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Branched Covers of Tangles in Three-balls

Makiko Ishiwata, Józef H. Przytycki and Akira Yasuhara

Abstract. We give an algorithm for a surgery description of a p-fold cyclic branched cover of B^3 branched along a tangle. We generalize constructions of Montesinos and Akbulut-Kirby.

Tangles were first studied by Conway [4]. They were particularly useful for analyzing prime and hyperbolic knots. A branched cover of the three-ball branched along a tangle (succinctly a branched cover of a tangle) is an indispensable tool for understanding tangles. Hence it is important to give practical presentations of branched covers of tangles. Recall that a p-fold cyclic branched cover of a link or tangle (oriented for $p > 2$) is uniquely defined by an epimorphism of the fundamental group of the complement onto Z_p which sends meridians to 1. A p-fold branched cover of an n-tangle is a three-manifold, the boundary of which is a connected surface of genus $(n-1)(p-1)$. Such a manifold can be obtained from the genus $(n-1)(p-1)$ handlebody by a surgery. We provide an algorithm for a surgery description of a p-fold cyclic branched cover of B^3 branched along a tangle. The construction generalizes that of Montesinos [9] and Akbulut and Kirby [1]. It is strikingly simple in the case of a two-fold branched cover. We also discuss the related Heegaard decomposition of a p-fold branched cover of an n-tangle.

1 Surgery Descriptions

A *tangle* is a one-manifold properly embedded in a three-ball. An n-tangle is a tangle with $2n$ boundary points. Let T be an n-tangle and T_0 a trivial n-tangle diagram[1] (Figure 1). Let $D_1 \cup \cdots \cup D_n$ be a disjoint union of disks bounded by T_0 and let $b_1, \ldots, b_m$ be mutually disjoint disks in B^3 such that $b_i \cap \bigcup_j D_j = \partial b_i \cap T_0$ are two disjoint arcs in $\partial b_i (i = 1, \ldots, m)$ (see Figure 2). We denote by $\Omega(T_0; \{D_1, \ldots, D_n\}, \{b_1, \ldots, b_m\})$ the tangle $T_0 \cup \bigcup_i \partial b_i - \operatorname{int}(T_0 \cap \bigcup_i \partial b_i)$ and call it a *disk-band representation* of a tangle. A disk-band representation is called *bicollared* if the surface $\bigcup_i D_i \cup \bigcup_j b_j$ is orientable. We will see that any n-tangle has a bicollared disk-band representation (Proposition 5).

A *framed link* is a disjoint union of embedded annuli in a three-manifold. Framed links in S^3 can be identified with links whose each component is assigned an integer. Such links are also called *framed links*. Let M be a three-manifold and $\mathcal{L}$ a framed link in M. We denote by $\Sigma(\mathcal{L}, M)$ the manifold obtained from M by the surgery along $\mathcal{L}$ [10].

[1]Tangles are considered up to ambient isotopy but in practice we will often use the word tangle for a tangle diagram or an actual embedding of a one-manifold.

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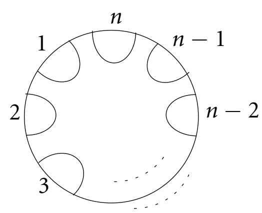

Figure 1

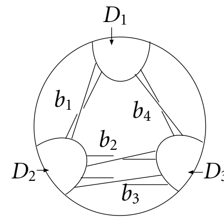

Figure 2

The case of two-fold branched covers is easy to visualize so we will formulate it first.

Theorem 1 *Let $\Omega(T_0; \{D_1, \dots, D_n\}, \{b_1, \dots, b_m\})$ be a disk-band representation of an n-tangle T in B^3. Let $\varphi\colon H_0 \to B^3$ be the two-fold branched cover of B^3 by a genus $n-1$ handlebody H_0 branched along T_0. Then the two-fold branched cover of B^3 branched along T has a surgery description $\Sigma\big(\varphi^{-1}(\bigcup_i b_i), H_0\big)$ (see Figure 3).*

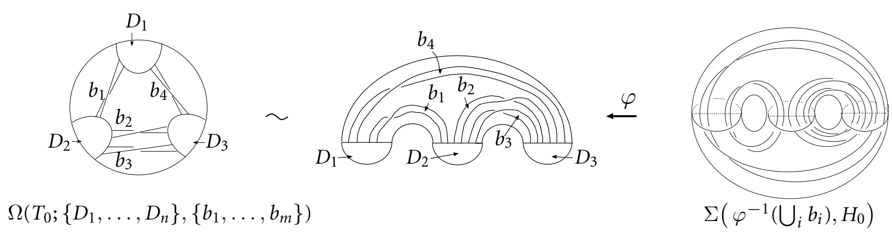

Figure 3

Proof Let X be $B^3 - \bigcup_i D_i$ compactified with two copies, $D_i^{\pm}$, of D_i ($i = 1, 2, \dots, n$) (Figure 4). Let X_1 and X_2 be two copies of X, and let $D_{i,k}^{\pm} \subset X_k$ denote copies of $D_i^{\pm}$ ($i = 1, 2, \dots, n, k = 1, 2$) (Figure 5). Then H_0 is obtained from $X_1 \cup X_2$ by identifying $D_{i,1}^{\varepsilon}$ with $D_{i,2}^{-\varepsilon}$ ($\varepsilon \in \{-, +\}$). Let $b_{j,k} = \varphi^{-1}(b_j) \cap X_k$ and let Y be $H_0 - \bigcup_{j,k} b_{j,k}$ compactified with two copies $b_{j,k}^{\pm}$ of $b_{j,k}$ in $X_k (j = 1, 2, \dots, m, k = 1, 2)$. Here, $+$ or $-$ sides of $D_{i,k}$ and $b_{j,k}$ are not necessarily compatible. We note, and it is the key observation of the construction, that the two-fold branched cover H of B^3 branched along T is obtained from Y by identifying $b_{j,1}^{\varepsilon}$ with $b_{j,2}^{-\varepsilon}$ ($\varepsilon \in \{-, +\}$). Note that each $b_{j,1}^{+} \cup b_{j,2}^{-} \cup b_{j,1}^{-} \cup b_{j,2}^{+}$ is a torus. Let c_j be the core of the annulus $b_{j,1}^{+} \cup b_{j,2}^{-}$. The manifold obtained from Y by identifying $b_{j,1}^{\varepsilon}$ with $b_{j,2}^{-\varepsilon}$ is homeomorphic to the one obtained from Y by attaching tori $D_j^2 \times S^1$ ($j = 1, 2, \dots, m$) so that $\partial D_j^2 = c_j$. Hence H is homeomorphic to the manifold with the surgery description $\Sigma\big(\varphi^{-1}(\bigcup_j b_j), H_0\big)$. ∎

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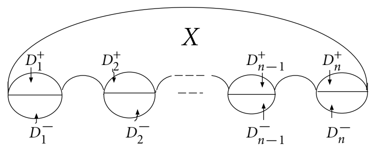

Figure 4

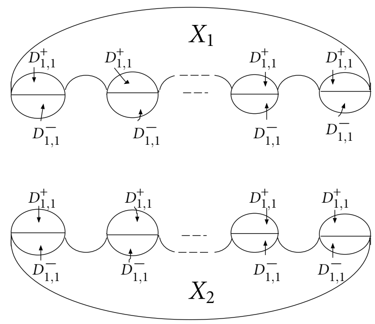

Figure 5

Example 2 (a) The two-fold branched cover $M^{(2)}(T_1)$ branched along a tangle T_1 in Figure 6 is the Seifert manifold with the base a disk and two special fibers of type $(2,1)$ and $(2,-1)$. Furthermore $M^{(2)}(T_1)$ is a twisted I-bundle over the Klein bottle (for example see [7]). In particular, $\pi_1\big(M^{(2)}(T_1)\big) = \langle a, b \mid aba^{-1}b = 1\rangle$.

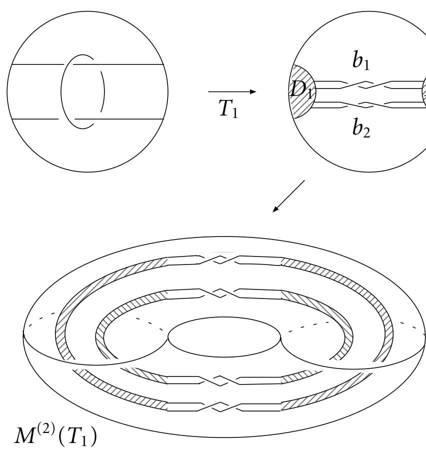

Figure 6

(b) If we glue together two copies of T_1 as in Figure 7, we get Borromean rings L. Thus our previous computation shows that the two-fold branched cover $M^{(2)}(L)$ of S^3 branched along L is a "switched" double of the twisted I-bundle over the Klein bottle (see Figure 8 for a surgery description). The fundamental group $\pi_1\big(M^{(2)}(L)\big) = \langle x, a \mid x^2ax^2a^{-1}, a^2xa^2x^{-1}\rangle$ is a three-manifold group which is torsion-free but not left orderable [11].

(c) If we take the double of the tangle T_1, we obtain the link in Figure 9. The two-fold branched cover of S^3 branched along this link is the double of twisted I-bundle over Klein bottle. A surgery description of this manifold is shown in Figure 10. Thus this manifold is the Seifert manifold with the base S^2 and four special fibers of type

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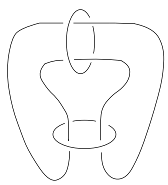

Figure 7

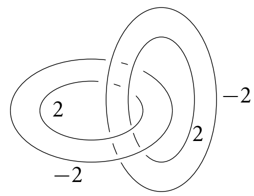

Figure 8

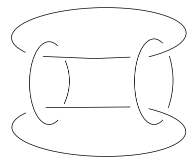

Figure 9

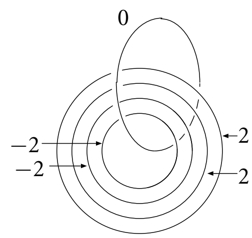

Figure 10

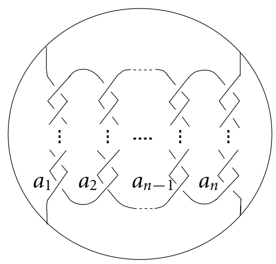

Figure 11

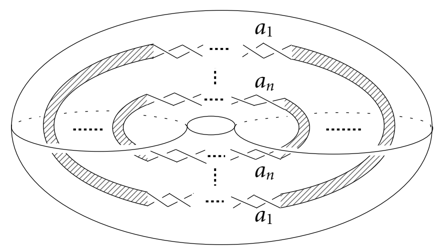

Figure 12

$(2,1), (2,1), (2,-1)$ and $(2,-1)$. This manifold also has another Seifert fibration, which is a circle bundle over the Klein bottle.

Example 2(a) was motivated by the fact that the tangle T_1 yields a virtual Lagrangian of index 2 in the symplectic space of the Fox $\mathbf{Z}$-colorings of the boundary of our tangle [5].

More generally we have:

Example 3 Consider a tangle T_2 in Figure 11, called a *pretzel tangle* of type $(a_1, a_2, \dots, a_n)$, where each a_i is an integer indicating the number of half-twists ($i = 1, 2, \dots, n$). The two-fold branched cover $M^{(2)}(T_2)$ branched along the tangle T_2 is a Seifert fibered manifold with the base a disk and n special fibers of type $(a_1, 1), (a_2, 1), \dots, (a_n, 1)$ (Figure 12).

Theorem 1 can be generalized to a p-fold cyclic branched cover assuming that an n-tangle is oriented whose disk-band representation is bicollared, where p is any positive integer greater than 2. We proceed as follows:

Let $T = \Omega(T_0; \{D_1, \dots, D_n\}, \{b_1, \dots, b_m\})$ be a bicollared disk-band representation of an n-tangle. Then $\bigcup_i D_i \cup \bigcup_j b_j$ has a bicollar neighborhood $(\bigcup_i D_i \cup \bigcup_j b_j) \times [-1, 1]$. Let $X = \overline{B^3 - \big((\bigcup_i D_i) \times [-1, 1]\big)}$ and $D_i^{\pm} = (D_i \times [\pm 1, 0]) \cap \partial X$. Let X_k be a copy of X and $D^{\pm}_{i,k} \subset \partial X_k$ a copy of $D^{\pm}_i (k = 1, 2, \dots, p)$. Then the p-fold cyclic branched cover $\varphi \colon H_0 \to B^3$ branched along T_0 is obtained from $X_1 \cup \dots \cup X_p$ by identifying $D^{+}_{i,k}$ with $D^{-}_{i,k+1} (k = 1, \dots, p)$, where k is considered modulo p. Let $b^{\pm}_{j,k} = \varphi^{-1}(b_j \times \{\pm 1\}) \cap X_k$. Note that each $b^{+}_{j,k} \cup b^{-}_{j,k+1}$ is an annulus in H_0 for any

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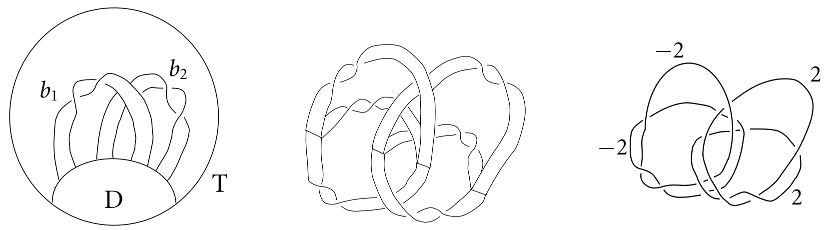

(a) (b) (c)

Figure 13

j and k. Then we obtain the p-fold cyclic branched cover of B^3 branched along T in a similar way as in Theorem 1.

Theorem 4 *Let $\Omega(T_0; \{D_1, \dots, D_n\}, \{b_1, \dots, b_m\})$ be a bicollared disk-band representation of an n-tangle T in B^3. Then $\Sigma\Big(\bigcup_{j=1}^{m}\big(\bigcup_{k=1}^{p-1}(b^+_{j,k} \cup b^-_{j,k+1})\big), H_0\Big)$ is the p-fold cyclic branched cover of B^3 branched along T.*

Note that we do not use the annuli $b^+_{j,p+1} \cup b^-_{j,1}$ $(j = 1, 2, \dots, m)$ in the theorem above. In fact the cores of these annuli bound mutually disjoint 2-disks in $\Sigma\Big(\bigcup_{j=1}^{m}\big(\bigcup_{k=1}^{p-1}(b^+_{j,k} \cup b^-_{j,k+1})\big), H_0\Big)$.

Proof Let $Y = \overline{H_0 - \bigcup_j \varphi^{-1}(b_j \times [-1, 1])}$, $V^{\pm}_{j,k} = \varphi^{-1}(b_j \times [\pm 1, 0]) \cap X_k$ and $\beta^{\pm}_{j,k} = V^{\pm}_{j,k} \cap Y (= \partial V^{\pm}_{j,k} \cap \partial Y)$. Note that $\varphi^{-1}(b_j \times [-1, 1])$ is a genus $p - 1$ handlebody. Then the p-fold cyclic branched cover of B^3 branched along T is homeomorphic to a manifold H that is obtained from Y by identifying $\beta^+_{j,k}$ with $\beta^-_{j,k+1} (k = 1, \dots, p)$, where k is taken modulo p. Moreover H is homeomorphic to a manifold obtained from $\overline{H_0 - \bigcup_j \varphi^{-1}(b_j \times [-1, 1])} \cup \bigcup_j \big(V^-_{j,1} \cup V^+_{j,p} \cup \varphi^{-1}(b_j \times \{0\})\big)$ by identifying $\beta^+_{j,k}$ and $b_{j,k}$ with $\beta^-_{j,k+1}$ and $b_{j,k+1}$ $(j = 1, \dots, m, k = 1, \dots, p - 1)$ respectively, where $b_{j,k} = \varphi^{-1}(b_j \times \{0\}) \cap X_k$. Note that $\beta^+_{j,k} \cup b_{j,k} \cup \beta^-_{j,k+1} \cup b_{j,k+1}$ is a torus. By an argument similar to that in the proof of Theorem 1, we have the required result. ∎

Example 5 Let $T = \Omega(T_0; \{D\}, \{b_1, b_2\})$ be a tangle as in Figure 13(a) and $\varphi \colon H_0 \to B^3$ the three-fold cyclic branched cover of B^3 branched along T_0. Note that H_0 is a three-ball and $\varphi^{-1}(b_1 \cup b_2)$ is as shown in Figure 13(b). By Theorem 4, the three-fold cyclic branched cover of B^3 branched along T is obtained from H_0 by the surgery along a framed link in Figure 13(c). Note that Figure 13(c) is ambient isotopic to Figure 14(a). Since the figure eight knot has a tangle decomposition into T and a trivial 1-tangle, the three-fold cyclic branched cover $M^{(3)}(4_1)$ of S^3 branched along the figure eight knot has a surgery description shown in Figure 14(a). The framed link in Figure 14(a) can be deformed into the link in Figure 14(b) by an ambient isotopy and a second Kirby move. The link in Figure 14(b) is ambient isotopic to the

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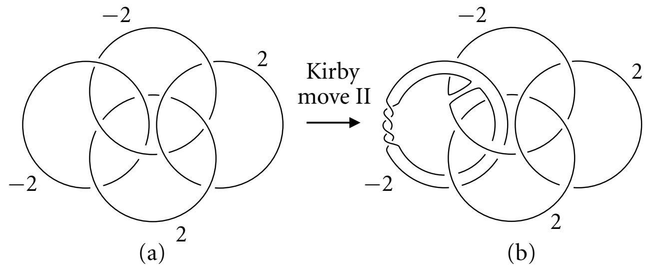


Figure 14

link in Figure 8. Hence $M^{(3)}(4_1)$ is homeomorphic to the two-fold branched cover of S^3 branched along the Borromean rings [8, 6] (*cf.* Example 2(b)).

Proposition 6 *Any n-tangle* (B^3, T) *has a (bicollared) disk-band representation.*

Proof We attach a trivial n-tangle T_0 to the n-tangle T by a homeomorphism $\varphi\colon \partial(B, T_0) \to \partial(B, T)$. We obtain a link $L = T_0 \cup T$ in a three-sphere $B \bigcup_\varphi B$ with a diagram $D(T_0 \cup T)$ as in Figure 15. We may assume that the diagram $D(T_0 \cup T)$ is connected. We color, in checkerboard fashion, the regions of the plane cut by the diagram $D(T_0 \cup T)$ and choose n points $\{v_1, v_2, \dots, v_n\}$ as in Figure 16. Since $D(T_0 \cup T)$ is connected, there is a spine G of the black surface with the vertex set $V(G)$ containing $\{v_1, v_2, \dots, v_n\}$. Deforming G on the surface by edge contractions, we have a new spine H with $V(H) = \{v_1, v_2, \dots, v_n\}$. By retracting the black regions into the neighborhood of H and restricting to B^3, we have a required surface. For an example, see Figure 17.

When we use the Seifert algorithm instead of checkerboard coloring, we always obtain a bicollared disk-band representation. ■

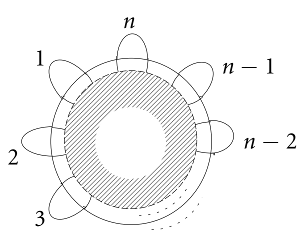


Figure 15

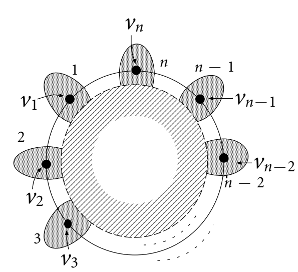


Figure 16

2 Heegaard Decompositions

In addition to the surgery presentation, it is also useful to have another presentation of a p-fold cyclic branched cover. Our construction leads straightforwardly to a Heegaard decomposition, that is a decomposition into a *compression body* [2, 3] and a handlebody, of a p-fold cyclic branched cover.

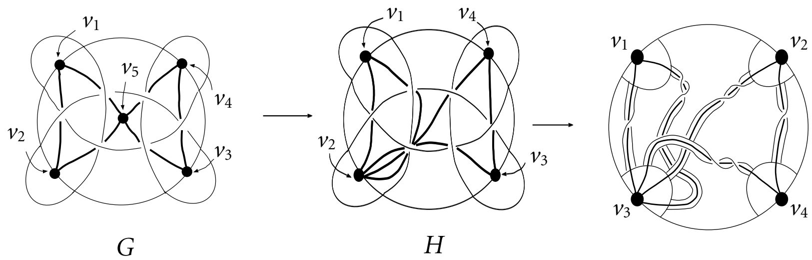
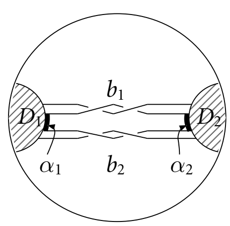
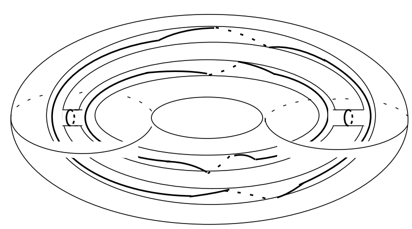


Figure 17

Let F be a connected surface in B^3 bounded by an n-tangle T and n arcs in ∂B^3. The surface F is defined to be *free* if the exterior of F is homeomorphic to $(S_n \times I) \cup$ (1-handles), where S_n is an n-punctured sphere and the all attaching points of the 1-handles are contained in $S_n \times \{0\}$. As we observed before, any connected surface has disk-band decomposition. A disk-band representation $\Omega(T_0; \{D_1, \ldots, D_n\}, \{b_1, \ldots, b_m\})$ is defined to be *free* if the surface $\bigcup_i D_i \cup \bigcup_j b_j$ is free. It is obvious that any disk-band representations constructed as in the proof of Proposition 6 are free. So any n-tangle has a free disk-band representation.

First we consider the case of two-fold branched covers. The following algorithm gives a Heegaard decomposition of $M^{(2)}(T)$. Start from a free disk-band representation of T. Then we have a surgery description $\Sigma\big(\varphi^{-1}(\bigcup_i b_i), H_0\big)$ of $M^{(2)}(T)$ (Figure 3). Let $\alpha_1, \alpha_2, \ldots, \alpha_{2m-n}$ be the $2m-n$ arcs which are the connected components of $T_0 - \bigcup_j b_j$ contained in the interior of B^3 (Figure 18). Then the complement of $\varphi^{-1}(\bigcup_i b_i \bigcup_l \alpha_l)$ is a compression body. Then $M^{(2)}(T)$ is obtained from the compression body by gluing the handlebody as follows: (i) adding meridian disks of the arcs, and (ii) filling the rest according to the surgery description. For the tangle T_1 in Example 2, a Heegaard decomposition of $M^{(2)}(T_1)$ is given in Figure 19.





Heegaard decomposition of $M^{(2)}(T_1)$

Figure 18

Figure 19

A similar method gives a Heegaard decomposition of a p-fold cyclic branched cover. Our construction is a modification of the construction in the proof of Theorem 4. The handlebody part of the decomposition is obtained from the handlebodies $\bigcup_j \varphi^{-1}(b_j \times [-1, 1])$ by connecting them using $2m - n$ "tubes" along $\varphi^{-1}(T_0)$. We get genus $mp - n + 1$ handlebody.

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From the observation above it follows that:

Theorem 7 *Let $\Omega(T_0; \{D_1, \dots, D_n\}, \{b_1, \dots, b_m\})$ be a free, bicollared disk-band representation of an n-tangle T in B^3. Let $\varphi\colon H_0 \to B^3$ be the p-fold cyclic branched cover of branched along T_0. Let $\alpha_1, \alpha_2, \dots, \alpha_{2m-n}$ be the $2m-n$ arcs which are the connected components of $T_0 - \bigcup_j b_j$ contained in the interior of B^3. Then the following holds.*

(a) *The complement $W = \overline{H_0 - \varphi^{-1}\big(\bigcup_j b_j \times [-1,1] \cup \bigcup_l N(\alpha_l)\big)}$ is a compression body, where $N(\alpha_l)$ is the tubular neighborhood of α_l in B^3.*

(b) *The p-fold cyclic branched cover of B^3 branched along T has a Heegaard decomposition into the compression body W and a genus $mp - n + 1$ handlebody.*

(c) *The gluing map is given by the curves $c_{j,k}$ $(j = 1, 2, \dots, m, k = 1, 2, \dots, p-1)$ and m_l $(l = 1, 2, \dots, 2m-n)$ in ∂W, where $c_{j,k}$ is the core of the annulus $b^+_{j,k} \cup b^-_{j,k+1}$ in Theorem 4 and m_l is the meridian curve of $\varphi^{-1}\big(N(\alpha_l)\big)$.* ■

In the theorem above, the assumption that a disk-band representation is bicollared is not necessary in the case that $p = 2$. The curves $c_{j,k}$ $(j = 1, 2, \dots, m, k = 1, 2, \dots, p-1)$, m_l $(l = 1, 2, \dots, 2m-n)$ are essential, $mp - n + 1$ of them are nonseparating and $m - 1$ curves, the m_l s, are separating.

Remark 8 Since the surfaces given in the proof of Proposition 6 are connected and free, we can use them to find Heegaard decompositions of branched cyclic covers. Let c denote the crossing number of a connected diagram $D(T_0 \cup T)$, b the number of the black regions and s the number of the Seifert circles of $D(T_0 \cup T)$. Then we have a Heegaard decomposition of $M^{(2)}(T)$ (resp. $M^{(p)}(T)$) of the genus $n + 2c - 2b + 1$ (resp. $p(n + c - s) - n + 1$).

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Department of Mathematics
Tokyo Woman's Christian University
Zempukuji 2-6-1, Suginamiku
Tokyo 167-8585
Japan
e-mail: mako@twcu.ac.jp

Department of Mathematics
The George Washington University
Washington, DC 20052
USA
e-mail: przytyck@gwu.edu

Department of Mathematics
Tokyo Gakugei University
Nukuikita 4-1-1, Koganei
Tokyo 184-8501
Japan
e-mail: yasuhara@u-gakugei.ac.jp