 := \gamma_0(n)$ such that for any $Q \in \mathscr{W}$, one has $\sharp N(Q) \le \gamma_0$ and*

$$\frac{1}{|Q|}\int_U \varphi_Q(x)\,dx \le \frac{1}{|Q|}\int_U \chi_{\frac{9}{8}Q}(x)\,dx \le \sum_{P\in\mathscr{W}} \frac{1}{|Q|}\int_P \chi_{\frac{9}{8}Q}(x)\,dx \le \sum_{P\in N(Q)} \frac{|P|}{|Q|} \le 4^n\gamma_0. \tag{3.1}$$

Next we recall the reflected quasi-cubes of Whitney cubes when Ω is a global n-regular domain. For $\epsilon > 0$, we set $\mathscr{W}_\epsilon := \left\{Q \in \mathscr{W} : l_Q < \frac{1}{\epsilon}\operatorname{diam}\Omega\right\}$.

Obviously, $\mathscr{W} = \mathscr{W}_\epsilon$, for all $\epsilon > 0$, if $\operatorname{diam}\Omega = \infty$, and $\mathscr{W}_\epsilon \subsetneq W$, for any $\epsilon > 0$, if $\operatorname{diam}\Omega < \infty$. Write

$$\mathcal{A}_Q^\varepsilon := \left\{P \in \mathscr{W}_\varepsilon : Q(x_P^*, \epsilon l_P) \cap Q(x_Q^*, \epsilon l_Q) \ne \emptyset, l_P \le \epsilon l_Q\right\},$$

where $x_Q^* \in \Omega$ is a point nearest to Q on Ω. Let

$$Q^{*,\epsilon} := [Q(x_Q^*, \epsilon l_Q) \cap \Omega] \setminus (\cup\{Q(x_P^*, \epsilon l_P) : P \in \mathcal{A}_Q^\epsilon\}). \tag{3.2}$$

We have the following result, which is essentially given by Shvartsman [16]; see also [13].

Lemma 3.4 *If Ω is a global n-regular domain, then there exists $\epsilon_0 \in (0,1)$ and γ_1, $\gamma_2 \in (1,\infty)$ depending only on θ and n such that the following hold.*

(i) $Q^{*,\epsilon_0} \subset (10\sqrt{n}Q) \cap \Omega$, *for all* $Q \in \mathscr{W}_{\epsilon_0}$.
(ii) $|Q| \le \gamma_1|Q^{*,\epsilon_0}|$, *whenever* $Q \in \mathscr{W}_{\epsilon_0}$.
(iii) $\sum_{Q\in\mathscr{W}_{\epsilon_0}} \chi_{Q^{*,\epsilon_0}} \le \gamma_2\chi_\Omega$.

In the case that Ω is a global n-regular domain, for each $Q \in \mathscr{W}_{\epsilon_0}$, following [16], we call $Q^* := \widetilde{Q}^{*,\epsilon_0}$ as the reflected quasi-cube of Q. If $\operatorname{diam}\Omega < \infty$ additionally, for any $Q \in \mathscr{W} \setminus \mathscr{W}_{\epsilon_0}$, we call $Q^* = \Omega$ as the reflected quasi-cube of Q.

Set $\mathscr{W}_{\epsilon_0}^{(k)} := \left\{Q \in N(P) : P \in \mathscr{W}_{\epsilon_0}^{(k-1)}\right\}$, for $k \ge 1$ and $\mathscr{W}_{\epsilon_0}^{(0)} = \mathscr{W}_{\epsilon_0}$. Let

$$V^{(k)} := \{x \in Q : Q \in \mathscr{W}_{\epsilon_0}^{(k)}\}$$

for $k \ge 0$. If Ω is a bounded (global) n-regular domain, we have the following result.

Lemma 3.5 ([13]) *If Ω is a bounded (global) n-regular domain, then*

$$\sum_{Q\in\mathscr{W}_{\epsilon_0}^{(k)}} \chi_{Q^*} \le [\gamma_2 + (\epsilon_0 + 4^{k+2}\sqrt{n})^n]\chi_\Omega, \tag{3.3}$$

$$|Q| \le (\gamma_1 + \theta^{-1}4^{kn}\epsilon_0^{-n})|Q^*|, \quad \textit{for all } Q \in \mathscr{W}_{\epsilon_0}^{(k)}. \tag{3.4}$$

Proof Recall that (3.3) was proved in [13, §3]. For every $P \in \mathscr{W}_{\epsilon_0}$, $|P| \le \gamma_1|P^*|$. For $Q \in \mathscr{W}_{\epsilon_0}^{(k)} \setminus W_{\epsilon_0}$, we have $Q^* = \Omega$ and $l_Q \le \frac{4^k}{\epsilon_0}\operatorname{diam}\Omega$. Since Ω is global n-regular, $|Q| \le \frac{4^{kn}}{\epsilon_0^n}(\operatorname{diam}\Omega)^n \le \theta^{-1}4^{kn}\epsilon_0^{-n}|Q^*|$ for all $Q \in \mathscr{W}_{\epsilon_0}^{(k)} \setminus W_{\epsilon_0}$, and hence (3.4) holds. ■

Remark 3.6 If Ω is an unbounded n-regular domain, we define quasi-cubes for Whitney cubes with side-length less than 1. Set $\widetilde{\mathscr{W}} = \{Q \in \mathscr{W} : l_Q \le 1\}$. Given any $\epsilon \in (0,1)$, define the reflected quasi-cube $Q^{*,\epsilon}$ similarly as in (3.2) for any $Q \in \widetilde{\mathscr{W}}$. Following Shvartsman [16, Theorem 2.4], we also get the same estimates about these reflected quasi-cubes as in Lemma 3.4 with $\mathscr{W}_{\epsilon_0}$ replaced by $\widetilde{\mathscr{W}}$.

Finally, we state the following well-known result for (global) n-regular domains, proved in [5, Lemma 9], [17, Lemma 2.1], and [24, §2].

Lemma 3.7 *If $\Omega \subset \mathbb{R}^n$ is a (global) n-regular domain, then* $|\overline{\Omega} \setminus \Omega| = 0$.

4 (Global) n-regular Domains Are Orlicz–Besov Extension Domains

Theorem 4.1 shows that global n-regular domains are $\dot{\mathbf{B}}^{\alpha,\phi}$-extension domains; Theorem 4.2 shows that n-regular domains are $\mathbf{B}^{\alpha,\phi}$-extension domains.

Theorem 4.1 *Let α and ϕ be as in Theorem 1.1. If Ω is a global n-regular domain, then Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-extension domain.*

Proof Recall the reflected quasi-cube Q^* of a cube $Q \in \mathscr{W} = \mathscr{W}_{\epsilon_0}$ given as in Section 3 and $U = \mathbb{R}^n \setminus \overline{\Omega}$. By Lemma 3.7, we can assume, without loss of generality, that Ω is closed. Define the extension operator E by

$$Eu(x) \equiv \begin{cases} u(x) & x \in \Omega, \\ \sum_{Q \in \mathscr{W}} \varphi_Q(x) u_{Q^*} & x \in U, \end{cases} \tag{4.1}$$

for any $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$. Obviously, E is linear and $Eu = u$ in Ω. It is sufficient to show the boundedness of $E\colon \dot{\mathbf{B}}^{\alpha,\phi}(\Omega) \to \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$. This is further reduced to finding a constant $M > 0$ depending only on α, ϕ, n, and θ such that

$$H(\lambda) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi\left(\frac{|Eu(x) - Eu(y)|}{\lambda |x-y|^\alpha} \right) \frac{dydx}{|x-y|^{2n}} \le 1,$$

whenever $\lambda > M$ and $\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} = 1$.

Let $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ and assume $\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} = 1$. Write

$$\begin{aligned} H(\lambda) &= \left[\int_\Omega \int_\Omega + 2 \int_U \int_\Omega + \int_U \int_U \right] \phi\left(\frac{|Eu(x) - Eu(y)|}{\lambda |x-y|^\alpha} \right) \frac{dydx}{|x-y|^{2n}} \\ &=: H_1(\lambda) + 2H_2(\lambda) + H_3(\lambda). \end{aligned}$$

We claim that there exist constants $L_i \ge 1$ such that

$$H_i(\lambda) \lesssim H_1(\lambda / L_i) \quad \text{for } i = 2, 3. \tag{4.2}$$

Assume that (4.2) holds for the moment. Denote by $M_i > 1$ the constant in (4.2) for $i = 2, 3$. Letting $M = 8(L_2 M_2 + L_3 M_3)$, by the monotonicity and convexity of ϕ and $\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} = 1$, for $\lambda > M$ we have

$$H_i(\lambda) \le M_i H_1(\lambda / L_i) \le M_i H_1(8 M_i) \le H_1(8) \quad \text{for } i = 2, 3.$$

The convexity of ϕ then yields $H(\lambda) \le 4 H_1(8) \le H_1(2) \le 1$ as desired.

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To prove claim (4.2), we consider cases $\operatorname{diam}\Omega = \infty$ and $\operatorname{diam}\Omega < \infty$ separately.

Case 1: $\operatorname{diam}\Omega = \infty$. To bound $H_2(\lambda)$, by Lemma 3.2(iii), one has

$$Eu(x) - u(y) = \sum_{Q\in\mathscr{W}} \varphi_Q(x)[u_{Q^*} - u(y)] \quad \text{for all } x \in U, y \in \Omega.$$

Using Jensen's inequality twice, we have

$$\phi\left(\frac{|Eu(x) - u(y)|}{\lambda|x-y|^\alpha}\right) \le \sum_{Q\in\mathscr{W}} \varphi_Q(x) \fint_{Q^*} \phi\left(\frac{|u(z) - u(y)|}{\lambda|x-y|^\alpha}\right) dz. \tag{4.3}$$

If $\varphi_Q(x) \ne 0$, by Lemma 3.2(i), $x \in \frac{17}{16}Q$. For any $z \in Q^*$, by $Q^* \subset 10\sqrt{n}Q$ given in Lemma 3.4(i), we have $|x-z| \le 10nl_Q$. For $x \in \frac{17}{16}Q$, by Lemma 3.3(i), there is $P \in N(Q)$ such that $x \in P$. So by Lemma 3.1(ii)–(iii), for any $y \in \Omega$ one has

$$|x-y| \ge \operatorname{dist}(x,\Omega) \ge l_P \ge \frac{1}{4} l_Q.$$

Hence

$$|y-z| \le |y-x| + |x-z| \le |y-x| + 10nl_Q \le 41n|x-y|. \tag{4.4}$$

When $\alpha \in (-n, 0)$, set $s = \frac{|y-z|}{41n|x-y|}$, which is larger than 1 by (4.4). Then by (2.1),

$$\begin{aligned}\phi\left(\frac{|u(z)-u(y)|}{\lambda|x-y|^\alpha}\right) &\lesssim \phi\left(\frac{|u(z)-u(y)|}{\lambda|y-z|^\alpha/(41n)^\alpha}\right)\frac{|x-y|^n}{|y-z|^n}\\ &\lesssim \phi\left(\frac{|u(z)-u(y)|}{\lambda|y-z|^\alpha/(41n)^\alpha}\right)\frac{|x-y|^{2n}}{|y-z|^{2n}}.\end{aligned}$$

When $\alpha \in (0,1)$, by (4.4) and the the monotonicity of ϕ, one has the same estimates. From these and (4.3) it follows that

$$H_2(\lambda) \lesssim \int_U \sum Q \in \mathscr{W} \varphi_Q(x) \fint_{Q^*} \int_\Omega \phi\left(\frac{|u(z)-u(y)|}{\lambda|y-z|^\alpha/(41n)^\alpha}\right)\frac{dydz}{|y-z|^{2n}}\, dx.$$

Since Lemma 3.4(ii)–(iii) and (3.1) give $|Q| \le \gamma_1|Q^*|$, $\sum_{Q\in\mathscr{W}_{\epsilon_0}} \chi_{Q^*} \le \gamma_2\chi_\Omega$, and $\frac{1}{|Q|}\int_U \varphi_Q(x)\,dx \le 4^n\gamma_0$, one has

$$\begin{aligned}H_2(\lambda) &\lesssim \sum Q \in \mathscr{W}\left[\frac{1}{|Q|}\int_U \varphi_Q(x)\,dx\right]\int_{Q^*}\int_\Omega \phi\left(\frac{|u(z)-u(y)|}{\lambda|y-z|^\alpha/(41n)^\alpha}\right)\frac{dzdy}{|y-z|^{2n}}\\ &\lesssim \int_\Omega\int_\Omega \phi\left(\frac{|u(z)-u(y)|}{\lambda|y-z|^\alpha/(41n)^\alpha}\right)\frac{dzdy}{|y-z|^{2n}} \lesssim H_1(\lambda/(41n)^\alpha)\end{aligned}$$

as desired.

To bound $H_3(\lambda)$, let

$$\begin{aligned}A_1 &:= \left\{(x,y) \in U\times U : |x-y| < \frac{1}{163\sqrt{n}}\max\{\operatorname{dist}(x,\Omega), \operatorname{dist}(y,\Omega)\}\right\},\\ A_2 &:= (U\times U)\setminus A_1.\end{aligned}$$

Write

$$H_3(\lambda) = \left[\iint_{A_1} + \iint_{A_2}\right]\phi\left(\frac{|Eu(x)-Eu(y)|}{\lambda|x-y|^\alpha}\right)\frac{dydx}{|x-y|^{2n}} =: H_{31}(\lambda) + H_{32}(\lambda).$$

Note that it is enough to find constants $L_{3i} \geq 1$ such that $H_{3i}(\lambda) \lesssim H_1(\lambda/L_{3i})$ for $i = 1, 2$. Indeed, if it is true, letting $L_3 = L_{31} + L_{32}$, by the monotonicity of ϕ we have $H_3(\lambda) \lesssim H_1(\lambda/L_3)$.

Write

$$H_{31}(\lambda) = \sum_{P_1\in\mathscr{W}} \sum_{P_2\in\mathscr{W}} \iint_{A_1\cap(P_1\times P_2)} \phi\bigg(\frac{|Eu(x) - Eu(y)|}{\lambda|x-y|^\alpha}\bigg) \frac{dydx}{|x-y|^{2n}}.$$

Given any $(x, y) \in A_1 \cap (P_1 \times P_2)$, observe that

$$(4.5) \qquad |x-y| \leq \frac{1}{32} l_{P_1}, \quad \text{hence } y \in \frac{17}{16} P_1, \quad \text{and then } P_1 \in N(P_2).$$

Indeed, choosing $\bar{x} \in \Omega$ to satisfy $|x - \bar{x}| = \operatorname{dist}(x, \Omega)$, we have

$$\operatorname{dist}(y,\Omega) \leq |y-\bar{x}| \leq |y-x| + |x-\bar{x}| \leq \frac{1}{163\sqrt{n}} \operatorname{dist}(y,\Omega) + \operatorname{dist}(x,\Omega),$$

which implies $\operatorname{dist}(y,\Omega) \leq \frac{163\sqrt{n}}{163\sqrt{n}-1} \operatorname{dist}(x,\Omega)$. Thus, by this and $(x, y) \in A_1$,

$$(4.6) \qquad |x-y| \leq \frac{1}{163\sqrt{n}} \max\big\{\operatorname{dist}(x,\Omega), \operatorname{dist}(y,\Omega)\big\} \leq \frac{1}{163\sqrt{n}-1} \operatorname{dist}(x,\Omega).$$

For $x \in P_1$, Lemma 3.1(ii) gives $\operatorname{dist}(x, \Omega) \leq 5\sqrt{n} l_{P_1}$. From this and (4.6) we conclude $|x - y| \leq \frac{1}{32} l_{P_1}$ as desired.

Next, by Lemma 3.2(iii), for $(x, y) \in A_1 \cap (P_1 \times P_2)$, we rewrite

$$Eu(x) - Eu(y) = \sum_{Q\in\mathscr{W}} \big[\varphi_Q(x) - \varphi_Q(y)\big]\big[u_{Q^*} - u_{P_2^*}\big].$$

Moreover, we claim that

$$(4.7) \qquad \text{if } \varphi_Q(x) + \varphi_Q(y) \neq 0, \text{ then } Q \in N(P_2).$$

To see this, if $\varphi_Q(y) \neq 0$, by Lemma 3.2(i), we have $y \in \frac{17}{16}Q$, and hence by $y \in P_2$ and Lemma 3.3(i), $Q \in N(P_2)$ as desired.

Assume that $\varphi_Q(y) = 0$ and $\varphi_Q(x) \neq 0$. By Lemma 3.2(i), $x \in \frac{17}{16}Q$. Lemma 3.1(ii) implies that

$$\operatorname{dist}(x,\Omega) \leq \operatorname{dist}(x,Q) + \max_{a\in Q} \operatorname{dist}(a,\Omega) \leq \frac{1}{16}\sqrt{n} l_Q + 5\sqrt{n} l_Q \leq \frac{81}{16}\sqrt{n} l_Q.$$

This, together with (4.6), gives $|x - y| \leq \frac{1}{32} l_Q$. From this and $x \in \frac{17}{16}Q$, it follows that $y \in \frac{9}{8}Q$. Again by Lemma 3.3(i) and $y \in P_2$, we have $Q \in N(P_2)$ as desired.

By (4.7), Lemma 3.2(ii) and Lemma 3.1(iii), one gets

$$|Eu(x)-Eu(y)| \leq \sum_{Q\in N(P_2)} |\varphi_Q(x)-\varphi_Q(y)|\,|u_{Q^*}-u_{P_2^*}| \lesssim \sum_{Q\in N(P_2)} 4L\frac{|x-y|}{l_{P_2}} |u_{Q^*}-u_{P_2^*}|.$$

By (4.5) and Lemma 3.1(iii), one has $|x - y| \leq \frac{1}{8} l_{P_2} < l_{P_2}$. Note that $\sum_{Q\in N(P_2)} 1 \leq \gamma_0$. Thus, by Jensen's inequality,

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$$I(A_1 \cap (P_1 \times P_2)) := \iint_{A_1\cap(P_1\times P_2)} \phi\left(\frac{|Eu(x)-Eu(y)|}{\lambda|x-y|^\alpha}\right)\frac{dydx}{|x-y|^{2n}}$$
$$\le \gamma_0^{-1}\sum_{Q\in N(P_2)}\int_{P_1}\int_{|x-y|\le l_{P_2}}\phi\left(\frac{|x-y|^{1-\alpha}|u_{Q^*}-u_{P_2^*}|}{l_{P_2}\lambda/(4L\gamma_0)}\right)\frac{dydx}{|x-y|^{2n}}.$$

Via a change of a variable, applying (1.1) and $|P_1| \sim |P_2|$ as indicated by (4.5), one gets

$$I(A_1\cap(P_1\times P_2)) \le \gamma_0^{-1}\sum_{Q\in N(P_2)}\int_{P_1}\frac{\omega_n}{l_{P_2}^n}\int_0^1\phi\left(\frac{s^{1-\alpha}|u_{Q^*}-u_{P_2^*}|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right)\frac{ds}{s^{n+1}}dx$$
$$\lesssim \sum_{Q\in N(P_2)}\frac{|P_1|}{l_{P_2}^n}\phi\left(\frac{|u_{Q^*}-u_{P_2^*}|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right) \lesssim \sum_{Q\in N(P_2)}\phi\left(\frac{|u_{Q^*}-u_{P_2^*}|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right).$$

From this, (4.5), and Lemma 3.3(ii), it follows that

$$H_{31}(\lambda) \lesssim \sum_{P_1\in\mathscr{W}}\sum_{P_2\in N(P_1)}\sum_{Q\in N(P_2)}\phi\left(\frac{|u_{Q^*}-u_{P_2^*}|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right) \lesssim \sum_{P_2\in\mathscr{W}}\sum_{Q\in N(P_2)}\phi\left(\frac{|u_{Q^*}-u_{P_2^*}|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right).$$

Using Jensen's inequality twice, one gets

$$H_{31}(\lambda) \lesssim \sum_{P_2\in\mathscr{W}}\sum_{Q\in N(P_2)}\frac{1}{|P_2^*||Q^*|}\iint_{P_2^*\times Q^*}\phi\left(\frac{|u(z)-u(w)|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right)dzdw.$$

Note that $Q \in N(P_2)$ and Lemmas 3.1 and 3.4 give

$$Q \subset 10P_2,\quad P_2 \subset 10Q,\quad Q^* \subset 10\sqrt{n}Q,\quad P_2^* \subset 10\sqrt{n}P_2,$$
$$|P_2| \sim |Q| \sim |P_2^*| \sim |Q^*|. \tag{4.8}$$

Thus for any $(z,w) \in P_2^* \times Q^*$, one has

$$|z-w| \le 100nl_{P_2}. \tag{4.9}$$

If $\alpha \in (-n,0)$, set $s = \frac{l_{P_2}}{|z-w|/100n}$, by (4.9), $s \ge 1$. By (2.1), one then has

$$\phi\left(\frac{|u(z)-u(w)|}{l_{P_2}^\alpha\lambda/(4L\gamma_0)}\right) \lesssim \phi\left(\frac{|u(z)-u(w)|}{|z-w|^\alpha\lambda/[4L\gamma_0(100n)^\alpha]}\right)\frac{l_{P_2}^n}{|z-w|^n}$$
$$\lesssim \phi\left(\frac{|u(z)-u(w)|}{|z-w|^\alpha\lambda/[4L\gamma_0(100n)^\alpha]}\right)\frac{l_{P_2}^{2n}}{|z-w|^{2n}}.$$

If $\alpha \in (0,1)$, by the monotonicity of ϕ and (4.9), the same estimate also holds. By this estimate, (4.8), and Lemma 3.4(iii), it follows that

$$H_{31}(\lambda) \lesssim \sum_{P_2\in\mathscr{W}}\sum_{Q\in N(P_2)}\iint_{P_2^*\times Q^*}\phi\left(\frac{|u(z)-u(w)|}{|z-w|^\alpha\lambda/[4L\gamma_0(100n)^\alpha]}\right)\frac{dzdw}{|z-w|^{2n}}$$
$$\lesssim \int_\Omega\int_\Omega\phi\left(\frac{|u(z)-u(w)|}{|z-w|^\alpha\lambda/[4L\gamma_0(100n)^\alpha]}\right)\frac{dzdw}{|z-w|^{2n}} = H_1(\lambda/[4L\gamma_0(100n)^\alpha])$$

as desired.

Regarding $H_{32}(\lambda)$, by Lemma 3.2(iii), we have

$$\begin{aligned} Eu(x) - Eu(y) &= \sum_{P\in\mathscr{W}} \sum_{Q\in\mathscr{W}} \varphi_Q(x)\varphi_P(y)[u_{Q^*} - u_{P^*}] \\ &= \sum_{P\in\mathscr{W}} \sum_{Q\in\mathscr{W}} \varphi_Q(x)\varphi_P(y) \fint_{Q^*} \fint_{P^*} [u(z) - u(w)]\, dw dz. \end{aligned}$$

Applying Jensen's inequality twice, one gets

$$\phi\Big(\frac{|Eu(x) - Eu(y)|}{\lambda|x-y|^\alpha}\Big) \le \sum_{Q\in\mathscr{W}} \sum_{P\in\mathscr{W}} \varphi_Q(x)\varphi_P(y) \fint_{Q^*} \fint_{P^*} \phi\Big(\frac{|u(z)-u(w)|}{\lambda|x-y|^\alpha}\Big)\, dw dz.$$

For any $(x,y) \in A_2$ with $\varphi_Q(x)\varphi_P(y) \ne 0$, by Lemma 3.2(ii), we have $x \in \frac{17}{16}Q$ and $y \in \frac{17}{16}P$. For any $z \in Q^*$, by $Q^* \subset 10\sqrt{n}Q$ given in Lemma 3.4(i) and by Lemma 3.1(ii)–(iii), one has

$$\begin{aligned} |x - z| &\le 10nl_Q \le \min\big\{40nl_{\widetilde{Q}} : \widetilde{Q} \in N(Q)\big\} \le 40n \min\big\{\operatorname{dist}(\widetilde{Q}, \Omega), \widetilde{Q} \in N(Q)\big\} \\ &\le 40n \operatorname{dist}(x, \Omega). \end{aligned}$$

Similarly, for $w \in P^*$, we have $|y - w| \le 40n \operatorname{dist}(y, \Omega)$. Since $(x,y) \in A_2$, we obtain

$$|z - w| \le |x - z| + |x - y| + |y - w| < 13041n^2|x-y|. \tag{4.10}$$

If $\alpha \in (-n, 0)$, set $s = \frac{|x-y|}{|z-w|/(13041n^2)}$, which is larger than 1 by (4.10). Then by (2.1),

$$\begin{aligned} \phi\left(\frac{|u(z)-u(w)|}{\lambda|x-y|^\alpha}\right) &\lesssim \phi\left(\frac{|u(z)-u(w)|}{\lambda|z-w|^\alpha/(13041n^2)^\alpha}\right) \frac{|x-y|^n}{|z-w|^n} \\ &\le \phi\left(\frac{|u(z)-u(w)|}{\lambda|z-w|^\alpha/(13041n^2)^\alpha}\right) \frac{|x-y|^{2n}}{|z-w|^{2n}}. \end{aligned}$$

If $\alpha \in (0,1)$, by (4.10) and the monotonicity of ϕ, one also has the same estimates. Thus, by Lemma 3.4(ii), (3.1), and Lemma 3.4(iii), we obtain

$$\begin{aligned} H_{32}(\lambda) &\lesssim \sum_{Q\in\mathscr{W}} \sum_{P\in\mathscr{W}} \left[\frac{1}{|Q|}\int_U \varphi_Q(x)\, dx \frac{1}{|P|}\int_U \varphi_P(y)\, dy\right] \\ &\quad\times \int_{Q^*}\int_{P^*} \phi\left(\frac{|u(z)-u(w)|}{\lambda|z-w|^\alpha/(13041n^2)^\alpha}\right) \frac{dwdz}{|z-w|^{2n}} \\ &\lesssim \int_\Omega\int_\Omega \phi\left(\frac{|u(z)-u(w)|}{\lambda|z-w|^\alpha/(13041n^2)^\alpha}\right) \frac{dwdz}{|z-w|^{2n}} = H_1(\lambda/(13041n^2)^\alpha) \end{aligned}$$

as desired.

Case 2: $\operatorname{diam}\Omega < \infty$. Recall the definitions of $V^{(i)}$ and $\mathscr{W}_{\epsilon_0}^{(i)}$ in Section 3. Write

$$\begin{aligned} H_2(\lambda) &= \left[\int_{V^{(2)}}\int_\Omega \phi + \int_{U\setminus V^{(2)}}\int_\Omega\right] \phi\left(\frac{|Eu(x)-u(y)|}{\lambda|x-y|^\alpha}\right) \frac{dydx}{|x-y|^{2n}} \\ &=: \widehat{H}_{21}(\lambda) + \widehat{H}_{22}(\lambda). \end{aligned}$$

It is enough to find constants L_{2i} such that $\widehat{H}_{2i} \lesssim H_1(\lambda/L_{2i})$, $i = 1,2$. Regarding $\widehat{H}_{21}(\lambda)$, observe that $\sum_{Q\in\mathscr{W}} \varphi_Q(x) = \sum_{Q\in\mathscr{W}_{\epsilon_0}^{(3)}} \varphi_Q(x) = 1$, for all $x \in V^{(2)}$. Since (3.4) and (3.3) give

$$|Q| \le (\gamma_1 + \theta^{-1} 4^{3n} \epsilon_0^{-n}) |Q^*|, \quad \text{for all } Q \in \mathscr{W}_{\epsilon_0}^{(3)},$$
$$\sum_{Q \in \mathscr{W}_{\epsilon_0}^{(3)}} \chi_{Q^*} \le (\gamma_2 + (\epsilon_0 + 4^5\sqrt{n})^n) \chi_\Omega,$$

using the same argument as $H_2(\lambda)$ of case $\operatorname{diam}\Omega = \infty$, we will have $\widehat{H}_{21}(\lambda) \lesssim H_1(\lambda/\hat{L}_{21})$. Here we omit the details.

Regarding $H_{22}(\lambda)$, we first note that

$$Eu(x) = u_\Omega \quad \text{for all } x \in U \setminus V^{(2)}. \tag{4.11}$$

Indeed, for $x \in U \setminus V^{(2)}$, there exists $Q \in \mathscr{W} \setminus \mathscr{W}_{\epsilon_0}^{(2)}$ such that $x \in Q$. Thus,

$$N(Q) \cap \mathscr{W}_{\epsilon_0} = \varnothing,$$

and $P^* = \Omega$ for any $P \in N(Q)$. By Lemma 3.2 (i) and (iii), we have

$$\sum_{P \in \mathscr{W}} \varphi_P(x) = \sum_{P \in N(Q)} \varphi_P(x) = 1,$$

and hence $Eu(x) = \sum_{P \in \mathscr{W}} \varphi_P(x) u_{P^*} = \sum_{P \in N(Q)} \varphi_P(x) u_\Omega = u_\Omega$.

Next, for $x \in U \setminus V^{(2)}$ and $y \in \Omega$, one has

$$|x - y| \ge \operatorname{dist}(x, \Omega) \ge l_Q \ge \frac{1}{\epsilon_0} \operatorname{diam}\Omega > \operatorname{diam}\Omega,$$

where $x \in Q \in \mathscr{W} \setminus \mathscr{W}_{\epsilon_0}^{(2)}$ as above. By (4.11), Jensen's inequality, and a change of variables, we have

$$\begin{aligned}
\widehat{H}_{22}(\lambda) &\le \int_{U \setminus V^{(2)}} \int_\Omega ⨍_\Omega \phi\left(\frac{|u(z) - u(y)|}{\lambda |x - y|^\alpha} \right) dz \frac{dydx}{|x - y|^{2n}} \\
&\le \frac{1}{|\Omega|} \int_\Omega \int_\Omega \int_{|x-y| \ge \operatorname{diam}\Omega} \phi\left(\frac{|u(z) - u(y)|}{\lambda |x - y|^\alpha} \right) \frac{dx}{|x - y|^{2n}} dydz \\
&= \frac{\omega_n}{|\Omega|} \int_\Omega \int_\Omega |\operatorname{diam}\Omega|^n \int_1^\infty \phi\left(\frac{|u(z) - u(y)|}{\lambda (\operatorname{diam}\Omega)^\alpha} \right) \frac{dt}{t^{n+1}} dydz.
\end{aligned}$$

Since (1.2) implies

$$\int_1^\infty \phi\left(\frac{|u(z) - u(y)|}{\lambda (\operatorname{diam}\Omega)^\alpha} \right) \frac{dt}{t^{n+1}} \lesssim \phi\left(\frac{|u(z) - u(y)|}{\lambda (\operatorname{diam}\Omega)^\alpha} \right),$$

and $\operatorname{diam}\Omega < \infty$ gives $|\operatorname{diam}\Omega|^n \lesssim |\Omega|$, one has

$$\widehat{H}_{22}(\lambda) \lesssim \int_\Omega \int_\Omega \phi\left(\frac{|u(z) - u(y)|}{\lambda (\operatorname{diam}\Omega)^\alpha} \right) dydz.$$

If $\alpha \in (-n, 0)$, from this and (2.1) with $s = \frac{\operatorname{diam}\Omega}{|x-y|} \ge 1$, it follows that

$$\widehat{H}_{22}(\lambda) \lesssim \int_\Omega \int_\Omega \phi\left(\frac{|u(z) - u(y)|}{\lambda |y - z|^\alpha} \right) \frac{dydz}{|y - z|^{2n}} = H_1(\lambda).$$

If $\alpha \in (0, 1)$, using the monotonicity of ϕ, one also gets the same estimate as desired.

Regarding $H_3(\lambda)$, since

$$U \times U \subset [V^{(3)} \times V^{(3)}] \cup [V^{(2)} \times (U \backslash V^{(3)})] \cup [(U\backslash V^{(3)}) \times V^{(2)}] \\ \cup [(U\backslash V^{(2)}) \times (U\backslash V^{(2)})],$$

by (4.11), one has

$$H_3(\lambda) \le \left[\int_{V^{(3)}}\int_{V^{(3)}} +2\int_{V^{(2)}}\int_{U\backslash V^{(3)}}\right]\phi\left(\frac{|Eu(x)-Eu(y)|}{\lambda|x-y|^{\alpha}}\right)\frac{dydx}{|x-y|^{2n}} \\ =: \widehat{H}_{31}(\lambda) + 2\widehat{H}_{32}(\lambda).$$

It is enough to find $\hat{L}_{3i}$ such that $\widehat{H}_{3i} \lesssim H(\lambda/\hat{L}_{3i})$, $i = 1, 2$.

Regarding $\widehat{H}_{31}(\lambda)$, observe that $\sum_{Q\in\mathscr{W}}\varphi_Q(x) = \sum_{Q\in\mathscr{W}^{(4)}_{\epsilon_0}}\varphi_Q(x) = 1$, for all $x \in V^{(3)}$. By (3.4) and (3.3) with $k = 4$, using an argument similar to $H_3(\lambda)$ in case $\operatorname{diam}\Omega = \infty$, we will have $\widehat{H}_{31}(\lambda) \lesssim H_1(\lambda/L_{31})$. Here we omit the details.

Regarding $\widehat{H}_{32}(\lambda)$, by (4.11) we have $Eu(y) = u_\Omega$ for $y \in U\backslash V^{(3)}$. Note also that $|x-y| \ge \frac{1}{\epsilon_0}\operatorname{diam}\Omega > \operatorname{diam}\Omega$ for all $x \in V^{(2)}$, $y \in U\backslash V^{(3)}$. Then

$$\widehat{H}_{32}(\lambda) \le \int_{V^{(2)}}\int_{|x-y|\ge\operatorname{diam}\Omega}\left(\frac{|Eu(x)-u_\Omega|}{\lambda|x-y|^{\alpha}}\right)\frac{dy}{|x-y|^{2n}}dx \\ = \frac{\omega_n}{(\operatorname{diam}\Omega)^n}\int_{V^{(2)}}\int_1^{\infty}\phi\left(\frac{|Eu(x)-u_\Omega|}{\lambda(\operatorname{diam}\Omega)^{\alpha}}\frac{1}{t^{\alpha}}\right)\frac{dt}{t^{n+1}}dx.$$

By (1.2) and Jessen's inequality,

$$\widehat{H}_{32}(\lambda) \lesssim \int_{V^{(2)}}\phi\left(\frac{|Eu(x)-u_\Omega|}{\lambda(\operatorname{diam}\Omega)^{\alpha}}\right)dx \lesssim |\Omega|^{-1}\int_{V^{(2)}}\int_{\Omega}\phi\left(\frac{|Eu(x)-u(z)|}{\lambda(\operatorname{diam}\Omega)^{\alpha}}\right)dzdx.$$

Note that for any $x \in V^{(2)}$ and $z \in \Omega$, we have

$$|x-z| \le \frac{\operatorname{diam}\Omega}{\epsilon_0}(4^2+4+1)\sqrt{n} + \operatorname{diam}\Omega < \frac{22\sqrt{n}}{\epsilon_0}\operatorname{diam}\Omega.$$

If $\alpha \in (-n, 0)$, set $s = \frac{\operatorname{diam}}{|x-z|/(22\sqrt{n}/\epsilon_0)}$, which is larger than 1 by the above inequality. Then by (2.1),

$$\widehat{H}_{32}(\lambda) \lesssim \int_{V^{(2)}}\int_{\Omega}\phi\left(\frac{|Eu(x)-u(z)|}{\lambda|x-z|^{\alpha}/(22\sqrt{n}/\epsilon_0)^{\alpha}}\right)\frac{dzdx}{|z-x|^{2n}} \lesssim \widehat{H}_{21}(\lambda/[22\sqrt{n}/\epsilon_0]^{\alpha}).$$

If $\alpha \in (0,1)$, by the monotonicity of ϕ, the same estimate also holds. We then conclude that $\widehat{H}_{32}(\lambda) \lesssim H_1(\lambda/\hat{L}_{21}[22\sqrt{n}/\epsilon_0]^{\alpha})$. This completes the proof of Theorem 4.1. ∎

Theorem 4.2 *Let α and ϕ be as in Theorem 1.1. If Ω is an n-regular domain, then Ω is a $\mathbf{B}^{\alpha,\phi}$-extension domain.*

Proof Assume that Ω is n-regular. It suffices to find a linear bounded operator $\widetilde{E}: \mathbf{B}^{\alpha,\phi}(\Omega) \to \mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)$ such that $\widetilde{E}u(x) = u(x)$ in Ω for all $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$. We consider the cases $\operatorname{diam}\Omega < \infty$ and $\operatorname{diam}\Omega = \infty$ separately.

Case 1: $\operatorname{diam}\Omega < \infty$. In this case Ω is global n-regular. Thus the extension operator E in (4.1) is bounded from $\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ to $\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$.

Let $\widetilde{E}u = \eta E u$ for any $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$. Here $\eta \in C_c^1(V^{(3)} \cup \Omega)$ satisfying that $\eta = 1$ on $V^{(2)} \cup \Omega$, $0 \le \eta \le 1$ in $V^{(3)} \setminus V^{(2)}$ and $|\nabla \eta| \le L_\eta$. Note that $\widetilde{E}$ is linear and $\widetilde{E}u = Eu$ in $V^{(2)} \cup \Omega = u$, and hence $\widetilde{E}u = u$ in Ω. By (4.11), $Eu(x) = u_\Omega$ when $x \in U \setminus V^{(2)}$.

To get the boundedness of $\widetilde{E}\colon \mathbf{B}^{\alpha,\phi}(\Omega) \to \mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)$, it suffices to show that if $\|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)} = 1$, then $\|\widetilde{E}u\|_{\mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)} \lesssim 1$. This is further reduced to proving

$$\begin{aligned}\widetilde{J}(\lambda) + \widetilde{H}(\lambda) &:= \int_{\mathbb{R}^n} \phi\left(\frac{|\widetilde{E}u(x)|}{\lambda}\right) dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi\left(\frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|}{\lambda |x-y|^\alpha}\right), \frac{dx\,dy}{|x-y|^{2n}} \\ &\lesssim \int_\Omega \phi\left(\frac{|u(x)|}{\lambda/\widetilde{L}_1}\right) dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi\left(\frac{|Eu(x) - Eu(y)|}{\lambda |x-y|^\alpha/\widetilde{L}_2}\right), \frac{dx\,dy}{|x-y|^{2n}},\end{aligned}$$

where $\widetilde{L}_1, \widetilde{L}_2 \ge 1$ is a constant. Indeed, by this, the convexity of ϕ and the boundedness of $E\colon \dot{\mathbf{B}}^{\alpha,\phi}(\Omega) \to \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$, there exists a constant $\widetilde{M} > 1$ such for all $\lambda > \widetilde{M}$, $\widetilde{J}(\lambda) + \widetilde{H}(\lambda) \le 1$, that is, $\|\widetilde{E}u\|_{\mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)} \le \widetilde{M}$ as desired.

Let $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$ with $\|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)} = 1$. Since

$$\sum_{Q\in W} \varphi_Q(x) = \sum_{Q \in W_{\epsilon_0}^{(4)}} \varphi_Q(x) = 1 \quad \text{for all } x \in V^{(3)},$$

by (3.4), (3.1), and (3.3), one gets

$$\begin{aligned}\widetilde{J}(\lambda) &\le \int_{V^{(4)}\cup\Omega} \phi\left(\frac{|Eu(x)|}{\lambda}\right) dx \\ &= \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx + \int_{V^{(4)}} \phi\left(\frac{|Eu(x)|}{\lambda}\right) dx \\ &\le \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx + \sum_{Q\in \mathscr{W}_{\epsilon_0}^{(5)}} \int_{V^{(4)}} \varphi_Q(x)\, dx \fint_{Q^*} \phi\left(\frac{|u(z)|}{\lambda}\right) dz \\ &\lesssim \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx.\end{aligned} \tag{4.12}$$

Moreover, noting that for all $x, y \in \mathbb{R}^n \setminus [V^{(3)} \cup \Omega] \times \mathbb{R}^n \setminus [V^{(3)} \cup \Omega]$, $\widetilde{E}u(x) = \widetilde{E}u(y) = 0$, one can see that

$$\begin{aligned}\widetilde{H}(\lambda) &\le \int_{V^{(4)}\cup\Omega}\int_{V^{(4)}\cup\Omega} \phi\left(\frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|}{\lambda|x-y|^\alpha}\right) \frac{dx\,dy}{|x-y|^{2n}} \\ &\quad + 2\int_{V^{(3)}\cup\Omega}\int_{\mathbb{R}^n\setminus[V^{(4)}\cup\Omega]} \phi\left(\frac{|\widetilde{E}u(x)|}{\lambda|x-y|^\alpha}\right) \frac{dx\,dy}{|x-y|^{2n}} \\ &=: \widetilde{H}_1(\lambda) + 2\widetilde{H}_2(\lambda).\end{aligned}$$

Note that $|x-y| \ge \epsilon_0^{-1} \operatorname{diam}\Omega > \operatorname{diam}\Omega$, for all $x \in V^{(3)} \cup \Omega$, $y \in \mathbb{R}^n \setminus [V^{(4)} \cup \Omega]$. If $\alpha \in (-n, 0)$, by (1.2) one has

$$\widetilde{H}_2(\lambda) \lesssim \int_{V^{(3)}\cup\Omega} \phi\left(\frac{|Eu(x)|}{\lambda}\right) dx \lesssim \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx.$$

If $\alpha \in (0,1)$, by the monotonicity of ϕ, the same estimate also holds.

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Regarding $\widetilde{H}_1(\lambda)$, note that

$$\begin{aligned}|\widetilde{E}u(x) - \widetilde{E}u(y)| = |\eta(x)Eu(x) - \eta(y)Eu(y)| &\le |\eta(x) - \eta(y)|\,|Eu(x)| \\ &\quad + \eta(y)|Eu(x) - Eu(y)| \\ &\le L_\eta|x-y|\,|Eu(x)| + |Eu(x) - Eu(y)|.\end{aligned}$$

By the convexity of ϕ, we have

$$\begin{aligned}\widetilde{H}_1(\lambda) &\le \frac{1}{2}\int_{V^{(4)}\cup\Omega}\int_{V^{(4)}\cup\Omega}\phi\left(\frac{2L_\eta|Eu(x)||x-y|^{1-\alpha}}{\lambda}\right)\frac{dxdy}{|x-y|^{2n}} \\ &\quad + \frac{1}{2}\int_{V^{(4)}\cup\Omega}\int_{V^{(4)}\cup\Omega}\phi\left(\frac{2|Eu(x) - Eu(y)|}{\lambda|x-y|^{\alpha}}\right)\frac{dxdy}{|x-y|^{2n}} \\ &=: \frac{1}{2}\widetilde{H}_{11}(\lambda) + \frac{1}{2}\widetilde{H}_{12}(\lambda).\end{aligned}$$

Obviously,

$$\widetilde{H}_{12}(\lambda) \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi\left(\frac{2|Eu(x) - Eu(y)|}{\lambda|x-y|^{\alpha}}\right)\frac{dxdy}{|x-y|^{2n}}.$$

Observe that $|x-y| \le 4^5\sqrt{n}\epsilon_0^{-1}\operatorname{diam}\Omega$ for $x, y \in V^{(4)}\cup\Omega$. By (1.1) and (4.12), we have

$$\begin{aligned}\widetilde{H}_{11}(\lambda) &\le \int_{V^{(4)}\cup\Omega}\omega_n\int_0^{4^5\sqrt{n}\epsilon_0^{-1}\operatorname{diam}\Omega}\phi\left(\frac{2L_\eta|Eu(x)|t^{1-\alpha}}{\lambda}\right)\frac{dt}{t^{n+1}}\,dx \\ &\lesssim \int_{V^{(4)}\cup\Omega}\phi\left(\frac{2L_\eta|Eu(x)|}{\lambda/(4^5\sqrt{n}\epsilon_0^{-1}\operatorname{diam}\Omega)^{1-\alpha}}\right)dx \\ &\lesssim \int_\Omega\phi\left(\frac{|u(x)|}{\lambda/(4^5\sqrt{n}\epsilon_0^{-1}\operatorname{diam}\Omega)^{1-\alpha}}\right)dx\end{aligned}$$

as desired.

Case 2: $\operatorname{diam}\Omega = \infty$. For any $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$, define

$$\widehat{E}u := \begin{cases} u(x) & x \in \Omega, \\ \sum_{Q\in\widetilde{\mathscr{W}}}\phi_Q(x)u_{Q^*} & x \in U.\end{cases}$$

Recall that $\widetilde{\mathscr{W}}$ is as in Remark 3.6. Set $\widetilde{U} := \bigcup\{Q \in \mathscr{W}, l_Q \le 4^{-1}\}$. Observe that

$$\sum_{Q\in\widetilde{\mathscr{W}}}\varphi_Q(x) = \sum_{Q\in\mathscr{W}}\varphi_Q(x) = 1, \quad \text{for all } x \in \widetilde{U}. \tag{4.13}$$

Indeed, assume $x \in \widetilde{U}$ and $\varphi_Q(x) \ne 0$ for some $Q \in \mathscr{W}$. Then $x \in P$ for some $P \in \mathscr{W}$ with $l_P \le 1/4$ and $x \in \frac{9}{8}Q$. By Lemma 3.3(i) we know that $Q \in N(P)$ and hence by Lemma 3.1(iii), $l_Q \le 4l_P \le 1$, that is, $Q \in \widetilde{\mathscr{W}}$. Thus (4.13) holds. Since Ω is n-regular, with the aid of (4.13), Remark 3.6, and by arguments similar to those of the case $\operatorname{diam}\Omega = \infty$ in Theorem 4.1, we have

$$\|\widehat{E}u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\widetilde{U}\cup\Omega)} \le C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} \quad \text{for all } u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega). \tag{4.14}$$

Here we leave the details to the reader.

For $0 \le k \le 5$, set $\widetilde{V}^{(k)} := \bigcup\{Q : Q \in \mathscr{W}, l_Q \le 4^{-5+k}\}$. Let $\widetilde{\eta} \in C_c^1(\widetilde{V}^{(3)} \cup \Omega)$ such that $\widetilde{\eta} = 1$ in $\widetilde{V}^{(2)} \cup \Omega$, $0 \le \widetilde{\eta} \le 1$ and $|\nabla \widetilde{\eta}| \le L_{\widetilde{\eta}}$ in $\widetilde{V}^{(3)} \cup \Omega$. For each $u \in \mathbf{B}^{\alpha,\phi}(\Omega)$, define $\widetilde{E}u := \widetilde{\eta}\hat{E}u$. Obviously $\widetilde{E}$ is a linear operator and $\widetilde{E}u = u$ in Ω. Using (4.14) instead of the boundedness of $E: \dot{\mathbf{B}}^{\alpha,\phi}(\Omega) \to \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$ used in the case $\operatorname{diam}\Omega < \infty$ and by an argument very similar to the case $\operatorname{diam}\Omega < \infty$, we have $\|\widetilde{E}u\|_{\mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)} \lesssim \|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)}$ as desired; here we leave the details to the reader. ∎

5 Extension Implies Imbedding

Theorem 5.1 indicates that $\dot{\mathbf{B}}^{\alpha,\phi}$-extension domains are $\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domains; Theorem 5.3 indicates that $\mathbf{B}^{\alpha,\phi}$-extension domains are $\mathbf{B}^{\alpha,\phi}$-imbedding domains.

Theorem 5.1 *Let α and ϕ be as in Theorem 1.1. If Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-extension domain, then it is a $\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domain.*

To prove Theorem 5.1, we need the following lemma.

Lemma 5.2 *Let α and ϕ be as in Theorem 1.1.*

(i) *If $\alpha \in (0,1)$, there exists a constant $C \ge 1$ such that for all $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$, we can find $\hat{u} \in C(\mathbb{R}^n)$ satisfying $\hat{u} = u$ a.s. and $|\hat{u}(x) - \hat{u}(y)| \le C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}|x-y|^\alpha$ for all $x, y \in \mathbb{R}^n$.*

(ii) *If $\alpha \in (-n, 0)$, then there exists a constant $C \ge 1$ such that for any $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$, we have $\inf_{c\in\mathbb{R}} \|u - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \le C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}$.*

Proof Note that Lemma 5.2(ii) has already been proved in [19, Theorem 1.1]. To see (i), let $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$. Note that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ due to Lemma 2.2. Then almost all the points in $\mathbb{R}^n$ are the Lebesgue points of u. For any Lebesgue points $x, y \in \mathbb{R}^n$, we have

$$|u(x) - u(y)| \lesssim |x-y|^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}. \tag{5.1}$$

Indeed, write $|u(x)-u(y)| \le |u(x)-u_{B(x,2|x-y|)}| + |u(y)-u_{B(x,2|x-y|)}|$. By Lemma 2.3, we have

$$\begin{aligned}|u(x) - u_{B(x,2|x-y|)}| &= \sum_{j=0}^{\infty} |u_{B(x,2^{-j}|x-y|)} - u_{B(x,2^{-j+1}|x-y|)}| \\ &\lesssim \sum_{j=0}^{\infty} \fint_{B(x,2^{-j+1}|x-y|)} |u(z) - u_{B(x,2^{-j+1}|x-y|)}|\,dz \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\alpha}|x-y|^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(B(x,2^{-j+1}|x-y|))} \\ &\le C|x-y|^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}.\end{aligned} \tag{5.2}$$

Similarly to (5.2), we also have $|u(y) - u_{B(x,2|x-y|)}| \lesssim |x-y|^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}$. This and (5.2) give (5.1) as desired.

Using (5.1) and Lemma 2.3, similarly to the proof of (5.2), one has that for all $x \in \mathbb{R}^n$, $\{u_{B(x,2^{-j})}\}_{j>0}$ is a Cauchy sequence. Define $\hat{u}(x) := \lim_{j\to\infty} \fint_{B(x,2^{-j})} u(z)\,dz$

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for all $x \in \mathbb{R}^n$. Then $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ implies $\hat{u} = u$ almost everywhere, and (5.1) implies that

$$\begin{aligned} |\hat{u}(x) - \hat{u}(y)| &\le \lim_{i\to\infty} \lim_{j\to\infty} \fint_{B(x,2^{-i})} \fint_{B(y,2^{-j})} |u(z) - u(w)|\, dxdw \\ &\lesssim \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)} \lim_{i\to\infty} \lim_{j\to\infty} \fint_{B(x,2^{-i})} \fint_{B(y,2^{-j})} |z-w|^\alpha\, dzdw \\ &\le |x-y|^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)} \end{aligned}$$

as desired. This completes the proof of Lemma 5.2. ∎

Proof of Theorem 5.1 Let $E: \dot{\mathbf{B}}^{\alpha,\phi}(\Omega) \to \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$ be a bounded linear extension operator. For any $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$, we have $Eu \in \dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)$ with $Eu(x) = u(x), x \in \Omega$ and $\|Eu\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)} \le C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}$.

If $\alpha \in (0,1)$, by Lemma 5.2(i), there exists $\hat{u} \in C(\mathbb{R}^n)$ such that $\hat{u} = Eu$ almost everywhere and $|\hat{u}(x) - \hat{u}(y)| \lesssim |x-y|^\alpha \|Eu\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}$ for all $x, y \in \mathbb{R}^n$. Thus $\hat{u} = u$ almost everywhere in Ω, and $|\hat{u}(x) - \hat{u}(y)| \lesssim |x-y|^\alpha \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}$ for all $x, y \in \Omega$ as desired.

If $\alpha \in (-n,0)$, by Lemma 5.2(ii), we have $\inf_{c\in\mathbb{R}} \|Eu - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \lesssim \|Eu\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}$, which yields $\inf_{c\in\mathbb{R}} \|u - c\|_{L^{n/|\alpha|}(\Omega)} \lesssim \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}$ as desired. ∎

Theorem 5.3 *Let α and ϕ be as in Theorem 1.1. If Ω is a $\mathbf{B}^{\alpha,\phi}$-extension domain, then it is a $\mathbf{B}^{\alpha,\phi}$-imbedding domain.*

By an argument as in the proof of Theorem 5.1, we know that Theorem 5.3 follows from the following lemma.

Lemma 5.4 *Let α and ϕ be as in Theorem 1.1.*

(i) *If $\alpha \in (0,1)$, there exists a constant $C \ge 1$ such that for all $u \in \mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)$, we can find $\hat{u} \in C(\mathbb{R}^n)$ satisfying $\hat{u} = u$ a.s. and*

$$|\hat{u}(x) - \hat{u}(y)| \le C\|u\|_{\mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)} |x-y|^\alpha \quad \text{for all } x, y \in \mathbb{R}^n.$$

(ii) *If $\alpha \in (-n,0)$, then there exists a constant $C \ge 1$ such that for any $u \in \mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)$, we have $\|u\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \le C\|u\|_{\mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)}$.*

Proof When $\alpha \in (0,1)$, since $\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)} \le \|u\|_{\mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)}$, Lemma 5.2(i) implies Lemma 5.4(i). When $\alpha \in (-n,0)$, Lemma 5.2(ii) gives $\inf_{c\in\mathbb{R}} \|u - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \lesssim \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}$ and hence $\|u - c_0\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \lesssim \|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\mathbb{R}^n)}$ for some c_0. If $c_0 = 0$, then (ii) follows. Below, we prove $c_0 = 0$ by contradiction. If $c_0 \ne 0$, we assume without loss of generality that $c_0 > 0$. By

$$\left|\left\{x \in \mathbb{R}^n : |u(x) - c_0| > \frac{c_0}{2}\right\}\right| < \frac{2}{c_0} \|u - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} < \infty,$$

we have

$$\left|\left\{x \in \mathbb{R}^n : \frac{c_0}{2} \le u(x) \le \frac{3c_0}{2}\right\}\right| = \infty. \tag{5.3}$$

Since $u \in \mathbf{B}^{\alpha,\phi}(\mathbb{R}^n)$, then $u \in L^\phi(\mathbb{R}^n)$. Letting $\lambda = \|u\|_{L^\phi(\mathbb{R}^n)} + 1$, by the convexity of ϕ, we have

$$\left|\left\{x \in \mathbb{R}^n : u(x) \ge \frac{c_0}{2}\right\}\right| \le \int_{u(x)\ge \frac{c_0}{2}} \frac{u}{c_0/2}\, dx \le \int_{\mathbb{R}^n} \phi\Big(\frac{u}{\lambda}\Big) \frac{1}{\phi(c_0/(2\lambda))} < \infty,$$

which contradicts (5.3). The completes the proof of Lemma 5.4. ∎

6 Imbedding Implies (Global) n-regular

Theorem 6.1 shows that $\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domains are global n-regular. Theorem 6.2 shows that $\mathbf{B}^{\alpha,\phi}$-imbedding domains are n-regular.

Theorem 6.1 *Let α and ϕ be as in Theorem 1.1. If Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domain, then it is global n-regular.*

Proof If $\alpha \in (-n, 0)$ and Ω is bounded, then Theorem 6.1 was already proved [19, Theorem 1.2]. If $\alpha \in (-n, 0)$ and Ω is unbounded, Theorem 6.1 can be proved similarly to that in [19] for the case when $\operatorname{diam}\Omega < \infty$. Assume that $0 < \alpha < 1$ below. For $z \in \Omega$, $t < \operatorname{diam}\Omega/2$, and $0 < r < t/2$, set u as defined in (2.2). Then by Lemma 2.4, $u \in \dot{\mathbf{B}}^{\alpha,\phi}(\Omega)$ and

$$\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)} \le C(t-r)^{-\alpha}\left[\phi^{-1}\left(\frac{(t-r)^n}{|B_\Omega(z,t)|}\right)\right]^{-1} \le C\frac{2^\alpha}{t^\alpha}\left[\phi^{-1}\left(\frac{t^n}{2^n|B_\Omega(z,t)|}\right)\right]^{-1}.$$

Since Ω is a $\dot{\mathbf{B}}^{\alpha,\phi}$-imbedding domain, there exists $\hat{u}$ such that $\hat{u} = u$ a.s. and

$$|\hat{u}(x) - \hat{u}(y)| \le C\|u\|_{\dot{\mathbf{B}}^{\alpha,\phi}(\Omega)}|x-y|^\alpha.$$

Take $x \in B_\Omega(z,r)$ and $y \in B_\Omega(z,3t/2)\setminus B(z,t)$ satisfying $\hat{u}(x) = u(x)$ and $\hat{u}(y) = u(y)$. Then $|\hat{u}(x) - \hat{u}(y)| = 1$ and $|x-z| \le t/2$, $t \le |y-z| \le 3t/2$, and hence $t \le |x-y| \le 2t$. Therefore,

$$1 \le C2^\alpha t^{-\alpha}\left[\phi^{-1}\left(\frac{t^n}{2^n|B_\Omega(z,t)|}\right)\right]^{-1}|x-y|^\alpha \le C2^{2\alpha}\left[\phi^{-1}\left(\frac{t^n}{2^n|B_\Omega(z,t)|}\right)\right]^{-1},$$

which yields $|B_\Omega(z,t)| \ge [2^n\phi(C2^{2\alpha})]^{-1}t^n$ for all $z \in \Omega$ and $t < \frac{1}{2}\operatorname{diam}\Omega$. If $\operatorname{diam}\Omega = \infty$, this implies that Ω is global n-regular. If $\operatorname{diam}\Omega < \infty$, for $\frac{1}{2}\operatorname{diam}\Omega < t < 2\operatorname{diam}\Omega$, considering $|B_\Omega(z,t)| \ge |B_\Omega(z,t/4)|$, we also know that Ω is global n-regular as desired. This completes the proof of Theorem 6.1. ∎

Theorem 6.2 *Let α and ϕ be as in Theorem 1.1. If Ω is a $\mathbf{B}^{\alpha,\phi}$-imbedding domain, then it is n-regular.*

Proof We claim that there exists a constant C such that for any $z \in \Omega$, $t < \min\{1, \operatorname{diam}\Omega\}$, and $r < t$, $u := u_{z,r,t} \in \mathbf{B}^{\alpha,\phi}(\Omega)$ and

$$\|u\|_{\mathbf{B}^{\alpha,\phi}(\Omega)} \le C(t-r)^{-\alpha}\left[\phi^{-1}\left(\frac{(t-r)^n}{|B_\Omega(z,t)|}\right)\right]^{-1}. \tag{6.1}$$

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Here $u_{x,r,t}$ is defined in (2.2). Assume the claim is true for the moment. Similarly to the proof of Theorem 6.1 when $\alpha \in (0,1)$ and the proof of [19, Theorem 1.2] when $\alpha \in (-n,0)$, we know that Ω is n-regular.

To see (6.1), since $u = u_{z,r,t}$ is supported in $B_\Omega(z,t)$ and $0 \le u \le 1$, we have

$$\int_\Omega \phi\Big(\frac{|u(x)|}{\lambda}\Big)\,dx \le \phi(1/\lambda)|B_\Omega(z,t)| \le 1$$

whenever

$$\lambda > \left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)\right]^{-1}.$$

Hence, $\|u\|_{L^\phi(\Omega)} \le \big[\phi^{-1}\big(\frac{1}{|B_\Omega(z,t)|}\big)\big]^{-1}$. Thus (6.1) follows if there exists a constant $C > 1$ such that $\big[\phi^{-1}\big(\frac{1}{|B_\Omega(z,t)|}\big)\big]^{-1} \le C(t-r)^{-\alpha}\big[\phi^{-1}\big(\frac{(t-r)^n}{|B_\Omega(z,t)|}\big)\big]^{-1}$. Note that this is equivalent to

$$\frac{(t-r)^n}{|B_\Omega(z,t)|} \le \phi\left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)C(t-r)^{-\alpha}\right]. \tag{6.2}$$

But when $\alpha \in (-n,0)$, by (2.1) with $t < 1$ and the convexity of ϕ, we have

$$\begin{aligned}\frac{1}{|B_\Omega(z,t)|} &= \phi\left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)\right]\\ &\le 2^{3n}\overline{\Lambda}_\phi(\alpha)\Big(\frac{1}{t-r}\Big)^n\phi\left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)(t-r)^{-\alpha}\right]\\ &\le \Big(\frac{1}{t-r}\Big)^n\phi\left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)\frac{1+2^{3n}\overline{\Lambda}_\phi(\alpha)}{(t-r)^\alpha}\right],\end{aligned}$$

which gives (6.2). Moreover, when $\alpha \in (0,1)$, by $t < 1$ we have

$$\frac{(t-r)^n}{|B_\Omega(z,t)|} \le \frac{1}{|B_\Omega(z,t)|} = \phi\left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)\right] \le \left[\phi^{-1}\left(\frac{1}{|B_\Omega(z,t)|}\right)(t-r)^{-\alpha}\right],$$

which gives (6.2) as desired. This completes the proof of Theorem 6.2. ∎

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