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SUBSIMPLE, INJECTIVE, RETRACT

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Simple and subsimple objects were introduced in [6]. It was shown that if there are enough simple objects in a category $\mathscr{C}$, then there is no room for injectives in $\mathscr{C}$. This idea was exploited in [6] and [2] to show that several classes of groups, rings and classes belonging to other categories do not possess non-trivial injectives or retracts. In this note, the above results will be strengthened by introducing a weaker condition than subsimple of [6]. As a consequence, and by employing some embedding theorems, we show that some important classes do not possess non-trivial retracts.

All the categories are assumed to have a zero object.

Definition. Let $\mathscr{C}$ be a full subcategory of a category $\mathscr{D}$. An object A of $\mathscr{C}$ will be called *$\mathscr{D}$-subsimple* if there exist $S\in \operatorname{ob}\mathscr{D}$, $T\in\operatorname{ob}\mathscr{C}$ such that A is a proper subobject of S, S is a subobject of T, and S is simple in $\mathscr{D}$ [**6**, Definition (i)].

Obviously a subsimple object in a category $\mathscr{C}$, as defined in [6], is a $\mathscr{C}$-subsimple object.

Theorem 1 of [6] and Lemma 1 of [2] are extended as follows. (Let $\mathscr{C}$ and $\mathscr{D}$ be as in the definition above.)

Theorem 1. *If a non-zero $I\in\operatorname{ob}\mathscr{C}$ is an extremal quotient in $\mathscr{D}$ of* a *$\mathscr{D}$-subsimple object $A\in\operatorname{ob}\mathscr{C}$, then I is not injective in $\mathscr{C}$.*

Proof. Assume I is injective in $\mathscr{C}$. Let $A \overset{m}{\rightarrowtail} S \overset{h}{\rightarrowtail} T$, m non-invertible, S simple in $\mathscr{D}$, $T\in\operatorname{ob}\mathscr{C}$, and let $A \overset{e}{\twoheadrightarrow} I$ be extremal [**4**, 17.9]. As I is injective in $\mathscr{C}$ and $hm\in\mathscr{C}(A,T)$, there exists $f\in\mathscr{C}(T,I)$ such that $f(hm)=e$. Clearly fh is a monomorphism since $fh\neq 0$, as $I\neq 0$. Hence fh is invertible, since e is extremal. So m is an extremal epimorphism and a monomorphism, hence invertible. Contradiction.

Corollary 2. (i) *A full category of groups containing the free groups and the symmetric groups does not possess non-zero injectives.* (ii) *A full category of J-algebras, J an integral domain, containing the free J-algebras and the algebras of endomorphisms of J-modules does not possess non-zero injectives.*

Theorem 3. *A $\mathscr{D}$-subsimple object $A\in\operatorname{ob}\mathscr{C}$ cannot be a retract in $\mathscr{C}$.* (*$\mathscr{C}$ and $\mathscr{D}$ are as in the definition above.*)

Proof. Assume A is a retract in $\mathscr{C}$ and $A \overset{m}{\rightarrowtail} S \overset{h}{\rightarrowtail} T$ as in the proof of Theorem 1. Since $hm \in \mathscr{C}(A, T)$, there exists $g \in \mathscr{C}(T, A)$ such that $g(hm) = 1$. But $gh \neq 0$ since $A \neq 0$, so gh is a monomorphism, hence invertible. So m is invertible. Contradiction.

Theorem 4. *There are no non-trivial retracts in:*

(i) *the class of finitely generated groups,* $\mathscr{F}g\mathscr{G}r$; (ii) *the class of n-generator groups, n a positive integer,* $n\mathscr{G}r$; (iii) *the class of countable, locally finite groups,* $\mathscr{C}\mathscr{L}f\mathscr{G}r$; (iv) *the class of finitely generated groups with solvable word problem.*

Proof. (i): Let G be a finitely generated group. In particular G is countable, so by a theorem of Boone and Higman [1], there exists a simple countable group H such that $G \lneqq H$. Hence, by a theorem of Higman, B. H. Neumann and H. Neumann [5, Theorem 4], there exists a 2-generator group K with $H \leqq K$. It follows that G is $\mathscr{G}r$-subsimple in $\mathscr{F}g\mathscr{G}r$, so by Theorem 3, there are no non-trivial retracts in $\mathscr{F}g\mathscr{G}r$.

(ii): The case $n=1$ can be easily proved directly. Let $n>1$. Again by the theorems of [1] and [5] mentioned above, every group in $n\mathscr{G}r$ is $\mathscr{G}r$-subsimple in $n\mathscr{G}r$, hence there are no non-trivial retracts in $n\mathscr{G}r$, Theorem 3.

(iii): Let G be a countable, locally finite group. By a theorem of P. Hall [3], there exists a simple, countable, locally finite group H such that $G \lneqq H$. Put $K=H$, and apply Theorem 3 to obtain that there are no non-trivial retracts in $\mathscr{C}\mathscr{L}f\mathscr{G}r$.

(iv): Same proof as for (iii) but instread of P. Hall's theorem we employ a theorem of Thompson [7] to embed any finitely-generated group with solvable word problem into a simple group of the same sort.

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