The real solutions of $x = a^x$

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1. *Introduction*

We denote the real logarithm of a positive number a by $\ln a$, so that $a^x = \exp(x \ln a)$, and we shall discuss what is known about the real solutions x of the equation

$$x = a^x, \qquad a > 0. \tag{1}$$

First, as $\exp t > 0$ for all real t, each real solution x of (1) is positive. Just over fifty years ago in [1] L. J. Stroud showed (correctly) that (1) has a real solution x if, and only if, $0 < a \leqslant e^{1/e}$, even though there is an error in his analysis of (1). Next, for some values of a, (1) may have more than one solution; for example, if $a = \sqrt{2}$ then (1) has solutions $x = 2$ and $x = 4$. Finally, our analysis suggests that it is difficult to understand fully even the real solutions of (1) without invoking complex analysis, so our discussion illustrates Hadamard's famous assertion that 'the shortest path between two truths in the real domain passes through the complex domain'.

In this Article we present a different approach to that in [1]. Briefly, for all a for which solution x exists we let $y = \ln x$. Then $y \exp(-y) = \ln a$, so that $\ln a$ is expressible as a power series in $\ln x$. In essence, we shall 'invert' this power series, and so express $\ln x$, and therefore also x, as a power series in $\ln a$. For each such a we let $b = \ln(1/a) = -\ln a$, and we shall show that if $1/e^e \leqslant a \leqslant e^{1/e}$, then (1) has a solution x that may be expressed as power series in $\ln a$, namely

$$x = \sum_{m=0}^{\infty} \frac{[-(m+1)]^m}{(m+1)!} b^m = 1 - b + \frac{3}{2}b^2 - \frac{8}{3}b^3 + \frac{125}{24}b^4 + O(b^5). \tag{2}$$

For example, a calculation shows that if $a = 1.1$, then $b = -0.095310\ldots$ and, if x is calculated using only the terms up to and including b^4, we obtain $x = 1.1117$, and $a^x = 1.1118$. We also show that if $0 < a < 1/e^{1/e}$, then (1) has a solution x that be expressed as an infinite series whose terms are explicit (but more complicated) functions of b. This dichotomy arises because the power series in (8) has radius of convergence $1/e$.

As much of the earlier work on this topic comes from a different direction from ours (actually, from repeated exponentiation), we shall begin by discussing Stroud's paper. We then give the proof of our result, and we end the article with a brief historical survey of the origins of our method, and of the so-called Lambert W function. Although some properties of the real, and the complex, solutions of (1) have been available since the time of Lambert (1728–1777) and Euler (1707–1783), one of the results stated above is derived from information that has only recently been found.

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2. *The existence of solutions of* $x = a^x$

For completeness, we briefly describe the proof (taken from [1]) of the existence of solutions of (1) when $0 < a \leqslant e^{1/e}$. First, as any solution x is positive, (1) is equivalent to the equation $\ln x = x \ln a$, and the graph of $(\ln x)/x$ is illustrated in Figure 1 (in which the axes have different scales).

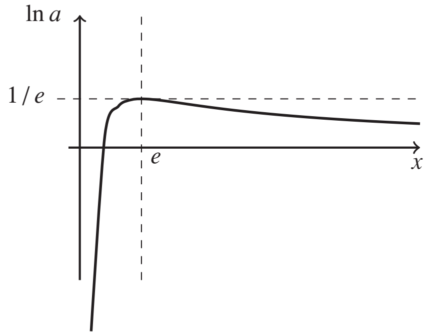

FIGURE 1: The graph of $y = (\ln x)/x$

This graph shows quite clearly that

(i) if $0 < a \leqslant 1$ then (1) has a unique positive solution;

(ii) if $1 < a \leqslant e^{1/e}$ then (1) has two positive solutions;

(iii) if $a > e^{1/e}$ then (1) has no solution.

In Case (ii) where (1) has two solutions, say u and v, we have $u = a^u$ and $v = a^v$. Then $u^{1/u} = a = v^{1/v}$ so that $u^v = v^u$. Conversely, if $u = a^u$ and $u^v = v^u$, then $a = u^{1/u} = v^{1/v}$ so that we also have $v = a^v$. In fact, the equation $x^y = y^x$ has also been studied since the time of Euler, and for more information on solutions of this equation see the recent article [2] in the *Gazette* and the references therein. Note that if $x^y = y^x$, then with $a = y^{1/y}$ we have $x = a^x$ so that if we think of x as a function of y, then x is a power series in $y^{-1} \ln y$.

In the article [3], the equation

$$a^{a^{\cdot^{\cdot^{\cdot}}}} = 8 \tag{3}$$

(in the variable a) is discussed. Clearly if a is a solution of this equation then $a^8 = 8$, so the only possible solution is $8^{1/8}$. However, it is shown in [3] that even though the left-hand side of (3) converges when $a = 8^{1/8}$, it does *not* converge to 8, so there are no (real) solutions of (3). The explanation of this is as follows. There is a number r with $8^r = r^8$ (and r is approximately 1.462). Now if a satisfies (3) then the only possible value of a is given by $a = 8^{1/8} = r^{1/r} = 1.2968\ldots$. With this value of a we have both $r = a^r$ and $8 = a^8$, and the left-hand side of (3) converges to r. In conclusion,

$$a = r^{1/r} = 8^{1/8}, \qquad a^{a^{\cdot^{\cdot^{\cdot}}}} = r \neq 8.$$

3. *Approximate solutions of* (1)

Having dealt with the question of existence, Stroud turned his attention to the problem of finding the smallest positive solution x of (1) when $0 < a \leqslant e^{1/e}$. Nowadays the problem of finding an approximate numerical solution of (1) is a straightforward exercise on a computer, so here we focus on a more theoretical discussion. Now if $x = a^x$, then $x = a^x = a^{(a^x)} = \dots$. With this in mind, Stroud asserted that the sequence

$$a,\ a^a,\ a^{(a^a)},\ \dots \tag{4}$$

(defined inductively by $a_1 = a$ and $a_{n+1} = a^{a_n}$) converges to the smallest solution x of (1), and then proceeded to give a proof of this. Unfortunately, his proof contains an error, as it must since the assertion that the sequence (4) converges when $0 < a \leqslant e^{1/e}$ is itself false. Indeed, Euler [4] had already shown in 1777 that the sequence (4) is convergent if, and only if,

$$0.06598\ldots = 1/e^e \leqslant a \leqslant e^{1/e} = 1.4446\ldots, \tag{5}$$

so Stroud's proposed proof must fail, at least when $0 < a < 1/e^e$. Of course, the fact that the sequence (4) diverges when $0 < a < 1/e^e$ does not invalidate the existence of a solution of (1) for these a; it merely says that, in this case, the (unique) solution of (1) is definitely not the (non-existent) limit of the sequence in (4). We shall not discuss the error in Stroud's proof for the issues involved are carefully worked through in the case of real a in the article [3] in the *Gazette*, and in the case of complex a in [3, 5, 6, 7], although none of these refer to Stroud's paper.

4. *The Lambert function* W_0

We have seen that (1) is equivalent to the equation $\ln x = x \ln a$. From now on we shall exclude the trivial case $a = 1$; then (1) is equivalent to the equation

$$t \exp t = b, \qquad t = bx,\ b = \ln(1/a) = -\ln a. \tag{6}$$

We shall now focus on the equation $t \exp t = b$, and this is where complex analysis enters the discussion for it raises the more general question of solving the equation $z \exp z = w$ in the complex variable z, where w is a given complex number, and this takes us back to Lambert and Euler. The function E, where $E(z) = z \exp z$, is holomorphic throughout the complex plane $\mathbb{C}$, and (since exp is never zero) we see that $E(z) = 0$ if, and only if, $z = 0$. However, some deep complex analysis (which we shall not use, and which we omit here) shows that for all *non-zero* w the equation $z \exp z = w$ has *infinitely many complex solutions* z. Once we know this we must acknowledge that, by restricting ourselves to the positive solutions of (1), we will only ever see a fraction of a much bigger picture and, inevitably, much will be hidden from us. For example, although $x = (\sqrt{2})^x$ has two real solutions, it actually has infinitely many complex solutions.

Nevertheless, let us continue with the original problem, and consider the graph of $x \to x \exp x$; see Figure 2. As $E'(x) = (x + 1) \exp x$, we see that

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the function E is a strictly increasing map of $[-1, +\infty)$ onto $[-1/e, +\infty)$, and it follows from this that this strictly increasing map has an inverse which is a strictly increasing map of $[-1/e, +\infty)$ onto $[-1, +\infty)$. By convention, this latter map is denoted by W_0. Obviously the graph of W_0 is obtained by reflecting the 'solid' part of the curve in Figure 2 in the line $y = x$. It follows that, for every real number b with $b \geqslant -1/e$ (equivalently, $a \leqslant e^{1/e}$), there is a unique t with $t \geqslant -1$ such that $E(t) = b$ or, equivalently, $W_0(b) = t$. In other words, $W_0(b)$ is a *solution* of $t \exp t = b$. It then follows that $x = a^x$, where

$$x = \frac{W_0(\ln \frac{1}{a})}{\ln \frac{1}{a}} = \frac{W_0(b)}{b}. \qquad (7)$$

Unfortunately, there is no simple 'closed' formula for the function W_0, and this is why we must resort to approximate solutions. However, although there is no 'closed' formula for W_0, if we use standard methods in complex analysis we can express W_0 as a Taylor series about 0, and this gives us the 'theoretical' solution (2) to our problem. First, we have the following result.

Theorem 1: Suppose that z is a complex number, and $|z| < \frac{1}{e}$. Then

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n, \qquad (8)$$

where this power series has radius of convergence $\frac{1}{e}$.

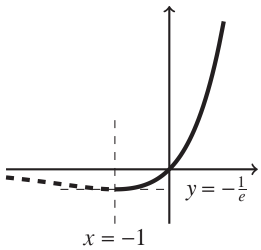

FIGURE 2: The graph of the function $y = x \exp x$

This result is classical but, for a simple proof, see [8]. Let us briefly explain the ideas behind Theorem 1 in terms of complex analysis. First, a simple application of the ratio test shows that this series has radius of convergence $1/e$. As $E(0) = 0$, the non-constant holomorphic function E maps a neighbourhood $\mathcal{N}$ of 0 (in its domain) onto the neighbourhood $E(\mathcal{N})$ of 0 (in its codomain). As $E'(0) = 1$, we can choose $\mathcal{N}$ so that the restriction of E to $\mathcal{N}$ is a (bijective) conformal map which (necessarily) *has a holomorphic inverse* which (for the moment) we denote by $E^{-1} : E(\mathcal{N}) \to \mathcal{N}$. Now the holomorphic function E^{-1} has a Taylor expansion at 0 and, after some work (and recalling that W_0 is the inverse of E in the real setting), we find that we can identify E^{-1} with W_0, and thereby obtain the Taylor expansion of W_0 given in Theorem 1. We omit the details (see [8, 9]). It now follows that, by virtue of (6), (7) and (8), for each a in the interval

$\left[1/e^{1/e}, e^{1/e}\right]$, there is a positive solution x of $x = a^x$ that is given by (2).

There remains the question of what happens in those cases in which $0 < a < 1/e^{1/e}$, or (equivalently) $b > 1/e$. In these cases the Taylor series in (8) diverges so we need a different approach. In fact, there is another (more complicated) expression for W_0 available, but with the added advantage that it is valid for all real b with $b > -1/e$; equivalently, for all a in the interval $\left(0, e^{1/e}\right)$ which is the entire range of a that is of interest to us. This formula (which is proved by using conformal mapping; see [8]) is as follows.

Theorem 2: Suppose that $b \in \left(-1/e, +\infty\right)$; then

$$W_0(b) = \sum_{m=1}^{\infty} c_m \left(\frac{\sqrt{eb+1}-1}{\sqrt{eb+1}+1}\right)^m, \tag{9}$$

where

$$c_m = \sum_{n=1}^{m} \frac{(-n)^{n-1}}{n!} \left(\frac{4}{e}\right)^n \binom{m+n-1}{m-n}.$$

Exactly as before we find that if $0 < a < e^{1/e}$, then $x = a^x$, where $x = \dfrac{W_0(b)}{b}$, $b = \ln\left(\frac{1}{a}\right)$ and $W_0(b)$ is given by (9).

5. *Some historical remarks*

Johann Heinrich Lambert was born on 26 August 1728, and died on 25 September 1777. He did important work in number theory (he was the first to prove that π is irrational), and in non-Euclidean geometry (the Lambert quadrilateral is important in the study of the parallel postulate), and also in statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. In 1758 Lambert considered the equation $x = q + x^m$, and his ideas were taken further by Leonhard Euler (1707–1783). Eventually their work led to what is now called the *Lambert W function* (namely the multi-valued inverse of the holomorphic function $z \to z \exp z$), and the function W_0 which we have used above is the principal branch of this function. Because of the power of modern computers, the Lambert W function is now an important part of mathematics, and it has applications in, for example, acoustics, astrophysics, biochemistry, biology, ecology, electronics, engineering, geology, geophysics, general relativity, graph theory, information theory, optics, particle physics, radiation, risk theory, stellar structures and technological systems. Indeed, it is argued by some that the function W should be studied alongside the complex logarithm, trigonometric functions, and the other so-called elementary functions. For more information about the Lambert W function, we recommend [9].

We end with a simple example to illustrate the use of the Lambert W function in another situation. The first mathematical model of population growth of a species ever considered was given by Thomas Robert Malthus in 1798 in his essay *An essay on the principle of population*, and was essentially as follows. Let $P(t)$ be the population of the species at time t, and let us make the simple (but unrealistic) assumption that $P'(t) = kP(t)$, where k is a positive constant. A more realistic model is one in which individuals can only give birth to new individuals after a fixed period of time, say t_0, has elapsed (so t_0 is the time to maturity), and this leads to what is known as a *delay differential equation*, say $P'(t) = kP(t - t_0)$. If we assume that this differential equation has a solution $P(t) = Ae^{\lambda t}$, we find that $\lambda e^{\lambda t_0} = k$; thus $E(\lambda t_0) = kt_0$ or, equivalently, $\lambda W_0(kt_0)/t_0$.

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