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FORECASTING BUSINESS CYCLES USING DEVIATIONS FROM LONG-RUN ECONOMIC RELATIONSHIPS

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We introduce a new index that explores the linkage between business-cycle fluctuations and deviations from long-run economic relationships. This index is virtually a measure of the distance between an attractor, a space spanned by the associated cointegrating vectors, and a point in the n-dimensional Euclidean space. The index is applied to U.S. quarterly data to demonstrate its association with an economy's vulnerability state. We find that the average of the index during expansions negatively correlates with the average contraction in output during recessions. A nonlinear error correction model based on a revised version of the index reveals a forecasting gain as compared to the linear error correction model.

Keywords: Business Cycles, Cointegration, Nonlinear Error Correction Model, Forecast

1. INTRODUCTION

The research on business-cycle indicators started with the work by Wesley Mitchell and Arthur Burns in 1938 at the National Bureau of Economic Research. Their work has generated a sequence of contributions ever since, including the leading-indicator approach of Hymans (1973), Neftci (1982), Stock and Watson (1989, 1993), and Swanson et al. (1999) and the simulation approach of Wecker (1979), Kling (1987), and Fair (1993), among many others. Most suggestions, however, have relied heavily on data analysis and rather little on economic theory and have concentrated on forecasting the cycles' timing rather than their severity. In this paper, using another type of "measurement without theory" approach, we hope

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to obtain a measure that could serve as an early warning variable in representing an economy's fragility. The measure we have developed in this paper is virtually a Euclidean metric and is based on a simplistic viewpoint about cointegration, a notion first introduced in Granger (1981) and advanced by Engle and Granger (1987).

To provide a vulnerability index for an economy, we develop a distance measure, based solely on cointegrations and the corresponding error correction terms. It is the authors' intention to use this measure as a warning indicator to tell us how far the economy is from equilibrium and the likely fragility degree when encountering a negative impact. Note, however, that the proposed index is virtually orthogonal in concept to the leading index introduced by Stock and Watson (1993), who exclude cointegrations from their exploration of possible terms to be included. Suppose a vector $\boldsymbol{X}_t$ has all components that are I(1) variables and let the column(s) of $\boldsymbol{\beta}$ represent the vector(s) of cointegrations linking these components. The hyperplane $\boldsymbol{Z}_t = \boldsymbol{\beta}'\boldsymbol{X}_t = 0$ can be considered as the system's attractor, which for convenience is also called the "long-run equilibrium" in our paper, even though we realize that this term is now rather controversial. A norm applied to $\boldsymbol{Z}_t$ is used to measure the distance to this attractor. A collection of nonzero errors in $\boldsymbol{Z}_t$ is considered as an indication that the economy is out of equilibrium and the norm will measure the extent that this occurs. Such a distance measure is named as the *disequilibrium index* throughout this paper.

What motivates this study is the concept that when a system is further from its long-run equilibrium, an external shock to it will cause a stronger rebound toward the attractor. This concept is quite appealing, according to a suggestion from the structural approach of the Cowles Commission that the effects of economic shocks would depend on the extent to which output deviated from its full employment level. On the other hand, the ratchet effect investigated in a number of nonlinear studies of output is close to what we try to explore here.[1]

The ratchet effect says that when output growth falls below (rises above) its average level, then the further it falls (rises) the larger the pressure will be for it to return to the average level. If the ratchet effect is an appropriate mechanism explaining the fluctuations in output, then the distance to the attractor should be an economy's appropriate measure for the vulnerability degree to shocks at downturns and for the degree of reflection force to shocks at upturns. Thus, large values of the index will indicate that a business-cycle turning point is likely, and the magnitude of the index suggests the possible size of the change in output if a large shock should occur.

We consider the disequilibrium index as a reasonable candidate that could help us detect likely turning points over business cycles, because the cointegration relations employed in formulating the index are suggested by economic theories. For example, the real-business-cycle literature of King et al. (1988) predicts that real consumption, real investment, and real output share a common stochastic trend. This implies that the ratios of consumption to output and investment to output (in logarithm) can be characterized as stationary processes. The observed cyclical behaviors of consumption and investment indeed are compatible with the mean-reversion property implied by stationarity; namely, increases in investment boost valuation, leading to further investment increases until there is overcapacity, strains in the economy, higher interest rates and borrowing costs, and then a slowdown. As to the cycle led by consumer spending, low prices and high income encourage purchases and borrowing and help sustain economic growth until rising inflation and interest rates depress consumer sentiment, and then a downturn follows. The above two long-run relations are best known as the balanced-growth cointegrations. The other cointegration relation examined in this paper is concerned with financial variables: the long-run money demand, a long-run relation often investigated by macroeconomists.

Two important features about this measure are, first, that it is scale-invariant, and second, that it is still valid even with changing economic fundamentals. The invariance property, expounded later in the paper, keeps us in check if we intend to give cardinal interpretations. The second feature is important in that even in the face of changing equilibria (i.e., the levels of the variables in equilibrium change), as long as the cointegrating vectors remain the same, then our measure of disequilibrium is valid. In other words, the measure only depends on the "attractor" between variables of interest and by how much these variables are displaced from their cointegrating relation by shocks to the economy. Consequently, this approach has an advantage over the impulse response analysis in assessing the effects of shocks. The idea behind impulse responses is to see how long variables in question will return to their steady-state growth path after shocks hit the system. However, if the steady-state path changes in the process, we then might be tempted, using impulse responses, to interpret this as a delay in returning to the steady state. Using our distance measure gets around this nuance in the interpretations, as we will be correct in saying that the variables are away from their steady-state path if the distance is not zero.

Care should be taken when one gives a cardinal interpretation to this measure. Although the disequilibrium index is invariant to the scale of the cointegrating vectors, it is subject to change in the measurement unit of the variables under study. Therefore, any inference drawn from the index has to be in relation to another point in time over the sample period. To render additional information of the measure to readers, we provide a simulated statistical distribution to it. From an ex post perspective, we can make inferences about the likelihood of an abnormally high or low measure and record any systematic pattern between the measure's shape and the business-cycle turning points.

Around 1968, L.R. Klein and others organized a conference "Is the Business Cycle Obsolete?" [mentioned in Klein (1987, p. 435)], following an earlier conference in London suggesting that the policies of the Kennedy and Johnson administrations had abolished the world and national business cycles. A few years later, the first oil price shock occurred. Just a few quarters prior to the most recent recession that has begun, the United States had experienced the longest expansion since World War II. Similar sentiments about business-cycle obsolescence have been raised.

As will be seen, the U.S. economy had high measures of the disequilibrium index before the peak dates of the expansions of the 1960's and the 1990's. Hence, the proposed index seems to reveal a useful alarming signal for the vulnerability of the economy.

The remainder of this paper is organized as follows. Section 2 introduces the measure of disequilibrium, where its structure and practical concerns are discussed in details. In Section 3, the disequilibrium index is applied to U.S. quarterly data over the period of 1953–2001 to demonstrate its association with an economy's vulnerability state. We examine the time-series properties of the proposed index and its usefulness in forecasting macroeconomic variables. Some nonlinear error correction models that employ the index or a revised version of it are presented. Section 4 concludes.

2. A BUSINESS-CYCLE INDICATOR BASED ON LONG-RUN RELATIONSHIPS

2.1. A Disequilibrium Index for Economic Vulnerability

In this section we introduce the mathematical details of the index we propose for economic vulnerability as an alternative indicator for predicting economic turning points. This index is constructed by the concept of cointegration among economic variables and its error correction form. Consider an n-variable economy whereby some mechanism exists such that a subset of these n variables is linked in an equilibrium that moves on an attractor, a space spanned by the associated cointegrating vector(s). Any shock to the economy may pull it away from the attractor, with the tendency of moving toward it. The extent to which the economy is away from the attractor, or in disequilibrium, can be measured by the error correction term. If more than one long-run relationship exists, then the measure indeed is the distance between the attractor and the point this economy represents in the n-dimensional Euclidean space. This provides a measure of how large the overall equilibrium error is in the system.

To be specific, let us consider an n-variable economic system in which $\boldsymbol{X}_t$ is an $n \times 1$ vector of I(1) variables, and $\boldsymbol{\beta}$ is the corresponding $n \times r$ full column-rank cointegrating matrix. Let $\boldsymbol{Z}_t = \boldsymbol{\beta}'\boldsymbol{X}_t$ be the vector of cointegration errors or, alternatively, called equilibrium errors. For simplicity, we assume here that each element of $\boldsymbol{Z}_t$ has a zero mean; that is,

$$E(\boldsymbol{\beta}'\boldsymbol{X}_t) = 0. \tag{1}$$

This assumption can be relaxed and is discussed later in Section 2.3. The economy's equilibrium condition can be represented as a hyperplane:

$$\boldsymbol{Z}_t = \boldsymbol{\beta}'\boldsymbol{X}_t = 0. \tag{2}$$

We would like to develop an index that can account for the amount of equilibrium errors in the economic system, that is, the size that the economy deviates from these

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r long-run equilibrium relations. The space that describes a situation in which all long-run relations in the economy are satisfied (also known as an attractor) is the nullspace of $\boldsymbol{\beta}$. Given an observation $\boldsymbol{X}_t$ in the n-dimensional Euclidean space, the distance between $\boldsymbol{X}_t$ and the nullspace of $\boldsymbol{\beta}$ would be a natural measure for this purpose.

It is known that the distance between a given point $\boldsymbol{X}_t$ and the null space of $\boldsymbol{\beta}$ is the length (or norm) of the projection of $\boldsymbol{X}_t$ onto the column space of $\boldsymbol{\beta}$. We denote the projection as $P_\beta \boldsymbol{X}_t = \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\boldsymbol{X}_t$. In Appendix A, we provide more details about the construction of this distance measure and its geometric interpretation. The suggested measure is formulated in the following definition.

DEFINITION 1. *For an n-dimensional cointegrated system, whereby the columns of* $\boldsymbol{\beta}$ $(n \times r)$ *are the cointegrating vectors, the disequilibrium index associated with a given observation* $\boldsymbol{X}_t$ *is proposed as*

$$d(\boldsymbol{\beta}, \boldsymbol{X}_t) = \| P_\beta \boldsymbol{X}_t \|, \tag{3}$$

where $P_\beta = \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'$. *This measure has a geometric interpretation as the distance of* $\boldsymbol{X}_t$ *in an n-dimensional Euclidean space to the hyperplane* $\boldsymbol{\beta}'\boldsymbol{X}_t = 0$, *which is an attractor that represents the system's long-run equilibrium.*

The proposed index has an invariance property with respect to the scale in the column of $\boldsymbol{\beta}$. This property is due to the fact that the space spanned by the cointegrating vectors is uniquely determined, although the cointegrating matrix $\boldsymbol{\beta}$ is not. Therefore, the index proposed, given a set of cointegration relationships, is also uniquely determined. The invariance property is formally shown in the next lemma.

LEMMA 1 (Scale-invariance property). *The disequilibrium index proposed in* (3) *is invariant to the scale in the columns of* $\boldsymbol{\beta}$.

Proof. See Appendix B. ■

2.2. Disequilibrium Index and Equilibrium Errors

In this section we make an effort to understand more about the connection between the proposed index and the role each individual equilibrium-error (cointegration-error) series plays in the formula.

Because P_β is an idempotent matrix, the definition of $d(\boldsymbol{\beta}, \boldsymbol{X}_t)$ immediately implies that

$$d(\boldsymbol{\beta}, \boldsymbol{X}_t) = \| P_\beta \boldsymbol{X}_t \| = (\boldsymbol{X}_t' P_\beta \boldsymbol{X}_t)^{1/2} = (\boldsymbol{Z}_t'(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{Z}_t)^{1/2},$$

where $\boldsymbol{Z}_t \equiv \boldsymbol{\beta}'\boldsymbol{X}_t$ is an $r \times 1$ vector of equilibrium errors. In a special case in which there exists only one cointegrating vector in the n-variable system, the error $\boldsymbol{Z}_t = \boldsymbol{\beta}'\boldsymbol{X}_t$ is a scalar variable. Thus, the measure of $d(\boldsymbol{\beta}, \boldsymbol{X}_t)$ is proportional to the absolute value of equilibrium errors. Namely, when there is only one cointegrating relation the measure can be simplified as

$$d(\boldsymbol{\beta}, \boldsymbol{X}_t) = \frac{|\boldsymbol{Z}_t|}{\|\boldsymbol{\beta}\|}.$$

In a general case in which the number of cointegrating relations, r, is greater than 1, the disequilibrium index can be expressed as a square root of the linear combinations of the error correction elements of $\boldsymbol{Z}_t$. To explore this linkage, we re-express the index as

$$d(\boldsymbol{\beta}, \boldsymbol{X}_t) = [(\boldsymbol{Z}_t'(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{Z}_t)]^{1/2} = (\boldsymbol{Z}_t'\boldsymbol{C}\boldsymbol{C}'\boldsymbol{Z}_t)^{1/2} = (\boldsymbol{e}_t'\boldsymbol{e}_t)^{1/2}, \tag{4}$$

where $\boldsymbol{e}_t = \boldsymbol{C}'\boldsymbol{Z}_t$ and matrix $\boldsymbol{C}$ are defined such that $(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1} = \boldsymbol{C}\boldsymbol{C}'$. In essence, each element of $\boldsymbol{e}_t$ is a linear combination of these r errors in $\boldsymbol{Z}_t$; therefore, it has the interpretation of an equilibrium error to r cointegration relations. The index is virtually the square root of the sum of squares of objects that are linear combinations of $\boldsymbol{Z}_t$. In other words, the distance measure nonlinearly transforms r cointegration errors into a scalar.

Note that although the disequilibrium index is uniquely determined, the choice of such $\boldsymbol{C}$ or, equivalently, error vector $\boldsymbol{e}_t$, is not identified. This point can be seen easily by observing that the decomposition in (4) remains valid when we replace $\boldsymbol{C}$ with $\boldsymbol{CH}$. Here, $\boldsymbol{H}$ is any matrix such that $\boldsymbol{HH}' = \boldsymbol{I}$, an identity matrix.

2.3. Practical Issues

Several issues arise upon applying the proposed index in practice. The first issue concerns whether we should include appropriate deterministic terms in the space. This concern in essence relates to the fallibility of the assumption of zero mean in the error correction term, that is, the assumption of (1).

Let $\boldsymbol{\beta}^0$ be the cointegrating matrix associated with $\boldsymbol{X}_t$, a vector of stochastic variables. The assumption of $E(\boldsymbol{\beta}^{0'}\boldsymbol{X}_t) = 0$ may not hold in many circumstances; as a result, $\boldsymbol{\beta}^{0'}\boldsymbol{X}_t = 0$ is no longer the appropriate hyperplane to characterize the system's attractor. Suppose that $E(\boldsymbol{\beta}^{0'}\boldsymbol{X}_t) = \boldsymbol{\Psi}\boldsymbol{D}_t$, where $\boldsymbol{\Psi}$ is a coefficient matrix and $\boldsymbol{D}_t$ is a vector of nonstochastic variables that may contain constant and deterministic trend components. Let $\boldsymbol{\beta}' = [\boldsymbol{\beta}^{0'}, -\boldsymbol{\Psi}]$ and $\boldsymbol{X}_t^* = [\boldsymbol{X}_t{}', \boldsymbol{D}_t']'$. The zero mean condition, $E(\boldsymbol{\beta}'\boldsymbol{X}_t^*) = 0$, then holds and the distance index formula is ready to be applied.

The disequilibrium index for a cointegration space that includes constant and deterministic trends generally returns different values from one that does not. For instance, suppose $E(\boldsymbol{\beta}^{0'}\boldsymbol{X}_t) = \boldsymbol{\delta}t + \boldsymbol{\mu}$, where $\boldsymbol{\delta}$ and $\boldsymbol{\mu}$ are $r \times 1$ vectors of coefficients. Therefore, $\boldsymbol{\beta}' = [\boldsymbol{\beta}^{0'}, -\boldsymbol{\delta}, -\boldsymbol{\mu}]$ would be the transposed cointegrating matrix associated with $\boldsymbol{X}_t^* = [\boldsymbol{X}_t', t, 1]'$ such that $E(\boldsymbol{\beta}'\boldsymbol{X}_t^*) = 0$. When the cointegration errors $\{\boldsymbol{\beta}'\boldsymbol{X}_t^*\}$ are compared with $\{\boldsymbol{\beta}^{0'}\boldsymbol{X}_t\}$, it is clear that including a constant term shifts the errors horizontally, whereas including a time trend turns an upward- or downward-sloping line of errors into a flat one. Obviously, the pattern of cyclical movements in $\{\boldsymbol{\beta}^{0'}\boldsymbol{X}_t\}$ would be well preserved in calculating $d(\boldsymbol{\beta}, \boldsymbol{X}_t^*)$. As explained earlier in this paper, the main perception of an equilibrium error as a potentially effective indicator of the underlying economic condition is that the error series drifts around its attractor of the cointegrated system. Hence, we include appropriate deterministic components such as constant and time trends in the space in order to ensure that the error series are zero-mean without-trend stationary processes.

Given that all equilibrium errors have equal mean zero, the second concern about the index $d(\boldsymbol{\beta}, \boldsymbol{X}_t)$ is whether it would be dominated by the long-run error terms with a larger variability. A natural way to deal with this issue is to adjust the r equilibrium-error terms such that they have an equal spread. However, this turns out to be an unnecessary adjustment step, because the distance measure is unaffected by the standardizing procedure.

To see the above point, recall that $\boldsymbol{Z}_t \equiv \boldsymbol{\beta}'\boldsymbol{X}_t$ is an $r \times 1$ vector of equilibrium errors with the ith element denoted by $z_{i,t}$. Let $\boldsymbol{\beta}_s$ be computed by dividing the ith column of $\boldsymbol{\beta}$ by the standard deviation of $z_{i,t}$, for $i = 1, \ldots, r$. By construction, each element of $\{\boldsymbol{\beta}_s'\boldsymbol{X}_t\}$ is a standardized equilibrium-error series that crosses a zero-mean line frequently and varies around it with equal spread. According to Lemma 1, $d(\boldsymbol{\beta}_s, \boldsymbol{X}_t) = d(\boldsymbol{\beta}, \boldsymbol{X}_t)$ because the cointegrating space is invariant to this scaling procedure. Therefore, we can consider that the index $d(\boldsymbol{\beta}, \boldsymbol{X}_t)$ already accounts for an error standardization.[2]

The third issue concerns how we interpret the size of the disequilibrium index. Although the index is invariant to the scale of the cointegrating matrix, it is subject to change in the measurement unit of the variables under study. It would be helpful to comprehend the message delivered via the index if we have knowledge of the statistical distribution associated with it. The next lemma develops that knowledge under some assumptions.

LEMMA 2. *Suppose* $\boldsymbol{Z}_t = \boldsymbol{\beta}'\boldsymbol{X}_t$ *is a stochastic* $(r \times 1)$ *vector generated from a Gaussian distribution* $N(0, \boldsymbol{\Sigma})$ *and* $\boldsymbol{\Sigma}$ *is invertible. The square of the index proposed in this paper,* $d^2(\boldsymbol{\beta}, \boldsymbol{X}_t)$, *is a weighted sum of* r *independent* χ^2*-distributed variables where each has degrees of freedom* 1. *The weights are the eigenvalues of* $(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}$, *where* $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}\boldsymbol{\Sigma}^{-1/2}$.

Proof. See Appendix B. ■

Lemma 2 therefore implies that the disequilibrium index has a statistical distribution of the square root of the weighted sum of r independent $\chi^2(1)$ random variables, provided that the cointegration errors are normally distributed with zero mean and finite variance. In practice, we can calculate the probability bounds of this nonstandard distribution from Monte Carlo simulations based on a consistent estimate of $\boldsymbol{\Sigma}$, provided that the cointegration errors satisfy the assumptions of Lemma 2. Later in the empirical section, the normality assumption is tested and supported by our data.

It is noteworthy that yielding the result in Lemma 2 does not require any further assumption about the time dependence of $\boldsymbol{Z}_t$. The index is a nonlinear transformation of the elements in $\boldsymbol{Z}_t$. Because the elements of $\boldsymbol{Z}_t$ must be covariance-stationary and are often serially correlated, whether the index itself is an I(0) variable and whether the time dependence is present are empirical questions that need to be investigated. In the next section we apply the disequilibrium index to U.S. data. Discussions and empirical examinations are used to illustrate how to apply the index.

3. EMPIRICAL RESULTS USING U.S. DATA

3.1. Key Cointegrations Among Macroeconomic Variables

In this section we examine post-World War II, U.S. quarterly data over the period of 1953:2–2001:4. The U.S. postwar economy has been characterized by long expansions and short recessions. We are further interested in understanding the linkage between an economy's disequilibrium and the severity of recessions. Most business-cycle research investigates the fluctuations of economic time series over business-cycle frequencies, usually ranging between six quarters and eight years. There are, however, some prominent relationships among macroeconomic variables that are expected to hold over long horizons, and which might also provide useful information on short-term fluctuations. We look over three such empirical relationships for the United States that have been previously suggested by King et al. (1988, 1991), namely, the balanced-growth relations and the long-run money demand.[3]

For the U.S. quarterly data, six macroeconomic variables are considered. Among them, the three real aggregate variables are per-capita real nondurable consumption (c), per-capita real gross investment (i), and per-capita real gross domestic product (y), all in logarithm. The per-capita real balances measure (m) is the log of M1 per-capita minus the log of the implicit price deflator. The inflation rate (π) is the first difference of the log of the implicit price deflator. The nominal interest rate is the three-month Treasury bills rate (R). Except for the interest rate, all measures are seasonally adjusted.[4] The sample period is divided into 1953:2–2000:1 for in-sample estimation and 2000:2–2001:4 for forecast evaluations.

Variables c, i, and y are often characterized as I(1) processes with drift, whereas m, R, and π are often assumed to be I(1) processes without drift. The results of applying the augmented Dickey–Fuller (ADF) tests to individual series and their first differences over the whole sample period support this view.[5]

We next identify several cointegrating vectors by examining whether the hypothesized long-run relations are valid. If necessary, a constant and a deterministic time trend will be included in the cointegrating space to ensure that the condition of (1) holds. Appendix C summarizes in detail how we obtain these vectors using the data over the in-sample period 1953:2–2000:1.[6] In principle, we are able to

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obtain the following three long-run relations:

$$z_{1,t} = c_t - y_t + 0.292,$$
$$z_{2,t} = i_t - y_t - 0.002t + 1.928,$$
$$z_{3,t} = m_t - 0.436y_t + 0.039R_t + 1.718\pi_t + 0.001\ t + 0.009,$$

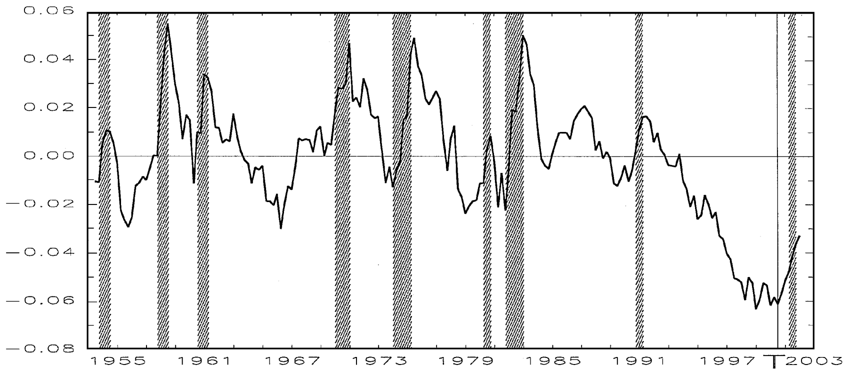
where z's are cointegration errors. The ADF test for unit roots also confirms that the above relationship transforms six I(1) time series into three stationary processes. Figure 1 displays the three cointegration error series, and each shows a tendency of mean reverting. In the figure, the mark "T" on the time horizon indicates the last quarter of the in-sample period (i.e., 2000:1). The shaded areas in all figures are NBER recessions, where each starts at a peak date and ends at a trough date.

The first two long-run relations are known as the balanced-growth relations. These are predictions from standard real-business-cycle models suggesting that income, consumption, and investment share a common stochastic trend when moving along a steady-state growth path.[7] The errors of the consumption–income relation, $\{z_{1,t}\}$, have been found to exhibit a significant increase during the NBER recessions. This finding is consistent with the permanent-income hypothesis studies in that a high consumption-to-income ratio is associated with low current output growth and can forecast high output growth in the near future.[8] The hypothesis implies that people use savings or borrow money to maintain a smooth pattern of consumption when hit by negative transitory income shocks. Hence, the ratio of consumption relative to income is higher in a recession.

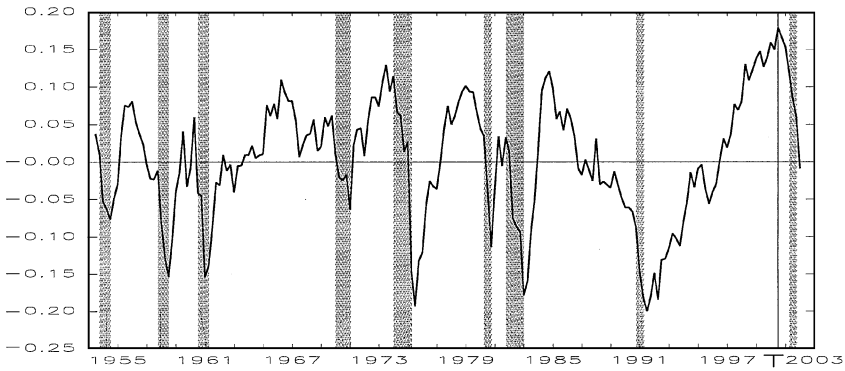
On the contrary, the errors of the investment–income relation (adjusted by the rate of inflation), $\{z_{2,t}\}$, always decline at economic downturns. This is consistent with the typical observation of business cycles where investment is highly procyclical and volatile.[9] Because profitability of investment depends on the general level of economic activity, entrepreneurs have an incentive to cut back on investments during a recession.

The third relation is known as the long-run money demand, which has long been a center of interest in macroeconomics. The estimated cointegrating vector that we yield for this relation is consistent with the theoretical suggestion of a positive income elasticity and a negative interest elasticity.[10] We find that dramatic drops in the cointegration error series, $\{z_{3,t}\}$, often coexist with recession periods, with an obvious exception of the 1995–1999 period. The drops are probably due to the Federal Reserve's response to downturns with prompt and large reductions in interest rates, as noted by Romer and Romer (1994). This point is further confirmed by observing that the movements in the errors of long-run money demand are mainly determined by swings in interest rates, whereas real money, real income, and the inflation rate are smoother time series.[11]

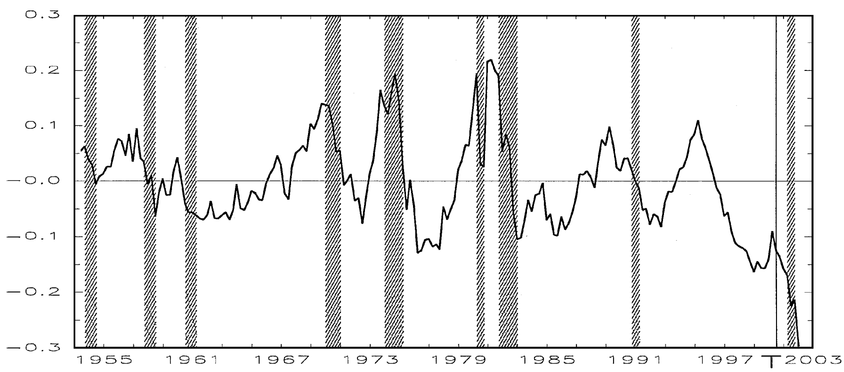
In summary, the deviation of these three long-run equilibrium relationships seems to be quite informative about the underlying economic structure. In the



(A) Cointegration error $z_{1,t}$



(B) Cointegration error $z_{2,t}$



(C) Cointegration error $z_{3,t}$

Figure 1. Cointegration errors. T marks the last quarter of the in-sample period. The shaded areas are NBER recessions.

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following empirical analysis, we calculate the disequilibrium index based on these cointegrations in order to build our forecasting models.

3.2. Disequilibrium Index Based on Three Long-Run Relationships

The corresponding cointegrating matrix with respect to $\boldsymbol{X}_t^* = [c_t, i_t, y_t, m_t, R_t, \pi_t, t, 1]'$ is

$$\boldsymbol{\beta}' = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0.292 \\ 0 & 1 & -1 & 0 & 0 & 0 & -0.002 & 1.928 \\ 0 & 0 & -0.436 & 1 & 0.039 & 1.718 & 0.001 & 0.009 \end{bmatrix}. \qquad \textbf{(5)}$$

By construction, the error correction elements of $\boldsymbol{Z}_t$ $(=\boldsymbol{\beta}'\boldsymbol{X}_t^*)$ are three cointegration error series with zero sample means over the in-sample period. Under the assumption that these error series have a joint Gaussian distribution with zero mean and finite covariance $\boldsymbol{\Sigma}$, Lemma 2 implies that $d(\boldsymbol{\beta}, \boldsymbol{X}_t^*)$ has a statistical distribution of the square root of the weighted sum of three independent $\chi^2(1)$ random variables. The weights are the eigenvalues of $(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}$, where $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}\boldsymbol{\Sigma}^{-1/2}$.

The normality assumption is tested using the Jarque–Bera skewness–kurtosis statistic.[12] The statistics are 2.28, 1.61, and 0.84 for $z_{1,t}$, $z_{2,t}$, and $z_{3,t}$, respectively. We thus conclude that the null hypothesis of normality is not rejected at the 5% level for each individual cointegration error series.

We compute the MLE estimate of $\boldsymbol{\Sigma}$ using the in-sample data. The weights are then computed on the basis of this estimate. The 5% and 10% probability upper bounds for this nonstandard distribution are generated from Monte Carlo simulations based on 50,000 observations. These bounds offer us a view from the ex post perspective of how likely an extreme-valued disequilibrium measure will appear at some point in time.[13]

Figure 2A plots the index $d(\boldsymbol{\beta}, \boldsymbol{X}_t^*)$ and the probability bounds associated with it, with 2000:2–2001:4 extended as the forecast period (the seven quarters following the end-of-in-sample quarter T in the figure). Figure 2B plots the annual rate of real output growth, computed as $g_t \equiv y_t - y_{t-4}$. The index therein often shows a rising trend not long before recessions, which is consistent with our conjecture that the economy is fragile at the point of time when its deviation to an economic equilibrium enlarges and could be hit hard by negative shocks and even result in serious recessions.

For the two notorious recessions caused by energy crises (1974–1975 and 1980), the observed values of $d(\boldsymbol{\beta}, \boldsymbol{X}_t^*)$ are exceptionally high (above the 10% statistical probability upper bound) not long before the downturn of these two contractions. In addition, the worst annual growth rates of output during these two contractions are below -5% and the durations are as long as 16 months. This indicates an interesting feature that the further the economy is from the attractor prior to a recession, the more severe the contraction might be. In addition, given the high

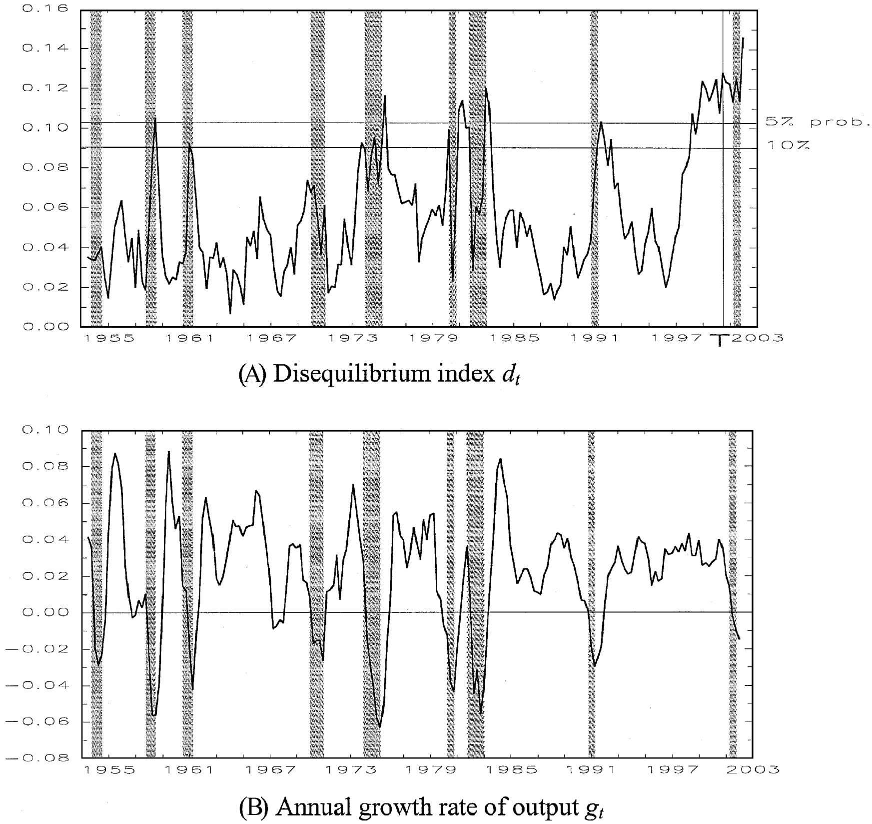

Figure 2. Disequilibrium index and growth rate of output. T marks the last quarter of the in-sample period.

measures of the disequilibrium index in the late 1990's, one would have foreseen some danger in the U.S. economy that a business-cycle turning point would be likely to occur once the economy was hit by negative shocks.

Although the index would not have signaled a large-sized disequilibrium for the downturns in 1960, 1969, and 1990, the index indeed reveals some danger by exhibiting a rising tendency prior to recessions. Interestingly, some local maxima of the index reached during expansion periods seem to be able to signal the economy's vulnerability to tackle negative shocks. For instance, the index hikes in 1955 and in 1966 are both followed by sudden decreases in output growth.

One notices that the index in general hits a high around a trough date, which is when the economy is recovering. The common scenario in these episodes was that the consumption-to-income ratio and the money-demand error were above the attractor whereas the investment-to-income ratio was below it. On the contrary,

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TABLE 1. Business cycles between 1954 and 1991[a]

Business cycles			Sample mean of d in expansion $\bar{d}$	Sample mean of g in recession $\bar{g}$ (%)	Minimum g in recession min[g] (%)	Duration of recession (months)
Troughs	Peaks	Troughs				
1954:05	1957:08	1958:04	0.037	−4.6	−5.6	8
1958:04	1960:04	1961:02	0.034	−1.6	−4.2	10
1961:02	1969:12	1970:11	0.038	−1.9	−2.7	11
1970:11	1973:11	1975:03	0.047	−4.1	−6.3	16
1975:03	1980:01	1980:07	0.066	−4.1	−4.4	6
1980:07	1981:07	1982:11	0.106	−3.5	−5.6	16
1982:11	1990:07	1991:03	0.041	−2.4	−3.0	8

Correlation coefficient between $\bar{d}$ and $\bar{g}$: −0.34
Correlation coefficient between $\bar{d}$ and $\bar{g}$ without the data of the 1980–1982 cycle: −0.52

[a] The disequilibrium index d_t is computed using the cointegrating matrix of (5), estimated from the in-sample data. The annual rate of real output growth, g_t, is computed as $y_t - y_{t-4}$.

when the economy was truly vulnerable to negative shocks before the peak dates were reached, the common feature was that the consumption-to-income ratio and money-demand error had been moving toward the attractor from below whereas the investment-to-income ratio had been moving downward toward it. This finding suggests that the movements of cointegration errors $z_{1,t}$, $z_{2,t}$, and $z_{3,t}$ were in opposite directions during expansions versus during contractions, and thus the economy must have bounced back to its attractor before shifting from an expansion phase to a contraction one. Indeed, the index we propose here is not sensitive to the direction of the deviation from the long-run equilibrium. In the subsequent section, an alternative index that accounts for this asymmetric phenomenon in business cycles will be further investigated via nonlinear forecasting models.

In an attempt to investigate whether the size of an expansion's disequilibrium is associated with the economy's vulnerability, we compute the sample correlation coefficient between the index average in each expansion ($\bar{d}$) and the average of the output growth rate in the following recession ($\bar{g}$) over the sample period of 1954–1991.[14] The results are reported in Table 1. We find that $\bar{d}$ and $\bar{g}$ are negatively correlated with coefficient −0.34. It is noteworthy that the recession of 1981–1982 came abruptly after a brief recovery in 1980 and very likely created an outlier in this data set. After dropping the data of $\bar{d}$ and $\bar{g}$ corresponding to the 1980–1982 cycle, the sample correlation coefficient is −0.52, which is quite an impressive degree of negative association between the index and the output growth. This somewhat confirms the conjecture we made in the introduction that the further the system is from the attractor, the more vulnerable and severely hit by negative shocks the system will be.

3.3. Forecasting Macroeconomic Variables: The Disequilibrium Index and Nonlinear Adjustment

In this section we evaluate the potential benefit of using the disequilibrium index in forecasting the six stationary macroeconomic variables: $\Delta \boldsymbol{X}_t = \{\Delta c_t, \Delta i_t, \Delta y_t, \Delta m_t, \Delta R_t, \Delta \pi_t\}'$. Continuing from the previous section, 1953:2–2000:1 is the in-sample estimation period and 2000:2–2001:4 is the forecast period. The evaluation is based on the fixed estimate of the cointegrating matrix $\boldsymbol{\beta}$ that is reported in the preceding section. Namely, the index used in the following forecast models is the one plotted in Figure 2A.

To simplify our notations, the disequilibrium index is denoted by d_t hereafter. Recall that d_t is a nonlinear function of the elements of $\boldsymbol{Z}_t$, which is a vector of I(0) variables. Therefore, d_t is itself an I(0) variable, provided that the definition of I(0) satisfies those specified by White (1984).[15] However, it is possible that nonstationarity emerges via nonlinear transformation when the stationarity definition of $\boldsymbol{Z}_t$ is less restrictive. Whether d_t is stationary or nonstationary is an empirical issue that needs to be tested.

The ADF unit root test via a first-order autoregressive model is now applied to the index over the in-sample period.[16] The t-statistic is -4.34, which is significant at the 5% level in rejecting a unit root in d_t. This indicates that d_t is a stationary series, which is consistent with the visual impression of the mean-reversion pattern in Figure 2A.

The first forecast model estimated is an error correction model (ECM) in which each equation has a constant, four lags of every differenced variable (suggested by the Akaike information criterion), and three error correction terms $z_{j,t-1}$ $(j = 1, 2, 3)$. In total, there are 28 coefficients in each equation. To yield a more parsimonious specification, we remove autoregressive components that are insignificant at the 10% level. That is,

Restricted ECM:

$$\Delta x_{j,t} = \phi_{10} + \phi_{11}' \boldsymbol{W}_{j,t-1} + \phi_{12}' \boldsymbol{Z}_{t-1} + \varepsilon_t, \quad \textbf{(6)}$$

where $\Delta x_{j,t}$ denotes the jth $(j = 1, 2, \ldots, 6)$ variable in the vector of $\Delta \boldsymbol{X}_t$, and $\boldsymbol{W}_{j,t-1}$ is a vector of the subset of $\{\Delta \boldsymbol{X}_{t-1}, \Delta \boldsymbol{X}_{t-2}, \Delta \boldsymbol{X}_{t-3}, \Delta \boldsymbol{X}_{t-4}\}$.

At the 10% level, $z_{1,t-1}$ is significant in the equation of Δy; $z_{2,t-1}$ is significant in the equations of Δi and Δy; and $z_{3,t-1}$ is significant in the equations of Δc, Δi, Δy, and Δm. None of the error correction terms are significant in the equations of ΔR or $\Delta \pi$. To save space, these estimated coefficients are not reproduced here, but we report some summary measures of the out-of-sample forecasting ability of the linear error correction equations in Table 2A. Shortly, these equations will be compared with other equations from nonlinear ECMs (discussed later).

As opposed to the above linear ECM, some researchers argue that the short-run adjustment toward the long-run equilibrium is nonlinear. The proposition that we would like to examine in this paper adequately fits at vision. That is, we wonder whether the strength of an attraction can be different depending on the magnitude

Table 2. Comparisons of forecasting models[a]

Variable	Δc	Δi	Δy	Δm	ΔR	$\Delta \pi$
			(A) Restricted ECM[b]			
$\bar{R}^2$	0.23	0.36	0.33	0.59	0.32	0.27
AIC	−987.84	−234.34	−738.20	−837.60	815.32	−687.14
SBC	−965.34	−205.41	−706.05	−818.31	863.54	−658.21
Log-likelihood	719.61	344.86	597.79	643.49	−173.97	571.26
$\hat{\sigma}$	0.0048	0.0371	0.0094	0.0073	0.6229	0.0108
RMSE	0.0068	0.0290	0.0066	0.0113	0.9716	0.0102
			(B) NLECM1[c]			
Transition var.	d^2_{t-2}	NA	d^2_{t-2}	d^2_{t-2}	d^2_{t-1}	d^2_{t-1}
$\bar{R}^2$	0.25		0.35	0.59	0.35	0.29
AIC	−991.12		−743.76	−836.12	807.79	−691.80
SBC	−965.40		−708.40	−813.62	859.23	−659.65
Log-likelihood	722.25		601.57	643.75	−169.21	574.59
$\hat{\sigma}$	0.0048		0.0092	0.0073	0.6069	0.0107
RMSE	0.0070		0.0076	0.0113	1.0434	0.0109
			(C) NLECM2[d]			
Transition var.	$\tilde{d}_{t-2}$	$\tilde{d}_{t-1}$	$\tilde{d}_{t-1}$	$\tilde{d}_{t-1}$	$\tilde{d}_{t-1}$	$\tilde{d}_{t-1}$
Transition function	logistic	logistic	logistic	logistic	logistic	expon.
$\bar{R}^2$	0.28	0.39	0.37	0.57	0.37	0.29
AIC	−995.19	−238.86	−743.47	−823.73	804.51	−687.54
SBC	−956.61	−193.85	−695.25	−788.37	868.81	−642.53
Log-likelihood	728.29	352.12	605.42	641.56	−163.56	576.46
$\hat{\sigma}$	0.0046	0.0357	0.0090	0.0074	0.5886	0.0105
RMSE	0.0042	0.0206	0.0058	0.0110	0.8096	0.0106

[a] The values of $\bar{R}^2$ (adjusted R^2), AIC, SBC, log-likelihood, and $\hat{\sigma}$ (the residual standard deviation) are calculated using the data of the in-sample period 1953:2–2000:1. The out-of-sample root-mean-square forecast errors (RMSE) are calculated using one-step-ahead forecasts for the period 2000:2–2001:4.
[b] Each equation is the regression that removes insignificant autoregressive components (at the 10% level) from a standard fourth-order error correction equation.
[c] The error correction terms adjust nonlinearly as in (7).
[d] The model includes (9) and (10) when the transition function is logistic, and includes (9) and (11) when the transition function is exponential.

of the disequilibrium. A smooth transition mechanism is usually incorporated in an ECM to allow for a nonlinear or asymmetric adjustment,[17] but the existing method of nonlinear ECMs is problematic when there are multiple equilibrium relationships. This is because the final models will depend on how the error correction terms are incorporated. The proposed measure is an ideal building block for nonlinear ECMs, because it is invariant to the normalization of cointegrating vectors.

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We hereby propose a model in which the error correction terms react to exogenous shocks in a nonlinear fashion. Specifically, we consider a single equation from a nonlinear ECM that uses the disequilibrium measure as the transition variable as follows.[18]

NLECM1:

$$\Delta x_{j,t} = \phi_{20} + \phi_{21}' \mathbf{W}_{j,t-1} + \phi_{22}' \mathbf{Z}_{t-1} F(d_{t-l}; \gamma) + \varepsilon_t, \qquad (7)$$

where

$$F(d_{t-l}; \gamma) = \frac{1}{1 + \exp\{-\gamma d_{t-l}^2\}} - \frac{1}{2}, \quad \gamma > 0, \qquad (8)$$

where $F(d_{t-l}; \gamma)$ is specified as a logistic function such that $F(0; \gamma) = 0$.

This model would result in a gradually increasing strength of adjustment to the error correction terms for larger deviations (i.e., larger d_{t-l}^2) from the equilibrium. A higher parameter γ would indicate faster speed of the transition. Note that to ensure smoothness of transition at $\mathbf{Z}_{t-1} = 0$, the index variable enters the transition function $F(d_{t-l}; \gamma)$ in a squared form.

Before attempting to estimate the above nonlinear equation, we adopt Teräsvirta's (1994) test for the null hypothesis of linearity, that is, $H_0 : \gamma = 0$, to the in-sample data.[19] The nonlinearity test is implemented for d_{t-l}^2, $l = 1, 2$. The most appropriate delay parameter l is determined by the minimum p-value rule of rejecting the null hypothesis of linearity, as suggested by Teräsvirta (1994). When linearity for a given equation is rejected, we estimate it by the nonlinear least-squares method. As proved by Teräsvirta (1994), under appropriate regularity conditions, the estimators will be consistent and asymptotically normally distributed.

The null hypothesis of linearity, using d_{t-l}^2 as a transition variable in the alternative, is rejected at the 10% level for the equations of Δc, Δm, ΔR, and $\Delta \pi$. The test statistic is marginally significant for the equation of Δy. Table 2B presents the forecasting results for these nonlinear error correction equations. The out-of-sample root-mean-square errors (RMSE) are calculated using a one-step-ahead forecast for the period of 2000:2–2001:4. Except for the equation of Δm, the in-sample fitness measure $\bar{R}^2$ and the model selection criteria AIC (Akaike information criterion) and SBC (Schwartz's Bayesian criterion) all favor NLECM1 over ECM. However, no equation outperforms its linear counterparts based on the RMSE of the one-period-ahead forecast.

The lack of forecasting power of the proposed index possibly comes with one fact. That is, the usefulness of the disequilibrium index as a proxy for the status of economic vulnerability is asymmetric in response to positive and negative shocks in these cointegration relationships. Over the whole sample period, the annual output growth rate is negatively correlated with $z_{1,t}$ and $z_{3,t}$, and positively correlated with $z_{2,t}$.[20] This suggests that cointegration errors $z_{1,t}$ and $z_{3,t}$ are countercyclical, while $z_{2,t}$ is procyclical.

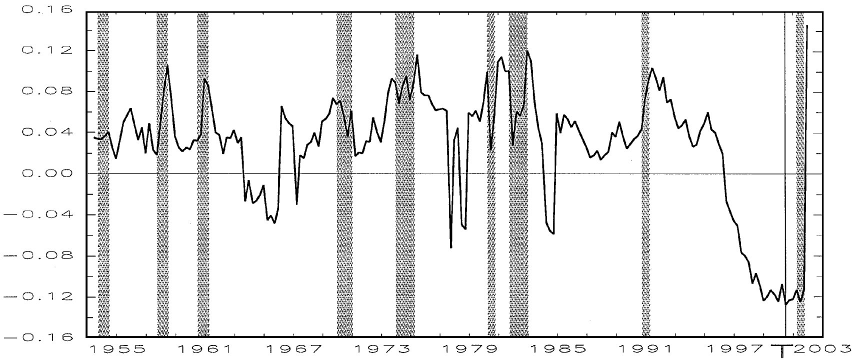

FIGURE 3. Signed index $\tilde{d}_t$. T marks the last quarter of the in-sample period.

To allow our index to reveal information about the status of the business cycle, we construct a signed index that accounts for the signs of the z's as follows:

$$\tilde{d}_t = \begin{cases} -d_t & \text{for } z_{1,t} < 0, z_{2,t} > 0, \text{ and } z_{3,t} < 0 \\ d_t & \text{otherwise.} \end{cases}$$

Consequently, the signed index is negative when $z_{1,t}$ and $z_{3,t}$ are below the attractor but $z_{2,t}$ is above it, and is positive otherwise. See Figure 3 for the movements in $\tilde{d}_t$.

We now consider an alternative nonlinear ECM that has the signed index as the transition variable, as follows.

NLECM2:

$$\Delta x_{j,t} = \phi_{30} + \phi_{31}' \boldsymbol{W}_{j,t-1} + \phi_{32}' \boldsymbol{Z}_{t-1} + \phi_{33}' \boldsymbol{Z}_{t-1} F(\tilde{d}_{t-l}; \gamma, c) + \varepsilon_t. \quad \mathbf{(9)}$$

Here we consider the transition function to be either a logistic function,

$$F(\tilde{d}_{t-l}; \gamma, c) = \frac{1}{1 + \exp\{-\gamma(\tilde{d}_{t-l} - c)\}}, \qquad \gamma > 0, \quad \mathbf{(10)}$$

or to be an exponential function,

$$F(\tilde{d}_{t-l}; \gamma, c) = 1 - \exp\{-\gamma(\tilde{d}_{t-l} - c)^2\}, \qquad \gamma > 0. \quad \mathbf{(11)}$$

While γ determines the speed of the transition, c is the centrality (or threshold) parameter. When the smooth transition is taken to be the logistic form of (10), the strength of the reversion of $\boldsymbol{Z}_{t-1}$ to its attractor changes monotonically from ϕ_{32} to $\phi_{32} + \phi_{33}$ for increasing values of $\tilde{d}_{t-l}$. Such a nonlinear ECM therefore allows for different adjustments to deviations at different directions from c.

When the smooth transition function is taken to be the exponential form of (11), the resulting nonlinear ECM has the strength of adjustment changing from

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$\phi_{32} + \phi_{33}$ to ϕ_{32} and back to $\phi_{32} + \phi_{33}$ with increasing values of $\tilde{d}_{t-l}$, with the change symmetric around c. Such a model can distinguish between small and large deviations.

The null hypothesis of linearity, using $\tilde{d}_{t-l}$ ($l = 1, 2$) as a transition variable in the alternative model, is rejected at the 10% level for the equation of $\Delta\pi$ in favor of the alternative model of (9) and (11). It is also rejected for the remaining five equations in favor of the alternative model of (9) and (10).[21] This suggests that the adjustment to long-run deviations is asymmetric to different phases of business cycles in these five macroeconomic variables.

We follow the suggestion of Teräsvirta (1994) to rescale the speed parameter γ by the standard deviation of the transition variable, $\hat{\sigma}(\tilde{d}_{t-l})$. The estimated equations are reported in Appendix D and the forecasting results are summarized in Table 2C. Recall that no error correction terms are significant in the linear error correction equations of ΔR and $\Delta\pi$. By contrast, at least one error correction term has a significant nonlinear adjustment (at the 10% significance level) in all six equations. The estimated speed parameters $\hat{\gamma}$ in these nonlinear error correction equations are rather large so that the model is quite similar to the two-regime threshold autoregressive (TAR) model discussed by Tong (1990).

Except for the equation of Δm, the remaining five equations have a better in-sample fit than their linear counterparts (in terms of $\bar{R}^2$), while AIC (but not SBC) also suggests that model NLECM2 is more appropriate than the restricted ECM. In addition, the smaller estimates of the residual standard error ($\hat{\sigma}$) indicate a more accurate estimation for the NLECM2 model. Based on RMSE, the out-of-sample forecasting capacity of NLECM2 is impressively favorable for five out of six equations, with significant improvement found in predicting the three real variables: Δc, Δi, and Δy.

4. CONCLUSION

Ever since the classic study of Mitchell and Burns (1938), the prediction of business-cycle turning points has been one of the core tasks of business-cycle analysis. Existing methods have been designed specifically to predict the turning points but not their severity, nor have these methods employed economic theory for this purpose. In light of the above caveats, we propose in this paper a new measure as an indicator for vulnerability of an economy. We have demonstrated that it can be measured using the deviations from long-run economic relationships, the notion of cointegration. We analyze quarterly U.S. data for the period of 1953:2–2001:4.

We do find some links between the disequilibrium index we propose and economic vulnerability. The average index during expansions is negatively correlated with the average contraction in output during recessions. Particularly, the U.S. economy had a high measure of the index before the 2001 recession started. We have explored the usefulness of the index in forecasting six prominent macroeconomic variables. We find that the capacity of the index in predicting business cycles seems limited in that as an indicator for vulnerability, the index has asymmetric interpretation power in response to positive and negative shocks. A nonlinear ECM that includes a revised disequilibrium index as a transition variable performs better than the linear ECM in predicting the most recent downturn. This leads us to encourage future research on business cycles to incorporate deviations from long-run economic relationships in nonlinear models.

NOTES

1. For the ratchet effect, see Beaudry and Koop (1993) and Pesaran and Potter (1997) for details.

2. Although the r elements in $\{\boldsymbol{\beta}'\boldsymbol{X}_t\}$ are standardized errors, the linear combinations of these r errors that appear in the formula of $d(\boldsymbol{\beta}, \boldsymbol{X}_t)$ generally do not have unit standard deviations.

3. Stock and Watson (1999) also investigate these relationships in a business-cycle paper; however, they do not further study their linkage with business cycles. The interest-rate spread cointegration is not considered in this paper. In an earlier investigation, we found that the spread series contained too much short-run noise, and therefore could not help in forecasting business-cycle turning points if included.

4. All data were collected from the Web site www.stls.frb.org, maintained by the Federal Reserve Bank of St. Louis. Nondurable consumption is computed as total consumption minus durable consumption. Investment is gross private domestic investment. Private real output is defined as GDP minus government expenditures and gross investment. These are real variables in chained 1996 prices. The monthly money supply M1 series for 1959:1–2001:4 was collected from the Web site. Data on M1 prior to 1959:1 were formed by splicing the M1 series reported in *Banking and Monetary Statistics, 1941–1970* [Board of Governors of the Federal Reserve System (1976)] to the data in January 1959. The monthly data were averaged to obtain quarterly observations. Consumption, investment, output, and money are expressed in per-capita terms using total civilian noninstitutional population. Inflation rate and the three-month Treasury bill rates are all measured in annual percentages.

5. For c, i, and y, the ADF test statistics are the t-statistics computed from OLS regressions that contain an intercept, four distributed lags, and a time trend. For m, R, and π, the regressions are without a time trend. The null hypothesis of a unit root is not rejected for c, y, m, R, and π at the conventional levels. For i, a unit-root null hypothesis is not rejected at the 1% level. A unit root in each first difference of these variables is rejected at the 5% level.

6. An alternative method to obtain cointegrating vectors is Johansen's (1991) full-information maximum likelihood estimators from a multivariate system, but unfortunately the two test statistics (trace and maximum eigenvalue) give ambiguous results. The ML estimates are not stable with respect to whether deterministic terms (trend and constant) are included in the system and barely show exactly those three cointegrating vectors implied by the theoretical model. Given the large number of regressors in the Johansen method, we chose not to rely on it. Instead, we impose the long-run relationships implied by the theoretical model and use a simpler, but equally reliable, univariate test on the restricted long-run cointegrating vectors. More details are presented in Appendix C.

7. See King et al. (1988, 1991) for theoretical and empirical background.

8. See Campbell (1987) and Cochrane and Sbordone (1988).

9. Further evidence of this different cyclical behavior in consumption and investment can be found in King and Rebelo (2000). Investments are also found to be about three times more volatile than income, whereas consumption is actually smoother than income.

10. The U.S. long-run money demand function estimated in the context of cointegration can be also found in Hoffman and Rasche (1991) and Stock and Watson (1999), among many others.

11. The sample standard deviation is equal to 0.11 for real money, 0.27 for real income, 0.03 for the inflation rate, and 2.75 for the interest rate.

12. The Jarque–Bera test statistic, $TS^2/6 + T(K-3)^3/24$, is asymptotically $\chi^2(2)$ distributed, where T is the number of observations, S is sample skewness, and K is sample kurtosis. The 10% critical value is 4.61.

13. Note that these probability bounds are computed on the basis of our estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$. To give an accurate inference about the uncertainty of these bounds would require further Monte Carlo simulations. We leave that part to future research.

14. Up to the present, the March 2001 peak was the most recent decision made by the Business Cycle Dating Committee of the NBER. The trough date of the 2001 recession has not yet been determined.

15. See White (1984, Theorem 3.35).

16. Both the Akaike information criterion and Schwartz Bayesian criterion choose the lag order to be one.

17. This type of nonlinear extensions of the ECMs can be found in Teräsvirta and Anderson (1992), Granger and Swanson (1996), and Anderson (1997).

18. A more general specification is to allow the transitory components among those that would nonlinearly react to exogenous shocks. However, we do not find that this type of model outperforms the models reported in this paper.

19. To save space, the results of the linearity test are not reported but are available from the authors upon request. For details of the linearity test, see Luukkonen et al. (1988) and Teräsvirta (1994).

20. The sample coefficients of correlation between $z_{1,t}$, $z_{2,t}$, and $z_{3,t}$ and the annual output growth rates are -0.42, 0.52, and -0.13, respectively.

21. The results of the linearity test are not reported but are available from the authors upon request.

REFERENCES

Anderson, H.M. (1997) Transaction costs and non-linear adjustment toward equilibrium in the US treasury bill market. *Oxford Bulletin of Economics and Statistics* 59, 465–484.

Beaudry, P. & G. Koop (1993) Do recessions permanently affect output? *Journal of Monetary Economics* 31, 149–163.

Campbell, J.Y. (1987) Does saving anticipate declining labor income? An alternative test of the permanent income hypothesis. *Econometrica* 55, 1249–1273.

Cochrane, J. & A.M. Sbordone (1988) Multivariate estimates of permanent components of GNP and stock prices. *Journal of Economic Dynamics and Control* 12, 255–296.

Engle, R.F. & C.W.J. Granger (1987) Co-integration and error correction: Representation, estimation and testing. *Econometrica* 55, 313–328.

Fair, R.C. (1993) Estimating event probabilities from macroeconometric models using stochastic simulation. In J.H. Stock & M.W. Watson (eds.), *Business Cycles, Indicators, and Forecasting*, pp. 157–176. Chicago: University of Chicago Press.

Granger, C.W.J. (1981) Some properties of time series data and their use in econometric model specification. *Journal of Econometrics* 16, 121–130.

Granger, C.W.J. & N. Swanson (1996) Future developments in the study of cointegrated variables. *Oxford Bulletin of Economics and Statistics* 58, 537–553.

Hansen, B.E. (1992) Tests for parameter instability in regressions with I(1) processes. *Journal of Business and Economic Statistics* 10, 321–335.

Hoffman, D.L. & R.H. Rasche (1991) Long run income and interest elasticities of money demand in the United States. *Review of Economics and Statistics* 73, 665–674.

Hymans, S. (1973) On the use of leading indicators to predict cyclical turning points. *Brookings Papers on Economic Activity* 2, 339–384.

Johansen, S. (1991) Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregression models. *Econometrica* 59, 1551–1580.

King, R.G. & S.T. Rebelo (2000) Resuscitating Real Business Cycles. NBER working paper 7534.

King, R.G., C.I. Plosser & S.T. Rebelo (1988) Production, growth, and business cycles: II, New directions. *Journal of Monetary Economics* 21, 309–342.

King, R.G., C.I. Plosser, J.H. Stock & M.W. Watson (1991) Stochastic trends and economic fluctuations. *American Economic Review* 81, 819–840.

Klein, L.R. (1987) The E.T. interview. *Econometric Theory* 3, 409–460.
Kling, J.L. (1987) Predicting the turning points of business and economic time series. *Journal of Business* 60, 201–238.
Luukkonen, R., P. Saikkonen & T. Teräsvirta (1988) Testing linearity against smooth transition autoregressive models. *Biometrika* 75, 491–499.
Mitchell, W.C. & A.F. Burns (1938) *Statistical Indicators of Cyclical Revivals*. New York: NBER.
Neftci, S.N. (1982) Optimal prediction of cyclical downturns. *Journal of Economic Dynamics and Control* 4, 225–241.
Park, J.Y. (1992) Canonical cointegrating regressions. *Econometrica* 60, 119–143.
Pesaran, M.H. & S.M. Potter (1997) A floor and ceiling model of US output. *Journal of Economic Dynamics and Control* 21, 661–695.
Phillips, P.C.B. & B. Hansen (1990) Statistical inference in instrumental variables regression with I(1) processes. *Review of Economic Studies* 57, 99–125.
Romer, C.D. & D.H. Romer (1994) What Ends Recessions? NBER working paper 4765.
Stock, J.H. & M.W. Watson (1989) New indexes of coincident and leading economic indicators. *NBER Macroeconomics Annual*, 351–394.
Stock, J.H. & M.W. Watson (1993) A procedure for predicting recessions with leading indicators: Econometric issues and recent experience. In J.H. Stock & M.W. Watson (eds.), *Business Cycles, Indicators, and Forecasting*, pp. 95–153. Chicago: University of Chicago Press.
Stock, J.H. & M.W. Watson (1999) Business cycle fluctuations in U.S. macroeconomic time series. In J.B. Taylor & M. Woodford (eds.), *Handbook of Macroeconomics*, vol. 1, pp. 3–64. New York and Oxford: Elsevier Science, North-Holland.
Swanson, N.R., E. Ghysels & M. Callan (1999) A multivariate time series analysis of the data revision process for industrial production and the composite leading indicator. In R.F. Engle & H. White (eds.), *Cointegration, Causality, and Forecasting*, pp. 45–75. New York: Oxford University Press.
Teräsvirta, T. (1994) Specification, estimation and evaluation of smooth transition autoregressive models. *Journal of American Statistical Association* 89, 208–218.
Teräsvirta, T. & H.M. Anderson (1992) Characterizing nonlinearities in business cycles using smooth transition autoregressive models. *Journal of Applied Econometrics* 7, S119–S136.
Tong, H. (1990) *Non-linear Time Series: A Dynamical System Approach*. Oxford: Oxford University Press.
Wecker, W.E. (1979) Predicting the turning points of a time series. *Journal of Business* 52, 35–50.
White, H. (1984) *Asymptotic Theory for Econometricians*. Orlando, FL: Academic Press.

APPENDIX A: GEOMETRIC INTERPRETATION OF THE DISEQUILIBRIUM INDEX

The idea of the proposed measure of disequilibrium in an economy is to calculate the distance of an observation $\boldsymbol{X}_t$ in the n-dimensional Euclidean space to its attractor $\boldsymbol{\beta}'\boldsymbol{X}_t = 0$, which represents multiple long-run equilibrium relations. Figure A.1 illustrates the geometric notion of the distance measure in terms of projection onto the attractor.

Let $\boldsymbol{\beta}_\perp$ denote any matrix that is orthogonal to $\boldsymbol{\beta}$. The attractor is the nullspace of $\boldsymbol{\beta}$, or the space spanned by $\boldsymbol{\beta}_\perp$, denoted as $S(\boldsymbol{\beta}_\perp)$. The projection of $\boldsymbol{X}_t$ onto $S(\boldsymbol{\beta}_\perp)$ is $P_{\boldsymbol{\beta}_\perp}\boldsymbol{X}_t$, where $P_{\boldsymbol{\beta}_\perp} = \boldsymbol{\beta}_\perp(\boldsymbol{\beta}_\perp'\boldsymbol{\beta}_\perp)^{-1}\boldsymbol{\beta}_\perp'$. The component in the orthogonal complement is $M_{\boldsymbol{\beta}_\perp}\boldsymbol{X}_t$, where $M_{\boldsymbol{\beta}_\perp} = \boldsymbol{I} - \boldsymbol{\beta}_\perp(\boldsymbol{\beta}_\perp'\boldsymbol{\beta}_\perp)^{-1}\boldsymbol{\beta}_\perp'$. The proposed distance index is the length (or norm) of

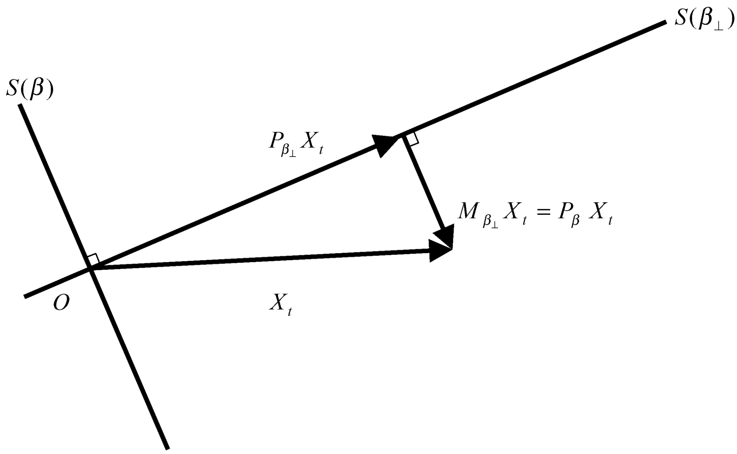

FIGURE A.1. Projection of $\boldsymbol{X}_t$ onto $S(\beta_\perp)$ and its orthogonal complement.

$M_{\beta_\perp}\boldsymbol{X}_t$, denoted by $\|M_{\beta_\perp}\boldsymbol{X}_t\|$. Because the column vectors of $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_\perp$ are orthogonal to each other, it is always true that $\boldsymbol{I} = \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}' + \boldsymbol{\beta}_\perp(\boldsymbol{\beta}'_\perp\boldsymbol{\beta}_\perp)^{-1}\boldsymbol{\beta}'_\perp$. Therefore, the index can be simplified to

$$\|M_{\beta\perp}\boldsymbol{X}_t\| = \|\boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\boldsymbol{X}_t\| = \|P_\beta\boldsymbol{X}_t\|,$$

where $P_\beta = \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'$. In brief, the index calculates the distance of the projection of $\boldsymbol{X}_t$ onto the space spanned by $\boldsymbol{\beta}$.

APPENDIX B: PROOF OF LEMMAS

B.1. PROOF OF LEMMA 1

Let $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}\boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is an $r \times r$ matrix with full rank. In this case the columns of $\tilde{\boldsymbol{\beta}}$ are linear combinations of the columns of $\boldsymbol{\beta}$ and are linearly independent of each other. Because $\boldsymbol{\Gamma}$ is a full-rank square matrix, its inverse exists. It is now straightforward to show that

$$\tilde{\boldsymbol{\beta}}(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}\tilde{\boldsymbol{\beta}}' = \boldsymbol{\beta}\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'\boldsymbol{\beta}'\boldsymbol{\beta}\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}'\boldsymbol{\beta}' = \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'.$$

In a special case where $\boldsymbol{\Gamma}$ is a diagonal matrix, the ith column of $\tilde{\boldsymbol{\beta}}$ is the ith column of $\boldsymbol{\beta}$ rescaled by the ith diagonal element of $\boldsymbol{\Gamma}$. ■

B.2. PROOF OF LEMMA 2

Define $\tilde{\boldsymbol{\beta}} \equiv \boldsymbol{\beta}\boldsymbol{\Sigma}^{-1/2}$. According to the invariance property of Lemma 1,

$$d^2(\boldsymbol{\beta}, \boldsymbol{X}_t) = \boldsymbol{X}_t'\boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\boldsymbol{X}_t = \boldsymbol{X}_t'\tilde{\boldsymbol{\beta}}(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}\tilde{\boldsymbol{\beta}}'\boldsymbol{X}_t = d^2(\tilde{\boldsymbol{\beta}}, \boldsymbol{X}_t).$$

Given that $(\boldsymbol{\beta}'\boldsymbol{X}_t)$ has a Gaussian distribution $N(0, \boldsymbol{\Sigma})$, it is easily verified that $(\tilde{\boldsymbol{\beta}}'\boldsymbol{X}_t)$ is $N(0, \boldsymbol{I}_r)$-distributed. Note that $(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}$ is a real symmetric matrix with rank r. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ be the orthonormal eigenvectors corresponding to the eigenvalues, denoted as $\lambda_1, \ldots, \lambda_r$, of $(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}$. We then have

$$(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1} = \sum_{i=1}^{r} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i'.$$

Combined with the fact that $\tilde{\boldsymbol{Z}}_t \equiv \tilde{\boldsymbol{\beta}}'\boldsymbol{X}_t$ is a stochastic r-dimensional random vector with distribution $N(0, \boldsymbol{I}_r)$, it follows that

$$d^2(\boldsymbol{\beta}, \boldsymbol{X}_t) = \tilde{\boldsymbol{Z}}_t' \left(\sum_{i=1}^{r} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i' \right) \tilde{\boldsymbol{Z}}_t = \sum_{i=1}^{r} \lambda_i \tilde{\boldsymbol{Z}}_t' \, \boldsymbol{u}_i \boldsymbol{u}_i' \tilde{\boldsymbol{Z}}_t = \sum_{i=1}^{r} \lambda_i y_{it}^2,$$

where $y_{it} = \boldsymbol{u}_i'\tilde{\boldsymbol{Z}}_t$ is a scalar variable. Because these $\boldsymbol{u}_i$'s constitute an orthonormal basis (i.e., $\boldsymbol{u}_i'\boldsymbol{u}_i = 1$ and $\boldsymbol{u}_i'\boldsymbol{u}_j = 0$ for $i \neq j$), y_{it} is $N(0, 1)$-distributed with $\mathrm{cov}(y_{it}, y_{jt}) = 0$, for $i \neq j$. Thus, $d^2(\boldsymbol{\beta}, \boldsymbol{X}_t)$ is the weighted sum of r independent χ^2 variables that each has degrees of freedom 1, with the eigenvalues of $(\tilde{\boldsymbol{\beta}}'\tilde{\boldsymbol{\beta}})^{-1}$ as weights. ■

APPENDIX C: ESTIMATING COINTEGRATING VECTORS

Since some long-run relations among $\{c_t, i_t, y_t, m_t, R_t, \pi_t\}$ are implied by macroeconomic theory, we test whether these relations are supported by our data for the in-sample period of 1953:2–2000:1. If a hypothesized cointegration exists, then the error to the cointegration relation should contain no unit root. The presence of a unit root is tested on the basis of the t-statistic of the augmented Dickey–Fuller test. We find that the OLS regressions with autoregressive order determined by the AIC and the SBC basically yield similar results. Below, we only report results for regression models using SBC lag length.

Applying the ADF test to the suggested consumption–income relation, the null hypothesis of a unit root in a demeaned $\{c - y\}$ sequence is rejected at the 5% level when the regression model has no drift and trend terms. The demeaned cointegration error series is $z_{1,t} = c_t - y_t + 0.292$.

For the suggested investment–income relation, the null hypothesis of a unit root in $\{i - y\}$ is rejected at the 5% level from a regression that includes a trend. This suggests $\{i - y\}$ is a trend-stationary process. Regressing $\{i - y\}$ on a constant and a time trend gives us the cointegration error: $z_{2,t} = i_t - y_t - 0.002\, t + 1.928$. Reestimating the cointegrating vector over subsamples of the data reveals no evidence of structural instability.

For the long-run money-demand cointegrating vectors, we have considered the two-step frequency-zero seemingly unrelated regression methods proposed by Phillips and Hansen (1990) and Park (1992). These two methods give us similar estimates. In this paper, our estimates were made by regressing m on y, R, π, and a trend, using the Fejer kernel. The estimated demeaned cointegration error series is $z_{3,t} = m_t - 0.436y_t + 0.039R_t + 1.718\pi_t + 0.001t + 0.009$. Three statistics proposed by Hansen (1992) for testing the stability of the money-demand cointegrating vector do not yield clean results. Tests based on the Mean$F$ statistic (=6.56) and the L_c statistic (=0.65), both insignificant at the 5% level, do not suggest instability. On the contrary, Hansen's Sup$F$ statistic (=33.03) is significant with the p-value smaller than 0.01.

APPENDIX D: NONLINEAR ERROR CORRECTION MODEL NLECM2

The specified and estimated NLECM2 model is listed below. The figures in parentheses are the estimated standard errors.

$$
\begin{array}{lllll}
\Delta\hat{c}_t = & 0.004 & -\,0.0004\Delta R_{t-1} & +\,0.205\Delta c_{t-3} & -\,0.100\Delta y_{t-4} \\
 & (0.001) & (0.0005) & (0.071) & (0.034) \\
 & -\,0.012z_{1,t-1} & +\,0.037z_{2,t-1} & +\,0.014z_{3,t-1} & \\
 & (0.032) & (0.014) & (0.015) & \\
 & +\,F_t & \times\,(0.023z_{1,t-1} & -\,0.038z_{2,t-1} & -\,0.050z_{3,t-1}) \\
 & & (0.058) & (0.018) & (0.018)
\end{array}
$$

$$F_t = \{1 + \exp[-6.67(\tilde{d}_{t-2} - 0.043)/\hat{\sigma}(\tilde{d}_{t-2})]\}^{-1}$$

$$
\begin{array}{llllll}
\Delta\hat{i}_t = & -\,0.010 & +\,2.755\Delta c_{t-1} & +\,0.024\Delta R_{t-1} & +\,0.481\Delta\pi_{t-1} & \\
 & (0.004) & (0.562) & (0.005) & (0.222) & \\
 & +\,0.653\Delta m_{t-2} & +\,0.014\Delta R_{t-3} & -\,1.209z_{1,t-1} & -\,0.343z_{2,t-1} & \\
 & (0.304) & (0.004) & (0.513) & (0.151) & \\
 & +\,0.100z_{3,t-1} & +\,F_t & \times\,(2.208z_{1,t-1} & +\,0.208z_{2,t-1} & -\,0.228z_{3,t-1}) \\
 & (0.203) & & (0.697) & (0.177) & (0.221)
\end{array}
$$

$$F_t = \{1 + \exp[-3.62(\tilde{d}_{t-1} - 0.016)/\hat{\sigma}(\tilde{d}_{t-1})]\}^{-1}$$

$$
\begin{array}{llllll}
\Delta\hat{y}_t = & 0.002 & +\,0.567\Delta c_{t-1} & +\,0.005\Delta R_{t-1} & +\,0.170\Delta\pi_{t-1} & +\,0.208\Delta m_{t-2} \\
 & (0.001) & (0.143) & (0.001) & (0.062) & (0.077) \\
 & +\,0.168\Delta\pi_{t-2} & +\,0.004\Delta R_{t-3} & -\,0.269z_{1,t-1} & -\,0.032z_{2,t-1} & +\,0.051z_{3,t-1} \\
 & (0.059) & (0.001) & (0.134) & (0.035) & (0.050) \\
 & +\,F_t & \times(0.572z_{1,t-1} & +\,0.018z_{2,t-1} & -\,0.116z_{3,t-1}) & \\
 & & (0.174) & (0.044) & (0.055) &
\end{array}
$$

$$F_t = \{1 + \exp[-3.05(\tilde{d}_{t-1} - 0.026)/\hat{\sigma}(\tilde{d}_{t-1})]\}^{-1}$$

$$\begin{array}{llllll}
\Delta\hat{m}_t = & -0.003 & +\,0.509\Delta m_{t-1} & -\,0.005\Delta R_{t-1} & +\,0.134z_{1,t-1} & +\,0.020z_{2,t-1} \\
 & (0.001) & (0.058) & (0.001) & (0.084) & (0.023) \\
 & -\,0.035z_{3,t-1} & +\,F_t & \times(-0.633z_{1,t-1} & -\,0.133z_{2,t-1} & +\,0.019z_{3,t-1}) \\
 & (0.016) & & (0.236) & (0.063) & (0.034)
\end{array}$$

$$F_t = \{1 + \exp[-2.85(\tilde{d}_{t-1} - 0.089)/\hat{\sigma}(\tilde{d}_{t-1})]\}^{-1}$$

$$\begin{array}{llllll}
\Delta\hat{R}_t = & 0.11 & +\,0.37\Delta R_{t-1} & +\,10.12\Delta\pi_{t-1} & -\,44.13\Delta c_{t-2} & -\,5.02\Delta i_{t-2} \\
 & (0.07) & (0.07) & (4.16) & (13.60) & (2.40) \\
 & +\,39.31\Delta y_{t-2} & -\,0.40\Delta R_{t-2} & +\,14.93\Delta\pi_{t-2} & +\,29.10\Delta m_{t-3} & +\,0.28\Delta R_{t-3} \\
 & (11.56) & (0.07) & (4.61) & (6.03) & (0.07) \\
 & +\,17.96\Delta\pi_{t-3} & -\,31.82\Delta m_{t-4} & -\,1.40z_{1,t-1} & +\,0.40z_{2,t-1} & +\,0.52z_{3,t-1} \\
 & (4.28) & (5.85) & (2.72) & (1.01) & (0.92) \\
 & +\,F_t & \times(15.57z_{1,t-1} & +\,3.92z_{2,t-1} & -\,5.52z_{3,t-1}) & \\
 & & (7.01) & (1.77) & (1.37) &
\end{array}$$

$$F_t = [1 + \exp\{-2329(\tilde{d}_{t-1} - 0.07)/\hat{\sigma}(\tilde{d}_{t-1})\}]^{-1}$$

$$\begin{array}{llllll}
\Delta\hat{\pi}_t = & -\,0.0004 & +\,0.004\Delta R_{t-1} & -\,0.570\Delta\pi_{t-1} & -\,0.332\Delta\pi_{t-2} & -\,0.273\Delta\pi_{t-3} \\
 & (0.0009) & (0.001) & (0.072) & (0.079) & (0.070) \\
 & +\,0.195\Delta y_{t-4} & -\,0.088z_{1,t-1} & -\,0.115z_{2,t-1} & -\,0.047z_{3,t-1} & \\
 & (0.073) & (0.120) & (0.079) & (0.067) & \\
 & +\,F_t & \times(0.209z_{1,t-1} & +\,0.191z_{2,t-1} & +\,0.041z_{3,t-1}) & \\
 & & (0.199) & (0.065) & (0.077) &
\end{array}$$

$$F_t = 1 - \exp[-0.86(\tilde{d}_{t-1} + 0.014)^2/\hat{\sigma}^2(\tilde{d}_{t-1})]$$

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