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POINT PROCESS APPROACH TO MODELING AND ANALYSIS OF GENERAL CASCADING FAILURE MODELS

HYUNJU LEE * ** AND
JI HWAN CHA,* *** *Ewha Womans University*

Abstract

A cascading failure is a failure in a system of interconnected parts in which the failure of a part can trigger the failure of successive parts. Although an initial and introductory approach for probabilistic modeling and analysis of the cascading failures was suggested in the literature, any general framework and fundamental results have yet to be reported. In this paper, applying the point process approach, we suggest a general framework for modeling and analysis of the cascading failures. Furthermore, a new concept of 'information-based residual lifetime' will be defined and discussed.

Keywords: Cascading failure; stochastic intensity; k-out-of-n system; risk measure; information-based residual lifetime

2010 Mathematics Subject Classification: Primary 60K10
Secondary 62P30

1. Introduction

A cascading failure is a failure in a system of interconnected parts in which the breakdown of one element can lead to the subsequent collapse of the others. Cascading failure is very common in many different areas such as power grids, computer networks (such as the Internet), and economic systems; see, e.g. Motter and Lai (2002) and Dobson *et al.* (2007).

A good initial and introductory approach for *'probabilistic' modeling and analysis* of the cascading failures was performed by Swift (2008). While some important specific case studies were given (for the exponential distribution case) in Swift (2008), there was no general result for the system survival function and its failure rate function. In this regard, the aim of this paper is to provide a general framework for modeling the cascading failures and to obtain fundamental results, including explicit formulas for the survival function and the corresponding failure rate. For this, we will take a new approach, which will be based on the point process theory.

The structure of this paper is as follows. In Section 2, a general framework for modeling and analysis of the cascading failures will be suggested, based on the concept of stochastic intensity of the point process. In Section 3, under the framework suggested in the previous section, general results on the lifetime distribution of the system will be obtained. Furthermore, the concept of the 'information-based residual lifetime' will be defined and discussed as well. In Section 4, some numerical results will be provided.

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* Postal address: Department of Statistics, Ewha Womans University, Seoul, 120-750, Korea.
** Email address: hyunjlee@ewhain.net
*** Email address: jhcha@ewha.ac.kr

2. A general stochastic model

For our discussions, we will briefly review the model discussed in Swift (2008). Suppose that there are two components in a 'parallel redundant system'. In the cascading failure model, the lifetimes of the two components behave as if they are independent, until one of the components fails, after which the remaining component suffers increased load for δ time units. The quantity $\delta > 0$ is assumed to be a constant and it is called the critical time (or the threshold time). To be more specific, suppose that the failure rates of the two components are all given by a constant λ. Suppose that the first component failure has occurred at $S_1 = s_1$. Then the failure rate of the surviving component changes at s_1 from λ to $\lambda + \eta$, but at time $s_1 + \delta$ it reverts back to λ, as illustrated in Figure 1.

Suppose now that there are three components with the same constant failure rate λ. In this case, in the system, we have two types of cascading model: two-valued cascading and many-valued cascading. We will first consider the two-valued cascading model. Upon the first failure at time $S_1 = s_1$, the failure rate of the remaining two components jumps to $\lambda + \eta$. If neither of the remaining components fails in the interval $(s_1, s_1 + \delta]$ then their failure rates drop back to λ (see Figure 2). This is also the same for the many-valued cascading model. However, for the two-valued cascading model, if the second failure occurs at $S_2 = s_2$ in the interval $(s_1, s_1 + \delta]$ then the failure of the last remaining component no longer drops back to λ at time $s_1 + \delta$ nor does it further jump; instead it remains at the level of $\lambda + \eta$ until $s_2 + \delta$ (see Figure 3). Thus, in the two-valued cascading model, the maximum level of the failure rate is $\lambda + \eta$ even though there are two overlapping 'effect-lasting periods'.

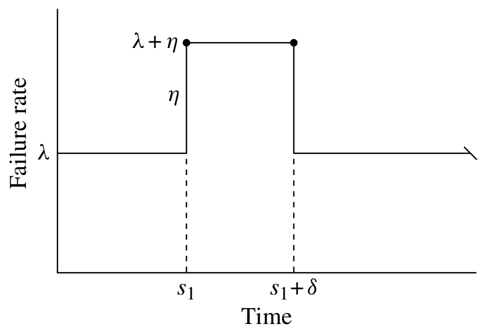

FIGURE 1: Failure rate of the component which fails last.

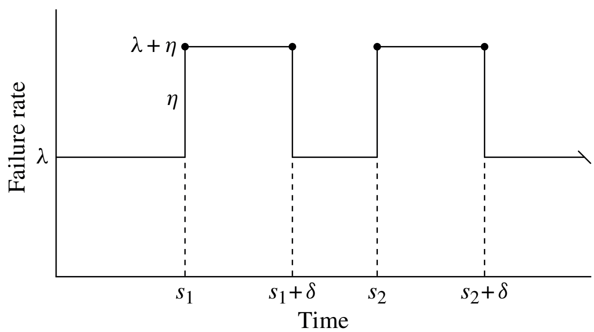

FIGURE 2: Failure rate of the component which fails last for both cascading models.

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On the other hand, for the many-valued cascading model, if the second failure occurs at $S_2 = s_2$ in the interval $(s_1, s_1 + \delta]$ then the failure of the last remaining component further jumps to $\lambda + 2\eta$ and it remains at the level of $\lambda + 2\eta$ until $s_1 + \delta$. Then it drops back to the previous level $\lambda + \eta$ at time $s_1 + \delta$ and it remains at the level of $\lambda + \eta$ until $s_2 + \delta$. Finally, after time $s_2 + \delta$, the effect of the increased load vanishes (see Figure 4). Thus, in the many-valued cascading model, the total jump size in the failure rate is proportional to the total number of overlapping 'effect-lasting periods'.

In Swift (2008), the main discussion was performed for the two-valued cascading model. However, when we consider the motivational examples suggested in Section 1, it might be more plausible to consider the many-valued cascading model. Therefore, in what follows, we will basically assume the many-valued cascading model holds. Nevertheless, it will be shown that our general modeling framework includes both types of cascading model as special cases. Furthermore, the main theoretical results of this paper will be stated for a general complex system with components having any kind of dependence structure (see Theorems 1 and 2).

In order to describe the general framework for the cascading failure models, we need to discuss the concept of stochastic intensity in the point process theory, which is crucial for a proper understanding of our model. Let $\{N(t),\ t \geq 0\}$ be an orderly point process and $\mathcal{H}_{t-} \equiv \{N(u),\ 0 \leq u < t\}$ be the history (internal filtration) of the process in $[0, t)$, i.e. the set of all point events in $[0, t)$. Observe that $\mathcal{H}_{t-}$ can equivalently be defined in terms of $N(t-)$ and the sequential arrival points of the events $0 \leq S_1 \leq S_2 \leq \cdots \leq S_{N(t-)} < t$ in $[0, t)$, where S_i is the time from 0 until the arrival of the ith event in $[0, t)$. A convenient mathematical description of the point processes can be obtained by using the concept of the stochastic intensity (the intensity process) λ_t, $t \geq 0$; see Aven and Jensen (1999), (2000).

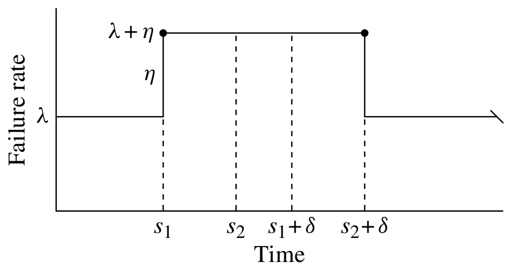

FIGURE 3: Failure rate of the component which fails last for two-valued cascading model.

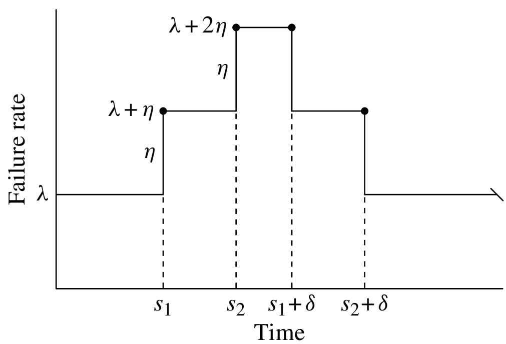

FIGURE 4: Failure rate of the component which fails last for the many-valued cascading model.

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As discussed in Cha and Finkelstein (2011) and Finkelstein and Cha (2013), the stochastic intensity λ_t of an orderly point process $\{N(t),\ t \geq 0\}$ is defined as the following limit:

$$\lambda_t \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}(N(t, t+\Delta t) = 1 \mid \mathcal{H}_{t-})}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{E}[N(t, t+\Delta t) \mid \mathcal{H}_{t-}]}{\Delta t},$$

where $N(t_1, t_2)$, $t_1 < t_2$, represents the number of events in $[t_1, t_2)$. Then the above stochastic intensity has the following heuristic interpretation: $\lambda_t \, \mathrm{d}t = \mathbb{E}[\mathrm{d}N(t) \mid \mathcal{H}_{t-}]$, which is very similar to the ordinary failure rate or hazard rate of a random variable; see Aven and Jensen (1999). In the following discussions, for convenience, the stochastic intensity λ_t, when $\mathcal{H}_{t-} = (N(t-) = m, S_1 = s_1, S_2 = s_2, \ldots, S_m = s_m)$, will be denoted by $\lambda_m(t \mid s_1, s_2, \ldots, s_m)$, $m = 0, 1, 2, \ldots$, and, when $m = 0$, the notation $\lambda_m(t \mid s_1, s_2, \ldots, s_m)$ will be written as $\lambda_0(t)$.

Now we are ready to discuss our general cascading failure model. While the parallel redundant systems are of main interest in the study of cascading failure model as in Swift (2008), we will consider k-out-of-n system, which is more general. A system that is functioning if at least k of the n components are functioning is called a k-out-of-n system. We assume that the n components in the system have the same general failure rate $\lambda(t)$. For our convenient description of the model, first, we will consider the ordinary k-out-of-n system 'without cascading effect'. Note that the stochastic failure model of a k-out-of-n system can be defined via a point process point of view. The system operates at time $t = 0$. Let $N(t)$ be the total number of component failures observed in $(0, t]$. Then the stochastic intensity which corresponds to the point process $\{N(t),\ t \geq 0\}$ is defined as

$$\lambda_m(t \mid s_1, s_2, \ldots, s_m) = (n - m)\lambda(t), \qquad m = 0, 1, 2, \ldots, n-k. \tag{1}$$

Note that in order to describe the stochastic failure model, it is sufficient to consider

$$\lambda_m(t \mid s_1, s_2, \ldots, s_m), \qquad m = 0, 1, 2, \ldots, n-k,$$

as the observation of the process that will be stopped at the $(n - k + 1)$th failure. Clearly, the stochastic intensities defined in (1) represent that there are no interactional effects among the components. Now we consider our cascading failure model. Then we need to modify the stochastic intensities defined in (1) in order to employ the dependence effects that exist among the components. In this case, we still have $\lambda_0(t) = n\lambda(t)$. When $\mathcal{H}_{t-} = (N(t-) = 1, S_1 = s_1)$, it is assumed that the stochastic intensity is given by $\lambda_1(t \mid s_1) = (n-1)(\gamma\alpha(t-s_1, \delta)+1)\lambda(t)$, where $\gamma > 0$ is a fixed constant and $\alpha(u, \delta)$ is defined by

$$\alpha(u, \delta) = \begin{cases} 0, & u > \delta, \\ \psi(u), & u \leq \delta, \end{cases}$$

and $\psi(u)$ is a nonincreasing function with $\psi(0) = 1$. Specifically, suppose that $\lambda(t) = \lambda$ for all t, and $\psi(u) = 1$ for all $u \geq 0$. Then, in this case, the stochastic intensity is given by

$$\lambda_1(t \mid s_1) = \begin{cases} (n-1)\lambda, & t > s_1 + \delta, \\ (n-1)(\gamma+1)\lambda, & s_1 < t \leq s_1 + \delta, \end{cases} \tag{2}$$

and, when $n = 2$, $k = 2$ (parallel system), and $\eta = \gamma\lambda$, the model in (2) represents the failure rate function in Figure 1 for $t > s_1$. Note that the nonincreasing function $\psi(u)$ is employed to model the case when the effect of increased load fades away gradually in time.

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Now we consider the stochastic intensity when $\mathcal{H}_{t-} = (N(t-) = 2, S_1 = s_1, S_2 = s_2)$. In this case, it is assume that

$$\lambda_2(t \mid s_1, s_2) = (n-2)\left(\gamma \sum_{i=1}^{2} \alpha(t - s_i, \delta) + 1\right)\lambda(t). \tag{3}$$

Then, when $n = 3$, $k = 3$ (parallel system), $\lambda(t) = \lambda$ for all t, $\psi(u) = 1$ for all $u \geq 0$, and $\eta = \gamma\lambda$, it is easy to see that the model in (3) represents the failure rate function in Figure 2 or Figure 4 for $t > s_2$, depending on whether $s_2 > s_1 + \delta$ or $s_1 < s_2 \leq s_1 + \delta$. Note that when $n = 3$, $k = 3$, $\lambda(t) = \lambda$ for all t, $\psi(u) = 1$ for all $u \geq 0$, and $\eta = \gamma\lambda$, the failure rate function of the component which fails last in Figure 2 or 4 'for the whole interval' can be described by

$$\tfrac{1}{3}\lambda_0(t)\,\mathbf{1}_{\{t<s_1\}} + \tfrac{1}{2}\lambda_1(t \mid s_1)\,\mathbf{1}_{\{(s_1 \leq t < s_2\}} + \lambda_2(t \mid s_1, s_2)\,\mathbf{1}_{\{t \geq s_2\}}. \tag{4}$$

The reason why we divide by 3 and 2 in the first and second terms in (4) is that there are three surviving components in $(0, s_1)$, and two surviving components in $[s_1, s_2)$, whereas there is only one component in $[s_2, \infty)$. Therefore, the cascading failure model under consideration corresponds to the many-valued cascading model.

As we are considering a general k-out-of-n system, in order to complete the stochastic description of the cascading failure model, we have to specify the stochastic intensity when $\mathcal{H}_{t-} = (N(t-) = j, S_1 = s_1, S_2 = s_2, \ldots, S_j = s_j)$, $j = 1, 2, \ldots, n-k$. Extending the above arguments, we can now generally specify it as

$$\lambda_j(t \mid s_1, s_2, \ldots, s_j) = \begin{cases} \lambda_0(t) = n\lambda(t), & j = 0 \\ (n-j)\left(\gamma \displaystyle\sum_{i=1}^{j} \alpha(t - s_i, \delta) + 1\right)\lambda(t), & j = 1, 2, \ldots, n-k. \end{cases} \tag{5}$$

Note that, when $\lambda(t) = \lambda$ for all t, $\psi(u) = 1$ for all $u \geq 0$, and $\eta = \gamma\lambda$, if we slightly modify the stochastic intensity in (5) to

$$\lambda_j(t \mid s_1, s_2, \ldots, s_j) = \begin{cases} \lambda_0(t) = n\lambda(t), & j = 0 \\ (n-j)\left(\gamma \displaystyle\coprod_{i=1}^{j} \alpha(t - s_i, \delta) + 1\right)\lambda(t), & j = 1, 2, \ldots, n-k, \end{cases} \tag{6}$$

where $\coprod_{i=1}^{j} x_i \equiv \max\{x_1, x_2, \ldots, x_j\} = 1 - \prod_{i=1}^{j}(1 - x_i)$, then the stochastic intensity defined in (6) corresponds to the two-valued cascading model. Therefore, two different types of cascading model (two-valued and many-valued cascading models) are unified into one general framework, which is based on the point process approach.

3. Main fundamental results

3.1. Survival function and system failure rate

First, we are interested in the survival function and the failure rate function of the system. Denote by T_S the lifetime of the system. Note that the following first theoretical result provides general formulas for the survival function and the failure rate function of a complex k-out-of-n system with components having 'arbitrary dependence structures' described by $\lambda_j(t \mid s_1, s_2, \ldots, s_j)$, $j = 1, 2, \ldots, n-k$. Thus, the application of the following theorem is not limited solely to the two types of cascading failure model discussed in Section 2.

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Theorem 1. *Suppose that the stochastic intensity functions of a k-out-of-n system are given by* $\lambda_j(t \mid s_1, s_2, \ldots, s_j)$, $j = 1, 2, \ldots, n-k$. *Then the survival function of the system is given by*

$$\mathbb{P}(T_S > t) = \exp\{-\Lambda_0(0, t)\} + \sum_{m=1}^{n-k} \int_0^t \int_0^{s_m} \cdots \int_0^{s_3} \int_0^{s_2} \prod_{j=0}^{m-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \ldots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \times \exp\{-\Lambda_m(s_m, t)\} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{m-1} \, \mathrm{d}s_m,$$

where $s_0 \equiv 0$, $\int_0^t \int_0^{s_m} \cdots \int_0^{s_3} \int_0^{s_2} (\cdot) \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{m-1} \, \mathrm{d}s_m = \int_0^t (\cdot) \, \mathrm{d}s_1$ *when* $m = 1$, $\sum_{m=1}^{n-k} (\cdot) \equiv 0$ *when* $n = k$, *and* $\Lambda_j(v_1, v_2) \equiv \int_{v_1}^{v_2} \lambda_j(u \mid s_1, s_2, \ldots, s_j) \, \mathrm{d}u$, $j = 0, 1, \ldots, n-k$.

Furthermore, the failure rate function of the system $r_S(t)$ *for* $n > k$ *is given by*

$$r_S(t) = \frac{1}{\mathbb{P}(T_S > t)} \times \bigg[\int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_3} \int_0^{s_2} \prod_{j=0}^{n-k-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \ldots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \times \lambda_{n-k}(t \mid s_1, s_2, \ldots, s_{n-k}) \times \exp\{-\Lambda_{n-k}(s_{n-k}, t)\} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{n-k-1} \, \mathrm{d}s_{n-k}\bigg],$$

where $\int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_3} \int_0^{s_2} (\cdot) \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{m-1} \, \mathrm{d}s_m = \int_0^t (\cdot) \, \mathrm{d}s_1$ *when* $n - k = 1$.

Proof. Note that

$$\mathbb{P}(T_S > t) = \mathbb{P}(N(t) \le n-k) = \sum_{m=0}^{n-k} \mathbb{P}(N(t) = m).$$

Clearly, $\mathbb{P}(N(t) = 0) = \exp\{-\int_0^t \lambda_0(u) \, \mathrm{d}u\}$. Now let us consider $\mathbb{P}(N(t) = m)$, $m \ge 1$. Note that the joint distribution of $(N(t) = m, S_1 = s_1, S_2 = s_2, \ldots, S_m = s_m)$ is given by

$$\prod_{j=0}^{m-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \ldots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \exp\{-\Lambda_m(s_m, t)\},$$

where $0 \le s_1 \le s_2 \le \cdots \le s_m \le t$. Thus, $\mathbb{P}(N(t) = m)$ can be obtained by

$$\mathbb{P}(N(t) = m) = \int_0^t \int_0^{s_m} \cdots \int_0^{s_3} \int_0^{s_2} \prod_{j=0}^{m-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \ldots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \times \exp\{-\Lambda_m(s_m, t)\} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{m-1} \, \mathrm{d}s_m.$$

Now the failure rate function will be derived. Differentiating

$$\int_0^t \int_0^{s_m} \cdots \int_0^{s_3} \int_0^{s_2} \prod_{j=0}^{m-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \ldots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \times \exp\{-\Lambda_m(s_m, t)\} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{m-1} \, \mathrm{d}s_m$$

with respect to t and by applying Leibnitz's rule (see, e.g. Casella and Berger (2002, p. 69)),we have

$$\begin{aligned}&\int_0^t\int_0^{s_{m-1}}\cdots\int_0^{s_3}\int_0^{s_2}\prod_{j=0}^{m-2}(\lambda_j(s_{j+1}\mid s_1,s_2,\dots,s_j)\exp\{-\Lambda_j(s_j,s_{j+1})\})\\&\qquad\times\lambda_{m-1}(t\mid s_1,s_2,\dots,s_{m-1})\\&\qquad\times\exp\{-\Lambda_{m-1}(s_{m-1},t)\}\,\mathrm{d}s_1\,\mathrm{d}s_2\cdots\mathrm{d}s_{m-1}\\&-\int_0^t\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2}\prod_{j=0}^{m-1}(\lambda_j(s_{j+1}\mid s_1,s_2,\dots,s_j)\exp\{-\Lambda_j(s_j,s_{j+1})\})\\&\qquad\times\lambda_m(t\mid s_1,s_2,\dots,s_m)\exp\{-\Lambda_m(s_m,t)\}\,\mathrm{d}s_1\,\mathrm{d}s_2\cdots\mathrm{d}s_{m-1}\,\mathrm{d}s_m.\end{aligned}$$

Summing these terms for $m = 1, 2, \dots, n-k$, and also with $(\mathrm{d}/\mathrm{d}t)\exp\{-\Lambda_0(0,t)\} = -\lambda_0(t)\exp\{-\Lambda_0(0,t)\}$ (the $m = 0$ case), we have

$$\begin{aligned}&\mathbb{P}(T_S>t)'\\&\quad=-\bigg[\int_0^t\int_0^{s_{n-k}}\cdots\int_0^{s_3}\int_0^{s_2}\prod_{j=0}^{n-k-1}(\lambda_j(s_{j+1}\mid s_1,s_2,\dots,s_j)\exp\{-\Lambda_j(s_j,s_{j+1})\})\\&\qquad\times\lambda_{n-k}(t\mid s_1,s_2,\dots,s_{n-k})\\&\qquad\times\exp\{-\Lambda_{n-k}(s_{n-k},t)\}\,\mathrm{d}s_1\,\mathrm{d}s_2\cdots\mathrm{d}s_{n-k-1}\,\mathrm{d}s_{n-k}\bigg].\end{aligned}$$

The failure rate function is then obtained by $r_S(t) = -(\mathbb{P}(T_S > t)')/(\mathbb{P}(T_S > t))$. □

Remark 1. The survival functions and failure rate functions of the many-valued and two-valued cascading models can be obtained by using the stochastic intensity functions in (5) and (6), respectively (see Section 4).

Remark 2. Note that the proof of Theorem 1 provides more insight into the risk assessment of the system. Clearly, the k-out-of-n system cannot fail in a short (infinitesimal) interval when more than k components are working. The system can fail only when exactly k components are working (i.e. when $n-k$ components have failed). Thus, the state when $n-k$ components have failed and only k components are working can be regarded as a 'risky state'. In this situation, our interest could be 'what is the probability that the system will be at the risky state when it is currently working?'. This measure can be expressed as

$$\begin{aligned}&\mathbb{P}(N(t)=n-k\mid N(t)\le n-k)\\&\quad=\frac{1}{\mathbb{P}(T_S>t)}\bigg[\int_0^t\int_0^{s_{n-k}}\cdots\int_0^{s_3}\int_0^{s_2}\\&\qquad\times\prod_{j=0}^{n-k-1}(\lambda_j(s_{j+1}\mid s_1,s_2,\dots,s_j)\exp\{-\Lambda_j(s_j,s_{j+1})\})\\&\qquad\times\exp\{-\Lambda_{n-k}(s_{n-k},t)\}\,\mathrm{d}s_1\,\mathrm{d}s_2\cdots\mathrm{d}s_{n-k-1}\,\mathrm{d}s_{n-k}\bigg].\end{aligned}\tag{7}$$

The measure defined in (7) can be used as a risk measure for the k-out-of-n system.

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In some cases, such as electrical systems, the failure history of the system is not observed, while it can be observed in some other cases (see Section 3.2 for detailed discussions of these cases). In the former case, the susceptibility to failure of a currently working system (at time t) is described by $r_{\mathrm{S}}(t)$. However, by inspection of the working system, the status, i.e. the total number of working components $n - N(t)$, can be observed even though the failure points in the interval $(0, t]$ cannot be observed. In this case, clearly, the susceptibility to failure of a currently working system (at time t) should be described by the following 'conditional failure rate':

$$r_{\mathrm{S}}(t \mid m) \equiv \lim_{\Delta t \to 0} \mathbb{P}(t \le T_{\mathrm{S}} < t + \Delta t \mid T_{\mathrm{S}} > t, N(t) = m), \qquad m = 0, 1, \dots, n - k.$$

As the k-out-of-n system cannot fail in a short (infinitesimal) interval when more than k components are working, obviously, $r_{\mathrm{S}}(t \mid m) = 0, m = 0, 1, \dots, n-k-1$. Thus, it would be of great interest to obtain $r_{\mathrm{S}}(t \mid n-k)$. Observe that the system failure rate can be expressed as

$$\begin{aligned} r_{\mathrm{S}}(t) &= \lim_{\Delta t \to 0} \mathbb{P}(t \le T_{\mathrm{S}} < t + \Delta t \mid T_{\mathrm{S}} > t) \\ &= \lim_{\Delta t \to 0} \sum_{m=0}^{n-k} \mathbb{P}(t \le T_{\mathrm{S}} < t + \Delta t \mid T_{\mathrm{S}} > t, N(t) = m) \mathbb{P}(N(t) = m \mid T_{\mathrm{S}} > t) \\ &= \sum_{m=0}^{n-k} r_{\mathrm{S}}(t \mid m) \mathbb{P}(N(t) = m \mid T_{\mathrm{S}} > t) \\ &= r_{\mathrm{S}}(t \mid n-k) \mathbb{P}(N(t) = n-k \mid N(t) \le n-k). \end{aligned} \tag{8}$$

Therefore, from the relationship in (8), and Theorem 1 and its proof, we have

$$\begin{aligned} & r_{\mathrm{S}}(t \mid n-k) \\ &= \bigg[\int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_3} \int_0^{s_2} \prod_{j=0}^{n-k-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \dots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \\ &\qquad \times \lambda_{n-k}(t \mid s_1, s_2, \dots, s_{n-k}) \exp\{-\Lambda_{n-k}(s_{n-k}, t)\} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{n-k-1} \, \mathrm{d}s_{n-k} \bigg] \\ &\quad \times \bigg[\int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_3} \int_0^{s_2} \prod_{j=0}^{n-k-1} (\lambda_j(s_{j+1} \mid s_1, s_2, \dots, s_j) \exp\{-\Lambda_j(s_j, s_{j+1})\}) \\ &\qquad \times \exp\{-\Lambda_{n-k}(s_{n-k}, t)\} \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_{n-k-1} \, \mathrm{d}s_{n-k} \bigg]^{-1}. \end{aligned}$$

3.2. Information-based residual lifetime

In this section we will discuss a new concept 'information-based residual lifetime'. Suppose that the system has survived until time u, i.e. the event $\{T_{\mathrm{S}} > u\}$ is given. In this situation, our main interest will be in determining how long the system will survive further into the future. To assess it, the residual lifetime $T_{\mathrm{S}}(u) \equiv (T_{\mathrm{S}} - u \mid T_{\mathrm{S}} > u)$ is defined and, then the survival function of $T_{\mathrm{S}}(u)$ is obtained as $\mathbb{P}(T_{\mathrm{S}}(u) > t) = \mathbb{P}(T_{\mathrm{S}} > t + u)/\mathbb{P}(T_{\mathrm{S}} > u)$. Its failure rate is just given by $r_{\mathrm{S}}(u + t), t \ge 0$.

However, suppose now that, at time u, in addition to the information $\{T_{\mathrm{S}} > u\}$, we have additional information on the process history of the system: $\mathcal{H}_{u-} = (N(u-) = l, S_1 = s_1, S_2 = s_2, \dots, S_l = s_l)$, $0 \le l \le n - k$. That is, the failure process of the system in the

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interval $(0, u)$ is observed. This is quite common in the practical examples highlighted in Section 1 (e.g. the power grid example), while the process history cannot be observed in some examples such as electrical systems. Clearly, in this situation, the measures $\mathbb{P}(T_S(u) > t)$ and $r_S(t+u)$, $t \geq 0$, do not properly assess the future reliability of the system. In this case, all the information that should be taken into account is '$\{T_S > u\}$ and $\mathcal{H}_{u-} = (N(u-) = l, S_1 = s_1, S_2 = s_2, \dots, S_l = s_l)$', and we need to define the information-based residual lifetime $T_S(u; \mathcal{H}_{u-}) \equiv (T_S - u \mid T_S > u, \mathcal{H}_{u-})$. Then the following conditional survival function:

$$\begin{aligned}&\mathbb{P}(T_S(u; \mathcal{H}_{u-}) > t)\\ &\qquad \equiv \mathbb{P}(T_S > t + u \mid \mathcal{H}_{u-} = (N(u-) = l, S_1 = s_1, S_2 = s_2, \dots, S_l = s_l)),\end{aligned} \tag{9}$$

and the corresponding failure rate function, denoted by $r(t \mid u; \mathcal{H}_{u-})$, should be considered. Note that in (9), $\{T_S > u\}$ is omitted in the condition part as it is redundant when $0 \leq l \leq n-k$. The following theorem provides these measures.

Theorem 2. *The survival function of* $T_S(u; \mathcal{H}_{u-})$ *is given by*

$$\begin{aligned}
&\mathbb{P}(T_S(u; \mathcal{H}_{u-}) > t)\\
&\quad = \exp\{-\Lambda_l(u, u+t)\}\\
&\qquad + \sum_{m=1}^{n-k-l} \int_u^{u+t} \int_u^{s_{l+m}} \cdots \int_u^{s_{l+3}} \int_u^{s_{l+2}} \lambda_l(s_{l+1} \mid s_1, s_2, \dots, s_l) \exp\{-\Lambda_l(u, s_{l+1})\}\\
&\qquad\qquad \times \prod_{j=1}^{m-1} (\lambda_{l+j}(s_{l+j+1} \mid s_1, s_2, \dots, s_{l+j}) \exp\{-\Lambda_{l+j}(s_{l+j}, s_{l+j+1})\})\\
&\qquad\qquad \times \exp\{-\Lambda_{l+m}(s_{l+m}, u+t)\}\, \mathrm{d}s_{l+1}\, \mathrm{d}s_{l+2} \cdots \mathrm{d}s_{l+m-1}\, \mathrm{d}s_{l+m},
\end{aligned}$$

where $s_0 \equiv 0$, $\int_u^{u+t} \int_u^{s_{l+m}} \cdots \int_u^{s_{l+3}} \int_u^{s_{l+2}} (\cdot)\, \mathrm{d}s_{l+1}\, \mathrm{d}s_{l+2} \cdots \mathrm{d}s_{l+m-1}\, \mathrm{d}s_{l+m} = \int_u^{u+t} (\cdot)\, \mathrm{d}s_l$ *when* $m = 1$, $\sum_{m=1}^{n-k-l} (\cdot) \equiv 0$ *when* $l = n-k$, *and* $\prod_{j=1}^{m-1} (\cdot) \equiv 1$ *when* $m = 1$.

Furthermore, for $n > k$, *the corresponding failure rate function* $r(t \mid u; \mathcal{H}_{u-})$ *is given by*

$$\begin{aligned}
&r(t \mid u; \mathcal{H}_{u-})\\
&\quad = \frac{1}{\mathbb{P}(T_S(u; \mathcal{H}_{u-}) > t)}\\
&\qquad \times \biggl[\int_u^{u+t} \int_u^{s_{n-k}} \cdots \int_u^{s_{l+3}} \int_u^{s_{l+2}} \lambda_l(s_{l+1} \mid s_1, s_2, \dots, s_l) \exp\{-\Lambda_l(u, s_{l+1})\}\\
&\qquad\quad \times \prod_{j=1}^{n-k-1} (\lambda_{l+j}(s_{l+j+1} \mid s_1, s_2, \dots, s_{l+j}) \exp\{-\Lambda_{l+j}(s_{l+j}, s_{l+j+1})\})\\
&\qquad\quad \times \lambda_{n-k}(u+t \mid s_1, s_2, \dots, s_{n-k})\\
&\qquad\quad \times \exp\{-\Lambda_{n-k}(s_{n-k}, u+t)\}\, \mathrm{d}s_{l+1}\, \mathrm{d}s_{l+2} \cdots \mathrm{d}s_{n-k-1}\, \mathrm{d}s_{n-k}\biggr],
\end{aligned}$$

where $\int_u^{u+t} \int_u^{s_{n-k}} \cdots \int_u^{s_{l+3}} \int_u^{s_{l+2}} (\cdot)\, \mathrm{d}s_{l+1}\, \mathrm{d}s_{l+2} \cdots \mathrm{d}s_{n-k-1}\, \mathrm{d}s_{n-k} = \int_u^{u+t} (\cdot)\, \mathrm{d}s_{l+1}$ *when* $n - k = 1$.

Proof. In order to calculate the survival function of the residual lifetime, we have to count the total number of component failures which occur after u, given that $\mathcal{H}_{u-} = (N(u-) = l, S_1 = s_1, S_2 = s_2, \dots, S_l = s_l)$. Clearly,

$$\mathbb{P}(T_S(u; \mathcal{H}_{u-}) > t) = \mathbb{P}(N(u+t) - N(u) \leq n - k - l \mid \mathcal{H}_{u-}).$$

Note that, given

$$\mathcal{H}_{u-} = (N(u-) = l, S_1 = s_1, S_2 = s_2, \dots, S_l = s_l),$$

the stochastic intensity at time $s > u$ in the $(l+j+1)$th stage (i.e. when $N(s-) = l+j$) is given by $\lambda_{l+j}(s \mid s_1, s_2, \dots, s_{l+j})$, $j = 0, 1, \dots, n-k-l$. Then, obviously, $\mathbb{P}(N(u+t) - N(u) = 0 \mid \mathcal{H}_{u-}) = \exp\{-\Lambda_l(u, u+t)\}$. By similar arguments as those given in the proof of Theorem 1, it can be shown that

$$\begin{aligned}
&\mathbb{P}(N(u+t) - N(u) = m \mid \mathcal{H}_{u-}) \\
&= \int_u^{u+t} \int_u^{s_{l+m}} \cdots \int_u^{s_{l+3}} \int_u^{s_{l+2}} \lambda_l(s_{l+1} \mid s_1, s_2, \dots, s_l) \mathrm{e}^{-\Lambda_l(u, s_{l+1})} \\
&\qquad \times \prod_{j=1}^{m-1} (\lambda_{l+j}(s_{l+j+1} \mid s_1, s_2, \dots, s_{l+j}) \\
&\qquad\qquad \times \mathrm{e}^{-\Lambda_{l+j}(s_{l+j}, s_{l+j+1})}) \\
&\qquad \times \mathrm{e}^{-\Lambda_{l+m}(s_{l+m}, u+t)} \, \mathrm{d}s_{l+1} \, \mathrm{d}s_{l+2} \cdots \mathrm{d}s_{l+m-1} \, \mathrm{d}s_{l+m}.
\end{aligned}$$

The corresponding failure rate function $r(t \mid u; \mathcal{H}_{u-})$ can also be obtained similarly. □

4. Numerical examples

In this section some numerical examples will be provided to illustrate the utility of the general results obtained in Section 3.

4.1. Survival function and system failure rate

First, we consider a numerical example for the system survival function and its failure rate. In this example, we consider the many-valued cascading model defined in (5). Suppose that $n = 5$, $k = 3$, and $\lambda(t) = 0.1t$, $t \geq 0$. Suppose further that the critical time is given by $\delta = 0.5$, $\gamma = 0.5$, and $\psi(u) = 1 - (1/\delta)u$. Thus, in this case, the effect of increased load disappears in a linear pattern. Applying the general results given in Theorem 1, the survival function and the corresponding failure rate function are obtained. The survival function and the corresponding failure rate function of the ordinary k-out-of-n system defined in (1) are also obtained and compared with those for the cascading failure model. They are illustrated in Figures 5 and 6, respectively.

It is observed that the survival functions and the failure rate functions are all ordered.

4.2. Information-based residual lifetime

Now let us consider the information-based residual lifetime defined in the previous section. We consider the cascading failure model considered above. Suppose that the k-out-of-n systems

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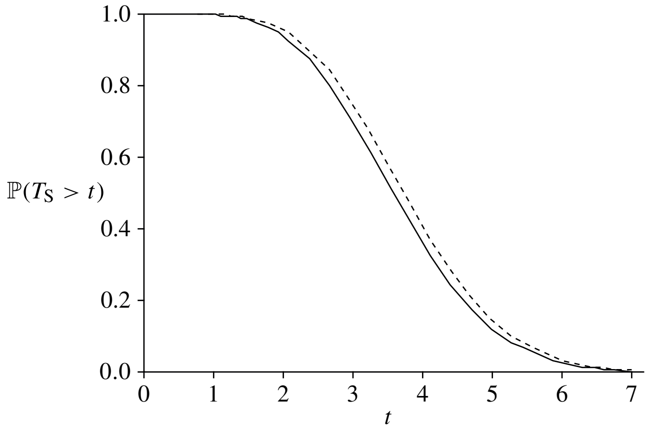

FIGURE 5: The system survival function $\mathbb{P}(T_S > t)$ for the cascading failure model (*solid line*) and that for ordinary k-out-of-n system (*dashed line*).

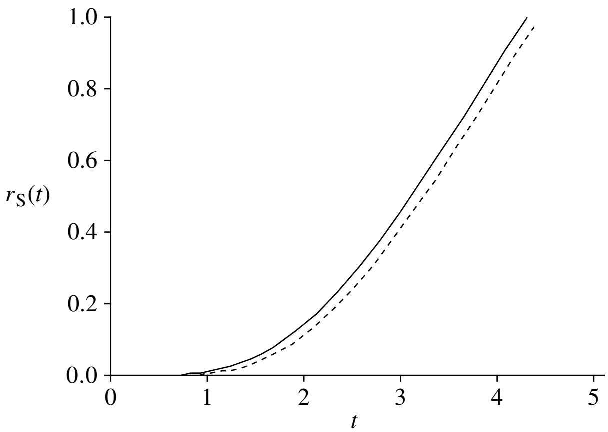

FIGURE 6: The system failure rate function $r_S(t)$ for the cascading failure model (*solid line*) and that for ordinary k-out-of-n system (*dashed line*).

have survived until time $u = 2.0$, but that they have the following different scenarios in the previous interval:

$$\mathcal{H}_{u-}^{(1)} \equiv (N(u-) = 0), \qquad \mathcal{H}_{u-}^{(2)} \equiv (N(u-) = 1,\, S_1 = 1.0),$$
$$\mathcal{H}_{u-}^{(3)} \equiv (N(u-) = 1,\, S_1 = 1.8), \qquad \mathcal{H}_{u-}^{(4)} \equiv (N(u-) = 2,\, S_1 = 1.0,\, S_2 = 1.5),$$
$$\mathcal{H}_{u-}^{(5)} \equiv (N(u-) = 2,\, S_1 = 1.0,\, S_2 = 1.8),$$
$$\mathcal{H}_{u-}^{(6)} \equiv (N(u-) = 2,\, S_1 = 1.8,\, S_2 = 1.9).$$

Note that the failure points in the interval $(0, u)$ affect the system survival in the next interval, and the survival probability of the system in the next interval should not only depend on the number of failures observed, but also depend on the failure points. For instance, in the scenario $\mathcal{H}_{u-}^{(4)}$, there is no 'lasting effect' at time $u = 2.0$ even though there were two failures in the

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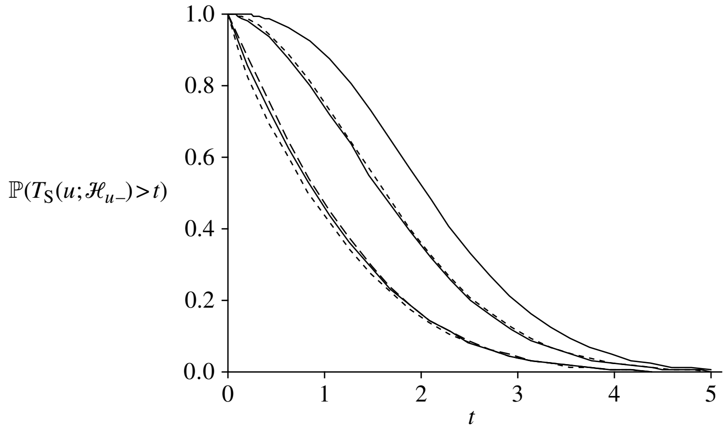

FIGURE 7: The survival function $\mathbb{P}(T_S(u; \mathcal{H}_{u-}) > t)$ for $\mathcal{H}_{u-}^{(i)}$, $i = 1, 2, \ldots, 6$, in order from top to bottom.

TABLE 1: The survival probability $\mathbb{P}(T_S(u; \mathcal{H}_{u-}^{(i)}) > t)$, $i = 1, 2, \ldots, 6$.

	t					
	0.5	1.0	1.5	2.0	2.5	3.0
$\mathcal{H}_{u-}^{(1)}$	0.979 19	0.885 71	0.721 06	0.522 45	0.336 47	0.193 23
$\mathcal{H}_{u-}^{(2)}$	0.924 56	0.753 82	0.549 06	0.358 99	0.211 74	0.113 17
$\mathcal{H}_{u-}^{(3)}$	0.914 37	0.741 72	0.538 43	0.351 23	0.206 82	0.110 41
$\mathcal{H}_{u-}^{(4)}$	0.713 55	0.472 37	0.290 11	0.165 30	0.087 38	0.042 85
$\mathcal{H}_{u-}^{(5)}$	0.693 61	0.459 16	0.281 20	0.160 68	0.084 94	0.041 65
$\mathcal{H}_{u-}^{(6)}$	0.658 98	0.436 25	0.267 92	0.152 66	0.080 70	0.039 58

previous interval. On the other hand, in the scenario $\mathcal{H}_{u-}^{(5)}$, there is one 'lasting effect' from the second failure (s_2=1.8), and, in the scenario $\mathcal{H}_{u-}^{(6)}$, there are two 'lasting effects' from the first (s_1=1.8) and second failures (s_2=1.9). Depending on these different histories, the survival functions $\mathbb{P}(T_S(u; \mathcal{H}_{u-}) > t)$ have been obtained in Figure 7.

As expected, the survival functions of the information-based residual lifetimes are ordered as

$$\mathbb{P}\Big(T_S(u; \mathcal{H}_{u-}^{(i)}\Big) > t\Big) > \mathbb{P}\Big(T_S\Big(u; \mathcal{H}_{u-}^{(i+1)}\Big) > t\Big), \qquad i = 1, 2, \ldots, 5.$$

The values of the survival probabilities for $t = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ are given in Table 1.

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