Comments on Y. O. Hamidoune's Paper 'Adding Distinct Congruence Classes'

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The main result in Y. O. Hamidoune's paper 'Adding distinct congruence classes' (*Combin. Probab. Comput.* **7** (1998) 81–87) is as follows. If S is a generating subset of a cyclic group G such that $0 \notin S$ and $|S| \geqslant 5$, then the number of sums of the subsets of S is at least $\min(|G|, 2|S|)$. Unfortunately, the argument of the author, who, sadly, passed away in 2011, relies on a lemma whose proof is incorrect; in fact, the lemma is false for all cyclic groups of even order. In this short note we point out this mistake, correct the proof, and discuss why the main result is actually true for all finite abelian groups.

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Let G be a finite abelian group, written additively. For a positive integer h and a subset A of G, we let $h\hat{}A$ denote the set of sums of the h-subsets of A:

$$h\hat{}A = \{\Sigma_{b \in B} b \mid B \subseteq A, |B| = h\}.$$

Additionally, we let ΣA denote the set of all (non-empty) subset sums of A:

$$\Sigma A = \cup_{h=1}^{|A|} h\hat{}A = \{\Sigma_{b \in B} b \mid \emptyset \neq B \subseteq A\}.$$

If G is cyclic and of order m, we identify it with $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. The main result in [3] is as follows.

Theorem 1 (Hamidoune [3]). *Let S be a generating subset of $\mathbb{Z}_m$ such that $0 \notin S$ and $|S| \geqslant 5$. Then the number of sums of the subsets of S is at least $\min(m, 2|S|)$.*

As was pointed out in [3], the result is best possible: if $m = 3k$ for some $k \geqslant 3$, then

$$S = \{3, 6, \ldots, 3(k-1)\} \cup \{1\}$$

has $|\Sigma S| = 2k$. (This example clearly generalizes to non-cyclic groups.)

The proof of Theorem 1 in [3] considers two cases: when $2|S| \leqslant m - 1$ and when $2|S| \geqslant m$. The proof provided for the first case is correct; in fact, it was delivered for an arbitrary abelian group of order m. However, the author derives the second case from the following lemma.

Lemma 2 (Hamidoune [3]). *Let A be a subset of $\mathbb{Z}_m \setminus \{0\}$ such that $2|A| \geqslant m$. Then $A \cup (2\hat{\,}A) = \mathbb{Z}_m$.*

Clearly, if Lemma 2 were true, it would immediately yield Theorem 1 in the case when $2|S| \geqslant m$. However, Lemma 2 is false for every even value of m: for example, with

$$A = \{1, 2, \ldots, m/2\},$$

we have $0 \notin A \cup (2\hat{\,}A)$. In fact, when $m \equiv 2 \bmod 4$, then there are subsets A of $\mathbb{Z}_m$ with the required properties for which $A \cup (2\hat{\,}A)$ misses two elements of $\mathbb{Z}_m$: for example, for

$$A = \{1, 2, \ldots, (m-2)/4\} \cup \{m/2, m/2 + 1, \ldots, (3m-2)/4\},$$

neither 0 nor $m/2 - 1$ is in $A \cup (2\hat{\,}A)$.

It turns out that for the conclusion of Lemma 2, one must assume that $2|A| \geqslant m + 2$. More generally, we have the following result.

Proposition 3. *Let G be a finite abelian group, and let G_2 be the subset – indeed, subgroup – of elements of order at most 2.*

(1) There is a subset $A \subseteq G \setminus \{0\}$ with $2|A| = |G| + |G_2| - 2$ for which $A \cup (2\hat{\,}A) \neq G$.
(2) If $A \subseteq G \setminus \{0\}$ satisfies $2|A| \geqslant |G| + |G_2|$, then $A \cup (2\hat{\,}A) = G$.

We should point out that $|G| + |G_2|$ is always even, hence our two statements are complementary.

Proof. To prove the first statement, partition $G \setminus G_2$ into disjoint parts K and $-K$ (with $-K$ consisting of the inverses of the elements in K). Then $A = (G_2 \setminus \{0\}) \cup K$ satisfies our requirements.

For our second statement, it suffices to prove that $(G \setminus A) \subseteq 2\hat{\,}A$. Let $g \in G \setminus A$ be arbitrary, and let

$$L_g = \{x \in G \mid 2x = g\}.$$

We show that if $L_g \neq \emptyset$, then $|L_g| = |G_2|$. To see this, we choose an element $x \in L_g$, and consider the set $x - L_g$. (Here and below, for an element z and a subset Y of G, we let $z + Y$ denote the set $\{z + y \mid y \in Y\}$ and $z - Y$ denote the set $\{z - y \mid y \in Y\}$.) Note that $x - L_g$ has size $|L_g|$ and is a subset of G_2, thus $|L_g| \leqslant |G_2|$. Similarly, $x + G_2 \subseteq L_g$, so $|G_2| \leqslant |L_g|$ as well.

Now let $A_0 = A \cup \{0\}$. Then

$$|A_0 \cap (g - A_0)| = |A_0| + |g - A_0| - |A_0 \cup (g - A_0)| \geqslant 2|A_0| - |G| \geqslant |G_2| + 2.$$

By the previous paragraph, we then must have an element $a_1 \in A_0 \cap (g - A_0)$ for which $a_1 \notin L_g$. Since $a_1 \in g - A_0$, we also have an element $a_2 \in A_0$ for which $a_1 = g - a_2$ and thus $g = a_1 + a_2$. But $a_1 \notin L_g$, and thus $a_2 \neq a_1$. Now if $a_1 = 0$, then $a_2 \neq 0$, so $g \in A$, contradicting our assumption. So $a_1 \in A$ and, similarly, $a_2 \in A$. Therefore, $g \in 2\hat{\ }A$, as claimed. □

Let us now turn to the proof of Theorem 1. We employ the following result.

Theorem 4 (Gallardo *et al.* [2]). *If $m \geqslant 12$ is even and $|A| \geqslant m/2 + 1$, then $3\hat{\ }A = \mathbb{Z}_m$.*

Proof of Theorem 1.   As we explained above, we only need to treat the case when $2|S| \geqslant m$. In the subcase when m is odd, this inequality is equivalent to $2|S| \geqslant m + 1$; since $G_2 = \{0\}$ in this subcase, the second statement of Proposition 3 implies that

$$|\Sigma S| \geqslant |S \cup (2\hat{\ }S)| = m = \min\{2|S|, m\}.$$

As the first statement of Proposition 3 shows, in the subcase when m is even, considering only $S \cup (2\hat{\ }S)$ is not sufficient. Luckily, when $m \geqslant 12$, we can take advantage of Theorem 4: with $A = S \cup \{0\}$, we have $|A| \geqslant m/2 + 1$, so

$$|\Sigma S| \geqslant |3\hat{\ }S| = m = \min\{2|S|, m\}.$$

This leaves only the cases of $m \in \{6, 8, 10\}$, which can be checked individually (or see Theorem 6 below). □

In closing, we mention the following generalization of Theorem 1.

Theorem 5. *Let S be a generating subset of an abelian group G, and suppose that $0 \notin S$ and $|S| \geqslant 5$. Then the number of sums of the subsets of S is at least $\min(|G|, 2|S|)$.*

Our proof relies on the following 1973 result on the so-called *critical number* $c(G)$ of G where

$$c(G) = \min\{s \in \mathbb{N} \mid A \subseteq G \setminus \{0\}, |A| = s \Rightarrow \Sigma A = G\}.$$

Theorem 6 (Diderrich and Mann [1]). *Let G be an abelian group of order $2k$ with $k \geqslant 2$.*
(1) If $k \geqslant 5$ or $G \cong \mathbb{Z}_2^3$, then $c(G) = k$.
(2) If $G \cong \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2^2,$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$, then $c(G) = k + 1$.

Proof Proof of Theorem 5.   As we mentioned above, the case when $2|S| \leqslant |G| - 1$ was completed in [3], so assume that $2|S| \geqslant |G|$. If $|G|$ is odd, then, as before, our claim follows from the second statement of Proposition 3. Finally, if $|G|$ is even, the claim follows from Theorem 6. □

References

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